

## A GENERAL DESCRIPTION OF TOTALLY GEODESIC FOLIATIONS

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(Received January 7, 1985)

**Abstract.** We give some general concepts and results for totally geodesic foliations  $\mathcal{F}$  on complete Riemannian manifolds. In particular, we reduce the problem to that of a generalization of the theory of principal connections. This enables us to show that the global geometry of  $\mathcal{F}$  is related to certain sheaves of germs of local Killing vector fields for the Riemannian structure along the leaves. Further, we define a cohomology group  $H_{i_0}^*$ , and natural mappings from  $H_{i_0}^*$  into the de Rham cohomologies of the leaves, such that the characteristic classes in  $H_{i_0}^*$  are mapped to the characteristic classes of the leaves.

**Introduction.** A foliation  $\mathcal{F}$ , on a Riemannian manifold  $M$ , is said to be totally geodesic if each geodesic of  $M$  is everywhere or nowhere tangent to  $\mathcal{F}$ . The different aspects of such foliations have been examined by many authors; for different approaches and more references, see for instance [14], [10], [1] and [4]. In particular, the codimension one case has been classified [12], and there is a homological classification of the dimension one case [28].

The purpose of this paper is to apply, to totally geodesic foliations of arbitrary dimension, the techniques developed in the theory of Riemannian foliations, and in particular, the works of Molino [16]  $\cdots$  [21]. Recall that a foliation  $\mathcal{F}$  on a manifold  $M$  is Riemannian if there exists a Riemannian metric on  $M$  such that  $\mathcal{F}$  can be defined by local Riemannian surmersions [23]. Equivalently,  $\mathcal{F}$  is Riemannian if  $M$  possesses a Riemannian metric whose geodesics are everywhere or nowhere perpendicular to  $\mathcal{F}$ . Riemannian foliations have been extensively studied, and in particular, there is a strong structure theorem [20]. Given the evident analogy between totally geodesic and Riemannian foliations, it is not surprising that there are many concepts and results from the Riemannian case that find similar expression in the totally geodesic situation. Indeed, by pursuing this approach one obtains a good geometric description and a useful cohomology group.

A first application of this work to the dimension  $\leq 3$  cases is given in [7]. In collaboration with E. Ghys, we have given a detailed account of totally geodesic foliations on 4-manifolds [9].

Fundamental to this study is the following result, proven in [2] and independently in [6]: if  $\mathcal{F}$  is a totally geodesic foliation on a complete Riemannian manifold  $M$ , and if  $\hat{\mathcal{F}}$  is the pull-back of  $\mathcal{F}$  to the bundle  $E$  of orthonormal frames tangent to the leaves of  $\mathcal{F}$ , then  $\hat{\mathcal{F}}$  possesses a parallelism along its leaves that preserves the distribution orthogonal to  $\hat{\mathcal{F}}$  (for the Riemannian metric lifted from  $M$  to  $E$ ). Furthermore, there is a locally trivial fibration  $\psi: E \rightarrow W$  such that the foliation induced by  $\hat{\mathcal{F}}$ , on each of the fibers of  $\psi$ , possesses a parallelism of Lie type. This situation is analogous to the case of Riemannian foliations, where one has a fibration of the bundle of orthonormal transverse frames. However, in the totally geodesic case, a Lie group  $G$  acts on the typical fiber  $N$  of  $\psi$  in such a way that  $G$  is transitive on the leaves of the foliation induced on  $N$  by  $\hat{\mathcal{F}}$ . The study of the typical fiber amounts to a generalization of the theory of connections on principal  $G$ -bundles (see Section 2). In particular, one can introduce invariants of  $\mathcal{F}$  that are analogous to such concepts as the holonomy algebra of a principal connection. Associated to these invariants, there are sheaves of germs of local Killing vector fields for the Riemannian structure along the leaves of  $\mathcal{F}$  (see Section 3).

In particular, this study gives information about the “sheets” of  $\mathcal{F}$ : if  $x$  is a point in  $M$ , the *sheet* of  $\mathcal{F}$  at  $x$  is the subset of  $M$  that can be joined to  $x$  by piecewise smooth paths perpendicular to  $\mathcal{F}$ . As a consequence of our main result (Theorem 3.1) we have the following: (cf. Corollary 3.4): *The intersections of the leaves and the closures of the sheets of  $\mathcal{F}$  are locally homogeneous Riemannian submanifolds of  $M$ .*

Analogous to the basic cohomology of a Riemannian foliation (see [24], [17], [11] and [27]), there is a cohomology group naturally associated to a totally geodesic foliation (see Section 4). We call this the *tangential cohomology*. It is constructed from the complex of forms, along the leaves, that are “invariant” in directions perpendicular to the leaves. There are homomorphisms from the tangential cohomology group into the de Rham cohomology of the leaves, and the images of the characteristic classes of the tangential cohomology are just the characteristic classes of the leaves. In particular, we give a result concerning the Euler class.

This paper elaborates and extends results announced in the short notes [6] and [8]. Finiteness and duality properties of the tangential cohomology, outlined in [8], will be proven in a separate paper.

It is a pleasure to acknowledge the assistance of Pierre Molino without whose help and encouragement this work would not have been possible. The author also thanks Etienne Ghys and Vlad Sergiescu for their friendly

advice.

In all the following,  $\mathcal{F}$  is a totally geodesic foliation on a connected complete Riemannian manifold  $(M, g)$ . The differentiable structures are  $C^\infty$ . In the last section we suppose that  $\mathcal{F}$  is orientable.

**1. Preliminaries.** Let  $T(M)$  be the tangent bundle of  $M$ , let  $A^*(M, \mathbf{R})$  be the complex of real valued forms on  $M$ , and let  $\mathcal{L}(M)$  be the  $A^0(M, \mathbf{R})$ -module of vector fields on  $M$ . The vector subbundle  $V(M, \mathcal{F})$  (resp.  $H(M, \mathcal{F})$ ) of  $T(M)$ , tangent (resp. orthogonal) to the leaves of  $\mathcal{F}$ , is called the *vertical* (resp. *horizontal*) *bundle*. The  $A^0(M, \mathbf{R})$ -module of sections of  $V(M, \mathcal{F})$  (resp.  $H(M, \mathcal{F})$ ) will be denoted  $\mathcal{L}_V(M, \mathcal{F})$  (resp.  $\mathcal{L}_H(M, \mathcal{F})$ ), and its elements referred to as *vertical* (resp. *horizontal*) *vector fields*. The orthogonal projection of  $\mathcal{L}(M)$  onto  $\mathcal{L}_V(M, \mathcal{F})$  will be denoted  $\pi_V$ . The *complex of vertical forms*  $A_V^*(M, \mathcal{F}; \mathbf{R})$  is the sub-complex of  $A^*(M, \mathbf{R})$  of forms that are non-zero only on vertical fields; that is,  $A_V^*(M, \mathcal{F}; \mathbf{R}) = \{\alpha \in A^*(M, \mathbf{R}) / i_X \alpha = 0, \text{ for all } X \in \mathcal{L}_H(M, \mathcal{F})\}$ . We will use the symbol  $\pi_V$  again to denote the orthogonal projection of  $A^*(M, \mathbf{R})$  onto  $A_V^*(M, \mathcal{F}; \mathbf{R})$ . When there is no risk of confusion, we will write  $V, \mathcal{L}_V, A_V^*$  etc., instead of  $V(M, \mathcal{F}), \mathcal{L}_V(M, \mathcal{F}), A_V^*(M, \mathcal{F}; \mathbf{R})$  etc.

Let  $\nabla^{LC}$  be the Levi-Civita connection on  $(M, g)$ . The following result is well known (see [25] or [14]).

**PROPOSITION 1.1.** *If  $X \in \mathcal{L}_H$  and if  $Y$  and  $Z$  belong to  $\mathcal{L}_V$ , then*

- (i)  $\pi_V \nabla_X^{LC} Y = \pi_V [X, Y]$ , and
- (ii)  $Xg(Y, Z) = g([X, Y], Z) + g(Y, [X, Z])$ .

As is also well known, these conditions are each sufficient for an arbitrary foliation to be totally geodesic.

Recall that a connection on the vertical bundle  $V$  is an element  $\nabla$  of  $\text{Hom}_{A^0}(\mathcal{L}, \text{Hom}_{\mathbf{R}}(\mathcal{L}_V, \mathcal{L}_V))$  such that  $\nabla(X)(fY) = X(f)Y + f\nabla(X)(Y)$  for all functions  $f \in A^0$  and all vector fields  $X \in \mathcal{L}$  and  $Y \in \mathcal{L}_V$ . As is usual, we denote  $\nabla(X)(Y)$  by  $\nabla_X Y$ . We will say that a connection  $\nabla$  on  $V$  is *vertical* if  $\nabla_X Y = \pi_V [X, Y]$  for all  $X \in \mathcal{L}_H$  and  $Y \in \mathcal{L}_V$  (cf. transverse or Bott connections [16], [3]). So, by Proposition 1.1 (i),  $\pi_V \nabla^{LC}$  defines a vertical connection on  $V$ , which we will denote by  $\nabla^t$  and call the *tangential Levi-Civita connection*.

