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SPECIAL DIVISORS AND VECTOR BUNDLES

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Introduction. Let X be a nonsingular, complete curve of genus $g \geqq 3$ over $\boldsymbol{C}$, the field of complex numbers and let J be the Jacobian of X, the space of isomorphism classes of line bundles of degree 0 on X. It is a complex torus of dimension g. If we denote by ϕ the Abel-Jacobi mapping of X_d, the d-fold symmetric product of the curve, into J, Abel's theorem assures us that $\phi^{-1}(\phi(D))$ is nothing but the projective linear system $P(H^0(L_D))$, associated to the line bundle L_D given by the effective divisor $D \in X_d$ on X. When d lies between 0 and $(g-1)$, $\phi(X_d)$ is a proper subvariety of J. This subvariety $\phi(X_d)$ admits a natural filtration by subvarieties

$$\phi(X_d) = W_d^0 \supsetneqq W_d^1 \cdots \supsetneqq W_d^r \cdots$$

defined in terms of the dimension of the fibre of ϕ. For example, $W_d^r = \{\phi(D) \mid \text{dimension of } \phi^{-1}(\phi(D)) \geqq r\}$. It is a classical problem to study the structure of these special linear systems.

Formally, one may define an effective divisor D on X to be special if $H^1(X, L_D) \neq 0$.

In 1874, A. Brill and M. Noether published their investigations on special linear systems and conjectured that on a very general curve X, the dimension of W_d^r is given by $\rho(r, d) = g - (r+1)(g-d+r)$. In 1980, Griffiths and Harris [G-H] settled this conjecture affirmatively.

Picking up the thread from here, we extend the notion of special divisors to stable vector bundles on X. Indeed, a vector bundle V on X is said to be stable, if for every subbundle $W \subsetneqq V$ with $W \neq 0$, $\mu(W) = (\text{degree } W)/(\text{rank } W) < \mu(V)$. Such a bundle with *nonnegative degree* is said to be *special if* $H^1(X, V) \neq 0$. Replacing J, the isomorphism classes of line bundles of degree zero, by $U_{n,d}$, the variety consisting of isomorphism classes of stable bundles of rank n and degree d we may define

$$W_d^r = \bar{U}_d^r\,, \quad U_d^r = \{V \in U_{n,d} \mid h^0(X, V) \geqq r+1\}$$

where W_d^r is the Zariski closure of U_d^r in $M_{n,d}$, the "natural compactification" of $U_{n,d}$. (See Section 1.12.2, Chapter I).

In this article, we undertake investigations of

(i) When is W_d^r non-empty?
(ii) What is the dimension of W_d^r for a general curve?
(iii) Determine the cohomology class of W_d^r.
(iv) Describe the singular set $(W_d^r)_s$ of W_d^r.
We prove:

THEOREM I.3.2. *For $0 \leqq d \leqq n(g-1)$, a generic $V \in U_{n,d}$ has no section.*

Following an argument of Kempf we observe:

THEOREM I.4.1. *For $0 < d \leqq n(g-1)$, if W_d^r is non-empty then the dimension of W_d^r is at least $\rho(r, d, n)$, where*

$$\rho(r, d, n) = n^2(g-1) + 1 - (r+1)(r+1-d-n+ng) .$$

We define W_d^r on a singular curve, in particular on curves whose only singularities are ordinary double points, and prove:

THEOREM II.3.1. *For $0 < d \leqq n(g-1)$, the dimension of W_d^0 is given by*

$$\rho(0, d, n) = n^2(g-1) + 1 - (ng - n - d + 1)$$

and it has a unique irreducible component of maximal dimension.

COROLLARY II.3.2. *Let $W_d^0(\rho)$ denote the unique irreducible component of dimension $\rho(0, d, n)$. Then a generic $F \in W_d^0(\rho)$ is locally free, F contains a trivial line sub-bundle and $h^0(F) = 1$.*

THEOREM II.4.2. *For d an odd integer, with $2 \leqq d \leqq 2g-2$, there exists $U'_{2,d}$, a Zariski open set in $U_{2,d}$ such that $W_d^1 \cap U'_{2,d}$ is non-empty and the dimension of this subvariety in $U'_{2,d}$ is $\rho(1, d, 2) = (2d-3)$.*

Next, we tie up W_d^r on smooth curves with W_d^r on integral curves.

THEOREM III.2.1. *Let $\bar{R}_{n,d} \subseteqq R^s_{n,d}$ be a Zariski open set satisfying*
(i) *$\bar{R}_{n,d}(t) = R^s_{n,d}(t)$ it $t \neq s_0$ in S.*
(ii) *$\bar{R}_{n,d}(s_0) \cap \bar{W}_d^r(S)$ is nonempty and*

$$\dim[\bar{R}_{n,d}(s_0) \cap \bar{W}_d^r(S)] = \rho(r, d, n) + \dim G_{s_0} - 1 ,$$

where

$$\bar{W}_d^r(S) = \{q \in R^s_{n,d} \mid h^0(X_{\sigma(q)}, V_{n,d}(q)) \geqq (r+1)\} .$$

Then $\dim[\bar{W}_d^r(S)] \geqq (\rho(r, d, n) + \dim S + \dim G_s - 1)$. *In fact every component is of dimension at least $(\rho(r, d, n) + \dim S + \dim G_s - 1)$.* (*See page* 203 *for definitions of $R^s_{n,d}$, $V_{n,d}$*).

Combining the above with the theorems on singular curves, we obtain:

THEOREM III.2.4. *For* $0 < d \leqq n(g-1)$, $W^0_d(X)$ *is nonempty and* $\dim W^0_d(X) = \rho(0, d, n)$ *on any smooth curve* X *of genus* g.

THEOREM III.2.5. *For* $2 \leqq d \leqq 2(g-1)$, d *an odd integer*, $W^1_d \subseteqq M_{2,d}$ *is nonempty and* $\dim W^1_d \geqq \rho(1, d, 2) = (2d-3)$ *on any smooth curve of genus* g.

In Chapter IV, an alternative proof of the existence of special bundles with a section is given and some properties of W^r_d are discussed. Here again we suppose that X is smooth.

THEOREM IV.1.1. *For* $\mu = (d/n) \in N = \{1, 2, \cdots, (g-1)\}$, *there exists* $V \in M_{n,d}$ *such that* V *is stable and* $h^0(V) \neq 0$.

COROLLARY IV.1.3. *For* $d = n(g-1)$, *if* $g \geqq (r+1)^2$, $r \geqq 1$, *then there exists* $V \in M_{n,d}$ *such that* V *is stable and* $h^0(V) \geqq (r+1)$; $\dim W^r_d \geqq n^2(g-1) + 1 - (r+1)^2$.

THEOREM IV.2.1. *For* $0 < d \leqq n(g-1)$, *the variety* $W^0_d \subseteqq M_{n,d}$ *is irreducible and* $W^0_d \setminus W^0_d(1)$ *is of codimension at least* $(n-1)$ *in* W^0_d *where*

$$W^0_d(1) = \{V \in W^0_d \mid \mathcal{O} \subseteqq V,\ V/\mathcal{O} \text{ is stable of degree } d \text{ and rank } (n-1)\}\,.$$

PROPOSITION IV.2.2. *For* $g \leqq d \leqq 2g-2$, d *an odd integer*, $W^1_d \subseteqq M_{2,d}$ *is irreducible on a general smooth curve of genus* $g \geqq 3$ *and* $\dim W^1_d = (2d-3)$; *a generic* $V \in W^1_d$ *contains* $\mathcal{O}^2$, *a trivial sheaf of rank* 2.

PROPOSITION IV.3.1. *For* $0 < d \leqq n(g-1)$, *the constructible set*

$$W^0_d(1, 1) = \{V \in W^0_d(1) \mid h^0(V) = 1\}\,,$$

where $W^0_d(1)$ *is as in Theorem* IV.2.1, *is Zariski dense in* $W^0_d \subseteqq M_{n,d}$. *Therefore, a generic* V *in* W^0_d *has exactly one section and it generates a trivial line subbundle of* V. *Moreover, for* $d < (n-1)(g-1)$, *the singular set* $\{W^0_d\}_s$ *of the reduced model of* W^0_d *is of codimension at least two.*

Chapter V contains a discussion of open problems.

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CHAPTER I. Bounded Families of Locally Free Sheaves. We recall a few facts about vector bundles over smooth, complete curves of genus g and then define the notation of special bundle. We then construct a Noetherian K-scheme of finite type parametrising vector bundles of specific types to show that a generic stable vector bundle of degree d and rank n with $0 \leqq d \leqq n(g-1)$ has no section. We then realise the varieties of special bundles with prescribed numbers of sections as preimages of Schubert varieties in suitable Grassmann bundles.

1. Notation and Preliminaries. 1.1. Most of the schemes considered here are Noetherian and of finite type. Accordingly, these phrases may often be omitted. We hope the context will make these points clear. All our schemes, with a few exceptions, are defined over $\boldsymbol{C}$, the field of complex numbers. We hope the interchangable use of symbols $\boldsymbol{C}$ and K will not cause the reader any confusion.

1.2. Let $f: X \to S$ be a morphism of schemes. For every S-scheme T, we denote by X_T, the base change $X \times_S T$; the symbols p_X and p_T stand for the natural projections of $X \times_S T \to X$ and $X \times_S T \to T$, respectively. If F is a sheaf of $\mathcal{O}_X$-modules on X, F_T denotes $F \otimes_{\mathcal{O}_S} \mathcal{O}_T$. We call elements of $\mathrm{Hom}_S(T, X)$ T-valued points of X. Closed points of a scheme are referred to as just points.

1.3. For $s \in S$, X_s denotes the fibre over $K(s)$, the residue field of the local ring at s; F_s denotes the restriction of a sheaf F to the fibre X_s.

1.4. By a sheaf, we mean a coherent sheaf. Let X be a projective scheme and $\mathcal{O}_X(1)$ a fixed ample invertible sheaf on X. Then $F(m)$ denotes $F \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes m}$. For a sheaf F, $\chi(F)$ and $h^i(X, F) = h^i(F)$ denote the Euler-Poincaré characteristic and dimension of $H^i(X, F)$, respectively.

1.5. When a K-scheme T parametrises a family of bundles on X, we often denote the bundle represented by $t \in T$ on X by the same letter.

1.6. K-Sch and Ens denote the category of Noetherian K-schemes and the category of sets, respectively.

1.7. For any scheme X, the stalk at $x \in X$ of the structure-sheaf $\mathcal{O}_X$ may be denoted by $\mathcal{O}_{X,x}$ or $\mathcal{O}_x$.

1.8. Let X denote a projective, integral curve. A torsion-free sheaf F of $\mathcal{O}_X$-modules is said to be stable (semistable) if for every subsheaf G of F with $G \neq 0$, $\mu(G) < \mu(F)$ ($\mu(G) \leqq \mu(F)$), where for example, the symbol $\mu(F)$ stands for (degree of F)/(rank of F), the degree of F being $\chi(F) - n(1-g)$ and n, the rank of F, being the dimension of the generic fibre of F as a vector space over $\boldsymbol{C}$.

1.9. Let X be smooth. If V is a semistable bundle of rank n and degree d over X, then it admits a filtration by sub-bundles

$$0 = V_{p+1} \subset V_p \cdots \subset V_1 \subset V_0 = V$$

such that for $0 \leqq i \leqq p$, V_i/V_{i+1} is stable and $\mu(V_i/V_{i+1}) = \mu(V)$. We refer to $\bigoplus_{i=0}^{p} (E_i/E_{i+1})$ as the associated graded sheaf and denote it by $\mathrm{Gr}(V)$. It is independent of the choice of the filtration satisfying the above conditions.

1.10. Let X be smooth. If V is a non-semistable bundle over X, then it has a *unique filtration* by subbundles

$$\{0\} = V_0 \subset V_1 \cdots \subset V_{s-1} \subset V_s = V$$

such that for $1 \leqq i \leqq s-1$, V_i/V_{i-1} is a maximal semistable subbundle of V/V_{i-1}. This unique filtration is called the Harder-Narasimhan filtration. It satisfies

$$\mu(V_1) > \mu(V_2/V_1) > \cdots > \mu(V_s/V_{s-1}) .$$

Let n_i and d_i be the rank and degree of V_i respectively. We refer to the data $\{\langle n_i, d_i\rangle\}_{i=1}^{s}$ as HN type.

1.11. Let X be as in 1.10. For V, a semistable vector bundle, $H^1(X, V) = 0$ if the degree of $V = d > n(2g-2)$, where n is the rank of V. This follows from Serre Duality Theorem.

1.12. Let X be as in 1.10. Denote by Z, a set of isomorphism classes of vector bundles on X.

DEFINITION 1.12.1. We say that a family of elements of Z is parametrised by a Noetherian K-scheme Y, if there exists a vector bundle V on $Y \times_K X$ such that the isomorphism classes of $\{V_y\}$ are in Z, where $\{y\}$ is a closed point of Y and $V_y = V|\{y\} \times_K X$, the restriction of V to $\{y\} \times_K X$. We do not insist that different points of Y correspond to non-isomorphic bundles V_y.

Two families V_1 and V_2 of elements of Z are said to be equivalent if there exists a line bundle L on Y such that $V_1 \cong V_2 \otimes p_Y^*(L)$.

1.12.2. *Moduli of stable bundles.* Let $S(n, d)$ and $S'(n, d)$ be, respectively, the isomorphism classes of semistable and stable bundles on X of rank $r \geqq 2$ and degree d.

MAIN THEOREM. *Let (n, d) be a pair of integers with $n \geqq 2$. Then there exists a coarse moduli space for $S'(n, d)$ whose underlying K-scheme is smooth and quasi projective, denoted by $U_{n,d}$.*

This variety possesses a natural compactification, denoted by $M_{n,d}$. The set of K-valued points of $M_{n,d}$ is isomorphic to the quotient of $S(n, d)$ under the equivalence relation: for every pair (E, F) of semistables on X of rank r and degree d, E and F are equivalent if and only if $\mathrm{Gr}(E) =$

Gr(F). (cf. Section 1.9). $M_{n,d}$ is a normal projective variety of dimension $n^2(g-1)+1$.

The compactification $M_{n,d}$ possesses the following "universal property": for every family E of semistable vector bundles over X of rank r and degree d parametrised by a Noetherian K-scheme T, there exists a unique morphism

$$f\colon T \to M_{n,d}$$

such that for t in $T(K)$, the point $f(t)$ of $M_{n,d}$ is $\mathrm{Gr}(E_t)$.

We refer the reader to [Se 1] for proofs of the above.

2. Bounded families of locally free sheaves. X is assumed to be smooth throughout the rest of this chapter.

PROPOSITION 2.1. *Let U_1 and U_2 be reduced K-schemes parametrising families of semistable bundles V_1 and V_2, respectively. Then there exists a reduced K-scheme Z parametrising $P(\mathrm{Ext}^1(u_2, u_1))$, the projectivisation of $\mathrm{Ext}^1(u_2, u_1)$ with $(u_1, u_2) \in U_1 \times_K U_2$.*

PROOF. Let p_1 and p_2 be the natural projections of $U_1 \times U_2 \times X$ to U_1 and U_2, respectively; $\bar{V}_1$ and $\bar{V}_2^*$ the pullbacks of V_1 and V_2^*, the dual of V_2, on $U_1 \times U_2 \times X$. Set $V = \bar{V}_1 \otimes \bar{V}_2^*$. The bundle V satisfies

$$V_{(u_1,u_2)} \cong u_1 \otimes u_2^*$$

over X.

Since all schemes under consideration are Noetherian and of finite type, and p the natural projection of $U_1 \times U_2 \times X$ to $U_1 \times U_2$ is proper, we conclude that $R^1p_*(V)$ is a coherent sheaf on $U_1 \times U_2$. By the semicontinuity theorem, there exists a finite stratification into locally closed sets $\{U_\alpha\}$ such that $\dim_{k(y)} H^1(X, V_y)$ is constant on every U_α. Since $U_1 \times_K U_2$ is reduced, we may suppose that U_α is reduced for every α, by taking the reduced subscheme. We note that U_α is Noetherian. Each U_α may be expressed as a finite union of connected components. By abuse of notation we denote them by the same symbol U_α. Let $\pi_\alpha\colon V_\alpha \to U_\alpha \times X$ denote the restriction of V to $U_\alpha \times X$.

Since p is flat, $R^1p_*(V_\alpha) = R^1p_*(V)|U_\alpha$, the restriction of $R^1p_* V$ to U_α, is locally free; let Z_α be the total space of the corresponding projective bundle. If we suppose, as we may, that U_α is affine, then there exists an extension of bundles on $Z_\alpha \times X$ with the required property (see, for example, [Se 1, p. 200]). Now we may take Z to be the disjoint union of the Z_α. q.e.d.

