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INTERPOLATION OF OPERATORS IN LEBESGUE SPACES WITH MIXED NORM AND ITS APPLICATIONS TO FOURIER ANALYSIS

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1. Intoroduction. In this paper we show interpolation theorems for linear operators in Lebesgue spaces with mixed norm and apply them to Fourier analysis.

Let M be a measure space and M^m be an m product space $\prod_{j=0}^{m-1} M_j$ where M_j are copies of M. Let $L^t(L^s)=L^t(M^n; L^s(M^m))$ be a Lebesgue space with mixed norm $\left(\int_{M^n}\left(\int_{M^m}|f|^s\right)^{t/s}\right)^{1/t}$. Let T be a linear operator in $L^t(L^s)$. Under the assumption that T is bounded in $L^t(L^s)$ for every permutation of $\{M_j; j = 0, 1, \cdots, d-1\}$, we discuss the boundedness of T in the space $L^u(M^{m+n})$, where

$$1/u = (m/s + n/t)/(m + n) .$$

Part I deals with interpolation problem. In §2 we introduce auxiliary holomorphic functionals W^z and F^z in $|z| < 1$. We divide the unit circle into several arcs and estimate these functionals by $L^t(M^n; L^s(M^m))$-norm for z in an arc, where the choice of permutation of the measure spaces $\{M_j\}$ depends on the arc. This is the idea to prove our interpolation theorems and they are given in §3. To get bounds of the functionals we restrict the domain of F^z to functions in $\prod_{j=0}^{m+n-1} L^u(M_j)$. As a consequence the domain of the linear operators of Remark 1 and Theorem 3 in §3 are restricted to $\prod L^u(M_j)$, but this condition is unremovable (cf. Remark 4 in §5).

In Part II we shall apply our interpolation theorems to two problems in Fourier analysis which are closely related. In §4 we consider the Riesz-Bochner summing opertor s^ε, $\varepsilon > 0$. For a function f on the d-dimensional Euclidean space $\boldsymbol{R}^d$ the Riesz-Bochner mean $s^\varepsilon(f)$ of order ε is defined by

$$s^\varepsilon(f)^\wedge(\xi) = (1 - |\xi|^2)^\varepsilon \hat{f}(\xi)$$

for $|\xi| < 1$ and $= 0$ otherwise, where $\hat{f}$ is the Fourier transform of f:

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$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}^d}\int_{R^d} f(x)e^{-i\xi x}dx \ .$$

For $d > 1$ the operator s^0 is not L^p-bounded if $p \neq 2$ (Fefferman [5]) but s^0 is L^p-bounded if it is restricted to radial functions and if $2d/(d+1) < p < 2d/(d-1)$ (Herz [8]). For $d = 2$, $\varepsilon > 0$ and $4/3 \leqq p \leqq 4$, the operator s^ε is L^p-bounded (Carleson and Sjölin [2]). Later, several proofs of the Carleson-Sjölin theorem were given (Hörmander [9], Fefferman [6] and Cordoba [4]).

In §§4.1–4.4 we shall prove that if $\varepsilon > 0$

$$\int_{R^2}\left(\int_{R^{d-2}}|s^\varepsilon(f)|^2dx(p)\right)^{4/2}dx(\complement p) \leqq \text{const}\int_{R^2}\left(\int_{R^{d-2}}|f|^2dx(p)\right)^{4/2}dx(\complement p) \tag{1.1}$$

for all reasonable function f on $\boldsymbol{R}^d$, where $x(p) = (x_0, \cdots, x_{d-3})$ and $x(\complement p) = (x_{d-2}, x_{d-1})$. Since the operator s^ε is rotation invariant (1.1) holds for any permutation of variables $(x_0, x_1, \cdots, x_{d-1})$. Therefore our interpolation theorem applies to s^ε and we get the inequality $\|s^\varepsilon(f)\|_p \leqq \text{const}\|f\|_p$ if $\varepsilon > 0$, $2d/(d+1) \leqq p \leqq 2$ and if f is of the form $f_0(x_0)f_1(x_1)\cdots f_{d-1}(x_{d-1})$.

In §5 we consider the restriction problem of Fourier transform.

In the following, C will denote constants which may be different in each occasion and $\mathscr{S}(\boldsymbol{R}^d)$ the set of functions in $\boldsymbol{R}^d$ infinitely differentiable and rapidly decreasing.

Part I. Interpolation of operators.

2. Notations and auxiliary functions. Let $(M, \mathscr{M}, \mu)$ be a σ-finite measure space. Let d be a positive integer and $(M_j, \mathscr{M}_j, \mu_j)$, $j = 0, 1, \cdots, d-1$, be copies of $(M, \mathscr{M}, \mu)$. Let $(\bar{M}, \bar{\mathscr{M}}, \bar{\mu})$ be the product measure space $\prod_{j=0}^{d-1}(M_j, \mathscr{M}_j, \mu_j)$. For a subset $p = \{p_0, p_1, \cdots, p_{m-1}\}$ of $\{0, 1, \cdots, d-1\}$ we denote $(M(p), \mathscr{M}(p), \mu(p)) = \prod_{j=0}^{m-1}(M_{p_j}, \mathscr{M}_{p_j}, \mu_{p_j})$. For a subset p of $\{0, 1, \cdots, d-1\}$ denote $\complement p = \{0, 1, \cdots, d-1\} - p$. Thus $d\mu(p)(x_{p_0}, \cdots, x_{p_{m-1}}) = d\mu_{p_0}(x_{p_0}) \cdots d\mu_{p_{m-1}}(x_{p_{m-1}})$ and $d\mu(p)\times d\mu(\complement p) = d\bar{\mu}$.

For $1 \leqq s < \infty$ $L^s(\bar{M})$ denotes the Lebesgue space with norm $\|f\|_s = \left(\int_{\bar{M}}|f|^s d\bar{\mu}\right)^{1/s}$. For $1 \leqq s, t < \infty$ and $p = \{p_0, p_1, \cdots, p_{m-1}\}$ $L^t(M(\complement p); L^s M(p))$ denotes the Lebesgue space with mixed norm

$$\|f\|_{(t,s:p)} = \left(\int_{M(\complement p)}\left(\int_{M(p)}|f|^s d\mu(p)\right)^{t/s} d\mu(\complement p)\right)^{1/t} .$$

The definition for $s = \infty$ or/and $t = \infty$ will be obvious.

Let $d = m + n$, where m and n are positive integers. Define u by

$$1/u = (m/s + n/t)/d \ .$$

For $1 \leqq s \leqq \infty$, s' will denote the conjugate exponent defined by $1/s + 1/s' = 1$.

For simple functions w and f in $(\bar{M}, \bar{\mathscr{M}}, \bar{\mu})$ we shall define holomorphic functions $W^z(x)$ and $F^z(x)$ and estimate them in mixed Lebesgue spaces. Our arguments are divided into two cases.

2.1. The case $m \geqq n$ and $\infty \geqq t \geqq s \geqq 1$. Let P be the family of index sets p of $\{0, 1, \cdots, d-1\}$ with $\operatorname{card}(p) = m$. Let $Q = \{p \in P; 0 \in p\}$ and $R = P - Q$. For $q \in Q$ put $R^q = \{r \in R; \operatorname{card}(q \cap r) = m - n\}$. Then we have

$$\operatorname{card}(P) = \binom{d}{m}, \quad \operatorname{card}(Q) = \binom{d-1}{m-1}, \quad \operatorname{card}(R) = \binom{d-1}{m}$$

and

$$\operatorname{card}(R^q) = \binom{m-1}{n-1}.$$

Divide the unit circle $\partial\boldsymbol{D}$ into $\binom{d}{m}$ congruent arcs I_p, $p \in P$.

Let $\alpha_0(z)$ and $\alpha_q(z)$, $q \in Q$, be functions in the Hardy space $H^2(\boldsymbol{D})$ in the unit disk $\boldsymbol{D}$ having the following properties:

$$\operatorname{Re}\alpha_0(e^{i\theta}) = \begin{cases} 1/s & \text{a.e. in } \bigcup_{q \in Q} I_q \\ 1/t & \text{a.e. in } \bigcup_{r \in R} I_r \end{cases}$$

and

$$\operatorname{Re}\alpha_q(e^{i\theta}) = \begin{cases} 1/t - 1/s & \text{a.e. in } I_q \\ (1/s - 1/t)/(\operatorname{card}(R^q)) & \text{a.e. in } \bigcup_{r \in R^q} I_r \\ 0 & \text{a.e. in } \partial\boldsymbol{D} - I_q - \bigcup_{r \in R^q} I_r \end{cases}$$

for each $q \in Q$. Furthermore we assume

$$\operatorname{Im}\alpha_0(0) = \operatorname{Im}\alpha_q(0) = 0 .$$

By the mean value theorem we have

$$\alpha_0(0) = [\operatorname{card}(Q)/s + \operatorname{card}(R)/t]\frac{1}{\operatorname{card}(P)} = (m/s + n/t)/d = 1/u \tag{2.1}$$

and

$$\alpha_q(0) = \frac{1/t - 1/s}{\operatorname{card}(P)} + \frac{1/s - 1/t}{\operatorname{card}(R^q)}\,\frac{\operatorname{card}(R^q)}{\operatorname{card}(P)} = 0 . \tag{2.2}$$

For a non zero function w in $L^u(\bar{M})$ and $z \in \boldsymbol{D}$ define

$$W^z(x) = A_w(z)e^{i\arg w(x)}|w(x)|^{u\alpha_0(z)}\prod_{q\in Q}\left(\int_{M(q)}|w(x)|^u d\mu(q)\right)^{\alpha_q(z)} \tag{2.3}$$

if $w(x)\neq 0$ and $=0$ otherwise, where

$$A_w(z) = \|w\|_u^{u\gamma(z)}$$

and

$$\gamma(z) = \frac{m}{n}\left(\alpha_0(z) - \frac{1}{s}\right) + \left(\frac{1}{u} - \frac{1}{t}\right).$$

