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A PROPERTY OF TRANSITIVE POINTS UNDER FUCHSIAN GROUPS

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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1. Let D be the unit disk in the complex plane and let ∂D be its boundary. We think of D as endowed with the Poincaré metric $ds = (1-|z|^2)^{-1}|dz|$, $z \in D$. By Möb(D) we denote the group consisting of all the fractional linear transformations which leave D invariant. A Fuchsian group G is a discrete subgroup of Möb(D).

For two points α and β $(\neq\alpha)$ in ∂D, we denote by $L(\alpha, \beta)$ the directed geodesic line from α to β. For a point ζ in ∂D and point z in D, we denote by $R(z, \zeta)$ the directed geodesic ray from z to ζ. If there exists an infinite sequence $\{g_n\}$ of elements in G such that $|g_n(z)-\alpha| + |g_n(\zeta)-\beta| \to 0$ as $n \to \infty$, then we say that the sequence $\{g_n(R(z, \zeta))\}_{n=1}^{\infty}$ converges to $L(\alpha, \beta)$ and that $\{g_n\}_{n=1}^{\infty}$ is a convergent sequence with respect to $R(z, \zeta)$ and $L(\alpha, \beta)$. We also say that ζ in ∂D is a transitive point under G if, for arbitrary $L(\alpha, \beta)$ and z, there exists a convergent sequence with respect to $R(z, \zeta)$ and $L(\alpha, \beta)$. In fact, the definition of a transitive point is independent of the choice of z. So, in this paper, we consider only the case $z=0$, the origin of the complex plane. In the language of the geodesic flow on the Riemann surface D/G, a transitive point under G can be rephrased as follows. If a point ζ is transitive under G, then the geodesic flow corresponding to $R(z, \zeta)$ is dense on $T_1(D/G)$, the unit tangent bundle to D/G. Let T_G be the set of all the transitive points under G. Clearly, T_G is invariant under G. In [4], it was proved that the measure of T_G is equal to 2π, if G is a Fuchsian group of divergence type. By H_G and P_G we denote the sets of all the hyperbolic fixed points and of all the parabolic fixed points of G, respectively. In this paper, we prove the following theorem on the set of transitive points under Fuchsian groups.

THEOREM. *Let Γ and G be Fuchsian groups. Suppose that G is a subgroup of Γ of finite index. Then $T_\Gamma = T_G$, $H_\Gamma = H_G$* and *$P_\Gamma = P_G$.*

We remark that T_Γ, T_G, P_Γ and P_G may be empty sets. The existence of Fuchsian groups Γ and G satisfying the condition of our Theorem was discussed in [2] if Γ and G are finitely generated Fuchsian groups.

2. Let g be a hyperbolic element of Möb(D) and let α and β be its attractive and repulsive fixed points, respectively. By A_g we denote the directed geodesic line $L(\alpha, \beta)$, which is often called the axis of g. Note that $g(A_g) = A_g$.

LEMMA 1. *Let Γ be a Fuchsian group and let g be a hyperbolic element of Γ with $A_g = L(\alpha, \beta)$. Then there exists a subset Γ_g of Γ with the following properties*: (i) *for each element h of Γ_g, there exists a positive integer N such that, for all $n \geqq N$, hg^n is a hyperbolic element of Γ and* (ii) *for all h in $\Gamma \setminus \Gamma_g$ and all positive integers n, hg^n is not a hyperbolic element of Γ. Moreover, if the axis A_{hg^n} is given by $L(\alpha_n, \beta_n)$ for the hyperbolic element hg^n, then* $\lim_{n\to\infty} \alpha_n = h(\alpha)$ *and* $\lim_{n\to\infty} \beta_n = \beta$.

PROOF. To show the first half, by taking a suitable conjugation, we may assume that the Fuchsian group Γ acts on the complex upper half plane and that the representations of g and h by matrices in $SL(2, \boldsymbol{R})$ are

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad (\lambda > 1) \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

respectively. Let E_g be the subset of Γ each of whose elements satisfies $a = d = 0$. If such elements do not exist, we regard E_g as empty. Each element of E_g is an elliptic element of order two and has a fixed point on A_g. For $h \in E_g$, hg^n also is contained in E_g, so hg^n is not a hyperbolic element. We set $\Gamma_g = \Gamma \setminus E_g$. Let h be an element of Γ_g. Then we have $|\operatorname{trace} hg^n| = |a\lambda^n + d\lambda^{-n}|$. Since Γ is a discrete group, we obtain $a \neq 0$. Hence there exists a positive integer N such that $|\operatorname{trace} hg^n| > 2$ for all $n \geqq N$. This shows that hg^n is a hyperbolic element.

To show the second half, we assume that Γ acts on D. Since Γ is a discrete group, every $h \in \Gamma_g$ satisfies $h(\alpha) \neq \beta$. Let I be an arbitrary open interval on ∂D with $h(\alpha) \in I$ and $\beta \notin \operatorname{Cl} I$, where $\operatorname{Cl} I$ denotes the closure of I. Since α is the attractive fixed point of g, there exists a positive integer M such that $g^n(\operatorname{Cl} I) \subset h^{-1}(I)$ for all $n \geqq M$. Hence we have $hg^n(\operatorname{Cl} I) \subset I$. Thus, for all $n \geqq \max(N, M)$, hg^n has an attractive fixed point α_n in $\operatorname{Cl} I$ (see [1, p. 96]). Since I is an arbitrary open interval containing $h(\alpha)$, we obtain $\lim_{n\to\infty} \alpha_n = h(\alpha)$. By using the same argument for β_n which is the attractive fixed point of $(hg^n)^{-1}$, we obtain $\lim_{n\to\infty} \beta_n = \beta$. This completes the proof of Lemma 1.