We will denote the curvature of  $\nabla^t$  by  $R^t$ ; that is,  $R^t$  is the element of  $\text{Hom}_{A^0}(\wedge^2 \mathcal{L}, \text{Hom}_{A^0}(\mathcal{L}_V, \mathcal{L}_V))$  given by  $R^t(X, Y)(Z) = \nabla_X^t \nabla_Y^t Z - \nabla_Y^t \nabla_X^t Z - \nabla_{[X, Y]}^t Z$ , for all  $Z \in \mathcal{L}_V$  and  $X$  and  $Y$  in  $\mathcal{L}$ . The following result is given in [13]. It can also be proven algebraically using Proposition 1.1 and the Jacobi identity (see [5]).

PROPOSITION 1.2. For all  $X \in \mathcal{L}_H$  and  $Y \in \mathcal{L}_V$ , one has  $R^t(X, Y) \equiv 0$ .

In general, we will say that a vertical connection is *tangential* if its curvature satisfies the condition expressed in the previous proposition.

For all  $X \in \mathcal{L}$ , the Lie derivation  $\mathcal{L}_X$  in  $A^*$  induces a derivation  $\pi_V \mathcal{L}_X$  in the complex  $A_V^*$  of vertical forms. We will say that a vertical form  $\alpha$  is *tangential* if  $\pi_V \mathcal{L}_X \alpha$  is zero for all  $X \in \mathcal{L}_H$ , and we denote by  $A_{i_g}^*$  the *complex of tangential forms*. Note that Proposition 1.1 (ii) says that  $g|_V$  is a "symmetric tangential 2-form".

If  $x \in M$ , we will call the *sheet* of  $\mathcal{F}$  at  $x$  the subset  $S(x)$  of  $M$  of points that can be joined to  $x$  by piecewise smooth horizontal arcs. So  $A_{i_g}^0$  is the ring of functions that are constant on the sheets of  $\mathcal{F}$ .

Recall that a vector field  $X \in \mathcal{L}$  is said to be *foliated* if  $[X, Y] \in \mathcal{L}_V$ , for all  $Y \in \mathcal{L}_V$  (foliated vector fields are also known as basic or foliate fields). Analogous to the concept of a foliated vector field, and dual to that of tangential 1-forms, we will say that a vertical vector field  $X$  is *tangential* if  $[X, Y] \in \mathcal{L}_H$ , for all  $Y \in \mathcal{L}_H$ . We denote by  $\mathcal{L}_{i_g}$  the  $A_{i_g}^0$ -module of tangential vector fields. Note that if  $X \in \mathcal{L}_{i_g}$  and if  $\alpha \in A_{i_g}^1$ , then  $i_X \alpha \in A_{i_g}^0$ . In particular, if  $X \in \mathcal{L}_{i_g}$  and  $\alpha \in A_{i_g}^1$ , then the function  $\alpha(X)$  is constant on the sheets of  $\mathcal{F}$ .

Let  $X \in \mathcal{L}_V$ . Then  $X \in \mathcal{L}_{i_g}$  if and only if  $X$  commutes with every local horizontal foliated vector field. Again,  $X \in \mathcal{L}_{i_g}$  if and only if the local one-parameter subgroups associated to  $X$  respect the horizontal distribution. If  $X \in \mathcal{L}_{i_g}$  is complete, then its one-parameter subgroup respects the sheets of  $\mathcal{F}$ .

If  $\mathcal{F}$  has dimension  $p$ , then a *tangential parallelism* of  $\mathcal{F}$  is a set  $\{X_1, \dots, X_p\}$  of  $p$  tangential vector fields of  $\mathcal{F}$  such that the vectors  $X_i(x)$  are linearly independent at each point  $x$  of  $M$ . Let  $\mathfrak{m}$  be the subspace of  $\mathcal{L}_{i_g}$  spanned by the  $X_i$ . If all the elements of  $\mathfrak{m}$  are complete vector fields, then the tangential parallelism is said to be *complete*. If  $\mathfrak{g}$  is a Lie algebra, then we say that  $\mathcal{F}$  is *tangentially g-Lie* if  $\mathfrak{m}$ , furnished with the Lie bracket, is a Lie algebra isomorphic to  $\mathfrak{g}$ .

Let  $B$  be the principal  $GL(p, \mathbf{R})$ -bundle of vertical frames; that is, the bundle of frames of  $V$ . There is a canonical equivalence between connections  $\nabla$  on  $V$  and connections  $\omega$  on  $B$ , where of course,  $\omega$  is a 1-form on  $B$  with values in the Lie algebra  $\mathfrak{gl}(p, \mathbf{R})$ . Let  $\omega^t$  be the connection on  $B$  corresponding to the tangential Levi-Civita connection  $\nabla^t$  on  $V$ . Let  $\hat{g}$  be the lift to  $B$ , determined by  $\omega^t$ , of the Riemannian metric  $g$  on  $M$ . Let  $\hat{\mathcal{F}}$  be the pull-back of  $\mathcal{F}$  to  $B$ . Then the horizontal bundle  $H(B, \hat{\mathcal{F}})$  of  $\hat{\mathcal{F}}$  depends only on the decomposition  $T(M) = H(M, \mathcal{F}) \oplus V(M, \mathcal{F})$ .

Indeed, there is a fundamental form  $\theta \in A^1(B, \mathbf{R}^p)$  defined by  $\theta_z(X) = z^{-1}(\pi_V \circ \pi_*(X_z))$ , for all  $z \in B$  and  $X \in \mathcal{X}(B)$ , where  $\pi$  is the natural projection from  $B$  to  $M$ . It is easy to see that  $\mathcal{H}_H(B, \hat{\mathcal{F}}) = \{X \in \mathcal{X}(B); \theta(X) = i_X d\theta = 0\}$ . Alternatively,  $H(B, \hat{\mathcal{F}})$  can be defined from the natural lifts to  $B$  of local foliated horizontal vector fields of  $\mathcal{F}$  (see [2]).

It is easy to verify that if  $\nabla$  is a connection on  $V$ , canonically associated to a connection  $\omega \in A^1(B, \mathfrak{gl}(p, \mathbf{R}))$  on  $B$ , then  $\nabla$  is vertical if and only if  $\omega \in A^1_\nabla(B, \hat{\mathcal{F}}; \mathfrak{gl}(p, \mathbf{R}))$ , and  $\nabla$  is tangential if and only if  $\omega \in A^1_{i_\theta}(B, \hat{\mathcal{F}}; \mathfrak{gl}(p, \mathbf{R}))$ . In particular,  $\omega^t$  is a  $\mathfrak{gl}(p, \mathbf{R})$ -valued tangential 1-form on  $B$ . One can easily check that the fundamental form  $\theta$  is tangential; that is,  $\theta \in A^1_{i_\theta}(B, \hat{\mathcal{F}}; \mathbf{R}^p)$ . It follows that the fundamental and basic vector fields on  $B$ , associated to  $\omega^t$  and  $\theta$ , define a *canonical tangential parallelism* of  $\hat{\mathcal{F}}$ . Since  $M$  is complete, the leaves of  $\mathcal{F}$  are complete, and so the parallelism of  $\hat{\mathcal{F}}$  is complete.

If  $G$  is a Lie subgroup of  $GL(p, \mathbf{R})$ , and if  $E$  is a  $G$ -reduction of the principal  $GL(p, \mathbf{R})$ -bundle  $B$ , then we will say that  $E$  is a *vertical  $G$ -structure* if, for every  $z \in E$ , the horizontal subspace  $H_z(B, \hat{\mathcal{F}})$  of  $T_z(B)$  is tangent to  $E$ . Then one has (see [5]):

**PROPOSITION 1.3.** *If  $E$  is a  $G$ -reduction of  $B$ , then  $E$  is a vertical  $G$ -structure if and only if there exists on  $B$  a vertical connection reducible to a connection on  $E$ .*

Since the tangential Levi-Civita connection  $\nabla^t$  on  $V$  is a metric connection, the associated connection  $\omega^t$  on  $B$  is reducible to a connection on the  $O(p, \mathbf{R})$ -subbundle  $E$  of  $B$  of orthonormal vertical frames. So by the previous proposition,  $E$  is a vertical  $O(p, \mathbf{R})$ -structure on  $B$ . We denote again by  $\hat{\mathcal{F}}$  and  $\hat{g}$  respectively the foliation and the metric induced on  $E$  by  $\hat{\mathcal{F}}$  and  $\hat{g}$ . Then  $(E, \hat{\mathcal{F}})$  inherits a complete tangential parallelism from  $B$ . In particular, we can apply the following result to the connected components of  $B$  and  $E$ . This theorem was announced for compact manifolds in [6]. Parts (i) and (ii) were proven independently in [2].