For a non zero function f in $L^{u'}(\bar{M})$ with $1/u' = 1 - 1/u$ and for $z \in \boldsymbol{D}$ define

$$F^z(x) = B_f(z)e^{i\arg f(x)}|f(x)|^{u'(1-\alpha_0(z))}\prod_{q\in Q}\left(\int_{M(q)}|f(x)|^{u'}d\mu(q)\right)^{-\alpha_q(z)} \tag{2.4}$$

if $f(x)\neq 0$ and $=0$ otherwise, where

$$B_f(z) = \|f\|_{u'}^{-u'\gamma(z)}.$$

LEMMA 1. *Let $d = m + n$, $m \geqq n \geqq 1$ and $\infty \geqq t \geqq s \geqq 1$. Let w and f be non zero functions $L^u(\bar{M})$ and $L^{u'}(\bar{M})$ respectively. We have the followings.*

(i) *$W^0(x) = w(x)$ and $F^0(x) = f(x)$.*

(ii) *Let $p \in P$ and $z \in \operatorname{int}(I_p)$. Then*

$$\|W^z\|_{(t,s:p)} \leqq \|w\|_u.$$

(iii) *Furthermore if f is of the form $f_0(x_0)f_1(x_1)\cdots f_{d-1}(x_{d-1})$,*

$$\|F^z\|_{(t',s';p)} = \|f\|_{u'}.$$

PROOF. (i) follows easily from (2.1) and (2.2). To prove (ii) and (iii) we assume $1 < s < t < \infty$. A proof for other cases is similar.

(ii) Assume $q \in Q$ and $z \in \operatorname{int}(I_q)$. Then

$$|W^z(x)|^s = |A_w(z)|^s|w(x)|^u\left(\int|w(x)|^u d\mu(q)\right)^{s/t-1}$$

from which we get (ii).

Next suppose $r \in R$ and $z \in \operatorname{int}(I_r)$. $\operatorname{Re}\alpha_q(z) = 0$ for q such that $r \notin R^q$ and $\operatorname{Re}\alpha_q(z) = v$ for q such that $r \in R^q$, where

$$v = (1/s - 1/t)\Big/\binom{m-1}{n-1}.$$

Therefore

$$|W^z(x)|^s = |A_w(z)|^s|w(x)|^{us/t}\prod_{q\in S}\left(\int|w(x)|^u d\mu(q)\right)^{sv},$$

where

$$S = \{q \in Q \,;\, r \in R^q\} . \tag{2.5}$$

Let $a \in r$ and put

$$A = \{q \in S \,;\, a \notin q\} \quad \text{and} \quad B = \{q \in S \,;\, a \in q\} . \tag{2.6}$$

Then

$$\operatorname{card}(S) = \binom{m}{n}, \quad \operatorname{card}(A) = \binom{m-1}{n-1} \tag{2.7}$$

and

$$\operatorname{card}(B) = \binom{m-1}{n} \tag{2.8}$$

if $m > n$ and $=0$ if $m = n$.

Applying Hölder's inequality with exponents $(s/t) + sv \cdot \operatorname{card}(A) = 1$ we have

$$\begin{aligned}
&\int |W^z(x)|^s d\mu_a \\
&\quad = |A_w(z)|^s \prod_{q \in B} \left(\int |w(x)|^u d\mu(q)\right)^{sv} \int |w(x)|^{us/t} \prod_{q \in A} \left(\int |w(x)|^u d\mu(q)\right)^{sv} d\mu_a \\
&\quad \leqq |A_w(z)|^s \prod_{q \in B} \left(\int |w(x)|^u d\mu(q)\right)^{sv} \prod_{q \in A} \left(\int |w(x)|^u d\mu(q) d\mu_a\right)^{sv} \left(\int |w(x)|^u d\mu_a\right)^{s/t} .
\end{aligned}$$

Iterating this process for all a in r we get

$$\int |W^z(x)|^s d\mu(r) \leqq |A_w(z)|^s \left(\int |w|^u d\mu(r)\right)^{s/t} \left(\int |w|^u d\bar{\mu}\right)^{sv\binom{m}{n}} .$$

Since $sv\binom{m}{n} = (1 - s/t)m/n$ and $|A_w(z)| = \|w\|_u^{1-u/t-u(1/s-1/t)m/n}$, we have

$$\int \left(\int |W^z|^s d\mu(r)\right)^{t/s} d\mu(\complement r) \leqq \left(\int |w|^u d\bar{\mu}\right)^{t/u} ,$$

which proves (ii).

(iii) Let $f(x) = f_0(x_0) f_1(x_1) \cdots f_{d-1}(x_{d-1})$ be a non zero function in $L^{u'}(\bar{M})$. If $q \in Q$ and $z \in \operatorname{int}(I_q)$, then

$$|F^z(x)|^{s'} = |B_f(z)|^{s'} |f(x)|^{u'} \left(\int |f(x)|^{u'} d\mu(q)\right)^{s'/t'-1} .$$

Since $|B_f(z)| = \|f\|_u^{1-u'/t'}$,

$$\int \left(\int |F^z|^{s'} d\mu(q)\right)^{t'/s'} d\mu(\complement q) = |B_f(z)|^{t'} \int |f(x)|^{u'} d\bar{\mu} = \|f\|_{u'}^{t'} .$$

Assume $r \in R$ and $z \in \operatorname{int}(I_r)$. Then

$$|F^z(x)|^{s'} = |B_f(z)|^{s'} |f(x)|^{u's'/t'} \prod_{q \in s} \left(\int |f(x)|^{u'} d\mu(q) \right)^{-s'v}, \tag{2.9}$$

where S is defined by (2.5). Let $r = \{r_0, r_1, \cdots, r_{m-1}\}$. Put $f(r)(x) = f_{r_0}(x_{r_0}) f_{r_1}(x_{r_1}) \cdots f_{r_{m-1}}(x_{r_{m-1}})$ and define $f(\complement r)$ for $\complement r = \{0, 1, \cdots, d-1\} - r$ similarly. Suppose $a \in r$. Since $\{q \in S ; a \notin q\} = A$ by definition, the exponent of $f_a(x_a)$ in (2.9) equals $u's'/t' - u's'v \operatorname{card}(A) = u'$. On the other hand since $\{q \in S ; a \in q\} = B$, the exponent of $\int |f_a|^{u'} d\mu_a$ is $-s'v \operatorname{card}(B) = (1 - s'/t')(m-n)/n$ by (2.8). Next consider the case $a \notin r$. We remark that $a \notin r$ implies that $a \in q$ for every $q \in S$. Thus the exponent of $f_a(x_a)$ in (2.9) is $u's'/t'$ and that of $\int |f_a|^{u'} d\mu_a$ is $-s'v \operatorname{card}(S) = (1 - s'/t')m/n$. Therefore we have

$$|F^z(x)|^{s'} = |B_f(z)|^{s'} |f(r)(x)|^{u'} \left(\int |f(r)|^{u'} d\mu(r) \right)^{(1-s'/t')(m-n)/n}$$
$$\times |f(\complement r)(x)|^{u's'/t'} \left(\int |f(\complement r)|^{u'} d\mu(\complement r) \right)^{(1-s'/t')m/n}.$$

Thus

$$\int \left(\int |F^z|^{s'} d\mu(r) \right)^{t'/s'} d\mu(\complement r) = |B_f(z)|^{t'} \left(\int |f|^{u'} d\bar{\mu} \right)^{1 + (t'/s' - 1)m/n}. \tag{2.10}$$

Since $|B_f(z)|^{t'} = \|f\|_{u'}^{-t'u'\gamma(z)} = \|f\|_{u'}^{-u' + t' - u'(t'/s' - 1)m/n}$, the right hand side of (2.10) equals $\|f\|_{u'}^{t'}$, which proves (iii).

2.2. The case $n > m$ and $\infty \geqq t \geqq s \geqq 1$. Let

$$d = mk + r$$

where $k \geqq 2$ and $m \geqq r > 0$, so that $n = m(k-1) + r$.

We define a family P of m integers $p^a = \{p_0^a, p_1^a, \cdots p_{m-1}^a\}$, $0 \leqq a < d$, as follows. If $0 \leqq j < k$ and $0 \leqq b < m - r$ or if $0 \leqq j < k - 1$ and $m - r \leqq b < m$, then

$$p_c^{mj+b} \equiv mj + b + c \mod d.$$

If $j = k - 1$ and $m - r \leqq b < m$ define

$$\begin{aligned} p^{m(k-1)+b} = {} & \{m(k-1) + b, m(k-1) + b + 1, \cdots, mk - 1\} \\ & \cup \{m(k-1) + b + r, m(k-1) + b + r + 1, \cdots, mk + b - 1\} \\ & \cup \{m(k+1), m(k+1) + 1, \cdots, m(k+1) + b - m + r - 1\} \\ & \mod d. \end{aligned}$$

For $mk \leqq a < d$ put

$$p^a = \{mk, mk+1, \cdots, mk+m-1\} \mod d .$$

We remark the followings.

(2.11) For each $l = 0, 1, \cdots, m-1$, $p^{mj+l}(j = 0, 1, \cdots, k-1)$ are mutually disjoint and $\operatorname{card}(\cup_{j=0}^{k-1} p^{mj+l} \cup p^a) = d$ if $mk \leqq a < d$.