Let $U_{n,d}$ be the moduli space of stable bundles of rank n and degree

d on X. It is well-known that there exists a finite Zariski open cover $\{U^i_{n,d}\}$ of $U_{n,d}$ such that for every i, there exists an algebraic variety $V^i_{n,d}$ with $V_i \to V^i_{n,d} \times X$ a vector bundle and an étale morphism $\Phi_i: V^i_{n,d} \to U^i_{n,d}$ which is surjective and satisfies the following condition: $\Phi_i(v) \in U^i_{n,d} \subseteqq U_{n,d}$ is represented by the bundle $V_i | \{v\} \times X$ for all $v \in V^i_{n,d}$. Let $Z_{n,d}$ be the disjoint union of $\{V^i_{n,d}\}$. Then $Z_{n,d}$ parametrises stable bundles of degree d and rank n on X.

It follows from 1.9 that every semistable bundle V on X determines a (up to order) unique data $\{\langle n_i, d_i \rangle\}_{i=1}^k$ such that $\sum_{i=1}^k n_i = n$, $n_i > 0$, $d_i/n_i = d/n$ for all i, d and n being the degree and rank of V.

Given $\{\langle n_i, d_i \rangle\}_{i=1}^k$ such that $\sum_{i=1}^k n_i = n$, $n_i > 0$, $d_i/n_i = d/n$ for all i, let $Z = \prod_{i=1}^k Z_{n_i, d_i}$. For $(V_1, \cdots, V_k) \in Z$, let $E(V_1, \cdots, V_k)$ be the space of k-fold extensions, i.e.,

$$E(V_1, \cdots, V_k) = \{E \mid E = W_k,\ V_1 = W_1,\ W_i \in P(\mathrm{Ext}^1(V_i, W_{i-1})),\ 2 \leqq i \leqq k\} .$$

A repeated application of Proposition 2.1 shows that there exists a reduced scheme $\bar{Z}$ which is noetherian and of finite type parametrising $E(V_1, \cdots, V_k)$ with $(V_1, \cdots, V_k) \in Z$. We note that every element of $E(V_1, \cdots, V_k)$ is semistable.

LEMMA 2.2. *Let V be a semistable of degree $d \geqq 0$ and rank n. Then* $h^0(V) = \dim H^0(V) \leqq d + n$.

PROOF. Let $T \subseteqq V$ be the subbundle generated by $H^0(V)$. If $H^0(V) \neq 0$, then T admits a filtration $V_1 \subseteqq V_2 \subseteqq \cdots \subseteqq V_k = T$ such that V_i/V_{i-1} is a line bundle of nonnegative degree. However, for a line bundle L of $\deg L = d(L) \geqq 0$, $h^0(L) \leqq d(L) + 1$ and hence

$$h^0(T) \leqq \sum_i h^0(V_i/V_{i-1}) \leqq \deg T + k \leqq \deg V + n . \qquad \text{q.e.d.}$$

LEMMA 2.3. *Let W and V be semistable vector bundles on X such that $\mu(W) \leqq \mu(V)$. Then $W^* \otimes V$ is semistable and* $\dim \mathrm{Hom}(W, V) \leqq \deg(W^* \otimes V) + \mathrm{rank}(W^* \otimes V)$, *where W^* is the dual of W.*

PROOF. We note that $W^* \otimes W$ and $V^* \otimes V$ are semistable. If V_1 and V_2 are semistable such that $\deg(V_1) = \deg(V_2) = 0$, then $V_1 \otimes V_2$ is semistable. Hence $W^* \otimes W \otimes V^* \otimes V$ is semistable. Let $T \subseteqq W^* \otimes V$ be a subbundle. Then $T \otimes W \otimes V^* \subseteqq W^* \otimes W \otimes V^* \otimes V$. Hence $\mu(T \otimes W \otimes V^*) = \mu(T) + \mu(W) - \mu(V) \leqq 0$. That is $\mu(T) \leqq \mu(V) - \mu(W) = \mu(W^* \otimes V)$. By Lemma 2.2, the second assertion follows. q.e.d.

By Lemma 2.3,

$$\dim E(V_1, \cdots, V_k) \leqq \sum_{i<j} h^1(V_j^* \otimes V_i) - (k-1) \leqq \sum_{i<j} n_i n_j g - (k-1) .$$

Hence

$$\dim \bar{Z} \leqq \dim Z + \sum_{i<j} n_i n_j g - (k-1) \leqq \sum_{i=1}^{k} n_i^2(g-1) + \sum_{i<j} n_i n_j g + 1 .$$

However

$$\sum_{i=1}^{k} n_i^2(g-1) + \sum_{i<j} n_i n_j g + 1 \leqq n^2(g-1) + 1 - (n-1) = \dim Z_{n,d} - (n-1) .$$

But $Z_{n,d}$ is étale over $U_{n,d}$. Therefore $\dim Z_{n,d} = \dim U_{n,d}$. This shows that

$$\dim \bar{Z} \leqq \dim U_{n,d} - (n-1) .$$

By taking all possible $\{\langle n_i, d_i \rangle\}_{i=1}^{k}$ such that $\sum_{i=1}^{k} n_i = n$, $n_i > 0$, $d_i/n_i = d/n$ for all i and constructing $\bar{Z}$ as above and taking the disjoint union of $Z_{n,d}$ and all the $\bar{Z}$ mentioned above, we observe that, this union being finite, there exists a reduced Noetherian K-scheme of finite type $\bar{M}_{n,d}$ parametrising all semistable bundles of rank n and degree d. We note that

$$\dim \bar{M}_{n,d} = \dim U_{n,d} = \dim Z_{n,d}$$

and

$$\dim(\bar{M}_{n,d} \setminus Z_{n,d}) \leqq \dim U_{n,d} - (n-1) .$$

From now on $\bar{M}_{n,d}$ refers to the above variety.

We shall now parametrise non-semistable bundles of rank n and degree d of fixed HN type

$$\{\langle n_i, d_{n_i} \rangle\}_{i=1}^{k} , \quad 0 < n_1 < n_2 < \cdots < n_k = n , \quad 0 < d_{n_1} < d_{n_2} < \cdots < d_{n_k} = d$$

with

$$\mu_1 > \mu_2 > \cdots > \mu_k > 0 , \quad \mu_i = (d_{n_i} - d_{n_{i-1}})/(n_i - n_{i-1}) , \quad d_{n_0} = n_0 = 0 .$$

Set $Y = \prod_1^k \bar{M}_{n_i - n_{i-1},\, d_{n_i} - d_{n_{i-1}}}$. For $(V_1, \cdots, V_k) \in Y$, let $E(V_1, \cdots, V_k)$ be the space of k-fold extensions

$$E(V_1, \cdots, V_k) = \{E \mid E = W_k,\ V_1 = W_1,\ W_i \in P(\mathrm{Ext}^1(V_i, W_{i-1})),\ 2 \leqq i \leqq k\} .$$

As in the earlier case by repeated application of Proposition 2.1, we get $\bar{M}_{n_1,\cdots,n_k}$, a reduced Noetherian K-scheme of finite type parametrising $E(V_1, \cdots, V_k)$ with $(V_1, \cdots, V_k) \in Y$.

By the same argument as above, we get

$$\dim \bar{M}_{n_1,\cdots,n_k} \leqq \dim U_{n,d} - (n-1) .$$

3. Special Vector Bundles.

DEFINITION 3.1. A stable vector bundle V of rank n and degree $d \geqq 0$ is said to be special if $H^1(X, V)$ has a nonzero element.

This definition is in analogy with the definition of special divisor on a curve. We define

$$W_d^r = \bar{U}_d^r\ , \quad U_d^r = \{V \in U_{n,d} \mid h^0(V) \geqq r + 1\}\ ,$$

where W_d^r is the Zariski closure of U_d^r in $M_{n,d}$, the natural compactification of $U_{n,d}$ (see 1.12.2 for notation).

THEOREM 3.2. *For $0 \leqq d \leqq n(g-1)$, a generic $V \in U_{n,d}$ has no section.*

PROOF. We shall construct a Noetherian K-scheme Z, a dominant morphism $\Phi\colon Z \to W_d^0$ and show that the dimension of Z is strictly less than the dimension of $M_{n,d}$. We may suppose that $d > 0$.

For $V \in U_{n,d}$, with $h^0(V) \neq 0$, let $L \subseteqq V$ be a sub-bundle generated by a section. Then $\deg(L) = d(L) < d/n$, and

$$0 \to L \to V \xrightarrow{\eta} V/L \to 0$$

For $F \subseteqq V/L$, a subsheaf of rank $r(F)$, let $\eta^{-1}(F)$ be the pull-up. Then

$$0 \to L \to \eta^{-1}(F) \to F \to 0$$

and $\deg(\eta^{-1}(F)) < (r(F) + 1)(d/n)$. Hence

$$\deg(F) = \deg(\eta^{-1}(F)) - d(L) < (r(F) + 1)(d/n) - d(L) \leqq d\ .$$

If V/L is non-semistable, then by the above observation the HN type of it is $\{\langle n_i, d_{n_i}\rangle\}_{i=1}^k$, $0 < d_{n_1} < \cdots < d_{n_k} = d - d(L)$ and $0 < n_1 < \cdots < n_k = n-1$, which is parametrised by the scheme $\bar{M}_{n_1,\dots,n_k}$. Since $0 \leqq d(L) < (d/n)$ and $0 \leqq d \leqq n(g-1)$, only finitely many HN types need be considered.

Set $Y_{d'} = X_{d'} \times \bar{M}_{n-1,d-d'}$, $0 \leqq d' < (d/n)$, where $X_{d'}$ is the d'-fold symmetric product of the curve X. If $d' = 0$, then $X_{d'} = \{\mathcal{O}\}$, the trivial line bundle. Let $\bar{W}_{d'} \subseteqq \bar{Y}_{d'}$ parametrise stable bundles where $\bar{Y}_{d'}$ parametrises $E(L, V) = P(\mathrm{Ext}^1(V, L))$ with $(L, V) \in Y_{d'}$. For $\bar{V} \in \bar{W}_{d'}$, let $\bar{V}$ denote the corresponding stable bundle. Then for some $(L, V) \in Y_{d'}$, we have

$$0 \to L \to \bar{V} \to V \to 0\ .$$

By the vanishing theorem for semi-stable bundles (see 1.11, Chapter I)

$$\dim \mathrm{Ext}^1(V, L) = \deg(K \otimes L^* \otimes V) + (n-1)(1-g)\ ,$$

where K is the canonical line bundle on X.

Hence

$$\begin{aligned}\dim \bar{W}_{d'} \leqq \dim Y_{d'} &+ (n-1)(1-g) + 2(n-1)(g-1)\\ &+ d - d' - (n-1)d' - 1\end{aligned}$$

which in turn is less than or equal to

$$(n-1)^2(g-1)+(n-1)(g-1)+d$$
$$-(n-1)d' < n^2(g-1)+1 = \dim U_{n,d} = \dim M_{n,d}\,.$$

Set $Y_{d',n_1,\cdots,n_k} = X_{d'} \times \bar{M}_{n_1,\cdots,n_k}$. Let $\bar{W}_{d',n_1,\cdots,n_k} \subseteqq \bar{Y}_{d',n_1,\cdots,n_k}$ parametrize stable bundles, where $\bar{Y}_{d',n_1,\cdots,n_k}$ parametrises $E(L, V) = P(\mathrm{Ext}^1(V, L))$ with $(L, V) \in Y_{d',n_1,\cdots,n_k}$. By the same argument as before, we obtain

$$\dim \bar{W}_{d',n_1,\cdots,n_k} \leqq (n-1)^2(g-1)+(n-1)(g-1)$$
$$+ d - (n-1)d' < \dim M_{n,d}\,.$$

Denote by Z the disjoint union $\cup_{d'}[\bar{W}_{d'} \cup \bar{W}_{d',n_1,\cdots,n_k}]$. Evidently it is a finite union. By the universal property (cf. 1.12.2 or [Se 1]) of the moduli space, there exists a morphism $\Phi: Z \to M_{n,d}$ and by the construction of Z, Φ dominates W_d^0. q.e.d.

REMARK 3.3. Let $Z' = \cup_{d'>0}[\bar{W}_{d'} \cup \bar{W}_{d',n_1,\cdots,n_k}]$ and $\Phi': Z' \to M_{n,d}$ the restriction of the morphism Φ to Z'. Then

$$\dim Z' \leqq (n-1)^2(g-1)+(n-1)(g-1)+d-(n-1)$$

and

$$\mathrm{Image}\,\Phi' \supseteqq \{V \in U_{n,d} \mid V \text{ has a nonzero section of positive degree}\}\,.$$

Denote by Z'' the union $\{V \in \bar{W}_0 \mid (V/\mathcal{O}) \text{ is nonstable}\} \cup \{\bar{W}_{0,n_1,\cdots,n_k}\}$ and $\Phi'': Z'' \to M_{n,d}$ the restriction of Φ to Z''. Then

$$\dim Z'' \leqq (n-1)^2(g-1)+(n-1)(g-1)+d-(n-1)$$

for $\dim(\bar{M}_{n,d} \setminus Z_{n,d}) \leqq \dim(U_{n,d}) - (n-1)$ and $\dim \bar{M}_{n_1,\cdots,n_k} \leqq \dim U_{n,d} - (n-1)$, where $Z_{n,d}$ is the parametrisation of all stables bundles of rank n and degree d (cf. Section 2, Chapter I). Moreover

$$\mathrm{Image}\,\Phi'' \supseteqq \{V \in U_{n,d} \mid 0 \to \mathcal{O} \to V \to (V/\mathcal{O}) \to 0,\ (V/\mathcal{O}) \text{ is non-stable}\}\,.$$

REMARK 3.4. We have $\dim W_d^0 \leqq (n-1)^2(g-1)+(n-1)(g-1)+d$.

4. An argument of Kempf. We follow an argument of Kempf (cf. [K-L]) to prove:

THEOREM 4.1. *For $0 < d \leqq n(g-1)$, if W_d^r is non-empty, then the dimension of W_d^r is at least $\rho(r, d, n)$, where*

$$\rho(r, d, n) = \dim U_{n,d} - h^0(V)\cdot h^1(V)$$

with $V \in U_d^r$ satisfying $h^0(V) = r+1$. Hence

$$\rho(r, d, n) = n^2(g-1)+1-(r+1)(r+1-d-n+ng)\,.$$

For rank 1, *we see that $\rho(r, d, n)$ reduces to the classical Brill-*Noether *number $g-(r+1)(g-d+r)$.*

PROOF. *Let* $d = \iota n + d_1$, where $\iota = [d/n]$. Let $V_{d_1} \to X \times Z_{n,d_1}$ parametrise the stable bundles of rank n and degree d_1 as in Section 2 (so that $\dim Z_{n,d_1} = \dim U_{n,d_1}$). Denote by $p\colon X \times Z_{n,d_1} \to Z_{n,d_1}$ and $q\colon X \times Z_{n,d_1} \to X$ the natural projections, so that p is proper and flat. Let $p_1, \cdots, p_{2g}$ be fixed points of X; we also write P for p_{2g}. For $D_1 = p_1 + \cdots + p_\iota$ and $D_2 = D_1 + P$, we have a bundle morphism $(-P) \to (D_1)$, where $(-P)$ and (D_1) are the line bundles corresponding to the divisors $-P$ and D_1. Thus we obtain on $X \times Z_{n,d_1}$ a short exact sequence

$$0 \to q^*(-P) \to q^*(D_1) \to q^*\mathcal{O}_{D_2} \to 0 \;.$$

Tensoring by V_{d_1} and writing for convenience

$$V(-P) := V_{d_1} \otimes q^*(-P) \;, \quad V(D_1) := V_{d_1} \otimes q^*(D_1) \;, \quad V_{D_2} := V_{d_1} \otimes q^*\mathcal{O}_{D_2} \;,$$

we obtain another short exact sequence

$$0 \to V(-P) \to V(D_1) \to V_{D_2} \to 0$$

We now consider the direct image of this exact sequence under p. Since the support of V_{D_2} is the disjoint union of $\iota + 1$ copies of Z_{n,d_1}, we see that $p_* V_{D_2}$ is the direct sum of $\iota + 1$ bundles of rank n, while $R^1 p_* V_{D_2} = 0$. Moreover, $V(-P)_{|u \times X}$ is a stable bundle of degree $d_1 - n < 0$; so $p_*(V(-P)) = 0$ and $Q = R^1 p_*(V(-P))$ is vector bundle of rank $n(g-1) - (d_1 - n) = ng - d_1$. Thus we obtain an exact sequence

$$0 \to p_*(V(D_1)) \to p_* V_{D_2} \xrightarrow{\delta} Q \to R^1 p_*(V(D_1)) \to 0 \;.$$

We now apply this to the case $D_1 = D_1' = p_1 + \cdots + p_{2g-1}$, $D_2' = D_1' + P$. The bundle $V(D_1')_{|u \times X}$ is stable of degree $\geqq n(2g-1)$; so $R^1 p_*(V(D_1')) = 0$, and the exact sequence becomes

$$0 \to p_*(V(D_1')) \to p_* V_{D_2'} \xrightarrow{\varepsilon} Q \to 0 \;.$$

Since $p_* V_{D_2'}$ is a bundle of rank $2ng$ and Q is a bundle of rank $ng - d_1$, this sequence defines a section α of the Grassmann bundle $B = \mathrm{Grass}_{ng+d_1}(p_* V_{D_2'})$ on Z_{n,d_1}.