**THEOREM 1.4.** *If  $\mathcal{F}$  possesses a complete tangential parallelism, then*

- (i) *the sheets of  $\mathcal{F}$  are the leaves of a foliation  $\mathcal{F}_s$  on  $M$ ,*
- (ii) *the closures of the sheets of  $\mathcal{F}$  are the fibers of a locally trivial fibration  $\psi: M \rightarrow W$ ,*
- (iii) *there is a Lie algebra  $\mathfrak{g}$  such that, for each  $z \in W$ , the foliation induced on  $\psi^{-1}(z)$  by  $\mathcal{F}$  is tangentially  $\mathfrak{g}$ -Lie.*

PROOF. (i) Let  $A$  be the set of complete vector fields on  $M$  whose orbits are contained in the sheets of  $\mathcal{F}$ . If  $x \in M$ , let  $A(x)$  be the subspace of  $T_x(M)$  spanned by  $A$ , and let  $f(x)$  be the dimension of  $A(x)$ . We first show that the function  $f$  is constant. Note that if  $X$  is a complete tangential vector field, then the one-parameter group  $(\phi_t)_{t \in \mathbf{R}}$ , associated to  $X$ , preserves the sheets of  $\mathcal{F}$ . So, for all  $t \in \mathbf{R}$  and  $Y \in A$ , the vector field  $\phi_{t*}Y$  is a member of  $A$ . Thus, as  $\mathcal{F}$  possesses a complete tangential parallelism,  $f$  is locally constant (hence constant) on the leaves of  $\mathcal{F}$ . Similarly, for all  $x \in M$  and every horizontal vector  $Z_x \in H_x(M, \mathcal{F})$ , there exists a complete horizontal vector field  $Z$ , on  $M$ , having  $Z_x$  as its value on  $x$ , and so  $f$  is constant on the sheets of  $\mathcal{F}$ . Thus, as every sheet meets every leaf, so  $f$  is constant on  $M$ . The set  $A$  therefore induces a smooth distribution  $D$  on  $M$ . Now let  $Y$  and  $Z$  be two elements of  $A$  and let  $(\varphi_t)_{t \in \mathbf{R}}$  be the one-parameter group of  $Y$ . For all  $t \in \mathbf{R}$ , one has  $\varphi_{t*}Z \in A$ , and so by differentiating, one has  $[Y, Z] \in D$ . Thus  $D$  is integrable, and clearly the sheets of  $\mathcal{F}$  are just the leaves of the foliation  $\mathcal{F}_s$  determined by  $D$ .

(ii) Once again, let  $X$  be a complete tangential vector field of  $\mathcal{F}$ , and let  $(\phi_t)_{t \in \mathbf{R}}$  be the one-parameter group associated to  $X$ . For all  $t \in \mathbf{R}$ , and  $Z \in A$ , one has  $\phi_{t*}Z \in A$  and so, by differentiating,  $[X, Z] \in D$ . In other words,  $X$  is a foliated vector field for  $\mathcal{F}_s$ . Then, since  $\mathcal{F}$  possesses a complete tangential parallelism, the family  $\mathcal{L}(M, \mathcal{F}_s)$  of complete foliated vector fields of  $\mathcal{F}_s$  is transitive on  $M$ ; that is, for each  $x \in M$  and  $X_x \in T_x(M)$ , there exists  $X \in \mathcal{L}(M, \mathcal{F}_s)$  such that the value of  $X$  at  $x$  is  $X_x$ . So, from [19], the closure of the leaves of  $\mathcal{F}_s$  are the fibers of a locally trivial fibration  $\psi: M \rightarrow W$ .

(iii) Let  $z \in W$ , and set  $N = \psi^{-1}(z)$ . Choose any point  $y \in N$ . Let  $\mathcal{F}(z)$  be the foliation defined on  $N$  by  $\mathcal{F}$ , and let  $C$  be a vector space, of dimension equal to  $\dim \mathcal{F}(z)$ , of complete tangential vector fields of  $\mathcal{F}$ , such that, at  $y$ ,  $C$  is tangent to  $\mathcal{F}(z)$ . The elements of  $C$  project to  $W$ , and so  $C$  is tangent to  $\mathcal{F}(z)$  at each point of  $N$ . Clearly,  $C$  induces on  $N$  a vector space  $\mathfrak{g}(z)$  of tangential vector fields of  $\mathcal{F}(z)$ . By Proposition 1.1 (ii), for all  $X$  and  $Y$  in  $C$ , the function  $g(X, Y)$  is tangential, and hence constant on  $N$ . So the elements of  $\mathfrak{g}(z)$  are linearly independent at each point of  $N$ . Again, by Proposition 1.1 (ii), for all  $X, Y$  and  $Z$  in  $C$ , the function  $g([X, Y], Z)$  is constant on  $N$ . In other words,  $\mathfrak{g}(z)$  is a Lie algebra. Thus  $\mathcal{F}(z)$  is tangentially  $\mathfrak{g}(z)$ -Lie. Finally, since  $\mathcal{F}$  possesses a complete tangential parallelism, so for all  $x \in W$ , there is a diffeomorphism of  $M$  that maps  $N$  onto  $\psi^{-1}(x)$  and preserves both  $\mathcal{F}$  and the horizontal bundle  $H$  of  $\mathcal{F}$ . Thus, up to isomorphism, the Lie

algebra  $\mathfrak{g}(z)$  is independent of  $z \in W$ . This completes the proof of the theorem.  $\square$

**2. Tangentially Lie Foliations.** In this section we suppose that  $\mathcal{F}$  is equipped with a complete Lie parallelism  $\{X_1, \dots, X_p\}$ , along its leaves, that preserves the horizontal bundle  $H$  of  $\mathcal{F}$ . In the notation of the previous section,  $\mathcal{F}$  is a tangentially  $\mathfrak{g}$ -Lie foliation, where  $\mathfrak{g}$  is the Lie subalgebra of  $\mathcal{L}_{i_0}$  spanned by the tangential vector fields  $X_i$ . For convenience, we suppose that the  $X_i$  are orthonormal.

Let  $G$  be a simply connected Lie group having  $\mathfrak{g}$  as its Lie algebra of left-invariant vector fields. Then there is a natural locally free right action of  $G$  on  $M$  whose orbits are the leaves of  $\mathcal{F}$  (see [22]). The action of  $G$  preserves the horizontal bundle and the sheets of  $\mathcal{F}$ . In this section, we draw an analogy between the behaviour of  $\mathcal{F}$  and that of principal  $G$ -bundles. In this analogy,  $H$  plays the role of a connection, and the elements of  $\mathfrak{g}$  correspond to the fundamental vector fields of the "connection". The sheets of  $\mathcal{F}$  are the "holonomy bundles" of  $H$ . To make explicit this analogy, we define the *transverse connection* of  $\mathcal{F}$  to be the  $\mathfrak{g}$ -valued 1-form  $\omega$  on  $M$  given by

$$\omega_x(X) = \sum_{i=1}^p g_x(X, X_i)X_i$$

for all  $x \in M$ , where, once again,  $g$  is the Riemannian metric on  $M$ . (In the terminology introduced by Molino [18],  $\omega$  is a "pseudo-connection"). We define the *transverse curvature* of  $\mathcal{F}$  to be the  $\mathfrak{g}$ -valued 2-form  $\Omega$  on  $M$  given by the "structural equation":

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

It is easy to see that  $\Omega$  is zero on vertical vector fields, and that  $\Omega$  is identically zero if and only if the horizontal bundle  $H$  is integrable. The following result, which expresses the pseudo-tensorial nature of  $\omega$  and  $\Omega$ , is a simple generalization of the classical situation:

**PROPOSITION 2.1.** *For all elements  $g \in G$ , one has  $g^*\omega = \text{Ad}(g^{-1})\omega$  and  $g^*\Omega = \text{Ad}(g^{-1})\Omega$ , where  $g^*\omega$  (resp.  $g^*\Omega$ ) is the pull-back of  $\omega$  (resp.  $\Omega$ ) by the action of  $g$ , and  $\text{Ad}$  is the standard adjoint representation of  $G$  in  $\mathfrak{g}$ .*