(2.12) If $a = mk + b \mod d$ for some $b = 0, 1, \cdots, m-1$, then

$$\operatorname{card}\{l;\, a \notin p^l \cup p^{m+l} \cup \cdots \cup p^{m(k-1)+l},\, 0 \leqq l < m\} = r .$$

Let $\alpha_0(z)$ and $\alpha_a(z)$, $m \leqq a < m(k+1)$, be functions in $H^2(\boldsymbol{D})$ which satisfy the following conditions: Divide the unit circle $\partial\boldsymbol{D}$ into d congruent arcs I_a, $a = 0, 1, \cdots, d-1$. We choose α_0 and α_a so that

$$\operatorname{Re}\alpha_0(e^{i\theta}) = \begin{cases} 1/s & \text{a.e. in } \bigcup_{l=0}^{m-1} I_l \\ 1/t & \text{a.e. in } \partial\boldsymbol{D} - \bigcup_{l=0}^{m-1} I_l , \end{cases}$$

for $1 \leqq j < k$ and $0 \leqq l < m$,

$$\operatorname{Re}\alpha_{mj+l}(e^{i\theta}) = \begin{cases} 1/t - 1/s & \text{a.e. in } I_{m(j-1)+l} \\ 1/s - 1/t & \text{a.e. in } I_{mj+l} \\ 0 & \text{a.e. in } \partial\boldsymbol{D} - I_{m(j-1)+l} \cup I_{mj+l} \end{cases}$$

and for $0 \leqq l < m$

$$\operatorname{Re}\alpha_{mk+l}(e^{i\theta}) = \begin{cases} 1/t - 1/s & \text{a.e. in } I_{m(k-1)+l} \\ (1/s - 1/t)/r & \text{a.e. in } \bigcup_{b=0}^{r-1} I_{mk+b} \\ 0 \quad \text{a.e. in } \partial\boldsymbol{D} - I_{m(k-1)+l} \cup \bigcup_{b=0}^{r-1} I_{mk+b} . \end{cases}$$

Furthermore we assmme that

$$\operatorname{Im}\alpha_0(0) = \operatorname{Im}\alpha_a(0) = 0 .$$

Then we have

$$\alpha_0(0) = 1/u \quad \text{and} \quad \alpha_a(0) = 0 \quad \text{for} \quad m \leqq a < m(k+1) .$$

Let γ be a function in $H^2(\boldsymbol{D})$ such that $\operatorname{Re}\gamma(e^{i\theta}) = 1/u - 1/t$ a.e. in $\cup_{a=0}^{mk-1} I_a$ and $=(1/u - 1/t) - (1/s - 1/t)m/r$ a.e. in $\cup_{a=mk}^{d-1} I_a$, and $\operatorname{Im}\gamma(0) = 0$. Then we have $\gamma(0) = 0$.

For a non zero function w in $L^u(\bar{M})$ define

(2.13)
$$W^z(x) = A_w(z)\,|w(x)|^{u\alpha_0(z)} e^{ia\arg w(x)} \times \prod_{l=0}^{m-1} \prod_{j=1}^{k} \Big(\int |w|^u d\mu(p^l \cup p^{m+l} \cup \cdots \cup p^{m(j-1)+l})\Big)^{\alpha_{mj+l}(z)}$$

if $w(x) \neq 0$ and $=0$ otherwise, where $A_w(z) = \|w\|_u^{u\gamma(z)}$. For a non zero function f in $L^{u'}(\bar{M})$ put

$$F^z(x) = B_f(z)|f(x)|^{u'(1-\alpha_0(z))}e^{i\arg f(x)} \times \prod_{l=0}^{m-1}\prod_{j=1}^{k}\left(\int|f|^{u'}d\mu(p^l \cup p^{m+l} \cup \cdots \cup p^{m(j-1)+l})\right)^{-\alpha_{mj+l}(z)} \tag{2.14}$$

if $f(x) \neq 0$ and $=0$ otherwise, where $B_f(z) = \|f\|_{u'}^{-u'\gamma(z)}$.

LEMMA 2. *Let $d = m + n$, $n > m \geqq 1$ and $\infty \geqq t \geqq s \geqq 1$. Let w and f be non zero functions in $L^u(\bar{M})$ and $L^{u'}(\bar{M})$ respectively. We have the followings.*

(i) $$W^0(x) = w(x) \quad \text{and} \quad F^0(x) = f(x)\,.$$

(ii) *Let $0 \leqq a < d$ and $z \in \operatorname{int}(I_a)$. Then*

$$\|W^z\|_{(t,s:p^a)} \leqq \|w\|_u\,.$$

(iii) *Furthermore if f is of the form $f_0(x_0)f_1(x_1)\cdots f_{d-1}(x_{d-1})$,*

$$\|F^z\|_{(t',s':p^a)} = \|f\|_{u'}\,.$$

PROOF. (i) is obvious. To prove (ii) and (iii) we assume $1 < s < t < \infty$. The other case is proved similarly.

(ii) First suppose that $0 \leqq b < m$ and $z \in \operatorname{int}(I_b)$. If $mj + b \neq 0$ and if $\alpha_{mj+l}(z) \neq 0$ then $j = 1$, $l = b$ and $\alpha_{m+b}(z) = 1/t - 1/s$. Thus

$$|W^z(x)|^s = |A_w(z)|^s|w(x)|^u\left(\int|w|^u d\mu(p^b)\right)^{s/t-1}\,.$$

Since $|A_w(z)| = \|w\|_u^{1-u/t}$,

$$\int\left(\int|W^z|^s d\mu(p^b)\right)^{t/s}d\mu(\complement p^b) = |A_w(z)|^t\int|w|^u d\mu(p^b)d\mu(\complement p^b) = \left(\int|w|^u d\bar{\mu}\right)^{t/u}\,.$$

Next suppose that $0 \leqq b < m$, $1 \leqq j < k$ and $z \in \operatorname{int}(I_{mj+b})$. Remark that $\operatorname{Re}\alpha_0(z) = 1/t$, $\operatorname{Re}\alpha_{m(j+1)+b}(z) = 1/t - 1/s$, $\operatorname{Re}\alpha_{mj+b}(z) = 1/s - 1/t$ and $\operatorname{Re}\alpha_a(z) = 0$ for $a \neq 0, mj + b, m(j + 1) + b$. Thus

$$|W^z(x)|^s = |A_w(z)|^s|w(x)|^{us/t}\left(\int|w|^u d\mu(p^b \cup p^{m+b} \cup \cdots \cup p^{m(j-1)+b})\right)^{1-s/t} \times \left(\int|w|^u d\mu(p^b \cup p^{m+b} \cup \cdots \cup p^{mj+b})\right)^{s/t-1}\,.$$

Integrate both sides with respect to $d\mu(p^{mj+b})$ and apply Hölder's inequality with exponents $s/t + (1 - s/t) = 1$. Then we get

$$\int|W^z|^s d\mu(p^{mj+b}) \leqq |A_w(z)|^s\left(\int|w|^u d\mu(p^{mj+b})\right)^{s/t},$$

which proves (ii) by the same way as above.

Finally suppose that $0 \leqq b < r$ and $z \in \operatorname{int}(I_{mk+b})$. By definition

$$(2.15) \qquad |W^z(x)|^s = |A_w(z)|^s |w(x)|^{us/t} \times \prod_{l=0}^{m-1} \left(\int |w|^u d\mu(p^l \cup p^{m+l} \cup \cdots \cup p^{m(k-1)+l})\right)^{(1-s/t)/r} .$$

Recall that $p^{mk+b} = \{mk, mk+1, \cdots, mk+m-1\} \bmod d$. Let $a \in p^{mk+b}$ and set $A = \{l; a \notin p^l \cup p^{m+l} \cup \cdots \cup p^{m(k-1)+l}, 0 \leqq l < m\}$ and $B = \{l \notin A; 0 \leqq l < m\}$. Then by (2.12) $\operatorname{card}(A) = r$. Furthermore we have $p^l \cup {}^{m+l} \cup \cdots \cup p^{m(k-1)+l} \cup p^{mk+b} = \{0, 1, \cdots, d-1\}$. Applying Hölder's inequality to (2.15) with exponents $s/t + r \cdot (1 - s/t)/r = 1$, we have

$$\int |W^z|^s d\mu_a \leqq |A_w(z)|^s \left(\int |w|^u d\mu_a\right)^{s/t} \times \prod_{l \in A} \left(\int |w|^u d\mu(p^l \cup p^{m+l} \cup \cdots \cup p^{m(k-1)+l} \cup \{a\})\right)^{(1-s/t)/r} \times \prod_{l \in B} \left(\int |w|^u d\mu(p^l \cup p^{m+l} \cup \cdots \cup p^{m(k-1)+l})\right)^{(1-s/t)/r} .$$

Iterating this process for all $a \in p^{mk+b}$ and using (2.11), we get finally

$$\int |W^z|^s d\mu(p^{mk+b}) \leqq |A_w(z)|^s \left(\int |w|^u d\mu(p^{mk+b})\right)^{s/t} \left(\int |w|^u d\bar{\mu}\right)^{(1-s/t)m/r} .$$

Integrating both sides with respect to $d\mu(\complement p^{mk+b})$ we get

$$\int \left(\int |W^z|^s d\mu(p^{mk+b})\right)^{t/s} d\mu(\complement p^{mk+b}) \leqq \left(\int |w|^u d\bar{\mu}\right)^{t/s} ,$$

since $|A_w(z)| = \|w\|_u^{(1-u/t)-u(1/s-1/t)m/r}$, which proves (ii).