The natural morphism $(D_1) \to (D_1')$ induces a diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & p_*(V(D_1)) & \longrightarrow & p_* V_{D_2} & \xrightarrow{\delta} & Q & \longrightarrow & R^1 p_*(V(D_1)) & \longrightarrow 0 \\ & \Big| & & \Big| & & \Big\| & & & \\ 0 \longrightarrow & p_*(V(D_1')) & \longrightarrow & p_* V_{D_2'} & \xrightarrow{\varepsilon} & Q & \longrightarrow & 0 \;. & \end{array}$$

Note that $p_* V_{D_2}$ is a subbundle of $p_* V_{D_2'}$, so we can define a subvariety σ_{r+1} of B by the condition

$$\sigma_{r+1} \cap B_u = \{\Lambda \in B_u; \dim(\Lambda \cap (p_* V_{D_2})_u) \geqq r+1\} .$$

Moreover, the codimension of σ_{r+1} in B is

$$(r+1)(ng - d_1 - \iota n - n + r + 1) = (r+1)(r+1-d-n+ng) ,$$

since this is the codimension of $\sigma_{r+1} \cap \mathrm{B}_u$ in B_u (this uses the assumption that $d = \iota n + d_1 \leqq n(g-1)$ in the form $n(\iota+1) \leqq ng - d_1$).

Now, for any $u \in Z_{n,d_1}$, $\alpha(u) = \ker(\varepsilon_u)$; so

$$\alpha(u) \cap (p_* V_{D_2})_u = p_*(V(D_1))_u = H^0(V(D_1))_{|u \times X}) .$$

Thus $h^0(V(D_1))_{|u\times X}) \geqq r+1$ if and only if $\alpha(u) \in \sigma_{r+1}$. In other words, if $\varphi\colon Z_{n,d_1} \to U_{n,d_1}$ is the natural morphism, we have

$$\varphi^{-1}(W_d^r) = \varphi^{-1}(W_{\iota n+d_1}^r) = \alpha^{-1}(\sigma_{r+1}) .$$

The result follows, since $U_{n,d}$ is irreducible and φ is étale. q.e.d.

REMARK 4.2. If W_d^r is non-empty, then every component of it has dimension at least $\rho(r, d, n)$.

REMARK 4.3. The same argument shows that, in any family of stable bundles parametrized by an irreducible variety (or by a connected complex manifold), the subvariety of the parameter space defined by the condition $h^0(V) \geqq r+1$, if non-empty, has codimension atmost $\rho(r, d, n)$.

CHAPTER II. Stable Sheaves on Nodal Curves. We describe stable torsion free sheaves on a Castelnuovo curve in terms of descent data on the normalisation of it, which is $\boldsymbol{P}^1$, and carry over the definition of W_d^r on smooth curves defined in the previous chapter, to stable sheaves on nodal curves and compute their dimensions in some special cases.

1. Some Facts. X is a projective integral curve of genus g. Let $S \subsetneqq X$ be the locus of singular points of X.

1.1. We quote [N, Chapter 5, Section 7]. There exists a quasi projective K-scheme $U_{n,d}$ which is a coarse moduli space for stable torsion free sheaves over X of rank n and degree d. Moreover,

(i) $U_{n,d}$ has a natural compactification to a projective K-scheme $\mathrm{M}_{n,d}$,

(ii) the points of $M_{n,d}$ are in a natural bijective correspondence with the classes of semistable torsion free sheaves over X under the relation $\sim$ given by

$$F \sim F'$$

if and only if

$$\mathrm{Gr}\, F \cong \mathrm{Gr}\, F'$$

(Gr F is defined analogously to Gr V for a torsion free sheaf V on smooth curves).

DEFINITION 1.2. A singular point $x \in S$ is said to be an ordinary double point if the completion of the local ring $\mathcal{O}_x$ is isomorphic to $K[[y, t]]/(y \cdot t)$.

Let $\tilde{X}$ be the normal model of X in the function field of X and π: $\tilde{X} \to X$, the normalisation morphism.

1.3. Let $\tilde{\mathcal{O}}_x$ be the normalisation of $\mathcal{O}_x$, for $x \in S$. Denote by δ_x the dimension of $(\tilde{\mathcal{O}}_x/\mathcal{O}_x)$ over $(\mathcal{O}_x/m_x)$, the residue field of $\mathcal{O}_x$. Then we have

$$\dim H^1(X, \mathcal{O}_X) = \dim H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) + \delta$$

where $\delta = \sum_{x \in S} \delta_x$ [D'S, Chapter I, 2.5].

1.4. For $x \in S$, if x is an ordinary double point, then $\delta_x = 1$ and $\pi^{-1}(x)$ consists of two distinct points in $\tilde{X}$.

DEFINITION 1.5. A Castelnuovo curve X is a singular curve of genus g with g ordinary double points.

BLANKET ASSUMPTION. Throughout this chapter, X will denote a Castelnuovo curve, $\pi: \boldsymbol{P}^1 \to X$, the normalisation morphism and $\{p_i, q_i\}_{i=1}^g$, g distinct pairs of points on $\boldsymbol{P}^1$ corresponding to the g nodes of X, $\pi(p_i) = \pi(q_i)$ for all i. In this case, the moduli scheme $M_{n,d}$ is an irreducible variety of dimension $n^2(g-1)+1$ (see [R]).

1.6. Let F be a torsion free, stable sheaf of rank n and degree d on X. Then for $d > n(2g-2+2n)$ the following hold:

(A) F is generated by its sections.

(B) $H^1(X, F) = 0$.

[N, Chapter 5, Section 7].

2. Description of Stable Sheaves on a Castelnuovo Curve. Let V be a vector bundle of rank n on $\boldsymbol{P}^1$ and V_p the fibre of V over $p \in \boldsymbol{P}^1$. Let $N_{p_i} \subsetneqq V_{p_i}$, $N_{q_i} \subsetneqq V_{q_i}$ be vector subspaces with $\dim N_{p_i} = \dim N_{q_i}$ for all i and $A_i: (V_{p_i}/N_{p_i}) \to (V_{q_i}/N_{q_i})$ be vector space isomorphisms for all i. Define $\bar{V}$, an $\mathcal{O}_X$ module as $\bar{V}(U) = \{s \in \pi_* V|U| A_i(s(p_i) + N_{p_i}) = s(q_i) + N_{q_i}$ for all i such that $\pi(p_i) \in U\}$. Then $\bar{V}$ is an $\mathcal{O}_X$-module with stalk at $\pi(p_i)$, the i-th node, isomorphic to $(n_{p_i}) \cdot m \oplus (n - n_{p_i}) \cdot \mathcal{O}_X$ where $n_{p_i} = \dim N_{p_i} = \dim N_{q_i} = n_{q_i}$, $(n_{p_i}) \cdot m$ and $(n - p_i)\mathcal{O}_X$ are the n_{p_i}-fold direct sum of m and the $(n - p_i)$-fold direct sum of $\mathcal{O}_X$, respectively, m being the maximal ideal at $\pi(p_i)$. We note that if $N_{p_i} = N_{q_i} = (0)$ for all i, then $\bar{V}$ is locally free. We shall refer to $\{N_{p_i}, N_{q_i}, A_i\}_{i=1}^g$ as descent data for $\bar{V}$. The data $\{N_{p_i}, N_{q_i}, A_i\}_{i=1}^g$ and $\{N'_{p_i}, N'_{q_i}, A'_i\}_{i=1}^g$ give rise to isomorphic sheaves if and only if there exists $\sigma \in \mathrm{Aut}(V)$ such that $\sigma(N_{p_i}) = N'_{p_i}$, $\sigma(N_{q_i}) = N'_{q_i}$ and

$$\begin{array}{ccc} (V_{p_i}/N_{p_i}) & \xrightarrow{A_i} & (V_{q_i}/N_{q_i}) \\ \downarrow \sigma & & \downarrow \sigma \\ (V_{p_i}/N'_{p_i}) & \xrightarrow{A'_i} & (V_{q_i}/N'_{q_i}) \end{array}$$

is commutative for all i. Thus $P(\mathrm{Aut}\, V)$, the projectivised group of automorphisms of V act on the descent data in a natural way.

LEMMA 2.1. *Let R be the completion of the local ring at a node of X. Then $R \cong K[[x, y]]/(xy)$ where $K[[x, y]]$ is the ring of formal power series in two variables x and y. Denote by m the maximal ideal. Then $\mathrm{Tor}_1(R/m, \bar{R}) \cong K$, where $\bar{R} = K[[t]]$, the ring of formal power series in t and $h: R \to \bar{R}$ is the ring homomorphism such that $h(x) = 0$, $h(y) = t$ making $\bar{R}$ an R-module.*

PROOF. We have the following exact resolution of R/m over R:

$$(**) \qquad \cdots \to R^2 \overset{A_3}{\to} R^2 \overset{A_2}{\to} R^2 \overset{A_1}{\to} R \to R/m \to 0 ,$$

where $A_1(1, 0) = x$, $A_1(0, 1) = y$ with respect to the standard basis in R^2. Tensoring (**) by $\bar{R}$, we get

$$\cdots \to R^2 \underset{R}{\otimes} \bar{R} \xrightarrow{A_2 \otimes \mathrm{Id}} R^2 \underset{R}{\otimes} \bar{R} \xrightarrow{A_1 \otimes \mathrm{Id}} R \underset{R}{\otimes} \bar{R} \to R/m \underset{R}{\otimes} \bar{R} \to 0 .$$

Therefore, $\mathrm{Tor}_1(R/m, \bar{R}) = \mathrm{Ker}(A_1 \otimes \mathrm{Id})/\mathrm{Im}(A_2 \otimes \mathrm{Id})$. Identifying the modules $R \otimes_R \bar{R}$ with $\bar{R}$ and $R^2 \otimes_R \bar{R}$ with $\bar{R}^2$, we see that $\mathrm{Ker}(A_1 \otimes \mathrm{Id}) \cong \bar{R}$ and $\mathrm{Im}(A_2 \otimes \mathrm{Id}) \cong (t)$, where (t) is the ideal generated by t in $\bar{R}$. Hence $\mathrm{Tor}_1(R/m, \bar{R}) \cong K$. q.e.d.

LEMMA 2.2. *Let V be a vector bundle of rank n and degree d on $\boldsymbol{P}^1$ and $\bar{V}$ a torsion free sheaf on X obtained from V with the descent data $\{N_{p_i}, N_{q_i}, A_i\}_{i=1}^g$ as per the description above. Then* $\deg \bar{V} = d + \sum_{i=1}^g n_{p_i}$, $n_{p_i} = \dim N_{p_i}$.

PROOF. It can be easily seen that if $\bar{V}$ is locally free, i.e., when $N_{p_i} = N_{q_i} = 0$ and A_i: $V_{p_i} \to V_{q_i}$ are isomorphisms for all i, then $\deg \bar{V} = \deg V$. Suppose $\bar{V}$ is not locally free. We may define B_i: $V_{p_i} \to V_{q_i}$, vector space isomorphisms, such that B_i induces A_i on $(V_{p_i}/N_{p_i}) \to (V_{q_i}/N_{q_i})$ for all i. Let $\bar{V}_1$ denote the locally free sheaf obtained with the descent data $\{B_i\}_{i=1}^g$. Clearly $\bar{V}_1 \subset \bar{V}$ and $\bar{V}/\bar{V}_1$ is a torsion module of $\mathcal{O}_X$ supported at the g nodes of total length $\sum_i n_{p_i}$. Hence the degree of $\bar{V}$ is $\deg \bar{V}_1 + \sum_i n_{p_i}$ which is the same as $d + \sum_i n_{p_i}$. q.e.d.

LEMMA 2.3. *Let F be a torsion free sheaf of rank n and degree d over X and $(F)_{\pi(p_i)} \cong n_{p_i} \cdot m \oplus (n - n_{p_i})\mathcal{O}_X$. Then there exists $\bar{V}$, a locally free sheaf on X of rank n and degree $(d + \sum_{i=1}^{g} n_{p_i})$ such that $F \subseteqq \bar{V}$ is an embedding.*

PROOF. We may suppose that F descends from W, a rank n bundle on $\boldsymbol{P}^1$ with the descent data $\{N_{p_i}, N_{q_i}, A_i\}_{i=1}^{g}$. We can construct a bundle V on $\boldsymbol{P}^1$ and Φ: $W \to V$ a bundle morphism with the following properties: (let Φ_p: $W_p \to V_p$ denote the induced maps of Φ at the fibres over $p \in \boldsymbol{P}^1$) Φ_p: $W_p \to V_p$ is an isomorphism for all p, $p \notin \{p_i, q_i\}_{i=1}^{g}$, $\operatorname{Ker}\Phi_{p_i} = N_{p_i}$, $\operatorname{Ker}\Phi_{q_i} = N_{q_i}$ for all i. Let $\bar{A}_i$: $V_{p_i} \to V_{q_i}$ be vector space isomorphisms such that there is a commutative diagram

$$\begin{array}{ccc} V_{p_i} & \xrightarrow{\bar{A}_i} & V_{q_i} \\ \cup\!\shortparallel & & \cup\!\shortparallel \\ \Phi_{p_i}(W_{p_i}) & \longrightarrow & \Phi_{q_i}(W_{q_i}) \end{array}$$

for all i. Let $\bar{V}$ be the locally free sheaf on X with the descent data $(\bar{A}_i)_{i=1}^{g}$. Clearly $F \hookrightarrow \bar{V}$ and by Lemma 2.2, $d = \deg W + \sum_{i=1}^{g} n_{p_i}$ and $\deg \bar{V} = \deg V = \deg W + 2\sum_{i=1}^{g} n_{p_i}$. q.e.d.

LEMMA 2.4. *Let F be a torsion free sheaf of rank n and degree d as in Lemma 2.3. Let $T \subseteqq F \otimes_{\mathcal{O}_X} \mathcal{O}_{\boldsymbol{P}^1}$ be the torsion part of the $\mathcal{O}_{\boldsymbol{P}^1}$-module $F \otimes_{\mathcal{O}_X} \mathcal{O}_{\boldsymbol{P}^1}$. Then $\deg(F \otimes_{\mathcal{O}_X} \mathcal{O}_{\boldsymbol{P}^1}/T) = d - \sum_{i=1}^{g} n_{p_i}$ and the length of T is $2\sum_{i=1}^{g} n_{p_i}$.*

PROOF. By Lemma 2.3 we can embed F in $\bar{V}$, a locally free sheaf of rank n and degree $d + \sum_{i=1}^{g} n_{p_i}$ on X. We have

$$0 \to F \to \bar{V} \to \bar{V}/F \to 0\ .$$

The sheaf $\bar{V}/F$ is supported at the nodes $\pi(p_i)$ and locally isomorphic to the n_{p_i}-fold direct sum of $\mathcal{O}_X/m$ at $\pi(p_i)$ for all i. Therefore tensoring the sequence by $\mathcal{O}_{\boldsymbol{P}^1}$, we get

$$0 \to \operatorname{Tor}_1(\bar{V}/F, \mathcal{O}_{\boldsymbol{P}^1}) \to F \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{\boldsymbol{P}^1} \to \bar{V} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{\boldsymbol{P}^1} \to \bar{V}/F \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{\boldsymbol{P}^1} \to 0$$

and appealing to Lemma 2.1 we see that the length of $T = \operatorname{Tor}_1(\bar{V}/F, \mathcal{O}_{\boldsymbol{P}^1})$ is $\sum_{i=1}^{g} 2n_{p_i}$ supported at $\{p_i, q_i\}_{i=1}^{g}$ and the length of $\bar{V}/F \otimes_{\mathcal{O}_X} \mathcal{O}_{\boldsymbol{P}^1}$ is $\sum_{i=1}^{g} 2n_{p_i}$. Since $\deg \bar{V} = \deg(\bar{V} \otimes_{\mathcal{O}_X} \mathcal{O}_{\boldsymbol{P}^1}) = d + \sum_{i=1}^{g} n_{p_i}$, we have $\deg(F \otimes_{\mathcal{O}_X} \mathcal{O}_{\boldsymbol{P}^1}/T) = d - \sum_{i=1}^{g} n_{p_i}$. q.e.d.