Now let  $x \in M$  and let  $S(x)$  be the sheet of  $\mathcal{F}$  at  $x$ . Let  $\Phi(x)$  be the subgroup  $\{g \in G; x \cdot g \in S(x)\}$  of  $G$ , and let  $\Phi^0(x)$  be the connected component of the identity of  $\Phi(x)$ . By Theorem 1.4, the sheets of  $\mathcal{F}$  define a foliation  $\mathcal{F}_s$  on  $M$ . It follows that every element of  $\Phi^0(x)$  can be joined

to the identity by a piecewise smooth path in  $G$  which lies in  $\Phi^0(x)$ . So  $\Phi^0(x)$  is a Lie group. Let  $\mathfrak{h}_x$  be the Lie algebra of  $\Phi^0(x)$ . We will call  $\mathfrak{h}_x$  the *transverse holonomy algebra* of  $\mathcal{F}$  at  $x$ . Clearly,  $\mathfrak{h}_x$  is the subalgebra of  $\mathfrak{g}$  tangent to  $\mathcal{F}_s$  at  $x$ . Since the elements of  $\mathfrak{g}$  are foliated for  $\mathcal{F}_s$ , so  $\mathfrak{h}_y = \mathfrak{h}_x$  for all points  $y$  in  $S(x)$ .

The following result can be verified by simply repeating the proof of the Ambrose-Singer theorem.

**PROPOSITION 2.2.** *If  $x \in M$ , then  $\mathfrak{h}_x$  is the subalgebra of  $\mathfrak{g}$  generated by the elements of the form  $\Omega_y(X, Y)$ , where  $y$  belongs to the sheet of  $\mathcal{F}$  at  $x$ , and  $X$  and  $Y$  are vector fields on  $M$ .*

It follows from Propositions 2.1 and 2.2, that for all  $x \in M$  and  $g \in G$ , one has  $\mathfrak{h}_{x \cdot g} = \text{Ad}(g^{-1})\mathfrak{h}_x$ .

Now, if  $x \in M$ , let  $\Psi(x)$  be the closure of  $\Phi(x)$  in  $G$ . Then  $\Psi(x)$  is a closed Lie subgroup of  $G$ . Let  $\mathfrak{s}_x$  be the Lie algebra of  $\Psi(x)$ . We will call  $\mathfrak{s}_x$  the *structural algebra* of  $\mathcal{F}$  at  $x$ . Evidently,  $\mathfrak{s}_x$  is the subalgebra of  $\mathfrak{g}$  tangent at  $x$  to the closure of the sheet  $S(x)$  of  $\mathcal{F}$  at  $x$ . Once again,  $\mathfrak{s}_y = \mathfrak{s}_x$  for all  $y \in S(x)$ , and  $\mathfrak{s}_{x \cdot g} = \text{Ad}(g^{-1})\mathfrak{s}_x$  for all  $g \in G$ . Since the elements of  $\mathfrak{s}_x$  are foliated for  $\mathcal{F}_s$ , so  $\mathfrak{h}_x$  is an ideal of  $\mathfrak{s}_x$ .

As we have seen, up to isomorphism, the Lie algebras  $\mathfrak{s}_x$  and  $\mathfrak{h}_x$  are independent of  $x \in M$ . We will call the isomorphism class  $\mathfrak{s}$  (resp.  $\mathfrak{h}$ ) of the Lie algebras  $\mathfrak{s}_x$  (resp.  $\mathfrak{h}_x$ ) the *structural* (resp. *transverse holonomy*) algebra of  $\mathcal{F}$ .

We now consider the geometric significance of these algebras. Let  $L$  be an arbitrary leaf of  $\mathcal{F}$ , and let  $u$  be a connected open subset of  $L$ . We will say that a vector field  $X$  on  $u$  is a *local commuting vector field* of  $\mathcal{F}$  on  $u$  if  $X$  commutes with the restriction to  $u$  of every tangential vector field of  $\mathcal{F}$ . The Lie algebra  $\mathfrak{e}(u)$ , of local commuting vector fields of  $\mathcal{F}$  on  $u$ , has dimension  $\leq \dim \mathcal{F}$ . Indeed, if  $X \in \mathfrak{e}(u)$  and if  $X_y = 0$  for some  $y \in u$ , then as  $X$  commutes with  $\mathfrak{g}$ , so  $X \equiv 0$ . In fact,  $\dim \mathfrak{e}(u) \leq \dim \mathfrak{s}$ , since if  $X \in \mathfrak{e}(u)$  then  $X(f) = 0$  for all  $f \in A_{i_g}^0$ , and so  $X$  is tangent on each point  $x$  of  $u$  to the closure of  $S(x)$ .

Let  $\mathcal{E}(L)$  be the sheaf of germs of local commuting vector fields of  $\mathcal{F}$  on subsets of  $L$ . We will call  $\mathcal{E}(L)$  the *commuting sheaf* of  $L$ . We have:

**THEOREM 2.3.** *For every leaf  $L$  of  $\mathcal{F}$ , the commuting sheaf  $\mathcal{E}(L)$  of  $L$  is locally trivial and has as typical fiber the Lie algebra  $\mathfrak{s}$  anti-isomorphic to the structural algebra  $\mathfrak{s}$  of  $\mathcal{F}$ . Furthermore, the orbits of  $\mathcal{E}(L)$  are the connected components of the intersections with  $L$  of the closures of the sheets of  $\mathcal{F}$ .*

**PROOF.** Let  $x \in L$ . The Lie group  $G$  acts transitively on  $L$  and, corresponding to the right-invariant vector fields on  $G$ , there exists, on some open connected neighbourhood  $u$  of  $x$  in  $L$ , a Lie algebra  $\mathfrak{g}^-$  of vector fields that commute with the restriction to  $u$  of all the elements of  $\mathfrak{g}$ . Let  $\mathfrak{m}$  be the vector subspace of  $\mathfrak{g}^-$  tangent at  $x$  to the closure of the sheet  $S(x)$  of  $\mathcal{F}$  at  $x$ . So  $\omega_x(\mathfrak{m}) = \mathfrak{s}_x$ . Since  $\omega_x$  defines a Lie algebra anti-isomorphism between  $\mathfrak{g}^-$  and  $\mathfrak{g}$ , so  $\mathfrak{m}$  is anti-isomorphic to  $\mathfrak{s}_x$ . By Proposition 2.1, one has  $\omega_y(\mathfrak{m}) = \mathfrak{s}_y$  for all  $y \in u$ . Thus, at each point of  $u$ , the elements of  $\mathfrak{m}$  are tangent to the closures of the sheets of  $\mathcal{F}$ . Then, since every tangential vector field of  $\mathcal{F}$  can be written as a linear combination of elements of  $\mathfrak{g}$ , with coefficients in the ring  $A_{i_g}^0$ , it follows that  $\mathfrak{m}$  is contained in the Lie algebra  $\mathfrak{e}(u)$  of local commuting vector fields of  $\mathcal{F}$  on  $u$ . But we have already seen that  $\mathfrak{e}(u)$  has dimension  $\leq \dim \mathfrak{s}_x$ . So  $\mathfrak{m} = \mathfrak{e}(u)$  and  $\mathcal{E}(L)$  is locally trivial and has as its typical fiber the Lie algebra  $\mathfrak{s}^-$ . Finally since  $\omega_y(\mathfrak{m}) = \mathfrak{s}_y$  for all  $y \in u$ , it is clear that the orbits of  $\mathcal{E}(L)$  are the connected components of the intersections with  $L$  of the closures of the sheets of  $\mathcal{F}$ .

**REMARK 2.4.** It is clear, by the same argument, that if  $L$  is a leaf of  $\mathcal{F}$ , then  $\mathcal{E}(L)$  has a locally trivial subsheaf, having as typical fiber the Lie algebra  $\mathfrak{h}^-$  anti-isomorphic to  $\mathfrak{h}$ , and having as orbits the connected components of the intersection with  $L$  of the sheets of  $\mathcal{F}$ .

It is pertinent to ask whether the commuting sheaves of the leaves extend to a sheaf on  $M$  of germs of local tangential vector fields that commute with all the (global) tangential vector fields of  $\mathcal{F}$ . In order to examine this question we introduce the “local transverse holonomy algebras” of  $\mathcal{F}$ .