To prove the equality (iii) we consider only the case $z \in \operatorname{int}(I_{mk+b})$ with $0 \leqq b < r$. A proof for other cases follows from similar arguments to (ii). By the definition (2.14)

$$|F^z(x)|^{s'} = |B_f(z)|^{s'} |f(x)|^{u's'/t'} \times \prod_{l=0}^{m-1} \left(\int |f|^{u'} d\mu(p^l \cup p^{l+m} \cup \cdots \cup p^{m(k-1)+l})\right)^{(1-s'/t')/r} .$$

Assume $a \in p^{mk+b}$ and A is the set defined in (ii). Since $\operatorname{card}(A) = r$ the exponent of $f_a(x_a)$ in $|F^z|^{s'}$ is $u's'/t' + r \cdot u'(1 - s'/t')/r = u'$ and the one of $\int |f_a|^{u'} d\mu_a$ equals $(m - r)(1 - s'/t')/r$. If $a \notin p^{mk+b}$ the term containing f_a in $|F^z(x)|^{s'}$ is

$$|f_a(x_a)|^{u's'/t'} \left(\int |f_a|^{u'} d\mu_a\right)^{m(1-s'/t')/r} .$$

Thus

$$\int |F^z|^{s'} d\mu(p^{mk+b}) = |B_f(z)|^{s'} |f(\complement p^{mk+b})(x)|^{u's'/t'}$$
$$\times\left(\int |f(p^{mk+b})|^{u'} d\mu(p^{mk+b})\right)^{1+(m-r)(1-s'/t')/r}$$
$$\times\left(\int |f(\complement p^{mk+b})|^{u'} d\mu(\complement p^{mk+b})\right)^{m(1-s'/t')r} .$$

Since $|B_f(z)| = \|f\|_{u'}^{u'[(1/u'-1/t')-(1/s'-1/t')m/r]}$, we get

$$\int\left(\int |F^z|^{s'} d\mu(p^{mk+b})\right)^{t'/s'} d\mu(\complement p^{mk+b}) = \left(\int |f|^{u'} d\bar{\mu}\right)^{t'/u'}$$

which proves (iii).

REMARK 1. If $d = mk$ for some $k \geqq 3$, then Lemma 2.2 (iii) holds for functions f of the form

$$f_0(x_0, \cdots, x_{m-1}) f_1(x_m, \cdots x_{2m-1}) \cdots f_{k-1}(x_{m(k-1)}, \cdots, x_{mk-1}) .$$

3. Interpolation theorems. Let $d = m + n$. Let P be the family of m integers defined in 2.1 or 2.2 according to $m \geqq n$ or $m < n$ and I_p, $p \in P$, be arcs in §2. Let $(M, \mathscr{M}, \mu)$ and $(N, \mathscr{N}, \nu)$ be σ-finite measure spaces. $(M(p), \mathscr{M}(p), \mu(p))$, $(\bar{M}, \bar{\mathscr{M}}, \bar{\mu})$, $(N(p), \mathscr{N}(p), \nu(p))$ and $(\bar{N}, \bar{\mathscr{N}}, \bar{\nu})$ will denote the product spaces defined in §2.

THEOREM 1. *Let T be a linear operator of simple functions on $(\bar{M}, \bar{\mathscr{M}}, \bar{\mu})$ to measurable functions on $(N, \mathscr{N}, \nu)$. Let $v(e^{i\theta})$ be a measurable function in $\partial \boldsymbol{D}$ such that $1 \leqq v(e^{i\theta}) \leqq \infty$. Define v by*

$$1/v = \int_{\partial D} (1/v(e^{i\theta})) \frac{d\theta}{2\pi} .$$

Let $1 \leqq u_0 \leqq u_1 \leqq \infty$ and

$$1/u = (m/u_0 + n/n_1)/d . \tag{3.1}$$

Suppose that

$$\| Tw \|_{v(e^{i\theta})} \leqq C(e^{i\theta}) \| w \|_{(u_1, u_0 : p)} \tag{3.2}$$

for simple functions w and $e^{i\theta} \in \operatorname{int}(I_p)$, $p \in P$, where $C(e^{i\theta})$ is measurable on $\partial \boldsymbol{D}$. Then

$$\| Tw \|_v \leqq C \|w\|_u , \tag{3.3}$$

where

$$C = \exp \int_{\partial D} \log C(e^{i\theta}) \frac{d\theta}{2\pi} .$$

REMARK 1. If $1 \leqq u_1 \leqq u_0 \leqq \infty$, (3.3) holds for w of the form $w_0(x_0)w_1(x_1) \cdots w_{d-1}(x_{d-1})$ under the assumption (3.2) with w as above. This is a consequence of Lemma 2 (see also [1]), but in general we cannot conclude (3.3) for all w in $L^u(\bar{M})$. We shall give a counter example in §5, Remark 4.

REMARK 2. The family of the spaces $L^{v(e^{i\theta})}(N)$ in Theorem 1 is replaced by more general family of Banach spaces $B[z]$, $z \in \partial \boldsymbol{D}$, which is introduced by Coifman, Cwikel, Rochberg, Sagher and Weiss [3].

PROOF. Let $P_z(e^{i\theta})$, $z = re^{i\tau} \in \boldsymbol{D}$, be the Poisson kernel $(1-r^2)/(1-2r\cos(\theta - \tau) + r^2)$. Denote by $v'(e^{i\theta})$ the conjugate exponent of $v(e^{i\theta})$. Let $V'(z)$ be a holomorphic function in $\boldsymbol{D}$ such that

$$\operatorname{Re} V'(e^{i\theta}) = 1/v'(e^{i\theta}) = 1 - 1/v(e^{i\theta}) \quad \text{a.e.}$$

and

$$\operatorname{Im} V'(0) = 0 \,.$$

Such a function exists in the space $H^2(\boldsymbol{D})$ and $V'(0) = 1 - 1/v = 1/v'$.

Suppose w and g are non null simple functions in N and $\|g\|_{v'} = 1$. Define W^z by (2.3) or (2.13) and $G^z = e^{i\arg g(x)}|g|^{v'V'(z)}$. Put

$$\Phi(z) = \int_N TW^z \cdot G^z d\nu \,.$$

$\Phi(z)$ belongs to the class $N_+(\boldsymbol{D})$, which consists of holomorphic functions ϕ in $\boldsymbol{D}$ such that $\sup_{0<r<1}\int_{\partial D} \log^+|\phi(re^{i\theta})|d\theta < \infty$ and

$$\log|\phi(z)| \leqq \int_{\partial D} \log|\phi(e^{i\theta})|P_z(e^{i\theta})\frac{d\theta}{2\pi} \tag{3.4}$$

for $z \in \boldsymbol{D}$ where $\log^+ x = \max(0, \log x)$ (see [3]).

Let $p \in P$ and assume $z = e^{i\theta} \in \operatorname{int}(I_p)$. Then by Hölder's inequality and (3.2)

$$|\Phi(z)| \leqq \|TW^z\|_{v(z)}\|G^z\|_{v'(z)} \leqq C(z)\|W^z\|_{(u_0,u_1:p)}\|G^z\|_{v'(z)} \,.$$

By Lemma 1 or 2 we have $\|W^z\|_{(u_1,u_0:p)} \leqq \|w\|_u$. Furthermore we have $\|G^z\|_{v'(z)} = \|g\|_{v'}^{v'/v'(z)} = 1$. Thus

$$|\Phi(z)| \leqq C(z)\|w\|_u \,.$$

Applying Jensen's inequality (3.4) to Φ and $z = 0$ we get

$$|\Phi(0)| \leqq \exp\int_{\partial D} \log C(e^{i\theta})\frac{d\theta}{2\pi}\|w\|_u \,.$$

Taking supremum over g such that $\|g\|_{v'} = 1$ we get (3.3).

We can generalize Theorem 1 for analytic family of operators $\{T^z\}$, $z \in \boldsymbol{D}$. $\{T^z\}$ is said to be an analytic family if

$$\Phi(z) = \int_N T^z W^z \cdot G^z d\nu$$

belongs to $N_+(\boldsymbol{D})$ for every simple function w and g on N.

THEOREM 1′. *Let $\{T^z\}$ be an analytic family of operators. Under the assumption of Theorem 1 if*

$$\| T^z w \|_{v(z)} \leqq C(z) \| w \|_{(u_1, u_0 : p)} \tag{3.2$'$}$$

for $z = e^{i\theta} \in \operatorname{int}(I_p)$, $p \in P$, then we have

$$\| T^0 w \|_v \leqq C \| w \|_u . \tag{3.3$'$}$$

THEOREM 2. *Let T be a linear operator of simple functions on $\bar{M}$ to measurable functions on $\bar{N}$. Let $1 \leqq u_0 \leqq u_1 \leqq \infty$ and $1 \leqq v_1 \leqq v_0 \leqq \infty$. Suppose that*

$$\| Tw \|_{(v_1, v_0 : p)} \leqq C(p) \| w \|_{(u_1, u_0 : p)}$$

for all w and $p \in P$.

If

$$1/u = (m/u_0 + n/u_1)/d \quad and \quad 1/v = (m/v_0 + n/v_1)/d \tag{3.5}$$

then

$$\| Tw \|_v \leqq C \| w \|_u$$

where $C = (\prod_{p \in P} C(p))^{1/\mathrm{card}(P)}$.