PROPOSITION 2.5. *For $0 \leqq k < n$, let $U_{n,k}(G) = \{\bar{V} \in M_{n,k} \mid \bar{V}$ is stable, locally free and $\pi^*\bar{V} \cong V = V_1 \oplus V_2$ on $\boldsymbol{P}^1\}$ where V_1 and V_2 are k-fold and $(n-k)$-fold direct sums of $\mathcal{O}(1)$ and $\mathcal{O}$,* *the structure sheaf of* $\boldsymbol{P}^1$, *respectively. Then* $U_{n,k}(G)$ *is a non-empty Zariski open set in* $M_{n,k}$. *(See Section* 1 *for notation).*

PROOF. Given a vector bundle W of rank r, $r < n$, and degree d on $\boldsymbol{P}^1$ with $d/r \geqq k/n$ and admitting an embedding in V as a sub-bundle, we define

$$E(W, V) = \{\sigma \in \mathrm{Hom}(W, V) \mid \sigma \text{ embeds } W \text{ as a sub-bundle in } V\}$$

$$G(l, V_p) = \text{Grassmannian of } l \text{ planes in the fibre } V_p \text{ of } V \text{ over } p \in \boldsymbol{P}^1$$

$$\mathrm{Iso}(V_{p_i}, V_{q_i}) = \text{Set of vector space isomorphisms of } V_{p_i} \text{ on } V_{q_i} .$$

Since $\dim H^0(W^* \otimes V) = dk + (n + k)(r - d)$ and $\dim E(W, V) = \dim H^0(W^* \otimes V)$, we have $\dim E(W, V) = dk + (n + k)(r - d)$. For

$$(*) \qquad 0 < l_1 \leqq r\,, \quad 0 < l_2 \leqq r\,, \quad [d - (r - l_1) - (r - l_2)]/r \geqq k/n$$

let $Z_{ij}(r, d, l_1, l_2)$ be

$$\{(A_1, A_2, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) \in I_{ij} \times G_{ij} \mid A_1(\Lambda_1) = \Lambda_2,\ A_2(\Lambda_3) = \Lambda_4 \text{ and for some }$$
$$\sigma \in E(W, V),\ \sigma(W_{p_i}) \supseteqq \Lambda_1,\ \sigma(W_{q_i}) \supseteqq \Lambda_2,\ \sigma(W_{p_j}) \supseteqq \Lambda_3,\ \sigma(W_{q_j}) \supseteqq \Lambda_4\}\,,$$

where $I_{ij} = \mathrm{Iso}(V_{p_i}, V_{q_i}) \times \mathrm{Iso}(V_{p_j}, V_{q_j})$ and G_{ij} denotes the variety $G(l_1, V_{p_i}) \times G(l_1, V_{q_i}) \times G(l_2, V_{p_j}) \times G(l_2, V_{q_j})$. Let $W_{ij}(r, d, l_1, l_2) \subseteqq I_{ij}$ be the image of $Z_{ij}(r, d, l_1, l_2)$ under the natural projection. Then the dimension of $Z_{ij}(r, d, l_1, l_2)$ is at most

$$[\{n^2 - l_1(n - l_1)\} + \{n_2 - l_2(n - l_2)\} + 2l_1(r - l_1) + 2l_2(r - l_2)] + \dim E(W, V)$$

and dimension of $W_{ij}(r, d, l_1, l_2)$ is at most

$$[\{n^2 - l_1(n - l_1)\} + \{n^2 - l_2(n - l_2)\} + l_1(r - l_1) + l_2(r - l_2)]$$
$$+ [\dim E(W, V) - \dim \mathrm{Aut}\, W]\,,$$

where $\mathrm{Aut}\, W$ is the group of bundle automorphisms of W and its dimension is at least r^2. Therefore the above expression is strictly less than $2n^2$ which is the dimension of I_{ij}. Hence

$$\dim W_{ij}(r, d, l_1, l_2) < \dim I_{ij} = 2n^2\,.$$

This shows that

$$\dim \bar{W}_{ij} < 2n^2$$

where

$$\bar{W}_{ij} = \text{Zariski closure of } [\bigcup_{l_1, l_2} W_{ij}(r, d, l_1, l_2) \mid l_1, l_2 \text{ satisfy } (*)]$$

in I_{ij}.

Define $\mathrm{NS}(i, j) = \bigcup_{[W] \in I} \bar{W}_{ij}$, where I is the set of isomorphism classes of subbundles W of V with $(\deg W/\operatorname{rank} W) \geqq (k/n)$. We note that this is actually a finite union. Hence $S(i, j) = I_{ij} \setminus \mathrm{NS}(i, j)$ is a non-empty Zariski open set.

Let $V(A_1, \cdots, A_g)$ be a vector bundle on X given by the discent data $\{A_i\}_{i=1}^g$ on V such that for at least two i's, say $i_1, i_2, (A_{i_1}, A_{i_2}) \in S(i_1, i_2)$. Then $V(A_1, \cdots, A_g)$ is stable, for let $F \subseteqq V(A_1, \cdots, A_g)$ be a subsheaf of rank r, $r < n$ and degree d such that $(d/r) \geqq (k/n)$. We may assume without loss of generality that $F|(X \setminus \{\pi(p_i)\}_{i=1}^g)$ is a subbundle of $V(A_1, \cdots, A_g)|(X \setminus \{\pi(p_i)\}_{i=1}^g)$. Suppose $(F)_{\pi(p_i)} = n_{p_i} m \oplus (r - n_{p_i})\mathcal{O}_X$. Then by Lemma 2.4, $\deg W = d - \sum_{i=1}^g n_{p_i}$ where $W = (F \otimes_{\mathcal{O}_X} \mathcal{O}_{P^1}/\text{Torsion})$. The inclusion morphism $j: F \to V(A_1, \cdots, A_g)$ on X induces a bundle morphism $j^*: W \to V$ on $\boldsymbol{P}^1$ such that rank of j^* at p_i equals rank of j^* at q_i which is at most $(r - n_{p_i})$, for all i, $j^*: W_{p_i} \to V_{p_i}$ being the map induced by j^*. Let W' be the subbundle generated by $j^*(W) \subseteqq V$. We have $\deg W' = d' = d + \sum_{i=1}^g n_{p_i} + \sum_{i=1}^g 2s_i$, where $n_{p_i} + s_i$ is the dimension of $\operatorname{Ker} j^*$ at W_{p_i}; we note that $n_{p_i} + s_i < r$, for, if $n_{p_i} + s_i = r$ then $d'/r \geqq (k/n) + 1$; since $d' \leqq \min(r, k)$ for any sub-bundle of V, this would be a contradiction. At i_1 and i_2, $A_{i_1}(j^*(W_{p_{i1}})) = j^*(W_{q_{i1}})$, $A_{i_2}(j^*(W_{p_{i2}})) = j^*(W_{q_{i2}})$ and $\dim j^*(W_{p_{i1}}) = l_1 = r - n_{p_{i1}} - s_{i_1} > 0$, $\dim j^*(W_{p_{i2}}) = l_2 = r - n_{p_{i2}} - s_{i_2} > 0$ and $[d' - (r - l_1) - (r - l_2)]/r \geqq d/r \geqq k/n$. Hence $(A_{i_1}, A_{i_2}) \in \mathrm{NS}(i_1, i_2)$, a contradiction.

Let $\bar{S}(i, j) = S(i, j) \times \prod_l \operatorname{Iso}(V_{p_l}, V_{q_l})$, $l \notin \{i, j\}$ and $\bar{U}_{n,k}(G) = \cup_{\substack{i,j \\ i \neq j}} \bar{S}(i, j)$. Since $\bar{U}_{n,k}(G)$ is Zariski open in the space $\prod_{i=1}^g \operatorname{Iso}(V_{p_i}, V_{q_i})$, $\dim \bar{U}_{n,k}(G) = n^2 g$. By our identification of the descent data, $\bar{U}_{n,k}(G)/P(\operatorname{Aut} V) = U_{n,k}(G)$ and $\dim U_{n,k}(G) = \dim M_{n,k}$. q.e.d.

REMARK 2.6. When $d \geqq n$, let L be a line bundle on X of degree $[d/n]$, $[d/n]$ being the integral part of d/n, $d = [d/n] \cdot n + k$. Then

$$\begin{aligned} U_{n,d}(G) &= \{L \otimes \bar{V}_1 \mid \bar{V}_1 \in U_{n,k}(G)\} \\ &= \{\bar{V} \in M_{n,d} \mid \bar{V} \text{ is stable, locally free and } \pi^* \bar{V} \cong V_1' \oplus V_2' \text{ on } \boldsymbol{P}^1\} \end{aligned}$$

where V_1' and V_2' are k-fold and $(n-k)$-fold direct sums of $\mathcal{O}([d/n] + 1)$ and $\mathcal{O}([d/n])$, respectively, is Zariski open in $M_{n,d}$.

3. Stable Sheaves with Sections.

Analogously to the definition of W_d^r on smooth curves, we define

$$W_d^r = \bar{U}_d^r, \quad U_d^r = \{F \in U_{n,d} \mid h^0(F) \geqq r + 1\}$$

where W_d^r is the Zariski closure of U_d^r in $M_{n,d}$ on the singular curve X.

We recall $U_{n,d} = \{F \in M_{n,d} \mid F \text{ in stable over } X\}$.

THEOREM 3.1. *For $0 < d \leqq n(g-1)$, the dimension of W_d^0 is given by*

$$\rho(0, d, n) = n^2(g-1) + 1 - (ng - n - d + 1)$$

and it has a unique irreducible component of maximal dimension.

PROOF. We define

$$U_{d,\mathrm{vec}} = \{V \in U_{n,d} \mid V \text{ is locally free on } X\},$$

$$U_d(i_j) = \{F \in U_{n,d} \mid F \text{ is locally free on } X \setminus \{\pi(p_{i_j})\}_{j=1}^{l}\}$$

and

$$U_d(\alpha) = \{V \in U_{d,\mathrm{vec}} \mid \pi^* V \cong \mathcal{O}(\alpha_1) \oplus \cdots \oplus \mathcal{O}(\alpha_n)\},$$

where $(\alpha) = (\alpha_1, \cdots, \alpha_n) \in \boldsymbol{Z}^n$ with $\sum \alpha_i = d$ and $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n$, $(i_j) = (i_1, \cdots, i_l) \in \boldsymbol{Z}^l$, $\boldsymbol{Z}$ being the ring of integers. Then

$$U_{n,d} = U_{d,\mathrm{vec}} \cup \bigcup_{(i_j)} U_d(i_j) = [\bigcup_{(\alpha)} U_d(\alpha)] \cup [\bigcup_{(i_j)} U_d(i_j)].$$

We shall show that

$$\dim[W_d^0 \cap U_d(\alpha)] < \dim[W_d^0 \cap U_d(\beta)]$$

for all (α), $(\alpha) \neq (\beta)$ where $\beta = (\beta_1, \cdots, \beta_n)$ satisfies

$$\beta_j = \begin{cases} [d/n] + 1, & j \leqq k \\ [d/n], & j > n. \end{cases} \qquad d = [d/n] \cdot n + k.$$

and

$$\dim[W_d^0 \cap U_d(i_j)] < \dim[W_d^0 \cap U_d(\beta)]$$

for all (i_j), and that

$$\dim[W_d^0 \cap U_d(\beta)] = \rho(0, d, n).$$

Parametrisation of $W_d^0 \cap U_d(\alpha)$: Let $V(\alpha)$ denote the vector bundle $\mathcal{O}(\alpha_1) \oplus \cdots \oplus \mathcal{O}(\alpha_n)$ on $\boldsymbol{P}^1$. Define

$$\bar{W}_d(\alpha) = \{((A_1, \cdots, A_g), s) \in \prod_{i=1}^{g} \mathrm{Hom}(V_{p_i}(\alpha), V_{q_i}(\alpha)) \times H^0(V(\alpha)) \mid A_i(s(p_i)) = s(q_i), \text{ for all } i, 1 \leqq i \leqq g\}$$

where $V_p(\alpha)$ is the fibre of $V(\alpha)$ over $p \in \boldsymbol{P}^1$, and

$$\bar{U}_d(\alpha) = \{(A_1, \cdots, A_g) \in \prod_{i=1}^{g} \mathrm{Iso}(V_{p_i}(\alpha), V_{q_i}(\alpha)) \mid (A_1, \cdots, A_g) \text{ is a descent datum for a stable bundle on } X\}.$$

Then it follows easily that

$$\bar{U}_d(\alpha)/P(\mathrm{Aut}\, V(\alpha)) = U_d(\alpha), \quad [\bar{U}_d(\alpha) \cap p(\bar{W}_d(\alpha))]/P(\mathrm{Aut}\, V(\alpha)) = U_d(\alpha) \cap W_d^0.$$

Note that $P(\mathrm{Aut}\, V(\alpha))$ acts freely on $\bar{U}_d(\alpha)$ because of the stability condition. Here p and q are the natural projections of $\prod_{i=1}^{g} \mathrm{Hom}(V_{p_i}(\alpha), V_{q_i}(\alpha)) \times H^0(V(\alpha))$ to $\prod_{i=1}^{g} \mathrm{Hom}(V_{p_i}(\alpha), V_{q_i}(\alpha))$ and $H^0(V(\alpha))$, respectively and $P(\mathrm{Aut}\, V(\alpha))$ is the projectivised group of bundle automorphisms of $V(\alpha)$.

Dimension of $\bar{W}_d(\alpha)$. We stratify $H^0(V(\alpha))$ as follows: let $Q = \{1, 2, \cdots, g\}$, 2^Q the power set of Q, $\Delta_k = [\{i_1, \cdots, i_k\} \in 2^Q]$ with $1 \leqq k \leqq g$. Define

$$S^{\alpha}_{i_1,\cdots,i_k} = H^0(V(\alpha)) \otimes \mathcal{O}\Big(-\sum_{j=1}^{k} (p_{i_j} + q_{i_j})\Big) \qquad \text{for} \quad \{i_1, \cdots, i_k\} \in \Delta_k \;;$$

Set

$$T^{\alpha}_k = \bigcup_{\{i_1,\cdots,i_k\} \in \Delta_k} S^{\alpha}_{i_1,\cdots,i_k}\,, \quad T^{\alpha}_0 = H^0(V(\alpha))\,.$$

Then

$$H^0(V(\alpha)) = T^{\alpha}_0 \supsetneqq T^{\alpha}_1 \supsetneqq \cdots \supsetneqq T^{\alpha}_g = 0\,.$$

Since $U_d(\alpha)$ is assumed to be non-empty, $T^{\alpha}_g = 0$ follows from the stability condition. If $T^{\alpha}_k \neq 0$ then

$$\dim(T^{\alpha}_k \setminus T^{\alpha}_{k+1}) = h^0\Big(V(\alpha) \otimes \mathcal{O}\Big(-\Big(\sum_{j=1}^{k} p_{i_j} + q_{i_j}\Big)\Big)\Big)\,,$$

which in turn equals $[(d - 2kn) + n + h^1(V(\alpha) \otimes \mathcal{O}(-2k))]$; moreover, the morphism

$$q\colon q^{-1}(T^{\alpha}_k \setminus T^{\alpha}_{k+1}) \cap \bar{W}_d(\alpha) \to T^{\alpha}_k \setminus T^{\alpha}_{k+1}$$

is a fibration and the dimension of the fibre is $[n^2k + (n^2 - n)(g - k)]$. Hence

$$\dim[p\{q^{-1}(T^{\alpha}_k \setminus T^{\alpha}_{k+1}) \cap \bar{U}_d(\alpha)\}/P(\mathrm{Aut}\, V(\alpha))] \leqq \rho^{\alpha}_k$$

where $\rho^{\alpha}_k = n^2k + (n^2 - n)(g - k) + \dim(T^{\alpha}_k \setminus T^{\alpha}_{k+1}) - \dim H^0(\mathrm{End}\, V(\alpha))$. For $(\alpha) \neq (\beta)$, $H^1(\mathrm{End}\, V(\alpha)) \neq 0$ and if $T^{\alpha}_k \neq 0$, then $\alpha_i \geqq 2k$ for some i, $1 \leqq i \leqq n$. Hence $\rho^{\alpha}_k < \rho(0, d, n)$ for all k and for all $(\alpha) \neq (\beta)$. For $(\alpha) = (\beta)$ and for $k > 0$

$$\dim \bar{W}_d(\beta) \cap q^{-1}(T^{\beta}_k \setminus T^{\beta}_{k+1}) = n^2k + (n^2 - n)(g - k) + h^0(V(\beta) \otimes \mathcal{O}(-2k))\,,$$

which in turn equals

$$n^2k + (n^2 - n)(g - k) + d - 2kn + n = \rho(0, d, n) + h^0(\mathrm{End}\, V(\beta)) - kn\;;$$

and for $k = 0$

$$\dim \bar{W}_d(\beta) \cap q^{-1}(T^{\beta}_0 \setminus T^{\beta}_1) = \rho(0, d, n) + h^0(\mathrm{End}\, V(\beta))\,.$$

We note that $T^{\beta}_0 \setminus T^{\beta}_1$ is a non-empty Zariski open subset of $H^0(V(\beta))$

and is therefore irreducible. Moreover, for any $s \in T_0^\beta \setminus T_1^\beta$, the fibre $q^{-1}(s) \cap \bar{W}_d(\beta)$ is a non-empty Zariski open subset of an affine space of dimension $(n^2 - n)g$. Therefore, $q^{-1}(T_0^\beta \setminus T_1^\beta) \cap \bar{W}_d(\beta)$ has exactly one component of maximal dimension $(n^2 - n)g + d + n$. It follows from the estimates above that the same is true of $\bar{W}_d(\beta)$. But, since $\bar{W}_d(\beta)$ is defined by ng equations, every component has dimension $\geqq (n^2 - n)g + d + n$; hence $\bar{W}_d(\beta)$ is irreducible.