Let  $x \in M$  and let  $O(x)$  be the set of open connected  $\mathcal{F}$ -saturated neighbourhoods of  $x$ . For each member  $u$  of  $O(x)$ , the foliation  $\mathcal{F}(u)$ , induced on  $u$  by  $\mathcal{F}$ , is equipped with a complete tangentially  $\mathfrak{g}$ -Lie parallelism. Let  $\mathfrak{h}_x(u)$  be the transverse holonomy algebra of  $\mathcal{F}(u)$  at  $x$ . Then the local transverse holonomy algebra of  $\mathcal{F}$  at  $x$  is the Lie algebra  $\mathfrak{h}_x^* = \bigcap_{u \in O(x)} \mathfrak{h}_x(u)$ . By construction,  $\mathfrak{h}_x^*$  is a subalgebra of  $\mathfrak{h}_x$ .

The following proposition is verified by the standard argument for the local holonomy algebras of a principal connection (see [15]).

**PROPOSITION 2.5.** *If  $x \in M$ , then there exists  $u \in O(x)$  such that  $\mathfrak{h}_x^* = \mathfrak{h}_x(u)$ .*

Now let  $w$  be an open connected subset of  $M$  and let  $\mathcal{F}(w)$  be the foliation induced on  $w$  by  $\mathcal{F}$ . In general, the  $A_{i_g}^0(w, \mathcal{F}(w); \mathbf{R})$ -module  $\mathcal{H}_{i_g}(w, \mathcal{F}(w))$  of tangential vector fields of  $\mathcal{F}(w)$  contains the restriction

$J$ , of  $\mathcal{L}_{i_g}(M, \mathcal{F})$  to  $w$ , as a proper subset. We will say that an element  $X$  of  $\mathcal{L}_{i_g}(w, \mathcal{F}(w))$  is a *local commuting tangential vector field of  $\mathcal{F}$  on  $w$*  if  $X$  commutes with all the elements of  $J$ , and we will denote the Lie algebra of these vector fields by  $c(w)$ . Since  $c(w)$  commutes with  $\mathfrak{g}$  and with all the local horizontal foliated vector fields of  $\mathcal{F}(w)$ , so  $c(w)$  has finite dimension. The elements of  $c(w)$  are tangent to the closures of the sheets of  $\mathcal{F}$ , and so  $\dim c(w) \leq \dim \mathfrak{s}$ . Now let  $x \in M$  and let  $c_x$  be the Lie algebra of germs at  $x$  of local commuting tangential vector fields of  $\mathcal{F}$ . We have:

**PROPOSITION 2.6.** *For all  $x \in M$ , the Lie algebra  $c_x$  contains a sub-algebra anti-isomorphic to the centralizer of  $\mathfrak{h}_x^*$  in  $\mathfrak{s}_x$ .*

**PROOF.** We proceed in a manner similar to the proof of Theorem 2.3. Let  $\mathfrak{Z}^*(x)$  be the centralizer of  $\mathfrak{h}_x^*$  in  $\mathfrak{s}_x$ , and let  $L$  be the leaf of  $\mathcal{F}$  containing  $x$ . There exists an open connected neighbourhood  $v$  of  $x$  in  $L$  and a Lie algebra  $\mathfrak{g}^-$  of vector fields on  $v$  that commute with  $\mathfrak{g}$ . Let  $\mathfrak{n}$  be the vector subspace of  $\mathfrak{g}^-$  tangent at  $x$  to  $\mathfrak{Z}^*(x)$ . By Proposition 2.5, there exists  $u \in O(x)$  such that  $\mathfrak{h}_x^* = \mathfrak{h}_u(u)$ . Let  $y \in v$ , and let  $\mathfrak{Z}(y)$  be the centralizer of  $\mathfrak{h}_y(u)$  in  $\mathfrak{s}_y$ . Then by Proposition 2.1, one has  $\omega_y(\mathfrak{n}) = \mathfrak{Z}(y)$ . Let  $Y \in \mathfrak{n}$  and  $Z \in \mathfrak{h}_y(u)$ . One has  $[\omega_y(Z), \omega_y(Y)] = [Z, \omega_y(Y)] = 0$ . Then, by the structural equation,  $Z_y \omega(Y) = \Omega_y(Z, Y) + Y_y \omega(Z) + \omega_y([Z, Y]) - [\omega_y(Z), \omega_y(Y)]$ , which is zero, term by term. But by definition,  $\omega(Y) = \sum_{i=1}^p g(Y, X_i) X_i$ . So, putting  $a_i = g(Y, X_i)$ , we have  $Z_y(a_i) = 0$  for all  $y \in v$  and  $Z \in \mathfrak{h}_y(u)$ . In other words, the functions  $a_i$  on  $v$  are constant on the connected components of the intersections with  $v$  of the sheets of the foliation  $\mathcal{F}(u)$  induced on  $u$  by  $\mathcal{F}$ . So it is clear that by choosing a sufficiently small neighbourhood  $w$  of  $x$  in  $M$ , we can extend the elements  $Y$  of  $\mathfrak{n}$  to vector fields  $Y'$  on  $w$ , by demanding that the functions  $g(Y', X_i)$  be constant on the connected components of the intersections with  $w$  of the sheets of  $\mathcal{F}(u)$ . By construction, we thus obtain a vector space  $\mathfrak{n}'$ , of dimension equal to  $\dim \mathfrak{Z}^*(x)$ , of local commuting tangential vector fields of  $\mathcal{F}$  on  $w$ . Finally, because the transverse connection  $\omega$  induces a Lie anti-isomorphism from  $\mathfrak{g}^-$  onto  $\mathfrak{g}$ , so  $\mathfrak{n}'$  is a Lie algebra anti-isomorphic to  $\mathfrak{Z}^*(x)$ . This completes the proof.  $\square$

We can now give the following:

**THEOREM 2.7.** *The following conditions are equivalent:*

(i) *There exists a sheaf  $\mathcal{C}$  on  $M$  of germs of local commuting tangential vector fields of  $\mathcal{F}$  such that for each leaf  $L$  of  $\mathcal{F}$  the restriction of  $\mathcal{C}$  to  $L$  is the commuting sheaf  $\mathcal{C}(L)$  of  $L$ .*

(ii) *The transverse holonomy algebra  $\mathfrak{h}$  of  $\mathcal{F}$  is contained in the centre of the structural algebra  $\mathfrak{s}$  of  $\mathcal{F}$ .*

**PROOF.** By the previous proposition, one has (ii)  $\Rightarrow$  (i). Conversely, suppose that the condition (i) is satisfied. Let  $x \in M$  and let  $S(x)$  be the sheet of  $\mathcal{F}$  at  $x$ . By Proposition 2.2, the algebra  $\mathfrak{h}_x$  is generated by the set  $\{\Omega_y(Z_1, Z_2); y \in S(x) \text{ and } Z_i \in \mathcal{X}\}$ . Let  $X \in \mathfrak{s}_x$ . We will show that  $X$  commutes with all the elements of  $\mathfrak{h}_x$ . Let  $y \in S(x)$ . By hypothesis, there exists a local commuting tangential vector field  $Y$  of  $\mathcal{F}$ , on a neighbourhood  $w$  of  $y$ , such that  $\omega_y(Y) = X$ . So one has  $[\Omega_y(Z_1, Z_2), X] = [\Omega_y(Z_1, Z_2), \omega_y(Y)] = (d\Omega)_y(Z_1, Z_2, Y)$ , for all  $Z_i \in \mathcal{X}$ . But we may restrict our attention to vector fields  $Z_i$  that are local horizontal foliated vector fields of  $\mathcal{F}$ . Then we have  $[\Omega_y(Z_1, Z_2), X] = Y_y \Omega(Z_1, Z_2) = -Y_y \omega([Z_1, Z_2])$ , which is zero, from the definition of  $\omega$ , because  $Y$  is a local Killing vector field for the Riemannian structure along the leaves and  $Y$  commutes with the local vector fields  $Z_i$ . Thus  $X$  commutes with all the elements of  $\mathfrak{h}_x$ , and the proof is completed.  $\square$

**3. Geometric Description of Totally Geodesic Foliations.** In this paragraph, we use the results of the previous section to study the behaviour of a totally geodesic foliation  $\mathcal{F}$ .