PROOF. Let w and g be non null simple functions on $\bar{M}$ and $\bar{N}$ respectively. Define W^z and G^z by (2.3) with respect to indices (u_0, u_1) and (v_0', v_1') respectively. Put

$$\Phi(z) = \int_{\bar{N}} T W^z \cdot G^z d\bar{\nu} .$$

Obviously $\Phi(z) \in N_+(\boldsymbol{D})$. If $p \in P$ and $z \in \operatorname{int}(I_p)$, by Hölder's inequality and our assumption

$$| \Phi(z) | \leqq \| T W^z \|_{(v_1, v_0 : p)} \| G^z \|_{(v_1', v_0' : p)} \leqq C(p) \| W^z \|_{(u_1, u_0 : p)} \| G^z \|_{(v_1', v_0' : p)} .$$

Since $1 \leqq v_0' \leqq v_1' \leqq \infty$, the last two terms are estimated by Lemma 1 or 2 and consequently the last term is bounded by $C(p) \| w \|_u \| g \|_{v'}$. By Jensen's inequality (3.4) we get

$$| \Phi(0) | \leqq \exp\left(\sum_p \int_{I_p} \log C(p) \frac{d\theta}{2\pi} \right) \| w \|_u \| g \|_{v'} ,$$

which proves our theorem,

Theorem 3. *Let T be a linear operator of simple functions on $\bar{M}$ of the form $f_0(x_0)f_1(x_1)\cdots f_{d-1}(x_{d-1})$ to measurable functions on $\bar{N}$. Let $1 \leqq u_1 \leqq u_0 \leqq \infty$ and $1 \leqq v_1 \leqq v_0 \leqq \infty$. Suppose that*

$$\|Tf\|_{(v_1,v_0:p)} \leqq C(p)\|f\|_{(u_1,u_0:p)}$$

for all simple function f on the above form and $p \in P$.

If u and v are defined by (3.5), then

$$\|Tf\|_v \leqq C\|f\|_u$$

for all f of product form, where $C = \prod_{p\in P} C(p)^{1/\mathrm{card}(P)}$.

Proof. Assume $m \geqq n$. For simple non null functions $w(x)$ on $\bar{M}$ and $f(x)$ on $\bar{N}$ of product form define W^z and F^z by (2.3) and (2.4) with (v_0', v_1') and (u_0, u_1) respectively. Remark that F^z is of product form too. Put $\Phi(z) = \int_{\bar{N}} TF^z\cdot W^z d\bar{\nu}$. If $p \in P$ and $z \in \mathrm{int}\,(I_p)$,

$$|\Phi(z)| \leqq \|TF^z\|_{(v_1,v_0:p)}\|W^z\|_{(v_1',v_0':q)} \leqq C(p)\|F^z\|_{(u_1,u_0:p)}\|W^z\|_{(v_1',v_0':p)}\,.$$

Since $1 \leqq v_0' \leqq v_1' \leqq \infty$, Lemma 1 is applied to the last term and we get our theorem by the same method as in the proof of Theorem 2.

A proof for the case $m < n$ proceeds similarly applying Lemma 2.

Part II. Applications to Fourier analysis.

4. Riesz-Bochner operator. Our aim in this section is to show Theorem 6 in §4.4. The idea of the proof is to estimate $s^\varepsilon(f)$ in the mixed Lebesgue space $L^4(\boldsymbol{R}^2; L^2(\boldsymbol{R}^{d-2}))$ applying the two dimensional argument due to Cordoba [4] and to use our interpolation theorem.

4.1. We introduce the operator s as follows. Let ϕ be a C^∞-function on the real line such that $\mathrm{supp}\,\phi \subset (-1, 1)$ and $\phi \geqq 0$. Fix $0 < \delta < 1/4$. For a function f in $\mathscr{S}(\boldsymbol{R}^d)$ define $s(f)$ by the Fourier transform;

$$s(f)\hat{}(\xi) = \phi((1-|\xi|^2)\delta^{-1})\hat{f}(\xi)\,.$$

Now we shall consider a decomposition of $s(f)$. In the following p denotes the set $\{0, 1, \cdots, d-3\}$ and use the notations $\bar{x} = x(p) = (x_0, x_1, \cdots, x_{d-3})$ and $\bar{\bar{x}} = x(\complement p) = (x_{d-2}, x_{d-1})$ for $x \in \boldsymbol{R}^d$. Let ψ be a C^∞-function on the real line such that $\mathrm{supp}\,\psi \subset (-5, 5)$, $\psi \geqq 0$ and $\psi = 1$ on $(-4, 4)$. For a positive integer k put $\rho_k = 1 - \delta k$ if $0 < \delta k \leqq (2-\sqrt{2})/2$ and $(1-(\sqrt{2}-1+\delta k)^2)^{1/2}$ if $(2-\sqrt{2})/2 < \delta k < 2-\sqrt{2}$. Let $\xi^{k,b}(\complement p) = \rho_k(\cos\delta^{1/2}\rho_k^{-1}b, \sin\delta^{1/2}\rho_k^{-1}b)$ for $b = 0, 1, \cdots$ and put

$$\psi_{k,b}(\xi) = \psi(|\xi(\complement p) - \xi^{k,b}(\complement p)|\delta^{-1/2})\psi((|\xi(\complement p)|^2 - \rho_k^2)\delta^{-1})\,.$$

If $\xi \in \mathrm{supp}\,\phi((1-|\cdot|^2)\delta^{-1})$, then $\psi_{k,b}(\xi) \neq 0$ for some k and b and the number of such (k, b)'s is uniformly bounded. If we put $\Psi_{k,b}(\xi) = \psi_{k,b}(\xi)/\sum_{l,j}\psi_{l,j}(\xi)$ where the denominator does not vanish, then $\{\Psi_{k,b}\}$ is a partition of unity of the support of $\phi((1-|\cdot|^2)\delta^{-1})$.

Let $\{\xi^a(p)\}$ be a sequence in $\boldsymbol{R}^{d-2}$ such that

$$\delta = \inf_{i \neq j} |\xi^i(p) - \xi^j(p)| .$$

Put

$$\psi^a(\xi) = \psi(|\xi(p) - \xi^a(p)|\delta^{-1})$$

and $\Psi^a(\xi) = \psi^a(\xi)/\sum_i \psi^i(\xi)$.

Let us define $s^a_{k,b}(f)$ by

$$s^a_{k,b}(f)\hat{\ }(\xi) = \phi((1-|\xi|^2)\delta^{-1})\Psi^a(\xi)\Psi_{k,b}(\xi)\hat{f}(\xi) .$$

Furthermore put

$$s_{k,b}(f) = \sum_a s^a_{k,b}(f) \quad \text{and} \quad s^a(f) = \sum_{k,b} s^a_{k,b}(f) .$$

We get a decomposition of $s(f)$;

$$s(f) = \sum_{k,b}\sum_a s^a_{k,b}(f) .$$

In the following we denote the Fourier transform with respect to $x(p)$ and $x(\complement p)$ by $\mathscr{F}_p$ and $\mathscr{F}_{\complement p}$ respectively. Thus $\hat{f} = \mathscr{F}_p\mathscr{F}_{\complement p}f$.

LEMMA 3. We have

$$\int_{R^2}\left(\int_{R^{d-2}}|s(f)|^2dx(p)\right)^2 dx(\complement p) \leqq C\int_{R^2}\left(\int_{R^{d-2}}\sum_a\sum_{k,b}|s^a_{k,b}(f)|^2dx(p)\right)^2 dx(\complement p)$$

for f in $\mathscr{S}(\boldsymbol{R}^d)$, where the constant C is independent of f and δ.

PROOF. Since $\mathscr{F}_p s^a_{k,b}(f) = \Psi^a\mathscr{F}_p s_{k,b}(f)$ and since $\operatorname{supp}\Psi^a$ $(a = 1, 2, \cdots)$ intersect at most 10^{d-2} times,

$$\int_{R^{d-2}}|s_{k,b}(f)(\bar{x}, \bar{\bar{x}})|^2d\bar{x} \leqq 10^{d-2}\sum_a\int_{R^{d-2}}|s^a_{k,b}(f)(\bar{x}, \bar{\bar{x}})|^2d\bar{x}$$

for all $\bar{\bar{x}}$. Therefore it suffices to show that

$$\int_{R^2}\left(\int_{R^{d-2}}|s(f)|^2d\bar{x}\right)^2 d\bar{\bar{x}} \leqq C\int_{R^2}\left(\int_{R^{d-2}}\sum_{k,b}|s_{k,b}(f)|^2d\bar{x}\right)^2 d\bar{\bar{x}} . \tag{4.2}$$

Dividing the sum $\sum_{k,b}s_{k,b}(f)$ into 100 sums we may assume $s(f) = \sum s_{100k,b}(f)$. If $\Psi_{100k,a}(\bar{\xi}, \bar{\bar{\xi}})\phi((1-|(\bar{\xi}, \bar{\bar{\xi}})|^2)\delta^{-1}) \neq 0$ for some $\bar{\xi}$, then $\Psi_{100j,b}(\bar{\xi}, \bar{\bar{\eta}})\phi((1-|(\bar{\xi}, \bar{\bar{\eta}})|^2)\delta^{-1}) = 0$ for any $j \neq k, b$ and $\bar{\bar{\eta}}$. Thus the support of $\mathscr{F}_p(\sum_b s_{100k,b}(f))(\cdot, \bar{\bar{x}})$, $k = 1, 2, \cdots$, are disjoint for each $\bar{\bar{x}}$ in $\boldsymbol{R}^2$. In order to prove (4.2) we may assume $s(f) = \sum s_{100k,100b}(f)$ where

$b < \delta^{-1/2}\rho_k/200$. We denote simply $s_{k,b}(f)$ for $s_{100k,100b}(f)$.