Dimension of fibres of p. We shall now take a look at the fibres of the projection morphism $p: \bar{W}_d(\beta) \to \prod_{i=1}^g \mathrm{Iso}(V_{p_i}(\beta), V_{q_i}(\beta))$. If V is a vector bundle on $\boldsymbol{P}^1$ and $(A_1, \cdots, A_g)$ a descent datum on V, we shall denote by $V(A_1, \cdots, A_g)$ the bundle on X given by $(A_1, \cdots, A_g)$. Let $V(\beta) = L \oplus V'$, where

$$L = \begin{cases} \mathcal{O}([d/n] + 1) & \text{if } [d/n] < d/n \\ \mathcal{O}([d/n]) & \text{if } [d/n] = d/n \, . \end{cases}$$

Then $0 < \deg L \leqq (g - 1)$ and $0 \leqq [\deg V'/(n - 1)] \leqq (g - 1)$. Hence there exists $(A_1', \cdots, A_g')$ a descent datum on V' such that $V'(A_1', \cdots, A_g')$ has no section. Let $(\lambda_1, \cdots, \lambda_g)$ be a descent datum on L such that $L(\lambda_1, \cdots, \lambda_g)$ has exactly one section. (This is possible because the degree of L is bounded by 0 and $(g - 1)$). Then for any descent datum $(A_1, \cdots, A_g)$ on $V(\beta)$ inducing $(\lambda_1, \cdots, \lambda_g)$ on L and the datum $(A_1', \cdots, A_g')$ on V', $V(\beta)(A_1, \cdots, A_g)$ has exactly one section. Hence the dimension of the generic fibre of p is one and

$$\dim(p(\bar{W}_d(\beta)) \subseteqq \mathrm{Iso}(V_{p_i}(\beta), V_{q_i}(\beta))) \quad \text{is} \quad \rho(0, d, n) + h^0(\mathrm{End}\, V(\beta)) - 1 \, .$$

Since a generic $s \in H^0(V(\beta))$ generates a trivial line sub-bundle, we note that for a generic $(A, \cdots, A_g) \in p(\bar{W}_d(\beta))$, $V(\beta)(A_1, \cdots, A_g)$ has exactly one section and it generates a trivial line sub-bundle in $V(\beta)$ $(A_1, \cdots, A_g)$.

Existence of a stable bundle on X with a section. We shall adopt the notation of Propsition 2.5. It is enough to show that the set of nonstable bundles with one section generating a trivial line subbundle on X is properly contained in $p(\bar{W}_d(\beta))$. If $\bar{V}$ is a stable bundle on X with $h^0(\bar{V}) \neq 0$, L a line bundle with $h^0(L) \neq 0$, then $h^0(L \otimes \bar{V}) \neq 0$. So we may restrict ourselves to $0 < d \leqq n$, i.e., $d = \sum_{i=1}^n \beta_i \leqq n$.

Let W be a vector bundle of rank r, with $2 \leqq r \leqq n - 1$ and degree d_1 on $\boldsymbol{P}^1$ admitting an embedding in $V(\beta)$.

For $0 < l_i \leqq r$, $i = 1, \cdots, g$, $[d_1 - \sum_{i=1}^g (r - l_i)]/r \geqq d/n$, let $Z(r, d_1, l_1, \cdots, l_g)$ denote

$$\Big\{((A_i), (\Lambda_i, \Lambda_i'), (\sigma, s)) \in \prod_{i=1}^g \mathrm{Iso}(V_{p_i}(\beta), V_{q_i}(\beta)) \times \Omega \,|\, A_i(\Lambda_i) = \Lambda_i' \, ,$$

$$\sigma(W_{p_i}) \supseteqq \Lambda_i,\ \sigma(W_{q_i}) \supseteqq \Lambda'_i,\ A_i s(p_i) = s(q_i) \text{ for all } i\Big\}\ ,$$

where $\Omega = \prod_{i=1}^{g} G(l_i, V_{p_i}(\beta)) \times G(l_i, V_{q_i}(\beta)) \times E(W, V) \times H^0(V(\beta), 1)$ and $H^0(V(\beta), 1) = \{s \in H^0(V(\beta)) \mid s$ generates a trivial line sub-bundle$\}$. We stratify Ω as follows:

$$\Omega_1 = \{((\Lambda_i, \Lambda'_i), (\sigma, s)) \in \Omega \mid s \notin H^0(\sigma(W))\}\ ,\quad \Omega_2 = \Omega \setminus \Omega_1\ .$$

Stratify Ω_1 and Ω_2 further by

$$\Omega^j_{i_1,\cdots,i_k} = \left\{((\Lambda_i, \Lambda'_i), (\sigma, s)) \in \Omega_j \,\middle|\, \begin{array}{l} s(p_i) \in \Lambda_i,\ s(q_i) \in \Lambda'_i \text{ if } i \in \langle i_1, \cdots, i_k\rangle \\ s(p_i) \notin \Lambda_i,\ s(q_i) \notin \Lambda'_i \text{ if } i \notin \langle i_1, \cdots, i_k\rangle \end{array}\right\}$$

and

$$\Omega^j_\phi = \{((\Lambda_i, \Lambda'_i), (\sigma, s)) \in \Omega_j \mid s(p_i) \notin \Lambda_i,\ s(q_i) \notin \Lambda'_i \text{ for all } i\}$$

where $j \in \{1, 2\}$, $\{i_1, i_2, \cdots, i_k\} \in 2^N$, $N = \{1, 2, \cdots, g\}$.
Let

$$Z^j_{i_1,\cdots,i_k} = \Big\{((A_i), (\Lambda_i, \Lambda'_i), (\sigma, s)) \in Z(r, d_1, l_1, \cdots, l_g) \cap \Big[\prod_{i=1}^{g} \operatorname{Iso}(V_{p_i}(\beta), V_{q_i}(\beta))\Big] \times \Omega^j_{i_1,\cdots,i_k}\Big\}$$

and

$$Z^j_\phi = \Big\{((A_i), (\Lambda_i, \Lambda'_i), (\sigma, s)) \in Z(r, d_1, l_1, \cdots, l_g) \cap \Big[\prod_{i=1}^{g} \operatorname{Iso}(V_{p_i}(\beta), V_{q_i}(\beta))\Big] \times \Omega^j_\phi\Big\}$$

and $W^j_{i_1,\cdots,i_k}$, W^j_ϕ be the images under natural projection to $\prod_{i=1}^{g} \operatorname{Iso}(V_{p_i}(\beta), V_{q_i}(\beta))$ of $Z^j_{i_1,\cdots,i_k}$ and Z^j_ϕ, respectively, for $j \in \{1, 2\}$. Set

$$\mathcal{O}_{i_1,\cdots,i_k} = \sum_{i=1}^{g}\Big[n^2 - l_i(n - l_i) - \sum_{j=1}^{k} l_{i_j}\Big] - n(g - k) + \sum_{i=1}^{g}\Big[l_i(r - l_i) - \sum_{j=1}^{k}(r - l_{i_j})\Big]\ ,$$

and

$$\mathcal{O}_\phi = \sum_{i=1}^{g}\Big[n^2 - l_i(n - l_i)\Big] - ng + \sum_{i=1}^{g} l_i(r - l_i)\ .$$

Then

$$\dim W^1_{i_1,\cdots,i_k} \leqq \mathcal{O}_{i_1,\cdots,i_k} + h^0(W) + \dim[\operatorname{Im} H^0(V) \cap H^0(V/W \otimes \mathcal{O}(-2k))] + \dim E(W, V) - \dim \operatorname{Aut} W$$

which in turn is strictly less than $(n^2g - ng + d + n - 1) = \dim p(\bar{W}_d(\beta))$. Here $\operatorname{Im} H^0(V)$ is the image of $H^0(V)$ given by the sequence

$$0 \to W \to V \to V/W \to 0$$

for some fixed embedding of W in V; $\dim E(W, V) = nr - nd_1 + dr$ and $\dim \operatorname{Aut} W = r^2 + h^1(W^* \otimes W)$.

$$\dim W^2_{i_1,\cdots,i_k} \leqq [\mathcal{O}_{i_1,\cdots,i_k} + h^0(W) + \dim E(W, V) - \dim \operatorname{Aut} W] < \dim p(\bar{W}_d(\beta)) .$$

Next we observe

$$\dim W^1_\phi \leqq [\mathcal{O}_\phi + d + n + \dim E(W, V) - \dim \operatorname{Aut} W] < \dim p(\bar{W}_d(\beta))$$

and

$$\dim W^2_\phi \leqq [\mathcal{O}_\phi + h^0(W) + \dim E(W, V) - \dim \operatorname{Aut} W] < \dim p(\bar{W}_d(\beta)) .$$

By an argument similar to the one given in Proposition 2.5 of the previous section, we conclude the existence of a stable bundle with a section. Since $p(\bar{W}_d(\beta))$ is irreducible and the set $p(\bar{W}_d(\beta)) \cap \bar{U}_d(\beta)$ is non-empty, $p(W_d(\beta)) \cap U_d(\beta)$ is irreducible and has dimension $\rho(0, d, n) + h^0(\operatorname{End} V(\beta)) - 1$. Hence $W^0_d \cap U_d(\beta)$ is irreducible and has dimension $\rho(0, d, n)$ as required.

Parametrisation of $W^0_d \cap U_d(i_j)$. We stratify $U_d(i_j)$ as $\cup_{(k_j)} U_d((i_j), (k_j))$, where $(k_j) = \{k_1, \cdots, k_l\}$ and

$$U_d((i_j), (k_j)) = \{F \in U_d(i_j) \mid (F)_{\pi(p_{ij})} \cong k_j m \oplus (n - k_j)\mathcal{O}_X\} .$$

We stratify $U_d((i_j), (k_j))$ further as $\cup_{(\alpha) \in Z^n} U_d((i_j), (k_j), (\alpha))$, where $(\alpha) = (\alpha_1, \cdots, \alpha_n)$ and

$$U_d((i_j), (k_j), (\alpha)) = \{F \in U_d((i_j), (k_j)) \mid F \text{ descends from } V(\alpha) = \mathcal{O}(\alpha_1) \oplus \cdots \oplus \mathcal{O}(\alpha_n) \text{ on } P^1\} .$$

We note that the union over (α) is actually a finite union because stability implies that $\alpha_j \leqq 2g$ for all j. Hence

$$W^0_d \cap U_d(i_j) = \bigcup_{k_j} [W^0_d \cap U_d((i_j), (k_j))] = \bigcup_{(k_j)} \bigcup_{(\alpha)} [W^0_d \cap U_d((i_j), (k_j), (\alpha))] .$$

Parametrisation of $W^0_d \cap U_d((i_j), (k_j), (\alpha))$. By Lemma 2.2, $\deg V(\alpha) = \sum_{j=1}^n \alpha_j = d - \sum_{j=1}^l k_j$. For notational convenience, we may take (i_j) to be $(j) = (1, 2, \cdots, l)$. We denote by $\bar{W}_d((j), (k_j), (\alpha))$ the set

$$\{((B_1, B_1', \cdots, B_l, B_l', A_{l+1}, \cdots, A_g), s) \in \bar{M} \times \bar{N} \times H^0(V(\alpha)) \mid A_i s(p_i) = s(q_i), l + 1 \leqq i \leqq g; B_j s(p_j) = B_j' s(q_j), 1 \leqq j \leqq l\} ,$$

where

$$\bar{M} = \prod_{j=1}^{l} \operatorname{Hom}(V_{p_j}(\alpha), C^{n-k_j}) \times \operatorname{Hom}(V_{q_j}(\alpha), C^{n-k_j})]$$

and

$$\bar{N} = \prod_{i=l+1}^{g} [\mathrm{Hom}(V_{p_i}(\alpha),\ V_{q_i}(\alpha))]\ ,$$

$V_p(\alpha)$ being fibre of $V(\alpha)$ over $p \in \boldsymbol{P}^1$.

Define $\bar{U}_d((j), (k_j), (\alpha))$ as

$$[(B_1, B_1', \cdots, B_l, B_l', A_{l+1}, \cdots, A_g) \in M \times N \mid \{A_i\}_{i=l+1}^{g} \cup \{\mathrm{Ker}\, B_j,\ \mathrm{Ker}\, B_j',\ \bar{B}_j'^{-1}\bar{B}_j\}$$

is a descent datum for a stable torsion free sheaf on X, where the morphisms $\bar{B}_j\colon (V_{p_j}(\alpha)/\mathrm{Ker}\, B_j) \to \boldsymbol{C}^{n-k_j}$ and $\bar{B}_j'^{-1}\colon \boldsymbol{C}^{n-k_j} \to (V_{q_j}(\alpha)/\mathrm{Ker}\, B_j')$ are induced by B_j and B_j']
where $M = \{(B_1, B_1', \cdots, B_l, B_l') \in \bar{M} \mid \mathrm{rank}\, B_j = (n - k_j)$ and $\mathrm{rank}\, B_j' = (n - k_j)$ for all $j\}$, and $N = \prod_{i=l+1}^{g} \mathrm{Iso}(V_{p_i}(\alpha),\ V_{q_i}(\alpha))$. The group $G = \prod_{i=1}^{l} \mathrm{GL}(\boldsymbol{C}^{n-k_j})$ acts on $\bar{U}_d((j), (k_j), (\alpha))$ as follows: for $(X_1, \cdots, X_l) \in G$

$$\begin{aligned}&((X_1, \cdots, X_l), (B_1, B_1', \cdots, B_l, B_l', A_{l+1}, \cdots, A_g))\\ &\quad \to (X_1 \circ B_1,\ X_1 \circ B_1',\ \cdots,\ X_l \circ B_l,\ X_l \circ B_l',\ A_{l+1}, \cdots, A_g)\ .\end{aligned}$$

Hence

$$U_d((j), (k_j), (\alpha)) = [\bar{U}_d((j), (k_j), (\alpha))/G]/P(\mathrm{Aut}\, V(\alpha))$$

and

$$\begin{aligned}&W_d^0 \cap U_d((j), (k_j), (\alpha)) \quad \text{is realised as}\\ &[\bar{U}_d((j), (k_j), (\alpha)) \cap p(\bar{W}_d((j), (k_j), (\alpha)))/G/P(\mathrm{Aut}\, V(\alpha))\ ,\end{aligned}$$

where p and q are the natural projections of $\bar{M} \times \bar{N} \times H^0(V(\alpha))$ to $\bar{M} \times \bar{N}$ and $H^0(V(\alpha))$, respectively.