Let  $E$  be the bundle of orthonormal vertical frames of  $\mathcal{F}$  and let  $\hat{\mathcal{F}}$  be the pull-back of  $\mathcal{F}$  to  $E$ . We equip  $E$  with the canonical lift  $\hat{g}$  of the Riemannian metric  $g$  on  $M$  (see Section 1). Then  $\hat{\mathcal{F}}$  possesses a complete tangential parallelism, and by Theorem 1.4, the closures of the sheets of  $\hat{\mathcal{F}}$  are the fibers of a locally trivial fibration  $\psi: E \rightarrow W$ , and there is a Lie algebra  $\mathfrak{g}$  such that, for each  $z \in W$ , the foliation  $\mathcal{F}(z)$  induced on  $\psi^{-1}(z)$  by  $\hat{\mathcal{F}}$  is tangentially  $\mathfrak{g}$ -Lie. The foliations  $\mathcal{F}(z)$  have dense sheets, and so their structural algebras are isomorphic to  $\mathfrak{g}$ . We will call  $\mathfrak{g}$  the *structural algebra* of  $\mathcal{F}$ . Similarly, the transverse holonomy algebras of the foliations  $\mathcal{F}(z)$  are all isomorphic; we call their isomorphism class the *transverse holonomy algebra* of  $\mathcal{F}$  and we denote it by  $\mathfrak{h}$ . By choosing any reference point in  $E$ , we can regard  $\mathfrak{h}$  as an ideal of  $\mathfrak{g}$ . Note that if  $\mathcal{F}$  is tangentially Lie, the tangential parallelism of  $\mathcal{F}$  defines a map from  $M$  to  $E$  which induces diffeomorphisms from the closures of the sheets of  $\mathcal{F}$  to the fibres of  $\psi$ . As these diffeomorphisms respect the foliation structures, the structural and transverse holonomy algebras of  $\mathcal{F}$  coincide with those defined in Section 2.

If  $G$  is a simply connected Lie group having  $\mathfrak{g}$  as its Lie algebra, then there is a locally free action of  $G$  on each of the fibres of  $\psi$ . In general, these actions do not define a smooth action of  $G$  on  $E$ . Never-

theless, for each  $z \in W$ , there is a neighbourhood  $u$  of  $z$ , trivializing  $\psi$ , such that  $G$  acts smoothly on  $\psi^{-1}(u)$ . (see Lemma 3.2 below).

The following theorem was announced for compact manifolds in [6].

**THEOREM 3.1.** *The closures of the sheets of  $\mathcal{F}$  are submanifolds of  $M$  and if  $L$  is an arbitrary leaf of  $\mathcal{F}$ , then there exists a locally trivial sheaf  $\mathcal{E}(L)$  on  $L$  of germs of local Killing vector fields of  $L$  (for the induced Riemannian metric) such that:*

- (i) *the orbits of  $\mathcal{E}(L)$  are the connected components of the intersections with  $L$  of the closures of the sheets of  $\mathcal{F}$ , and*
- (ii) *the typical fiber of  $\mathcal{E}(L)$  is a Lie algebra anti-isomorphic to the structural algebra  $\mathfrak{g}$  of  $\mathcal{F}$ .*

**PROOF.** First note that since the horizontal bundle  $H(E, \hat{\mathcal{F}})$  of  $\hat{\mathcal{F}}$  projects in  $M$  onto the horizontal bundle  $H(M, \mathcal{F})$  of  $\mathcal{F}$ , so the sheets of  $\hat{\mathcal{F}}$  project in  $M$  onto the sheets of  $\mathcal{F}$ . As the action of the orthogonal group  $O(p, \mathbf{R})$  on  $E$  preserves the fibers of  $\psi$ , the closures of the sheets of  $\hat{\mathcal{F}}$  project in  $M$  onto submanifolds of  $M$ . As  $E$  is a locally trivial fibration with compact fibers, the natural projection  $\pi: E \rightarrow M$  is a closed mapping. So the fibers of  $\psi$  project onto the closures of the sheets of  $\hat{\mathcal{F}}$ .

Now let  $L$  be a leaf of  $\mathcal{F}$ , and let  $\hat{L}$  be a leaf of  $\hat{\mathcal{F}}$  over  $L$ . Let  $\mathcal{E}(\hat{L})$  be the sheaf of germs of local vector fields on  $\hat{L}$  that commute with all the tangential vector fields of  $\hat{\mathcal{F}}$ . We will show that  $\mathcal{E}(\hat{L})$  is locally trivial, that its typical fiber is anti-isomorphic to  $\mathfrak{g}$ , and that its orbits are the connected components of the intersections with  $\hat{L}$  of the fibers of  $\psi$ . We first prove the following:

**LEMMA 3.2.** *If  $z \in W$  and if  $r$  is the dimension of  $W$ , then there exists an open connected neighbourhood  $u$  of  $z$  such that the foliation  $\mathcal{F}(u)$ , induced on  $\psi^{-1}(u)$  by  $\hat{\mathcal{F}}$ , is tangentially  $\mathfrak{g} \oplus \mathbf{R}^r$ -Lie, and has  $\mathfrak{g}$  as its structural algebra.*

**PROOF.** Let  $u$  be a sufficiently small neighbourhood of  $z$  such that  $u$  trivializes  $\psi$  and such that on  $u$  there exists  $r$  linearly independent commuting vector fields  $Y_i$ . If  $\varphi: \psi^{-1}(z) \times u \rightarrow \psi^{-1}(u)$  is such a trivialization, the vector fields  $\bar{Y}_i = \varphi_*(0 + Y_i)$  are tangential for  $\mathcal{F}(u)$ . If  $\mathcal{L}_{i\varphi}(\psi^{-1}(z), \mathcal{F}(z))$  is the  $(p-r)$ -dimensional Lie algebra of tangential vector fields of the foliation  $\mathcal{F}(z)$  induced by  $\hat{\mathcal{F}}$  on  $\psi^{-1}(z)$ , then the elements of the Lie algebra  $\mathfrak{a} = \varphi_*(\mathcal{L}_{i\varphi}(\psi^{-1}(z), \mathcal{F}(z)) \oplus \{0\})$  are tangential vector fields of  $\mathcal{F}(u)$  and they commute with the vector fields  $\bar{Y}_i$ . So there is a tangential  $\mathfrak{a} \oplus \mathbf{R}[\bar{Y}_1, \dots, \bar{Y}_r]$ -Lie parallelism of  $\mathcal{F}(u)$ , and

clearly the structural algebra of  $\mathcal{F}(u)$  is  $\mathfrak{a} \cong \mathfrak{g}$ . □

Returning to the proof of the theorem, let  $y$  be an arbitrary point in  $\hat{L}$  and let  $z = \psi(y)$ . By the previous lemma, there exists a connected open neighbourhood  $u$  of  $z$  such that  $\mathcal{F}(u)$  is tangentially  $\mathfrak{g} \oplus \mathbf{R}^r$ -Lie. Of course, a parallelism of  $\mathcal{F}(u)$  will not, in general, be complete. Nevertheless, since Theorem 1.4 holds for  $\mathcal{F}(u)$ , we may still apply the results of Section 2 to  $\mathcal{F}(u)$ . Let  $L(u)$  be the leaf of  $\mathcal{F}(u)$  containing  $y$ . So  $L(u)$  is an open subset of  $\hat{L}$ . The local sections of the commuting sheaf  $\mathcal{E}(L(u))$  of  $L(u)$  are also local sections of  $\mathcal{E}(\hat{L})$ . Conversely, the restrictions to  $L(u)$ , of local sections of  $\mathcal{E}(\hat{L})$ , define local sections of  $\mathcal{E}(L(u))$ . In other words,  $\mathcal{E}(L(u))$  is just the restriction of  $\mathcal{E}(\hat{L})$  to  $L(u)$ . Our claims concerning  $\mathcal{E}(\hat{L})$  are therefore verified by Theorem 2.3 applied to  $\mathcal{E}(L(u))$ .

Finally, identify  $\hat{L}$  with a connected component of the principal bundle of orthonormal frames of  $L$ . Then, as the elements of  $\mathcal{E}(\hat{L})$  commute with the fundamental and basic vector fields of  $\hat{L}$ , determined by the Levi-Civita connection on  $L$ , so  $\mathcal{E}(\hat{L})$  projects in  $L$  onto a sheaf  $\mathcal{E}(L)$  of germs of local Killing vector fields of  $L$ . Clearly,  $\mathcal{E}(L)$  has the desired properties. □

**COROLLARY 3.3.** *If, at any point, any leaf of  $\mathcal{F}$  has no non-zero germs of local Killing vector fields, then the sheets of  $\mathcal{F}$  are all closed and orthogonal to  $\mathcal{F}$ .*

**COROLLARY 3.4.** *The closures of the sheets of  $\mathcal{F}$  are complete Riemannian submanifolds of  $M$ , and on these submanifolds,  $\mathcal{F}$  induces totally geodesic foliations with dense sheets. Furthermore, the leaves of these foliations are locally homogeneous Riemannian submanifolds of  $M$ . In particular, if  $\mathcal{F}$  has a dense sheet, then the leaves of  $\mathcal{F}$  are locally homogeneous Riemannian submanifolds of  $M$ .*