By Parseval relation we have

$$\int_{R^{d-2}} |s(f)(\bar{x}, \bar{\bar{x}})|^2 d\bar{x} = \int_{R^{d-2}} \Big| \sum_k \mathscr{F}_p \Big(\sum_b s_{k,b}(f)\Big)(\bar{\xi}, \bar{\bar{x}})\Big|^2 d\bar{\xi}$$
$$= \sum_k \int_{R^{d-2}} \Big| \mathscr{F}_p\Big(\sum_b s_{k,b}(f)\Big)(\bar{\xi}, \bar{\bar{x}})\Big|^2 d\bar{\xi} \ .$$

Thus

$$\int_{R^2}\Big(\int_{R^{d-2}} |s(f)(x)|^2 d\bar{x}\Big)^2 d\bar{\bar{x}}$$
$$= \sum_{k,l} \int_{R^2} d\bar{\bar{x}} \int_{R^{d-2}} \Big|\mathscr{F}_p\Big(\sum_b s_{k,b}(f)\Big)(\bar{\xi}, \bar{\bar{x}})\Big|^2 d\bar{\xi} \int_{R^{d-2}} \mathscr{F}_p\Big(\sum_c s_{l,c}(f)\Big)(\bar{\eta}, \bar{\bar{x}})|^2 d\bar{\eta} \ .$$

Put

$$\sigma_{(k,b;l,c)}(\bar{\xi}, \bar{\eta}, \bar{\bar{\xi}}) = \frac{1}{2\pi}\int_{R^2} \hat{s}_{k,b}(f)(\bar{\xi}, \bar{\bar{\xi}} - \bar{\bar{\eta}})\hat{s}_{l,c}(f)(\bar{\eta}, \bar{\bar{\eta}})d\bar{\bar{\eta}} \ .$$

Then the last sum equals, by convolution relation,

$$\sum_k \sum_{b,b'} \sum_l \sum_{c,c'} \int d\bar{\xi} d\bar{\eta} \int \sigma_{(k,b;l,c)}(\bar{\xi}, \bar{\eta}, \bar{\bar{\xi}}) \overline{\sigma_{(k,b';l,c')}(\bar{\xi}, \bar{\eta}, \bar{\bar{\xi}})} d\bar{\bar{\xi}} \ .$$

SUBLEMMA. *If* $(b, c) \neq (b', c')$, *then*

$$\sigma_{(k,b;l,c)}(\bar{\xi}, \bar{\eta}, \bar{\bar{\xi}}) \cdot \sigma_{(k,b';l,c')}(\bar{\xi}, \bar{\eta}, \bar{\bar{\xi}})) \equiv 0 \ .$$

Granting for a moment this sublemma we have

$$\int\Big(\int |s(f)|^2 d\bar{x}\Big)^2 d\bar{\bar{x}} = \sum_{k,b}\sum_{l,c} \int d\bar{\xi} d\bar{\eta} \int |\sigma_{(k,b;l,c)}(\bar{\xi}, \bar{\eta}, \bar{\bar{\xi}})|^2 \, d\bar{\bar{\xi}} \ ,$$

which is carried by Fourier transform with respect to $\bar{\bar{\xi}}$ and then $\bar{\xi}, \bar{\eta}$ to the sum

$$\sum_{k,b}\sum_{l,c} \int d\bar{\bar{x}} \int |\mathscr{F}_p s_{k,b}(f)(\bar{\xi}, \bar{\bar{x}}) \mathscr{F}_p s_{l,c}(f)(\bar{\eta}, \bar{\bar{x}})|^2 d\bar{\xi} d\bar{\eta}$$
$$= \sum_{k,b}\sum_{l,c} \int d\bar{\bar{x}} \Big(\int |s_{k,b}(f)(x)|^2 d\bar{x}\Big)\Big(\int |s_{l,c}(f)(x)|^2 d\bar{x}\Big)$$
$$= \int_{R^2}\Big(\int_{R^{d-2}} \sum_{k,b} |s_{k,b}(f)|^2 d\bar{x}\Big)^2 d\bar{\bar{x}} \ .$$

Since $\mathscr{F}_p s^a_{k,b}(f) = \Psi^a \mathscr{F}_p s_{k,b}(f)$ and $\{\operatorname{supp} \Psi^a\}$ intersect finitely, the last term does not exceed

$$C\int_{R^2}\Big(\int_{R^{d-2}} \sum_a \sum_{k,b} |s^a_{k,b}(f)|^2 d\bar{x}\Big)^2 d\bar{\bar{x}} \ .$$

Thus a proof is complete.

PROOF OF SUBLEMMA. Fix $\bar{\xi}$ and $\bar{\eta}$ in $\boldsymbol{R}^{d-2}$. Then the sublemma follows from the fact that supp $\hat{s}_{k,b}(f)(\bar{\xi}, \cdot)$ + supp $\hat{s}_{l,c}(f)(\bar{\eta}, \cdot)$, $b, c = 0, 1, \cdots$, are disjoint. In fact, supp $\hat{s}_{k,b}(f)(\bar{\xi}, \cdot)$ is contained in the annulus $\{\rho_k - 5\delta\rho_k^{-1} \leqq |\bar{\bar{\xi}}| \leqq \rho_k + 5\delta\rho_k^{-1}\}$ and a disk of radius $< 5\delta^{1/2}$ centered at $\xi^{k,b}(\complement p)$. Therefore a proof is reduced to the two dimensional case which is well known if $k = l$ (see Fefferman [6]). If $k \neq l$, we can prove it by a similar way with more careful computation.

4.2. Let $M \geqq N_n \geqq 1$ and $\varepsilon_n > 0$ $(n = 1, 2, \cdots)$ and O_n be rotations in $\boldsymbol{R}^d$ which fix the first $d-2$ coordinates. Let $S_n = \{(\bar{x}, x_{d-2}, x_{d-1}) \in \boldsymbol{R}^d;$ $|\bar{x}| < \varepsilon_n M, |x_{d-2}| < \varepsilon_n N_n, |x_{d-1}| < \varepsilon_n\}$ and $R_n = O_n S_n$. For a function f on $\boldsymbol{R}^d$ define $M_n(f)$ by

$$M_n(f)(x) = \frac{1}{|R_n|}\int_{R_n} f(x-y)dy \,.$$

LEMMA 4. *There exists a constant C such that*

$$\int_{R^2}\left(\int_{R^{d-2}} \sum_n |M_n(f_n)|^2 dx(p)\right)^2 dx(\complement p) \leqq C(\log M)^8 \int_{R^2}\left(\int_{R^{d-2}} \sum_n |f_n|^2 dx(p)\right)^2 dx(\complement p)$$

for all f_n in $\mathscr{S}(\boldsymbol{R}^d)$ and $M > 2$.

PROOF. Let w be a non-negative function on $\boldsymbol{R}^2$ and put

$$I = \int_{R^2}\int_{R^{d-2}} \sum_n |M_n f_n(\bar{x}, \bar{\bar{x}})|^2 w(\bar{\bar{x}}) d\bar{x} d\bar{\bar{x}} \,.$$

By Schwarz's inequality

$$|M_n(f_n)(\bar{x}, \bar{\bar{x}})|^2 \leqq \frac{1}{|R_n|}\int |f_n(\bar{y}, \bar{\bar{y}})|^2 \chi_{R_n}(\bar{x}-\bar{y}, \bar{\bar{x}}-\bar{\bar{y}}) d\bar{y} d\bar{\bar{y}} \,.$$

Substituting this inequality we have

$$\int_{R^d} | M_n f_n(\bar{x}, \bar{\bar{x}})^2 w(\bar{\bar{x}}) d\bar{x} d\bar{\bar{x}}$$

$$\leqq \int |f_n(\bar{y}, \bar{\bar{y}})|^2 d\bar{y} d\bar{\bar{y}} \frac{1}{|R_n|}\int \chi_{R_n}(\bar{x}-\bar{y}, \bar{\bar{x}}-\bar{\bar{y}}) w(\bar{\bar{x}}) d\bar{x} d\bar{\bar{x}} \,.$$

Put

$$W(\bar{\bar{y}}) = \sup_n \frac{1}{|R_n|}\int_{R^2}\left(\int_{R^{d-2}} \chi_{R_n}(\bar{x}, \bar{\bar{x}}) d\bar{x}\right) w(\bar{\bar{y}} + \bar{\bar{x}}) d\bar{\bar{x}} \,.$$

Then the last integral is bounded by

$$\int_{R^2}\left(\int_{R^{d-2}} |f_n(\bar{y}, \bar{\bar{y}})|^2 d\bar{y}\right) W(\bar{\bar{y}}) d\bar{\bar{y}} \,.$$

Thus

$$I \leqq \int\Bigl(\int \sum_n |f_n(\bar{y}, \bar{\bar{y}})|^2 d\bar{y}\Bigr) W(\bar{\bar{y}}) d\bar{\bar{y}} ,$$

which does not exceed, by Schwarz's inequality,

$$\Bigl(\int\Bigl(\int\sum_n |f_n|^2 d\bar{y}\Bigr)^2 d\bar{\bar{y}}\Bigr)^{1/2}\Bigl(\int W^2(\bar{\bar{y}}) d\bar{\bar{y}}\Bigr)^{1/2} .$$

Therefore our proof is complete if we show that

$$\int W^2 d\bar{\bar{y}} \leqq C(\log M)^3 \int w^2 d\bar{\bar{y}} \tag{4.3}$$

with a constant independent of M or w.

In fact, put

$$\rho(\bar{\bar{x}}) = \frac{1}{|R_n|}\int_{R^{d-2}} \chi_{R_n}(\bar{x}, \bar{\bar{x}}) d\bar{x} .$$

Since O_n is a rotation which fixes the first $d-2$ coordinates,

$$\rho(\bar{\bar{x}}) \leqq \frac{1}{4\varepsilon_n^2 N_n} \chi_{I_n}(\bar{\bar{x}})$$

for some rectangle I_n of size $2\varepsilon_n \times 2\varepsilon_n N_n$. Therefore

$$W(\bar{\bar{y}}) \leqq \sup_I \frac{1}{|I|}\int_I w(\bar{\bar{y}} + \bar{\bar{x}}) d\bar{\bar{x}}$$

where the sup runs over all rectangles I in $\boldsymbol{R}^2$ of eccentricity $\leqq M$. Thus (4.3) follows from Cordoba's theorem ([4]).