Dimension of $\bar{W}_d((j), (k_j), (\alpha))$: Stratify $H^0(V(\alpha))$ as follows: for $Q = \{l+1, \cdots, g\}$ and $\{p_i, q_i\}_{i=1}^{g}$ we construct T_w^α as was done for $\bar{W}_d(\alpha)$; for a_x, b_y and c_z in $\boldsymbol{Z}$ with $a_x + b_y + c_z = l$, let $T(a_x, b_y, c_z)$ be the set

$$\left[s \in H^0(V(\alpha)) \,\middle|\, \begin{array}{l} \#(j \in Q' \mid s(p_j) \neq 0,\ s(q_j) \neq 0) = a_x \\ \#(j \in Q' \mid s(p_j) \neq 0 \text{ or } s(q_j) \neq 0) = b_y \\ \#(j \in Q' \mid s(p_j) = 0,\ s(q_j) = 0) = c_z \end{array} \right]$$

where $Q' = \{1, 2, \cdots, l\}$.
Then $\{T(w, a_x, b_y, c_z) = T(a_x, b_y, c_z) \cap T_w^\alpha\}$ is the required stratification. We have

$$q\colon Y(w, a_x, b_y, c_z) \to T(w, a_x, b_y, c_z)\ ,$$

where

$$Y(w, a_x, b_y, c_z) = [q^{-1}(T(w, a_x, b_y, c_z))] \cap \bar{W}_d((j), (k_j), (\alpha))\ .$$

Set

$$Z(w, a_x, b_y, c_z) = [p(Y(w, a_x, b_y, c_z)) \cap \bar{U}_d((j), (k_j), (\alpha))/G]/P(\mathrm{Aut}\, V(\alpha)) .$$

The dimension of $Z(w, a_x, b_y, c_z)$ is given by the expression

$$(*) \qquad \theta + n^2 w + (n^2 - n)(g - l - w) - \sum_{i=1}^{l} (n - k_i)^2 + h^0(V(\alpha) \otimes \mathcal{O}(-2w - 2c_z)) - h^0(\mathrm{End}\, V(\alpha)) ,$$

where θ is given by

$$\left(\sum_{i=1}^{x} [2n(n - k_{a_i}) - (n - k_{a_i})]\right) + \left(\sum_{i=1}^{y} [2n(n - k_{b_i}) - (n - k_{b_i})]\right) + \sum_{i=1}^{z} 2n(n - k_{c_i}) .$$

Since $h^1(V(\alpha) \otimes (-2w - 2c_z)) \leqq h^1(\mathrm{End}\, V(\alpha))$ when $T(w, a_x, b_y, c_z) \neq 0$, we have

$$\dim Z(w, a_x, b_y, c_z) < \rho(0, d, n) .$$

q.e.d.

COROLLARY 3.2. *Let $W_d^0(\rho)$ denote the unique irreducible component of dimension $\rho(0, d, n)$. Then a generic $F \in W_d^0(\rho)$ is locally free, F contains a trivial line subbundle and $h^0(F) = 1$.*

4. Rank Two Sheaves.

LEMMA 4.1. *Let $\{A_i\}_{i=1}^g$ be a descent datum on $V = \mathcal{O}(1) \oplus \mathcal{O}$ over $\boldsymbol{P}^1$ such that $V(A_1, \cdots, A_g)$ is a bundle on X. If for some i, $A_i(\mathcal{O}(1))_{p_i} \neq (\mathcal{O}(1))_{q_i}$, then $V(A_1, \cdots, A_g)$ is stable on X.*

PROOF. Let $i: F \hookrightarrow V(A_1, \cdots, A_g)$ be a subsheaf of rank 1 and degree $d > 0$. Then by lemma 2.4 of this chapter, the degree of $(\pi^* F/\mathrm{Torsion})$ is $(d - k)$, where k is the number of nodes at which F is not locally free. We assume without loss of generality that F_p is $\mathcal{O}_p$ for p smooth in X. Hence

$$i^*: (\pi^* F/\mathrm{Torsion}) = \mathcal{O}(d - k) \to \mathcal{O}(1) \oplus \mathcal{O} ,$$

the induced morphism on $\boldsymbol{P}^1$, vanishes at the $2k$ points on which F is not locally free. Therefore $k = 0$ and F is locally free with $\pi^* F = \mathcal{O}(1)$ and $A_i(\mathcal{O}(1))_{p_i} = \mathcal{O}(1)_{q_i}$ for all i, a contradiction. q.e.d.

THEOREM 4.2. *For d an odd integer, with $2 \leqq d \leqq 2g - 2$, there exists $U'_{2,d}$, a Zariski open set in $U_{2,d}$, such that $W_d^1 \cap U'_{2,d}$ is non-empty and the dimension of this subvariety in $U'_{2,d}$ is $\rho(1, d, 2) = (2d - 3)$.*

PROOF. By Remark 2.6 we know that the set

$$U_{2,d}(G) = [\bar{V} \in M_{2,d} \mid \bar{V} \text{ is stable, locally free and } \pi^* \bar{V} \cong \mathcal{O}([d/2]+1) \oplus \mathcal{O}([d/2]) = V(d)]$$

is Zariski open in $M_{2,d}$. By hypothesis $[d/2] \geqq 1$; let $\{s_1, s_1'\} \subseteqq H^0(\mathcal{O}[d/2])$, $s_2 \in H^0(\mathcal{O}[d/2]+1)$ satisfying the following:

s_1 vanishes at $p_1 \in \boldsymbol{P}^1$, and does not vanish at q_1.

s_1' does not vanish at any $\{p_i, q_i\}_{i=1}^g \subseteqq \boldsymbol{P}^1$.

s_2 does not vanish at any $\{p_i, q_i\}_{i=1}^g \subseteqq \boldsymbol{P}^1$.

Let $V(d)(A_1, \cdots, A_g)$ be the bundle on X_0 given by the descent datum $\{A_i\}_{i=1}^g$ satisfying $A_i s(p_i) = s(q_i)$, $A_i s'(p_i) = s'(q_i)$ for all i, where $s = (s_1, s_2) \in H^0(\mathcal{O}[d/2]) \oplus H^0(\mathcal{O}([d/2]+1))$ which is the same as $H^0(V(d))$, and $s' = (s_1', 0) \in H^0(V(d))$. Let $\{B_i\}_{i=1}^g$ be the descent datum on $\mathcal{O}(1) \oplus \mathcal{O} = V(1)$ such that $V(1)(B_1, \cdots, B_g) = L^{-1} \otimes V(d)(A_1, \cdots, A_g)$, where $L \subseteqq V(d)(A_1, \cdots, A_g)$, is the line bundle given by s'; note that $\pi^* L = \mathcal{O}([d/2])$ and by construction $B_1(\mathcal{O}(1))_{p_1} \neq (\mathcal{O}(1))_{q_1}$. Hence, by Lemma 4.1, the stability of $V(d)(A_1, \cdots, A_g)$ follows. Let

$$Z = \{(s_1, s_2) \in H^0(V(\mathrm{d})) \times H^0(V(d)) \mid s_1 \text{ and } s_2 \text{ are linearly independent}\}.$$

Set

$$\bar{W}_d^1 = \{((A_1, \cdots, A_g), s_1, s_2) \in \prod_{i=1}^{d} \operatorname{Hom}(V_{p_i}(d), V_{q_i}(d)) \times Z \mid A_i s_j(p_i) = s_j(q_i) \text{ for } j = 1, 2 \text{ and all } i\}$$

and

$$\bar{U}_{2,d}(G) = [(A_1, \cdots, A_g) \in \prod_{i=1}^{g} \operatorname{Iso}(V_{p_i}(d), V_{q_i}(d)) \mid \{A_i\}_{i=1}^g \text{ is a descent datum for a stable bundle on } X].$$

Then

$$[\bar{U}_{2,d}(G)/P(\operatorname{Aut} V(d))] = U_{2,d}(G)$$

and

$$[\bar{U}_{2,d}(G) \cap p(\bar{W}_d^1)]/P(\operatorname{Aut} V(d)) = U_{2,d}(G) \cap W_d^1,$$

where $p\colon \prod_{i=1}^g \operatorname{Hom}(V_{p_i}(d), V_{q_i}(d)) \times Z \to \prod_{i=1}^g \operatorname{Hom}(V_{p_i}(d), V_{q_i}(d))$ is the natural projection.

We denote by Z' the set

$$\left[(s_1, s_2) \in Z \,\middle|\, \begin{array}{l}\text{the dimension of linear span of } \langle s_1(p_i), s_2(p_i)\rangle \\ \text{is at most one for at least one } i.\end{array}\right]$$

Then Z' is a closed subset of Z.
Defining

$$\bar{W}_d^1(S) = \Big\{((A_1, \cdots, A_g), s_1, s_2) \in \prod_{i=1}^{g} \mathrm{Hom}(V_{p_i}(d), V_{q_i}(d)) \times Z' \mid A_i s_j(p_i) = s_j(q_i), j = 1, 2, 1 \leqq i \leqq g\Big\} .$$

We observe that $[\mathrm{Cl}\, p(\bar{W}_d^1 \setminus \bar{W}_d^1(S)) \cap \prod_{i=1}^{g} \mathrm{Iso}(V_{p_i}(d), V_{q_i}(d))]$ is an irreducible variety, where $\mathrm{Cl}\, p(\bar{W}_d^1 \setminus \bar{W}_d^1(S))$ is the Zariski closure of $p(\bar{W}_d^1 \setminus \bar{W}_d^1(S))$. By construction, $p(\bar{W}_d^1 \setminus \bar{W}_d^1(S)) \cap \bar{U}_{2,d}(G)$ is non-empty. By the theory of special divisors (cf. [G-H]) there exist descent data denoted by $(\lambda_i)_{i=1}^{g}$, $(\mu_i)_{i=1}^{g}$ on $\mathcal{O}([d/2])$ and $\mathcal{O}([d/2]+1)$, respectively, such that the corresponding line bundles have exactly one section and they do not vanish on any of the nodes of X. This shows the existence of a destent datum $(A_1, \cdots, A_g) \in \bar{U}_{2,d}(G) \cap p(\bar{W}_d^1 \setminus \bar{W}_d^1(S))$ such that the bundle $V(d)(A_1, \cdots, A_g)$ has exactly two linearly independent sections over X. Since $p(\bar{W}_d^1(S))$ is a closed subvariety of $p(\bar{W}_d^1)$, the theorem follows from Remark 2.2 to Theorem 2.1 of Chapter III. q.e.d.

CHAPTER III. Existence of Special Bundles. In this chapter, we adopt the well-known technique of degeneration of smooth curves to singular curves and reduce the problem of existence of W_d^r on a smooth curve to one on a singular curve. Then using results on singular curves proved in the previous chapter, we conclude the existence of W_d^r's in some cases and compute their dimensions.

1. Specialisation of Quot Scheme. We suppose that $q: X \to S$ is a flat projective morphism over $\boldsymbol{C}$ of a smooth surface X onto a smooth affine curve S with geometrically integral fibres. In addition, we assume that the generic fibre is a smooth curve of genus g and the special fibre X_{s_0} over $s_0 \in S$ is singular with g ordinary double points as its only singularities. We fix H, a relatively ample sheaf on X.

According to Grothendieck [FGA], for $\mathcal{F}$ a coherent sheaf on X, the quotients of $\mathcal{F}$ flat over S and having a fixed Hilbert polynomial P for all $s \in S$ form a projective algebraic scheme over S denoted by

$$\sigma: Q = Q(\mathcal{F}/P) \to S .$$

Moreover, there exists U, a coherent sheaf over $Q \times_S X$ and a surjective homomorphism $p_X^* \mathcal{F} \to U$ such that U is flat over Q and has the obvious universal property for flat families of quotients of $\mathcal{F}$ with Hilbert polynomial P.

We take for $\mathcal{F}$, $\mathcal{O}_X^N$, the free sheaf of rank N over X and $P(m) =$

$d + n(mh - g + 1) = N + nmh$, the Hilbert polynomial over S, where $h = \deg(H|X_s)$, $s \in S$.

We shall denote by $R''_{n,d}$, $R'_{n,d}$ and $R_{n,d}$ the following sets:

$$R''_{n,d} = \{q \in Q(\mathcal{O}_X^N/P) \mid U_q \text{ is locally free on } X_{\sigma(q)}\}$$
$$R'_{n,d} = \{q \in R''_{n,d} \mid H^1(X_{\sigma(q)}, U_q) = 0\}$$

and

$$R_{n,d} = \left\{ q \in R'_{n,d} \,\middle|\, \begin{array}{l} \text{the canonical map } H^0(\mathcal{O}^N_{X_{\sigma(q)}}) \to H^0(U_q) \\ \text{is an isomorphism.} \end{array} \right\},$$

where U_q is the sheaf U restricted to the fibre over q of the morphism $Q \times_S X \to Q$.

PROPOSITION 1.2. (i) *$R''_{n,d}$, $R'_{n,d}$ and $R_{n,d}$ are open subschemes of $Q(\mathcal{O}_X^N/P)$ stable under $G = \operatorname{Aut}(\mathcal{O}_X^N/S)$, the group of automorphisms of $\mathcal{O}_X^N$ over S.*

(ii) *The schemes $R'_{n,d}$ and $R_{n,d}$ are smooth.*

(iii) *If $R_{n,d}$ is non-empty then*

$$\dim R_{n,d} = n^2(g-1) + 1 + \dim S + \dim G_s - 1$$

where $G_s = \operatorname{Aut}(\mathcal{O}^N_{X_s})$, the group of automorphisms of $\mathcal{O}^N_{X_s}$ over the curve X_s.

Before we proceed with the proof we state

LEMMA 1.1. *Let A and B be Noetherian local rings, with m the maximal ideal of A, $\phi\colon A \to B$ a local homomorphism, and E a finite B-module which is flat over A. Then E is free over B if and only if E/mE is free over B/mB.*

PROOF. (cf. [N], Chapter 5, Lemma 5.4). It follows from Nakayama's lemma.

PROOF OF PROPOSITION 1.2.

$$\begin{array}{ccc} X \times_S Q & \xrightarrow{p_X} & X \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ Q & \longrightarrow & S \end{array}$$

The above diagram is commutative. We know that U is flat over Q. By the above lemma, $Q \setminus R''_{n,d}$ is the projection of the set of points of $X \times_S Q$ at which U is not locally free. Since p is proper, this set is closed; so $R''_{n,d}$ is open. Applying semicontinuity theorem openness of $R'_{n,d}$ follows.

We shall denote (by abuse of notation) by U, the vector bundle on $R''_{n,d} \times_S X$ given by U. For $q \in R'_{n,d}$, let H_q be the vector bundle defined by the exact sequence

$$0 \to H_q \to \mathcal{O}^N_{X_q} \to U_q \to 0 .$$

on $X_q = \{q\} \times_S X$.
Then we have the exact sequence

$$0 \to U^*_q \to (\mathcal{O}^N_{X_q})^* \to H^*_q \to 0 ,$$

where for example U^*_q is the dual of U_q on X_q. Tensoring this by U_q and writing the cohomology exact sequence, we get

$$\begin{aligned} 0 &\to H^0(U^*_q \otimes U_q) \to H^0((\mathcal{O}^N_{X_q})^* \otimes U_q) \to H^0(H^*_q \otimes U_q) \\ &\to H^1(U^*_q \otimes U_q) \to H^1((\mathcal{O}^N_{X_q})^* \otimes U_q) \to H^1(H^*_q \otimes U_q) \\ &\to H^2(U^*_q \otimes U_q) . \end{aligned}$$

We observe that $H^2(U^*_q \otimes U_q) = 0$, since X_q is a curve. Further $H^1((\mathcal{O}^N_{X_q})^* \otimes U_q) = 0$, for $(\mathcal{O}^N_{X_q})^* \otimes U_q$ is a direct sum of U_q. Hence $H^1(H^*_q \otimes U_q) = 0$. By results of Grothendieck (cf. [EGA, Chapter IV]) σ is smooth at q. By the above observation, it follows that σ is smooth at all $q \in R'_{n,d}$ over S. Since S is smooth, the smoothness of $R'_{n,d}$ follows.

In view of the semicontinuity theorem and the above fact that $R'_{n,d}$ is smooth, it follows that the sheaves $p_*(p^*_X(\mathcal{O}^N_X))$ and $p_* U$ are locally free on $R'_{n,d}$. For $q \in R_{n,d}$, the natural homomorphism of the locally free sheaves $p_*(p^*_X(\mathcal{O}^N_X))|R'_{n,d}$ and $p_* U|R'_{n,d}$ on $R'_{n,d}$ induces an isomorphism of the fibres of these sheaves at q by the definition of $R_{n,d}$. Hence there exists an open neighbourhood R of q in $R'_{n,d}$ such that the natural homomorphism induces an isomorphism of the fibres of these sheaves at every point q in R. This proves that $R_{n,d}$ is open in $R'_{n,d}$ and hence in $Q(\mathcal{O}^N_X/P)$.

We have shown that $\sigma|R'_{n,d}$ is smooth. Hence $\sigma|R_{n,d}$ is smooth. Therefore $R_{n,d}$ is smooth and $\dim R_{n,d} = \dim S + \dim(\text{fibre of } \sigma|R_{n,d})$ if it is non-empty.

For $s_0 \in S$, let X_{s_0} be the fibre over s_0 of $q: X \to S$. Let Q_{s_0} be the fibre of $\sigma: Q = Q(\mathcal{O}^N_X/P) \to S$. It is clear that $Q_{s_0} = Q(\mathcal{O}^N_{X_{s_0}}/P)$. Hence the dimension of the fibre of σ is $\dim Q_{s_0} = n^2(g-1) + 1 + \dim G_{s_0} - 1$ (cf. [Se 2, Chapter II]). q.e.d.

REMARK 1.3. We note that two elements q_1 and q_2 in $R_{n,d}(s_0) =$ fibre of $R_{n,d}$ over $s_0 \subseteq Q_{s_0}$ are isomorphic if and only if they are in the same orbit of $G_{s_0} = \operatorname{Aut}(\mathcal{O}^N_{X_{s_0}})$ (cf. [N, Theorem 5.3 and 5.3′]).