**PROOF.** Let  $N$  be the closure of a sheet of  $\mathcal{F}$ , and let  $\mathcal{F}(N)$  be the foliation induced on  $N$  by  $\mathcal{F}$ . Since  $M$  is complete, so also is  $N$ . Since  $\mathcal{F}(N)$  satisfies the condition of Proposition 1.1 (ii), so  $\mathcal{F}(N)$  is totally geodesic. Clearly,  $\mathcal{F}(N)$  has dense sheets. Then the corollary is verified by the theorem applied to the leaves of  $\mathcal{F}(N)$ . □

**COROLLARY 3.5.** *The union  $U$  of sheet closures of  $\mathcal{F}$  of maximal dimension is an open dense subset of  $M$ .*

**PROOF.** Let  $L$  be an arbitrary leaf of  $\mathcal{F}$ , and let  $(v_\alpha)_{\alpha \in I}$  be a covering of  $L$  by open connected sets that trivialize the commuting sheaf  $\mathcal{E}(L)$  of  $L$ . For each  $\alpha \in I$ , the set  $U \cap v_\alpha$  is the union of the orbits of maximal

dimension of a Lie algebra of Killing vector fields of  $v_\alpha$ . According to [21], the set  $U \cap v_\alpha$  is open and dense in  $v_\alpha$ . It follows that  $U \cap L$  is open and dense in  $L$ . Then, as  $U$  is the saturation of  $U \cap L$  by the closures of the sheets of  $\mathcal{F}$ , we have the desired result.  $\square$

If  $L$  is a leaf of  $\mathcal{F}$ , then we will call the sheaf  $\mathcal{E}(L)$ , constructed in the course of the proof of the previous theorem, the *commuting sheaf* of  $L$ .

REMARK 3.6. (i) To justify the notation, note that the elements of  $\mathcal{E}(L)$  commute with all the tangential vector fields of  $\mathcal{F}$ . This follows from the fact that the natural lifts of the local sections of  $\mathcal{E}(L)$ , to the bundle  $B$  of vertical frames of  $\mathcal{F}$ , commute with the canonical tangential parallelism of  $(B, \hat{\mathcal{F}})$  (see Section 1).

(ii) By a standard argument, if  $L$  is a simply connected leaf of  $\mathcal{F}$ , then  $\mathcal{E}(L)$  is globally trivial.

(iii) It is clear from Remark 2.4, that for all leaves  $L$  of  $\mathcal{F}$ , the sheaf  $\mathcal{E}(L)$  has a subsheaf whose orbits are the connected components of the intersections with  $L$  of the sheets of  $\mathcal{F}$ , and whose typical fiber is anti-isomorphic to the transverse holonomy algebra  $\mathfrak{h}$  of  $\mathcal{F}$ .

(iv) We have shown in [5], using the ideas of [21], that when  $M$  is compact, the space  $W$  of sheet closures of  $\mathcal{F}$  is compact and metrizable (for the quotient topology) and that the subset of  $Q$  of sheet closures of maximal dimension is equipped with a structure of a Satake manifold [26].

We now use Theorem 2.7. Let  $u$  be an open subset of  $M$  and let  $\mathcal{F}(u)$  be the foliation induced on  $u$  by  $\mathcal{F}$ . We will say that a tangential vector field  $X$  of  $\mathcal{F}(u)$  is a *local tangential Killing vector* of  $\mathcal{F}$  on  $u$  if  $X$  is a Killing vector field for the Riemannian structure along the leaves of  $\mathcal{F}$ .

Let  $z \in W$ , and let  $\mathcal{F}(z)$  again denote the foliation induced on  $\psi^{-1}(z)$  by  $\hat{\mathcal{F}}$ . Let  $Y$  be a local commuting tangential vector field of  $\mathcal{F}(z)$  (see Section 2). Then, using Lemma 3.2, we can extend  $Y$  locally in  $E$  to a local commuting tangential vector field  $Z$  of  $\hat{\mathcal{F}}$ . Clearly,  $Z$  projects in  $M$  onto a local tangential Killing vector field of  $\mathcal{F}$ . Thus the following theorem results without difficulty from Theorem 2.7.

**THEOREM 3.7.** *If the transverse holonomy algebra  $\mathfrak{h}$  of  $\mathcal{F}$  is contained in the centre of the structural algebra  $\mathfrak{g}$  of  $\mathcal{F}$ , then there exists a locally trivial sheaf  $\mathcal{E}$  on  $M$  of germs of local tangential Killing vector fields such that:*

- (i) the orbits of  $\mathcal{E}$  are the connected components of the intersections of the leaves and the closures of the sheets of  $\mathcal{F}$ ,
- (ii) the typical fiber of  $\mathcal{E}$  is anti-isomorphic to  $\mathfrak{g}$ ,
- (iii) the elements of  $\mathcal{E}$  commute with all the (global) tangential vector fields of  $\mathcal{F}$ .

REMARKS 3.8. (i) The condition that  $\mathfrak{h}$  be contained in the centre of  $\mathfrak{g}$  is, of course, quite strong. For example, if  $\mathfrak{g}$  is semi-simple, the condition is equivalent to  $\mathfrak{h}$  being zero; that is, to the horizontal bundle  $H$  being integrable.

(ii) If  $\tilde{M}$  is the universal covering space of  $M$ , then the pull-back  $\tilde{\mathcal{F}}$  of  $\mathcal{F}$  to  $\tilde{M}$  is totally geodesic and  $\tilde{M}$  is complete, for the lifted Riemannian metric. It is easy to see that the transverse holonomy algebra of  $\tilde{\mathcal{F}}$  is isomorphic to  $\mathfrak{h}$ . So the condition that  $M$  be simply connected does not imply that  $\mathfrak{h}$  is Abelian, as might have been thought by analogy with the structural algebra of a Riemannian foliation [19]. It follows from Proposition 2.6 however, that if  $\pi_1(M) = 0$ , then the centralizer of  $\mathfrak{h}$  in  $\mathfrak{g}$  is Abelian. Furthermore, it is easy to show that if  $\pi_1(M) = 0$  and if  $\mathfrak{h}$  is Abelian, then  $\mathfrak{g}$  is also Abelian. Using the arguments of [20], if  $\mathfrak{g}$  is Abelian, then the maximal dimension of the closures of the sheets is equal to  $\dim \mathfrak{g} + \text{codim } \mathcal{F}$ .

(iii) The structural and transverse holonomy algebras, and the commuting sheaves of the leaves depend only on the orthogonal decomposition  $T(M) = H \oplus V$ . This follows from the fact that the action of  $GL(p, \mathbf{R})$ , on the bundle  $B$  of vertical frames of  $\mathcal{F}$ , preserves the sheet closures of the pull-back  $\hat{\mathcal{F}}$  of  $\mathcal{F}$  to  $B$ . In particular, the local sections of the commuting sheaves  $\mathcal{E}(L)$ , and the local sections of  $\mathcal{E}$ , when it exists, are local Killing vector fields, along the leaves, for every Riemannian metric on  $M$  for which  $\mathcal{F}$  is totally geodesic and for which  $T(M) = H \oplus V$  is the orthogonal decomposition of the tangent bundle of  $M$ .

(iv) If, in addition,  $\mathcal{F}$  is a Riemannian foliation (for instance, if  $M$  and the leaves of  $\mathcal{F}$  are compact), then the local tangential Killing vector fields of  $\mathcal{F}$  are local Killing vector fields of  $M$ , for any Riemannian metric which makes  $\mathcal{F}$  both totally geodesic and Riemannian, and which preserves  $T(M) = H \oplus V$ .

**4. Characteristic Classes.** In this section, we suppose that  $\mathcal{F}$  is oriented.

Recall from Section 1 the definition of the complex  $A_{i_0}^*$  of tangential forms on  $M$ ; this is the complex of forms, along the leaves, that are "invariant" in the directions perpendicular to the leaves. The exterior

derivative  $d$  on  $A^*$  induces a differential operator  $d_\nu = \pi_\nu \circ d$  on  $A_{i_g}^*$ , thus making  $(A_{i_g}^*, d_\nu)$  a graded differential algebra. Let  $H_{i_g}^* = \bigoplus_{r=0}^p H_{i_g}^r$  be the associated cohomology group. We call  $H_{i_g}^*$  the *tangential cohomology group* of  $\mathcal{F}$ .