REMARK 3. Let R_n^μ be the set obtained from R_n expanding by the factor 2^μ and M_n^μ be the operator M_n defined by the set R_n^μ. Our proof shows that Lemma 4 holds if $M_n(f_n)$ is replaced by $\sum_{\mu=0}^\infty 2^{-\mu} M_n^\mu(f_n)$, in which form we apply it later.

4.3. Let $P_n = \{\bar{\bar{\xi}} = (\xi_{d-2}, \xi_{d-1});\ n\delta^{1/2} \leqq \xi_{d-1} < (n+1)\delta^{1/2}\}$ be strips in $\boldsymbol{R}^2$ and for $f \in \mathscr{S}(\boldsymbol{R}^d)$ f_n be the projection defined by

$$\mathscr{F}_{\mathfrak{C}p} f_n = \chi_n \mathscr{F}_{\mathfrak{C}p} f ,$$

where χ_n is the characteristic function of P_n.

LEMMA 5. *We have*

$$\int_{R^2}\Bigl(\int_{R^{d-2}} |s(f)|^2 dx(p)\Bigr)^2 dx(\mathfrak{C}p) \leqq C(\log \delta^{-1})^3 \int_{R^2}\Bigl(\int_{R^{d-2}} \sum_n |f_n|^2 dx(p)\Bigr)^2 dx(\mathfrak{C}p)$$

for f in $\mathscr{S}(\boldsymbol{R}^d)$, where C is a constant independent of f and δ.

PROOF. *Let* t_k^α be the operator defined by

$$t_k^a(f)\hat{}(\xi) = \psi((|\xi(p)|^2 - (1-\rho_k^2))/3\delta)\ \psi(|\xi(p) - \xi^a(p)|/3\delta)\hat{f}(\xi)\ .$$

We remark that

$$s_{k,b}^a(f) = t_k^a \circ s_{k,b}^a(f)\ .$$

In fact, if $\xi = (\bar{\xi}, \bar{\bar{\xi}}) = (\xi(p), \xi(\complement p)) \in \operatorname{supp} \hat{s}_{k,b}^a(f)$, then $-10\,\delta < |\xi(p)|^2 - (1-\rho_k^2) < 10\,\delta$, which implies $\psi((|\xi(p)|^2 - (1-\rho_k^2))/3\delta) = 1$. Obviously $\psi(|\xi(p) - \xi^a(p)|/3\delta) = 1$.

Since t_k^a is defined by a multiplier depending only on the first $d-2$ variables $\xi(p)$ and since $\sum_a \sum_k \psi((|\xi(p)|^2 - (1-\rho_k^2))/3\delta)\psi(|\xi(p) - \xi^a(p)|/3\delta)$ is uniformly bounded,

$$\sum_a \sum_k \int_{R^{d-2}} |t_k^a(f_n)(\bar{x}, \bar{\bar{x}})|^2 d\bar{x} \leqq C\int_{R^{d-2}} |f_n(\bar{x}, \bar{\bar{x}})|^2 d\bar{x}$$

for all $\bar{\bar{x}} \in \boldsymbol{R}^2$. Therefore by Lemma 3 it suffices to show that

$$\int_{R^2}\Big(\int_{R^{d-2}} \sum_a \sum_{k,b} |s_{k,b}^a(f)|^2 d\bar{x}\Big)^2 d\bar{\bar{x}} \leqq C\int_{R^2}\Big(\int_{R^{d-2}} \sum_{n,a,k} |t_k^a(f_n)|^2 d\bar{x}\Big)^2 d\bar{\bar{x}}\ . \tag{4.4}$$

Now $s_{k,b}^a(f) = \sum_n s_{k,b}^a(f_n)$. Since for each b the support of $\Psi_{k,b}$ intersects the support of χ_n at most 11 times, we have $\sum_b |s_{k,b}^a(f)|^2 \leqq 11 \sum_n \sum_b |s_{k,b}^a(f_n)|^2$. On the other hand for each n the support of χ_n intersects the support of $\Psi_{k,100b}(b = 0, 1, \cdots, \delta^{-1/2}\rho_k/200)$ at most a time. We denote such a b by $b(n)$. Thus

$$\sum_a \sum_{k,b} |s_{k,b}^a(f)|^2 \leqq 11 \sum_{n,a,k} |s_{k,b(n)}^a(f_n)|^2\ . \tag{4.5}$$

Next we remark that $s_{k,b(n)}^a(f_n)$ has a representation

$$s_{k,b(n)}^a(f_n) = K_{k,b(n)} * t_k^a(f_n)\ ,$$

where $\hat{K}_{k,b(n)}(\xi) = \Psi^a(\xi)\Psi_{k,b(n)}(\xi)\phi((1-|\xi|^2)\delta^{-1})$.

Assume for a moment $b(n) = 0$. Then by an elementary calculus

$$|K_{k,0}(x)| \leqq C_{s,t,u}\delta^{d-1/2}\rho_k^{-1}|\delta\bar{x}|^{-s}|\delta\rho_k^{-1}x_{d-2}|^{-t}|\delta^{1/2}x_{d-1}|^{-u}$$

for every $s, t, u \geqq 0$, where $C_{s,t,u} \leqq C\sum_{j=0}^{s+t+u} \|\phi^{(j)}\|_\infty$ with a constant C independent of ϕ.

Let $R_{k,0}^\mu$ be a set such that $R_{k,0}^\mu = \{(\bar{x}, x_{d-2}, x_{d-1}) \in \boldsymbol{R}^d;\ |\bar{x}| \leqq 2^\mu\delta^{-1},\ |x_{d-2}| \leqq 2^\mu\rho_k\delta^{-1},\ |x_{d-1}| \leqq 2^\mu\delta^{-1/2}\}$ and $R_{k,b}^\mu = O_{k,b}R_{k,0}^\mu$ where $O_{k,b}$ is a rotation such that $O_{k,b}(\bar{x}, 1, 0) = (\bar{x}, \cos\delta^{1/2}\rho_k^{-1}b, \sin\delta^{1/2}\rho_k^{-1}b)$. Then $K_{k,b}$ is bounded by

$$C\sum_{\mu=0}^\infty 2^{-\mu}\frac{1}{|R_{k,b}^\mu|}\chi_{R_{k,b}^\mu}\ .$$

Therefore

$$|s_{k,b(n)}^a(f_n)| \leqq C\sum_{\mu=0}^\infty 2^{-\mu}\frac{1}{|R_{k,b(n)}^\mu|}\chi_{R_{k,b(n)}^\mu} * |t_k^a(f_n)|\ .$$

By (4.5) and Lemma 4 with Remark 3 we have

$$\int_{R^2}\left(\int_{R^{d-2}}\sum_a\sum_{k,b}|s^a_{k,b}(f)|^2d\bar{x}\right)^2 d\bar{\bar{x}} \leqq C\int_{R^2}\left(\int_{R^{d-2}}\sum_{n,a,k}|s^a_{k,b(n)}(f_n)|^2d\bar{x}\right)^2 d\bar{\bar{x}}$$
$$\leqq C(\log\delta^{-1})^3\int_{R^2}\left(\int\sum_{n,a,k}|t^a_k(f_n)|^2d\bar{x}\right)^2 d\bar{\bar{x}}\ ,$$

which completes a proof.

LEMMA 6. *There exists a constant C such that*

$$\int_{R^2}\left(\int_{R^{d-2}}\sum_n|f_n|^2dx(p)\right)^2 dx(\complement p) \leqq C\int_{R^2}\left(\int_{R^{d-2}}|f|^2dx(p)\right)^2 dx(\complement p) \tag{4.6}$$

for all f in $\mathscr{S}(\boldsymbol{R}^d)$.

PROOF. Let $H = L^2(\boldsymbol{R}^{d-2})$. Then the left hand side of (4.6) is written as

$$\int_{R^2}\left(\sum_n\|f_n(\cdot, x(\complement p))\|^2_H\right)^2 dx(\complement p)\ .$$

Now we apply an H-valued version of Carleson's theorem (cf. Rubio de Francia [10]) to get a bound of the above integral

$$C\int_{R^2}\|f(\cdot, x(\complement p))\|^4_H dx(\complement p) = C\int_{R^2}\left(\int_{R^{d-2}}|f|^2dx(p)\right)^2 dx(\complement p)\ .$$

THEOREM 4. *Let $0<\delta<1/4$. Then*

$$\int\left(\int|s(f)|^2dx(p)\right)^2 dx(\complement p) \leqq C(\log\delta^{-1})^3\int\left(\int|f|^2dx(p)\right)^2 dx(\complement p)$$

for $f\in\mathscr{S}(\boldsymbol{R}^d)$, where C is a constant depending only on d, ϕ and ψ, more precisely $C\leqq C_{d,\psi}\sum_{j=0}^{d+1}\|\phi^{(j)}\|_\infty$.

Now choose a function ϕ' so that $\phi'\in C^\infty(-\infty,\infty)$, $\operatorname{supp}\phi'\subset(1/4, 1)$, $\phi'\geqq 0$ and $\sum_{k=-1}^\infty\phi'(2^kt)=1$ for $0<t<1$. Let $\varepsilon>0$ and put $\phi(t)=t^\varepsilon\phi'(t)$. Define $s^\varepsilon_k(f)$ by

$$\hat{s}^\varepsilon_k(f)(\xi) = (1-|\xi|^2)^\varepsilon\phi'((1-|\xi|)2^k)\hat{f}(\xi)\ .$$

Then

$$s^\varepsilon(f) = \sum_{k=-1}^\infty s^\varepsilon_k(f)\ .$$

By Theorem 4

$$\int\left(\int|s^\varepsilon_k(f)|^2dx(p)\right)^2 dx(\complement p) \leqq Ck^3 2^{-4\varepsilon k}\int\left(\int|f|^2dx(p)\right)^2 dx(\complement p)\ .$$

Summing over $k=-1, 0, 1, \cdots$ it follows

THEOREM 5. If $\varepsilon>0$, *then*

$$\int\left(\int|s^\varepsilon(f)|^2dx(p)\right)^2 dx(\complement p) \leqq C\int\left(\int|f|^2dx(p)\right)^2 dx(\complement p) \tag{1.1}$$

for $f \in \mathscr{S}(\boldsymbol{R}^d)$, *where* C *is a constant independent of* f.