We now suppose that $q: X \to S$ has a section which avoids the singular points of the special fibre X_{s_0}. Let L be the line bundle on X given by the divisor. Then $L|X_s$ is a line bundle of degree 1. In view of 1.6 of Chapter II and the fact that $\deg(V\otimes L^k)$ equals $\deg V + \operatorname{rank}(V)\cdot k\cdot \deg(L)$, we observe that for any rank n and degree d, we can construct $R_{n,d}$, with the following property: there exists $V_{n,d}$ a vector bundle on $R_{n,d}\times_S X$ and integers N and k such that

$$\text{for}\quad N = d' + n(1-g)\ ,\quad d' = d + nk > n(2g-2+2n)$$
$$\text{and}\quad P(m) = N + nhm\ ,\quad R_{n,d} = R_{n,d'}$$

where $R_{n,d'}$ is the open subscheme of $Q(\mathcal{O}_X^N/P)$ as constructed in Proposition 1.2; $R_{n,d'}$ is non-empty, hence has dimension $[n^2(g-1)+1+\dim S+\dim G_s - 1]$ where $G_s = \operatorname{Aut}(\mathcal{O}_{X_s}^N)$. Moreover $U = V_{n,d}\otimes p_X^* L^k$, where $U \to R_{n,d'}\times_S X$ is the bundle described in Proposition 1.2.

Throughout this chapter, $R_{n,d}$ with a universal bundle $V_{n,d}$ over $R_{n,d}\times_S X$ will refer to the space constructed as above and G and G_s to the corresponding $\operatorname{Aut}(\mathcal{O}_X^N/S)$ and $\operatorname{Aut}(\mathcal{O}_{X_s}^N)$.

The following was proved by Maruyama [M].

PROPOSITION 1.3. *Define $R_{n,d}^s$ as*

$$\{q\in R_{n,d}\,|\,V_{n,d}\ \textit{restricted to}\ X_{\sigma(q)}\ \textit{is stable}\}\ .$$

Then $R_{n,d}^s$ is open in $R_{n,d}$.

2. **Existence of Special Bundles.** We shall assume that $q\colon X\to S$ admits $2g$ sections $s_1, \cdots, s_{2g}$ (fixed throughout this section) away from the singular locus of the special fibre X_{s_0}.

THEOREM 2.1. *Let $\bar{R}_{n,d}\subseteqq R_{n,d}^s$ be a Zariski open set satisfying*
(i) *$\bar{R}_{n,d}(t) = R_{n,d}^s(t)$ if $t\neq s_0$ in S,*
(ii) *$\bar{R}_{n,d}(s_0)\cap \bar{W}_d^r(S)$ is non-empty and*

$$\dim[\bar{R}_{n,d}(s_0)\cap \bar{W}_d^r(S)] = \rho(r,d,n) + \dim G_{s_0} - 1\ ,$$

where $\bar{W}_d^r(S) = \{q\in R_{n,d}^s\,|\,h^0(X_{\sigma(q)}, V_{n,d}(q))\geqq (r+1)\}$. Then $\dim[\bar{W}_d^r(S)]\geqq (\rho(r,d,n)+\dim S+\dim G_s - 1)$. In fact every component of $\bar{W}_d^r(S)\cap \bar{R}_{n,d}$ is of dimension at least $\rho(r,d,n)+\dim G_s - 1 + \dim S$.

PROOF. We have $\pi\colon V_{n,d}\to R_{n,d}^s\times_S X$, a vector bundle. Let $P_1, \cdots, P_{2g}$ be the divisors on X given by the sections $s_1,\cdots, s_{2g}$ of $q\colon X\to S$. In this setting we carry over the argument of Theorem 4.1 of Chapter I verbatim. q.e.d.

REMARK 2.2. We note that the above proof can be adapted to show that if $\bar{W}_d^r(X_{s_0})\cap R_{n,d}^s(s_0)$ is non-empty, then every component of it has dimension at least $\rho(r,d,n)+\dim G_{s_0} - 1$.

REMARK 2.3. We observe that X_{s_0} can be replaced by any integral curve in the above theorem (cf. [N] or [Se 1] for generalised moduli).

Now we state:

THEOREM 2.4. *For* $0 < d \leqq n(g-1)$, $W_d^0(X)$ *is non-empty and* $\dim W_d^0(X) = \rho(0, d, n)$ *on any smooth curve* X *of genus* g.

PROOF. Combining Theorem 3.1 of Chapter II with the fact that there is a smooth curve X of genus g and a one-parameter family of curves $\{X_s\}$ with X_{s_0}, a Castelnuovo curve and for some $s_1 \neq s_0$ the curve X_{s_1} is X, satisfying the hypothesis of Theorem 2.1 and the fact that the moduli space of curves of genus g is irreducible along with Remark 3.4 of Chapter I that $\dim W_d^0 \leqq \rho(0, d, n)$ and Remark 4.2 of Chapter I, we observe that for a generic smooth curve X of genus g, the above theorem is true.

To complete the proof for all curves, we make the following additional observation:

For a smooth curve X, let

$$S_d^r(X) = \{F \in (M_{n,d} \setminus U_{n,d}) \,|\, h^0(\operatorname{Gr} F) \geqq r+1\}$$

($\operatorname{Gr} F$ denotes the associated graded sheaf for the semi-stable sheaf F).

When $r = 0$,

$$(*) \qquad \dim S_d^0(X) < \rho(0, d, n)\,.$$

This can be seen quite easily by using Remark 3.4 of Chapter I that

$$\dim W_d^0 \leqq \rho(0, d, n)\,,$$

for lower ranks.

Given any smooth curve X, let X_s be a one parameter family of smooth curves specialising to X such that for a generic X_s, the above theorem is valid. Then combining $(*)$ with the fact that $[W_d^r(X_s) \cup S_d^r(X)]$ specialises to $[W_d^r(X) \cup S_d^r(X)]$. (This follows from D'Souza's Thesis verbatim. See [D's], p. 69), we conclude the theorem for all curves.

THEOREM 2.5. *For* $2 \leqq d \leqq 2(g-1)$, *d an odd integer,* $W_d^1 \subseteqq M_{2,d}$ *is non-empty and* $\dim W_d^1 \geqq \rho(1, d, 2) = 2d-3$ *on any smooth curve of genus* g.

PROOF. This follows from Theorem 4.2 of Chapter II and Theorem 2.1.

CHAPTER IV. Properties of Special Bundles With Sections. We give an alternative proof of the existence of W_d^0, when the degree d is divisible by the rank. While many arguments of this article may be adapted to arbitrary fields, the argument given in this chapter for the existence of W_d^0 is valid only for $\boldsymbol{C}$.

Throughout this chapter, we shall deal only with smooth curves X.

1. Existence of W_d^r-a special case.

THEOREM 1.1. *For* $\mu = (d/n) \in N = \{1, 2, \cdots, (g-1)\}$, *there exists* $V \in M_{n,d}$ *such that* V *is stable and* $h^0(V) \neq 0$ (see 1.12.2 Chapter I for notation).

Before we proceed to prove the theorem we observe

LEMMA 1.2. *Let* $\bar{V} \in \operatorname{Ext}^1(V, L)$ *where* $\bar{V} \not\cong V \oplus L$, V *is stable of rank* $n-1$, L *a line bundle*, $h^0(V) = 0$, $h^0(L) \geqq 1$, $\mu(V) = \mu(L) = \mu \in N$. *Then* $\bar{V}$ *is simple.* (Theorem 3.2, Chapter I grants the existence of such a V).

PROOF. The bundle $\bar{V}$ is semi-stable and by choice nonsplit in $\operatorname{Ext}^1(V, L)$. Since any $\sigma \in \operatorname{End} \bar{V}$ is of constant rank and the associated graded for $\bar{V}$ is unique, it is indecomposable. Hence any $\sigma \in \operatorname{End} \bar{V}$ is of the form $\lambda \operatorname{Id}_{\bar{V}} + N_1$, where $\lambda \in \boldsymbol{C}$, $\operatorname{Id}_{\bar{V}}$ is the identity endomorphism and N_1 a nilpotent endomorphism. Since $h^0(V) = 0$, $N_1(L) \subseteqq L$, hence $N_1(L) = 0$ and the induced morphism $\bar{N}_1 \equiv 0$, $\bar{N}_1 \colon \bar{V}/L \to \bar{V}/L \cong V$ by virtue of the stability of V. Hence $\operatorname{Im} N_1 \subseteqq L$, i.e., $N_1 \colon \bar{V} \to L$. However, $h^0(V^* \otimes L) = 0$. Therefore $N_1 \equiv 0$. q.e.d.

PROOF OF THE THEOREM. Let $\bar{V}$ be as in the lemma above. Then $\dim H^0(\bar{V}) = h^0(\bar{V}) \geqq 1$. It is well-known (cf. [Se 1]) that there exists a complex analytic variety Y of dimension $n^2(g-1)+1$ and a vector bundle E on $Y \times X$ and a point $y_0 \in Y$ such that $E_{y_0} \cong \bar{V}$ and $E_y \not\cong E_x$ for $x \neq y$ in Y. Let $E(h^0(L)) = \{y \in Y \mid h^0(E_y) \geqq h^0(L)\}$. By Remark 4.3 of Chapter I,

$$\dim E(h^0(L)) \geqq n^2(g-1) + 1 - h^0(L)(h^0(L) - d - n + ng)\ .$$

Since $\bar{V}$ is semi-stable we may suppose that E_y is semi-stable for all $y \in Y$. Let $E(h^0(L), S) = \{y \in Y \mid h^0(E_y) \geqq h^0(L),\ E_y \text{ is non-stable}\}$. Then $E(h^0(L), S) \subseteqq E(h^0(L))$.

By the universal property of the moduli space, there exists $\Phi \colon Y \to M_{n,d}$, a morphism such that $\Phi(y) \in M_{n,d}$ is the associated graded of E_y, if E_y is nonstable. Let

$$M_{n,d}^0 = \{V \in M_{n,d} \mid V \text{ is stable or the associated graded of } V \text{ is of type } L_1 \oplus V_1,\ L_1 \text{ a line bundle, } V_1 \text{ a stable bundle such that } h^0(V_1) = 0\}\ .$$

It is easy to see that $M_{n,d}^0 \subseteqq M_{n,d}$ is Zariski open and $\Phi(\bar{V}) \in M_{n,d}^0$. Hence we may suppose that $\Phi(y) \in M^0_{n,d}$ for all $y \in Y$. Let Φ: $E(h^0(L)) \to M^0_{n,d}$ be the restriction of Φ. Then

$$\Phi(E(h^0(L), S)) \subseteqq S_{n,d}(h^0(L)) ,$$

where

$$S_{n,d}(h^0(L)) = \{V \in (M^0_{n,d} \setminus U_{n,d}) \mid V = L_1 \oplus V_1, h^0(L_1) \geqq h^0(L), h^0(V_1) = 0\} .$$

Set $\bar{X} = X_{d_1} \times U_{n-1,d_2}$, where X_{d_1} is the d_1-fold symmetric product of the curve X parametrising line bundles L, $h^0(L) \geqq 1$, $U_{n-1,d_2} \subseteqq M_{n-1,d_2}$, $d_1 = (d_2/n-1) = (d/n) = \mu \in N$.

Let $\bar{X}_{12} = \bar{X}_1 \cup \bar{X}_2$ where $\bar{X}_1$ and $\bar{X}_2$ parametrise

$\{(L, V) \in \bar{X}, P(\text{Ext}^1(V, L))\}$ and $\{(L, V) \in \bar{X}, P(\text{Ext}^1(L, V))\}$, respectively.

Then

$$\begin{aligned}\dim \bar{X}_{12} &\leqq [(n-1)^2(g-1) + d_1 + (n-1)(g-1)] \\ &< [(n-1)^2(g-1) + d + (n-1)(g-1)] \leqq \dim E(1) .\end{aligned}$$

Since $\dim \Phi^{-1}(S_{n,d}(1)) \leqq \dim \bar{X}_{12}$, $(E(1) \setminus E(1, S))$ is non-empty. That is, there exists $y \in Y$ such that E_y is stable and $h^0(E_y) \geqq 1$. q.e.d.

For $d = n(g-1)$, $d_1 = (g-1)$, $d_2 = (n-1)(g-1)$ let $W_{d_1} = \{L \in J_{g-1} \mid h^0(L) \geqq r+1\}$ where J_{g-1} is the set of isomorphism classes of line bundles of degree $g-1$ on X. By the theory of special divisors (cf. [G–H]) W_{d_1} is non-empty on any curve X if $g \geqq (r+1)^2$.

Let $\bar{X} = W_{d_1} \times U_{n-1,d_2}$, $\bar{X}_{12} = \bar{X}_1 \cup \bar{X}_2$, where $\bar{X}_1$ and $\bar{X}_2$ parametrise $\{(L, V) \in \bar{X}, \text{Ext}^1(V, L)\}$ and $\{(L, V) \in \bar{X}, \text{Ext}^1(L, V)\}$, respectively. For $r \geqq 1$, $\dim W_{d_1} \leqq (g-2)$ on any curve X. Therefore

$$\begin{aligned}\dim \bar{X}_{12} &\leqq [(n-1)^2(g-1) + (n-1)(g-1) + (g-2)] \\ &< [n^2(g-1) + 1 - (r+1)^2] \leqq \dim E(r+1) .\end{aligned}$$

Since $\dim \Phi^{-1}(S_{n,d}(r+1)) \leqq \dim \bar{X}_{12}$, $(E(r+1) \setminus E(r+1, S))$ is non-empty. That is, there exists $y \in Y$ such that E_y is stable and $h^0(E_y) \geqq r+1$ and combining with Theorem 4.1 of Chapter I, we have:

COROLLARY 1.3. *For $d = n(g-1)$, if $g \geqq (r+1)^2$, $r \geqq 1$,then there exists $V \in M_{n,d}$ such that V is stable and $h^0(V) \geqq (r+1)$; moreover,* $\dim(W^r_d) \geqq [n^2(g-1) + 1 - (r+1)^2]$.

2. Irreducibility of W^r_d.

THEOREM 2.1. *For $0 < d \leqq n(g-1)$, the subvariety W^0_d of the compactified moduli space $M_{n,d}$ over X is irreducible. Moreover, $(W^0_d \setminus W^0_d(1))$ is of codimension at least $n-1$ in W^0_d, where*

$$W_d^0(1) = \{V \in W_d^0 \mid \mathcal{O} \subseteqq V,\ (V/\mathcal{O}) \textit{ is stable}\} .$$

PROOF. Indeed, we merely collect the various remarks made earlier to see the proof.

(i) Theorem 2.4 of Chapter III shows the non-emptiness of W_d^0.

(ii) Remark 4.2 to Theorem 4.1 of Chapter I shows that all components of W_d^0 are of dimension at least $\rho(0, d, n)$.

(iii) Remark 3.3 to Theorem 3.2 of Chapter I.

We add one more observation to these, namely

$$\phi\colon R^s_{n-1,d} \to U_{n-1,d}$$

is a geometric quotient by a suitable $PGL(N)$ (See Chapter III for such details; here the family consists of just one smooth curve X). In addition

$$\psi\colon V_{n-1,d} \to R^s_{n-1,d} \times_K X$$

is the universal quotient sheaf.

Noting that $\dim H^1(V^*) = (n-1)(g-1) + d$ is of constant dimension for $V \in R^s_{n-1,d}$, we may construct a vector bundle $\bar{V}$ on $R^s_{n-1,d}$

$$\bar{\phi}\colon \bar{V} \to R^s_{n-1,d} ,$$

where the fibre of $\bar{V}$ over $V \in R^s_{n-1,d}$ is $H^1(V^*)$ and a vector bundle $\bar{\bar{V}}$ on $X \times_K T$, T being the total space of $\bar{V}$ such that

$$\bar{\bar{V}} \mid X \times \{t\} \cong \tilde{t} ,$$

where $\tilde{t}$ is the bundle on X given by $t \in T$ as an extension of $\bar{\phi}(t)$ by a trivial line bundle (see [Se 1], p. 199).

Denote by T^s the set $\{t \in T \mid (\bar{\bar{V}} \mid X \times \{t\}) \text{ is stable}\}$. Note that the non-emptiness of T^s follows from (iii) and (i). By the openness of the stability property, T^s is Zariski open in T. Since $R^s_{n-1,d}$ is a smooth variety, T^s is irreducible. By construction, T^s dominates $W_d^0(1)$. Now the irreducibility of W_d^0 follows. q.e.d.