We now define the characteristic classes in  $H_{i_g}^*$ . Let  $P^k$  be an invariant polynomial of degree  $k$  on the Lie algebra  $\mathfrak{so}(p, \mathbf{R})$ . Recall that the Pontrjagin class  $P^k(\mathcal{F})$  of  $\mathcal{F}$  is the element of  $H_{\text{deRham}}^{2k}(M)$  obtained by substituting, for the arguments of  $P^k$ , the curvature of a connection  $\nabla$  on the vertical bundle  $V$  of  $\mathcal{F}$ . Let us write  $P^k(\mathcal{F}) = [P^k(\nabla)]$ . The class thus obtained is independent of the connection  $\nabla$ . By restricting the choice of  $\nabla$  to the set of tangential connections (see Section 1) one obtains an element  $P_{i_g}^k(\mathcal{F}) = [\pi_\nu P^k(\nabla)]$  of the tangential cohomology. We call the elements of  $H_{i_g}^*$  thus obtained the *tangential characteristic classes* of  $\mathcal{F}$ . To see that these classes are well defined, note first that the set of tangential connections is not empty (see Section 1) and secondly that one has:

**PROPOSITION 4.1.** *The tangential characteristic classes do not depend upon the tangential connection used in their definition.*

**PROOF.** Let  $P^k$  be an invariant polynomial on  $\mathfrak{so}(p, \mathbf{R})$ , and suppose that  $\nabla^0$  and  $\nabla^1$  are two tangential connections on  $V$ . Let  $\mathcal{F}_1$  (resp.  $\mathcal{F}_2$ ) be the foliation on  $M \times \mathbf{R}$  whose leaves are of the form  $L \times \mathbf{R}$  (resp.  $L \times \{*\}$ ) whose  $L$  is a leaf of  $\mathcal{F}$ , and let  $i_0, i_1: M \rightarrow M \times \mathbf{R}$  be the injections given by  $i_0(x) = (x, 0)$  and  $i_1(x) = (x, 1)$ . If  $i_0^*, i_1^*: H_{i_g}^*(M \times \mathbf{R}, \mathcal{F}_1) \rightarrow H_{i_g}^*(M, \mathcal{F})$  are the cohomology mappings induced by  $i_0$  and  $i_1$ , then it is easy to show that  $i_0^* = i_1^*$  (see [3]). We will construct a connection  $\nabla$  on the vertical bundle  $V_2$  of  $\mathcal{F}_2$  such that  $\pi_\nu P^k(\nabla) \in A_{i_g}^{2k}(M \times \mathbf{R}, \mathcal{F}_1)$  and such that  $i_0^*([\pi_\nu P^k(\nabla)]) = [\pi_\nu P^k(\nabla^0)]$  and  $i_1^*([\pi_\nu P^k(\nabla)]) = [\pi_\nu P^k(\nabla^1)]$ .

In order to define  $\nabla$  it suffices to define its action on the sections  $Z$  of  $V_2$  that are constant in the  $\mathbf{R}$ -direction. Following [3], we set  $\nabla_{\partial/\partial t} Z = 0$  and  $\nabla_X Z = t\nabla_X^0 Z + (1-t)\nabla_X^1 Z$ , for all  $X \in T_{(x,t)}(M \times \{t\})$ . Then if  $R^0$  (resp.  $R^1$ , resp.  $R$ ) is the curvature of  $\nabla^0$  (resp.  $\nabla^1$ , resp.  $\nabla$ ), one has  $R(X, \partial/\partial t)Z = 0$ , and  $R(X, Y)Z = tR^0(X, Y)Z + (1-t)R^1(X, Y)Z$ , for all sections  $Y$  and  $Z$  of  $V_2$  that are constant in the  $\mathbf{R}$ -direction, and all  $X \in T_{(x,t)}(M \times \{t\})$ . It follows that if  $X$  is perpendicular to  $\mathcal{F}_1$ , then  $R(X, Y)Z = 0$ . It is easily verified then that  $\nabla$  has the required properties.  $\square$

We now consider the tangential Euler class  $\chi_{i_g}(\mathcal{F}) \in H_{i_g}^p$ . Recall that the Euler (or Pfaffian) class is calculated from the invariant polynomial  $(2\pi)^{-p/2} \det^{1/2}$ . Let  $E$  be the bundle of orthonormal vertical frames of  $\mathcal{F}$ .

Then, by the product formula for det, one sees that  $\chi_{ig}(\mathcal{F})$  is zero if there exists on  $E$  a tangential connection reducible to a  $SO(k, \mathbf{R}) \oplus SO(p - k, \mathbf{R})$ -reduction of  $E$ , for some odd integer  $k$ . Expressing this differently, we have the following:

**THEOREM 4.2.** *Suppose that there exists a vector subbundle  $K$  of the vertical bundle  $V$  of  $\mathcal{F}$ , such that, for all vector fields  $X$  tangent to  $K$  and  $Y$  tangent to the horizontal bundle  $H$  of  $\mathcal{F}$ , the vector field  $[X, Y]$  is tangent to  $K \oplus H$ . Then if  $K$  has odd rank, the tangential Euler class  $\chi_{ig}(\mathcal{F})$  of  $\mathcal{F}$  is zero.*

Before proving this result, we give the following immediate consequences:

**COROLLARY 4.3.** *If there exists a nowhere zero tangential vector field on  $M$ , then  $\chi_{ig}(\mathcal{F})$  is zero.*

**COROLLARY 4.4.** *If the sheets of  $\mathcal{F}$ , or their closures, define an odd codimensional foliation on  $M$ , then  $\chi_{ig}(\mathcal{F})$  is zero.*

**PROOF OF THEOREM 4.2.** Let  $K$  be as in the statement of the theorem. Let  $K^\perp$  be the orthogonal complement of  $K$  in  $V$ , and let  $\pi_K$  (resp.  $\pi_{K^\perp}$ ) be the orthogonal projection of  $V$  onto  $K$  (resp.  $K^\perp$ ). Then if  $\nabla^t$  is the tangential Levi-Civita connection on  $V$ , let  $\nabla$  be the connection on  $V$  defined by  $\nabla = \pi_K \nabla^t \pi_K + \pi_{K^\perp} \nabla^t \pi_{K^\perp}$ . Then to show that  $\chi_{ig}(\mathcal{F})$  is zero, it suffices to prove that  $\nabla$  is a tangential connection.

First note the following consequence of the hypothesis. Let  $X$  (resp.  $Y$ , resp.  $Z$ ) be a vector field on  $M$  tangent to  $H$  (resp.  $K$ , resp.  $K^\perp$ ). Then, since  $g([X, Y], Z) = 0$ , so by Proposition 1.1 (ii), one has  $g([X, Z], Y) = 0$ . Thus  $\nabla_X Y = \nabla_X^t Y = \pi_Y [X, Y]$ , and similarly,  $\nabla_X Z = \pi_Y [X, Z]$ . That is,  $\nabla$  is a vertical connection (see Section 1). It remains to show that if  $R$  is the curvature of  $\nabla$ , then  $R(X, Y) = R(X, Z) = 0$ . But we can easily verify this by calculating  $R(X, Y)\eta$  and  $R(X, Z)\eta$  where  $\eta$  is a vector field tangent to  $K$  or  $K^\perp$ , and using the fact that  $\nabla^t$  is tangential. □

Now let  $L$  be a leaf of  $\mathcal{F}$  and let  $i_L^*: H_{ig}^* \rightarrow H_{\text{de Rham}}^*(L)$  be the homomorphism induced by the natural injection of  $L$  into  $M$ . The tangential Levi-Civita connection  $\nabla^t$ , on the vertical bundle  $V$ , induces on  $L$  the Levi-Civita connection of  $L$  (with the induced Riemannian metric). It follows that the images under  $i_L^*$ , of the tangential characteristic classes, are just the standard characteristic classes of  $L$ .

The following theorem was announced in [8]. Its proof, which is rather long, will be presented in a subsequent paper.

**THEOREM 4.5.** (i) *If  $M$  is compact then  $H_{i_g}^*$  has finite dimension and  $H_{i_g}^p \cong 0$  or  $\mathbf{R}$ . Furthermore, if  $H_{i_g}^p \cong \mathbf{R}$ , then  $H_{i_g}^* \cong H_{i_g}^{p-*}$ .*

(ii) *If  $\mathcal{F}$  has a compact leaf  $L$ , then  $H_{i_g}^* H \cong H_{i_g}^{*-}$  and furthermore, the homomorphism  $i_L^*: H_{i_g}^* \rightarrow H_{\text{de Rham}}^*(L)$  is injective.*

Now, if  $L$  is a compact leaf and if a characteristic class  $P^k(L)$  of  $L$  is zero, then by part (ii) of the previous theorem,  $P_{i_g}^k(\mathcal{F})$  is zero. So we have immediately the following:

**THEOREM 4.6.** *If a Pontrjagin class  $P^k(L)$ , of a compact leaf  $L$  of  $\mathcal{F}$ , is zero, then for all leaves  $L'$  of  $\mathcal{F}$ , the class  $P^k(L')$  is zero.*

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