4.4. Let P be the family of subsets p of $\{0, 1, \cdots, d-1\}$ such that $\operatorname{card}(p) = d-2$. Since the operator s^ε is rotation invariant, (1.1) implies that

$$\| s^\varepsilon(f) \|_{(4,2:p)} \leqq C \| f \|_{(4,2:p)}$$

for $f \in \mathscr{S}(\boldsymbol{R}^d)$ and $p \in P$. By duality

$$\| s^\varepsilon(f) \|_{(4/3,2:p)} \leqq C \| f \|_{(4/3,2:p)} . \tag{4.7}$$

Applying Theorem 3 to (4.7) and then an interpolation theorem for multilinear operators (cf. [1]) we get

THEOREM 6. *Let* $\varepsilon > 0$ *and* $2d/(d+1) \leqq u \leqq 2$. *Then*

$$\| s^\varepsilon(f) \|_u \leqq C \| f \|_u$$

for all f *in* $\mathscr{S}(\boldsymbol{R}^d)$ *of the form* $f_0(x_0) f_1(x_1) \cdots f_{d-1}(x_{d-1})$.

5. **Restriction problem of Fourier transform.** We apply our interpolation theorem to a restriction problem of Fourier transform. By a theorem in Tomas [11], if $1 \leqq u < 2(d+1)/(d+3)$,

$$\left(\int_{S^{d-1}} |\hat{f}(\xi)|^2 d\sigma(\xi) \right)^{1/2} \leqq C \| f \|_u \tag{5.1}$$

for $f \in \mathscr{S}(\boldsymbol{R}^d)$ where $d\sigma(\xi)$ is the surface element and C is independent of f. The inequality fails for $u > 2(d+1)/(d+3)$ but a simple argument shows that (5.1) holds for $1 \leqq u < 2d/(d+1)$ if the functions f are radial.

THEOREM 7. *If* $d \geqq 2, 1 \leqq u \leqq 2d/(d+1)$ *and* f *is a function in* $\mathscr{S}(\boldsymbol{R}^d)$ *of the form* $f(x) = f_0(x_0) f_1(x_1) \cdots f_{d-1}(x_{d-1})$, *then*

$$\left(\int_{S^{d-1}} |\hat{f}(\xi)|^2 |\xi_0 \xi_1 \cdots \xi_{d-1}|^{1/d} d\sigma(\xi) \right)^{1/2} \leqq \frac{1}{\sqrt{2\pi}} \| f \|_u . \tag{5.2}$$

PROOF. We assume $d > 2$ but a careful reading shows that our proof applies to the case $d = 2$. Let $w, f \in \mathscr{S}(\boldsymbol{R}^d)$ and assume $\operatorname{supp} w(\xi) \subset \{|\xi_0| > 0\}$. By Fubini's theorem

$$\int_{S^{d-1}} |\xi_0|^{1/2} \hat{f}(\xi) w(\xi) d\sigma(\xi) = \int_{R^d} f(x) S w(x) dx \tag{5.3}$$

where

$$S w(x) = \frac{1}{\sqrt{2\pi}^d} \int_{S^{d-1}} |\xi_0|^{1/2} w(\xi) e^{-ix\xi} d\sigma(\xi) .$$

Therefore our problem reduces to an estimate of the following integral;

$$(5.4)\quad \int_{R^{d-1}} |Sw(x)|^2 dx_1 \cdots dx_{d-1}$$
$$= \frac{1}{(2\pi)^d}\int_{R^{d-1}} dx_1 \cdots dx_{d-1} \int_{S^{d-1}\times S^{d-1}} |\xi_0\eta_0|^{1/2} w(\xi)\bar{w}(\eta) e^{-i(\xi-\eta)x} d\sigma(\xi) d\sigma(\eta) .$$

We introduce the polar coordinates: $\xi_1 = \cos\theta_1$, $\xi_2 = \sin\theta_1\cos\theta_2$, $\xi_3 = \sin\theta_1\sin\theta_2\cos\theta_3, \cdots$, $\xi_{d-1} = \sin\theta_1\cdots\sin\theta_{d-2}\cos\theta_{d-1}$, $\xi_0 = \sin\theta_1\cdots\sin\theta_{d-2}\sin\theta_{d-1}$. Then the last integral is transformed to

$$(5.5)\quad \int_{S^{d-1}} d\sigma(\eta) \int_{R^{d-1}} dx_1 \cdots dx_{d-1} \int_D \omega(\theta, \eta) \exp -i[(\xi_1 - \eta_1)x_1 + \cdots + (\xi_{d-1} - \eta_{d-1})x_{d-1}] d\theta ,$$

where $\omega(\theta, \eta) = |\xi_0\eta_0|^{1/2} w(\xi)\bar{w}(\eta) \prod_{j=1}^{d-1} \sin^{d-j-1}\theta_j \exp[-i(\xi_0 - \eta_0)x_0]$ and D is the image of $S^{d-1}\cap \operatorname{supp} w$ by the mapping of ξ to θ.

Now fix η and introduce new variables

$$\rho_1 = \cos\theta_1 - \eta_1 ,$$
$$\rho_2 = \sin\theta_1\cos\theta_2 - \eta_2 ,$$
$$\cdots\cdots$$
$$\rho_{d-1} = \sin\theta_1\sin\theta_2\cdots\sin\theta_{d-2}\cos\theta_{d-1} - \eta_{d-1} .$$

Consider the Jacobian $|\partial\rho/\partial\theta| = |\sin^{d-1}\theta_1 \sin^{d-2}\theta_2 \cdots \sin\theta_{d-1}|$. The inner integral of (5.5) is transformed to

$$\int_{R^{d-1}} dx_1 \cdots dx_{d-1} \int_\Delta \omega(\theta, \eta) \Big/ \left|\frac{\partial\rho}{\partial\theta}\right| \exp[-i(\rho_1 x_1 + \cdots + \rho_{d-1}x_{d-1})] d\rho_1 \cdots d\rho_{d-1} ,$$

where Δ is the image of D by the mapping of θ to ρ. $\omega(\theta, \eta)/|\partial\rho/\partial\theta|$ is infinitely differentiable in ρ since $w(\xi)$ vanishes near $\xi_0 = 0$. Therefore by Fourier inversion formula the last integral equals

$$(2\pi)^{d-1}\omega(\theta, \rho) \Big/ \left|\frac{\partial\rho}{\partial\theta}\right|$$

at $\rho = 0$. Since $\xi = \eta$ if $\rho = 0$, the last term coincides with

$$(2\pi)^{d-1}|\xi_0|\,|w(\xi)|^2/|\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-1}| = (2\pi)^{d-1}|w(\xi)|^2 .$$

Thus by (5.4) and (5.5)

$$\int_{R^{d-1}} |Sw(x)|^2 dx_1 \cdots dx_{d-1} = \frac{1}{2\pi}\int_{S^{d-1}} |w(\xi)|^2 d\sigma(\xi) .$$

Applying the reversed Hölder's inequality to (5.3) we get

$$\left(\int_{S^{d-1}} |\xi_0|\,|\hat{f}(\xi)|^2 d\sigma(\xi)\right)^{1/2} \leqq \frac{1}{\sqrt{2\pi}}\int_R \left(\int_{R^{d-1}} |f|^2 dx_1 \cdots dx_{d-1}\right)^{1/2} dx_0 ,$$

from which we have

$$\left(\int_{S^{d-1}}|\xi_j|\,|\hat{f}(\xi)|^2 d\sigma(\xi)\right)^{1/2} \leqq \frac{1}{\sqrt{2\pi}}\int_R\left(\int_{R^{d-1}}|f|^2 dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_{d-1}\right)^{1/2} dx_j\,.$$

Let P be the set of $d-1$ indices in $\{0, 1, \cdots d-1\}$ and I_p, $p \in P$, be disjoint arcs in ∂D of length $2\pi/d$. Let $\delta_p(z)$ be functions in the Hardy class $H^2(\boldsymbol{D})$ such that $\operatorname{Re}\delta_p(e^{i\theta}) = 1$ a.e. in I_p and $=0$ a.e. in $\partial\boldsymbol{D} - I_p$, and $\operatorname{Im}\delta_p(0) = 0$. Then $\delta_p(0) = 1/d$. Identify p with j such that $j \notin p$ and define a mapping T^z by

$$T^z f(\xi) = \hat{f}(\xi)\prod_{j=0}^{d-1}|\xi_j|^{\delta_j(z)/2}\,.$$

Applying Theorem 1′ with $\bar{M} = \boldsymbol{R}^d$ and $N = S^{d-1}$ we get Theorem 7.

REMARK 4. If (5.2) is valid for every f in $\mathscr{S}(\boldsymbol{R}^d)$, then we have

$$\left(\int_{S^{d-1}}|\hat{f}(\xi)|^2 d\sigma(\xi)\right)^{1/2} \leqq C\|f\|_{2d/(d+1)} \tag{5.6}$$

for f such that $\operatorname{supp}\hat{f}\subset\{\xi;\,|\xi/|\xi| - (1, 1, \cdots, 1)/\sqrt{d}| < 1/2\sqrt{d}\}$. Thus by rotation (5.6) holds for all f in $\mathscr{S}(\boldsymbol{R}^d)$, which contradicts the optimality of Tomas's condition.

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