By the theory of special divisors, (cf. [G-H]), the number of linearly independent $g^1_{d_1}$'s on a general smooth curve of genus g is given by

$$\rho(1, d_1, 1) = g - 2(g - d_1 + 1) .$$

Therefore

$$\dim W_d^1(S) \leqq g + d - 4 ,$$

where

$$W_d^1(S) = \{V \in W_d^1 \subseteqq M_{2,d} \mid V \supseteqq L,\ \text{a line bundle such that } h^0(L) \geqq 2\} .$$

Since $\dim \mathrm{Ext}^1(T, \mathcal{O}^2) = 2d$, where T is a torsion sheaf of length d, combining with Theorem 2.5 of Chapter III, we observe

PROPOSITION 2.2. *For $g \leqq d \leqq (2g-2)$, d an odd integer, $W_d^1 \subseteq M_{2,d}$ is irreducible on a general smooth curve of genus $g \geqq 3$ and* $\dim W_d^1 =$ *$(2d-3)$; moreover, a generic $V \in W_d^1$ contains $\mathcal{O}^2$, the trivial sheaf of rank* 2.

3. Generic element of W_d^0. We retain the notation of the previous section.

$$\bar{\bar{\phi}}: \bar{\bar{V}} \to X \times_K T$$

induces a morphism,

$$\bar{\Psi}: T^s \to U_{n,d} .$$

If we denote by $P(T)$, the total space of the projectivised bundle of $\bar{\phi}$: $\bar{V} \to R^s_{n-1,d}$, we see that $\bar{\Psi}$ factors through $P(T^s)$, the image of T^s in $P(T)$.

$$\bar{\Psi}: P(T^s) \to U_{n,d} .$$

Evidently

$$\begin{aligned}\dim P(T^s) &= (n-1)^2(g-1) + d + (n-1)(g-1) + \dim \phi \\ &= \dim W_d^0 + \dim \phi ,\end{aligned}$$

where $\dim \phi$ is the dimension of the fibre of

$$\phi: R^s_{n-1,d} \to U_{n-1,d} .$$

If $d \leqq (n-1)(g-1)$, then by Theorem 3.2 of Chapter I, $h^0(V) = 0$ for a generic $V \in U_{n-1,d}$, hence in $R^s_{n-1,d}$ and the codimension of $\bar{\phi}^{-1}(\bar{U}^0_{n-1,d}) \cap P(T^s)$ is at least $1 + (n-1)(g-1) - d$ in $P(T^s)$ where $\bar{U}^0_{n-1,d} = \{V \in R^s_{n-1,d} \mid h^0(V) \geqq 1\}$. We note that for $V \in P(T^s) \setminus (P(T^s) \cap \phi^{-1}(\bar{U}^0_{n-1,d}))$, $\dim H^0(V) = 1$. If $d > (n-1)(g-1)$ then a generic $V \in U_{n-1,d}$ has at most $(g-1)$ sections; i.e., $h^0(V) \leqq g-1$ and hence for a generic $V \in R^s_{n-1,d}$, $h^0(V) \leqq (g-1)$.

For a generic $V \in R^s_{n-1,d}$ and $\bar{V} \in \operatorname{Ext}^1(V, \mathcal{O}) \cong H^1(V^*)$ we have a long exact sequence

$$0 \to H^0(\mathcal{O}) \to H^0(\bar{V}) \to H^0(V) \xrightarrow{\eta_{\bar{V}}} H^1(\mathcal{O}) \to \cdots$$

associated to

$$0 \to \mathcal{O} \to \bar{V} \to V \to 0 .$$

Let

$$\eta: H^0(V) \times H^1(V^*) \to H^1(\mathcal{O}) , \quad \eta(s, \bar{V}) = \eta_{\bar{V}}(s)$$

and

$$\eta_s\colon H^1(V^*) \to H^1(\mathcal{O}) , \quad \eta_s(\bar{V}) = \eta(s, \bar{V})$$

For $s \neq 0$ in $H^0(V)$, the dual morphism η_s^* is given by

$$\eta_s^*\colon H^0(K) \to H^0(K \otimes V) , \quad \eta_s^*(\omega) = s \otimes \omega$$

and hence it is injective. Therefore η_s is surjective for all non-zero s in $H^0(V)$. We define

$$\bar{S} = \bigcup_{\substack{s \in H^0(V) \\ s \neq 0}} \operatorname{Ker} \eta_s \subseteqq H^1(V^*) .$$

Since $\operatorname{Ker} \eta_s$ is of codimension g in $H^1(V^*)$ and $\dim H^0(V) \leqq (g-1)$, by dimension count $H^1(V^*) \setminus \bar{S}$ is a non-empty, Zariski open set. Therefore there exists $\bar{V} \in \operatorname{Ext}^1(V, \mathcal{O}) \cong H^1(V^*)$ such that

$$\eta_{\bar{V}}\colon H^0(V) \to H^1(\mathcal{O})$$

is injective; hence $h^0(\bar{V}) = 1$. We have thus shown that for some t in T, $\bar{\bar{V}}_t = \bar{\bar{V}} | X \times_K \{t\}$ has exactly one section. Since for every t, $h^0(\bar{\bar{V}}_t) \geqq 1$, by the semicontinuity theorem

$$T_1 = \{t \in T \,|\, h^0(\bar{\bar{V}}_t) = 1\}$$

is a non-empty, Zariski open set, and so is $T_1 \cap T^s$. We denote by $P(T_1 \cap T^s)$ the image of $T_1 \cap T^s$ in $P(T)$. The morphism

$$\bar{\psi}\colon P(T_1 \cap T^s) \to U_{n,d}$$

satisfies the following property:

$$\bar{\psi}([t]) = \bar{\psi}([t']) \quad \text{for } [t] \text{ and } [t'] \text{ in } \quad P(T_1 \cap T^s)$$

implies

$$\bar{\phi}(t) \cong \bar{\phi}(t') , \quad \bar{\phi}\colon \bar{V} \to R^s_{n-1,d}$$

for any t and t' lying above $[t]$ and $[t']$ in T, the total space of $\bar{V}$. This follows easily from the fact that $\bar{\bar{V}}_t$ has exactly one section for every $t \in T_1 \cap T^s$. Therefore the dimension of the fibre of $\bar{\psi}$ is the same as the dimension of the fibre of $\bar{\phi}$.

The above discussion proves:

PROPOSITION 3.1. *For $0 < d \leqq n(g-1)$, the constructible set*

$$W_d^0(1, 1) = \{V \in W_d^0(1) \,|\, h^0(V) = 1\} ,$$

where $W_d^0(1)$ is as in Theorem 2.1 of the previous section, is Zariski dense in $W_d^0 \subseteqq M_{n,d}$. Therefore, a generic V in W_d^0 has exactly one section and it generates a trivial line sub-bundle of V. Moreover, for $d < (n-1)(g-1)$, the singular set $\{W_d^0\}_s$ of the reduced model of W_d^0 is of *codimension at least two.*

CHAPTER V. Open Problems. Indeed, in this article, we have merely scratched the tip of an ice berg. Most of the questions that we raised in the introduction remain unanswered. We believe the reductions that follow, may throw some light on these problems.

For notational convenience, we shall stick to the case of rank 2 bundles.

1. Problem of nonemptiness. By Theorem 2.1 of Chapter III, it is enough to produce a Zariski open set W in $U_{d,\mathrm{vec}}(X_0)$, the smooth open subset consisting of all locally free sheaves of the generalised moduli space of stable torsion free sheaves of rank 2 and degree d over X_0, a Castelnuovo curve (more generally, any integral curve) and a stable bundle $V \in W$ such that $h^0(V) \geqq (r+1)$ and $\dim W = \rho(r, d, 2)$ to claim the nonemptiness of $W_d^r \subset M_{2,d}$.

We offer the following candidate for W; We shall stick to the notation of Section 3, Chapter II. We require W to be contained in $U_d(\beta)$, where

$$U_d(\beta) = \{V \in U_{d,\mathrm{vec}} \mid \pi^* V \cong V(\beta) = \mathcal{O}(\beta_1) \oplus \mathcal{O}(\beta_2)\}\ , \quad \beta_1 + \beta_2 = d\ .$$

We recall (β):

$$\beta_j = \begin{cases} [d/2] + 1\ , & j \leqq k \\ [d/2] & ,\ j > k \end{cases} \quad d = [d/2] \cdot 2 + k$$

and

$$\bar{U}_d(\beta) = \Big\{(A_1, A_2, \cdots, A_g) \in \prod_{i=1}^{g} \mathrm{Iso}(V_{p_i}(\beta), V_{q_i}(\beta)) \mid (A_1, \cdots, A_g) \text{ is a descent datum for a stable bundle on } X_0\Big\}\ .$$

Given $m \geqq 0$, an integer, we may take $\{1, t, \cdots, t^m\}$ as a basis of the vector space $H^0(\mathcal{O}(m))$ on $\boldsymbol{P}^1$. Thus given any $s \in H^0(\mathcal{O}(m))$

$$s(t) = \sum_{i=0}^{m} \alpha_i t^i \quad \text{for some} \quad \{\alpha_i\}_{i=0}^{m} \subseteqq \boldsymbol{C}\ .$$

For $(s_1, s_2) \in H^0(\mathcal{O}(\beta_1)) \oplus H^0(\mathcal{O}(\beta_2))$ to be a section of $V(\beta)(A_1, \cdots, A_g)$, the bundle on X_0 associated to the descent datum $(A_1, A_i, \cdots, A_g) \in \bar{U}_d(\beta)$, it should satisfy

$$\begin{aligned} a_i s_1(p_i) + b_i s_2(p_i) &= s_1(q_i) \\ c_i s_1(p_i) + d_i s_2(p_i) &= s_2(q_i) \end{aligned}$$

for all i, $1 \leqq i \leqq g$; here

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

(Note that we have trivialised the bundle $V(\beta)$ on $(\boldsymbol{P}^1 \setminus \{\infty\})$ and assumed that $\{p_i, q_i\}_{i=1}^g \subseteqq \boldsymbol{P}^1 \setminus \{\infty\}$).

We may rewrite the equation as

$$\gamma_0(a_i - 1) + \gamma_1(a_i p_i - q_i) + \cdots + \gamma_m(a_i p_i^m - q_i^m) + b_i\delta_0 + b_i\delta_1 q_i + \cdots + b_i\delta_n q_i^n = 0$$
$$c_i\gamma_0 + c_i\gamma_1 p_i + \cdots + c_i\gamma_m p_i^m + \delta_0(d_i - 1) + \delta_1(d_i p_i - q_i) + \cdots + \delta_p(d_i p_i^n - q_i^n) = 0\,,$$

where

$$s_1(t) = \sum_{i=0}^{m} \gamma_i t^i\,, \quad s_2(t) = \sum_{i=0}^{n} \delta_i t^i\,, \quad m = \beta_1\,, \quad n = \beta_2\,.$$

One can interpret

$$[(a_i - 1), (a_i p_i - q_i), \cdots, (a_i p_i^m - q_i^m), b_i, \cdots, b_i q_i^n]$$

and

$$[c_i, c_i p_i, \cdots, c_i p_i^m, (d_i - 1), (d_i p_i - q_i), \cdots, (d_i p_i^n - q_i^n)]$$

as homogeneous coordinates of points in the span of

$$[1, p_i, p_i^2, \cdots, p_i^m, 0, \cdots, 0]\,, \quad [1, q_i, q_i^2, \cdots, q_i^m, 0, \cdots, 0]$$
$$[0, \cdots, 0, 1, p_i, \cdots, p_i^n]\,, \quad [0, \cdots, 0, 1, q_i, \cdots, q_i^n]$$

in $\boldsymbol{P}^{n+m+1} = \boldsymbol{P}^{d+1}$.

Let $V = (H^0(\mathcal{O}(m)) \oplus H^0(\mathcal{O}(n)))^*$, the dual of $H^0(\mathcal{O}(m)) \oplus H^0(\mathcal{O}(n))$, which can be identified with $[H^0(\mathcal{O}(m))]^* \oplus [H^0(\mathcal{O}(n))]^*$, and the curves c_1 and c_2 in $\boldsymbol{P}(V)$ to be

$$c_1: \boldsymbol{P}^1 \to \boldsymbol{P}(V_1) \subseteqq \boldsymbol{P}(V)\,, \quad V_1 = [H^0(\mathcal{O}(m))]^*\,, \quad t \mapsto [1, t, \cdots, t^m, 0, \cdots, 0]$$
$$c_2: \boldsymbol{P}^1 \to \boldsymbol{P}(V_2) \subseteqq \boldsymbol{P}(V)\,, \quad V_2 = [H^0(\mathcal{O}(n))]^*\,, \quad t \mapsto [0, \cdots, 0, 1, t, \cdots, t^n]\,.$$

We denote by E_i the linear span of $\{c_1(p_i), c_1(q_i), c_2(p_i), c_2(q_i)\}$ in $\boldsymbol{P}(V)$. It has affine dimension 4.

We may parametrise the bundles $V(\beta)(A_1, \cdots, A_g)$ with $r + 1$ linearly independent sections in the following way:

$$\overline{\overline{W}}{}_d^r = \{((A_i), s_1, \cdots, s_{r+1}) \in \bar{U}_d(\beta) \times H^0(V(\beta)) \times \cdots \times H^0(V(\beta)) \,|\, A_i s_j(p_i) = s_j(q_i) \forall i, 1 \leqq i \leqq g, \forall j, 1 \leqq j \leqq r + 1\}\,.$$

Let p and q denote

$$p: \bar{U}_d(\beta) \times \prod_1^{r+1} H^0(V(\beta)) \to \bar{U}_d(\beta)$$

$$q\colon \bar{U}_d(\beta) \times \prod_1^{r+1} H^0(V(\beta)) \to \prod_1^{r+1} H^0(V(\beta))$$

the natural projections, where $\prod_1^{r+1} H^0(V(\beta))$ is the $(r+1)$-fold Cartesian product of $H^0(V(\beta))$. There is a natural map

$$\bar{q}\colon \prod_1^{r+1} H^0(V(\beta)) \to \mathrm{Gr}(d+2-r-1, d+2)$$

which sends $(s_1, \cdots, s_{r+1})$ to the $(d+2-r-1)$-plane in the dual V of $H^0(V(\beta))$ determined by $\{s_i\}_{i=1}^{r+1}$.

Let

$$\sigma^i = \{\Lambda \in \mathrm{Gr}(d+2-r-1, d+2) \mid \dim(\bar{\Lambda} \cap E_i) \geqq 1 \text{ in } \boldsymbol{P}(V),\ \bar{\Lambda} = \boldsymbol{P}(\Lambda) \subseteqq \boldsymbol{P}(V)\}\ .$$

Each σ^i is a Shubert variety of type $\sigma_{(r+1)-2,(r+1)-2}$ and of codimension $2[(r+1)-2]$ in $\mathrm{Gr}(d+2-r-1, d+2)$ (cf. [G-H-P]). We have

$$q_1 = \bar{q} \circ q\colon \bar{\bar{W}}_d^r \to \mathrm{Gr}(d+2-r-1, d+2)\ .$$

Denote by $\sigma \subseteqq \cap_{i=1}^g \sigma^i$, the open set

$$\sigma = \left\{x \in \bigcap_{i=1}^g \sigma^i \,\middle|\, \dim q_1^{-1}(x) \text{ is minimal for } q_1\colon q_1^{-1}\left(\bigcap_{i=1}^g \sigma^i\right) \to \bigcap_{i=1}^g \sigma^i\right\}\ .$$

Define $\bar{W} \subseteqq \bar{\bar{W}}_d^r$, and $W \subseteqq \bar{U}_d(\beta)$ as

$$\bar{W} = q_1^{-1}(\sigma)$$
$$W = \bar{U}_d(\beta) \setminus p(\bar{\bar{W}}_d^r \setminus q_1^{-1}(\sigma))\ .$$

QUESTIONS. (i) Does W have a bundle with $r+1$ sections?
(ii) Is $\cap_{i=1}^g \sigma^i$ a generically transversal intersection?

2. Problem of dimension. If $\cap_{i=1}^g \sigma^i$ is transversal, then one may hope to find a component of W_d^r of dimension $\rho(r, d, 2)$. We believe W_d^r is not equidimensional even on a general curve.

When the rank and degree are not coprime, there is an additional problem of determining whether, given a one-parameter family $\{X_t\}$ of smooth curves specialising to a Castelnuovo (more generally, any integral) curve, the $W_d^r(X_t)$ specialises to $W_d^r(X_0)$. What is true is that $[W_d^r(X_t) \cup S_d^r(X_t)]$ specialises to $[W_d^r(X_0) \cup S_d^r(X_0)]$, where $S_d^r(X) = \{F \in (M_{2,d} \setminus U_{2,d}) \mid h^0(\mathrm{Gr}\, F) \geqq r+1\}$ ($\mathrm{Gr}\, F$ denotes the associated graded sheaf for the semi-stable sheaf F).

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