We remark that the group $\langle E_g \rangle$ generated by E_g is an elementary group which leaves A_g invariant unless E_g is empty.

Let Γ and G be the Fuchsian groups in the Theorem. Suppose that

$[\Gamma: G] = S + 1$ and that the left coset decomposition of Γ with respect to G is $\Gamma = G \cup \gamma_1 G \cup \cdots \cup \gamma_S G$. We set $G = G_0$ and $\gamma_i G = G_i$ for $i = 1, 2, \cdots, S$, that is, $\cup_{i=0}^{S} G_i$ is the left coset decomposition of Γ with respect to G. It is easy to see that, for each $\gamma \in \Gamma$, there exists a positive integer n $(\leqq S + 1)$ such that $\gamma^n \in G_0$ and $\gamma^m \notin G_0$ for all $0 < m < n$.

LEMMA 2. *Let ζ be a transitive point under Γ and let g be a hyperbolic element of Γ. For all l $(0 \leqq l \leqq S)$, there exists a sequence $\{g_{li}\}_{i=1}^{\infty}$ in G_l such that the sequence $\{g_{li}(R(0, \zeta))\}_{i=1}^{\infty}$ converges to A_g.*

PROOF. We choose a natural number m which satisfies $\beta g^m \beta^{-1} \in G$ for all $\beta \in \Gamma$. We set $A_g = L(\alpha, \beta)$. By Lemma 1, for $h \in \Gamma_g$ and for a sufficient large n, the element hg^{nm} is hyperbolic. We set $A_{hg^{nm}} = L(\alpha_n, \beta_n)$. Since ζ is a transitive point under Γ, there exists a sequence $\{h_{ni}\}_{i=1}^{\infty}$ such that the sequence $\{h_{ni}(R(0, \zeta))\}_{i=1}^{\infty}$ converges to $A_{hg^{nm}}$. Let $\{\varepsilon_j\}$ be a sequence of positive numbers satisfying $\lim_{j\to\infty} \varepsilon_j = 0$. By Lemma 1, there exists an n such that $|h(\alpha) - \alpha_n| < \varepsilon_j/4$ and $|\beta - \beta_n| < \varepsilon_j/4$. Fixing such an n, we consider the sequence $\{h_{ni}\}_{i=1}^{\infty}$. Since the sequence $\{h_{ni}(R(0, \zeta))\}_{i=1}^{\infty}$ converges to $L(\alpha_n, \beta_n) = A_{hg^{nm}}$, we can choose an h_{ni} which satisfies $|h_{ni}(0) - \alpha_n| < \varepsilon_j/4$ and $|hg^{nm}h_{ni}(\zeta) - \beta_n| < \varepsilon_j/4$. Since β_n is the repulsive fixed point of hg^{nm}, it is clear that $|h_{ni}(\zeta) - \beta_n| < |hg^{nm}h_{ni}(\zeta) - \beta_n|$. Then we obtain $|h_{ni}(0) - h(\alpha)| + |h_{ni}(\zeta) - \beta| < \varepsilon_j$ and set $h_{ni} = h_j$. We can also obtain $|hg^{nm}h_j(0) - h(\alpha)| + |hg^{nm}h_j(\zeta) - \beta| < \varepsilon_j$ and set $hg^{nm}h_j = \bar{h}_j$. Hence two sequences $\{h_j(R(0, \zeta))\}_{j=1}^{\infty}$ and $\{\bar{h}_j(R(0, \zeta))\}_{j=1}^{\infty}$ both converge to $L(h(\alpha), \beta)$.

Suppose that the sequence $\{h_j\}_{j=1}^{\infty}$ is contained in a coset G_l. Then we have $(hh_j)^{-1}(hg^{nm}h_j) = (h_j^{-1}g^m h_j)^n \in G$ by the definition of m so that all the elements $hg^{nm}h_j$ are contained in the same left coset hG_l. Hence $\{\bar{h}_j\}_{j=1}^{\infty}$ is a sequence in hG_l. Let ε be an arbitrary positive number. Since $\{h_j(0)\}_{j=1}^{\infty}$ and $\{\bar{h}_j(0)\}_{j=1}^{\infty}$ converge to $h(\alpha)$ and since α is the attractive fixed point of g, for an arbitrary j_0, there exists a positive integer p such that $|g^p h_j(0) - \alpha| < \varepsilon$ and $|g^p \bar{h}_j(0) - \alpha| < \varepsilon$ for $j \geqq j_0$. Since $\{h_j(\zeta)\}_{j=1}^{\infty}$ and $\{\bar{h}_j(\zeta)\}_{j=1}^{\infty}$ converge to the repulsive fixed point β of g, for an arbitrary positive integer q, there exists $j(q)$ such that $|g^q h_j(\zeta) - \beta| < \varepsilon$ and $|g^q \bar{h}_j(\zeta) - \beta| < \varepsilon$ for $j \geqq j(q)$. Hence, for a sequence of positive numbers $\{\varepsilon_r\}$ satisfying $\lim_{r\to\infty} \varepsilon_r = 0$, there exist $j(r)$ and positive integers $t(r)$ and $t'(r)$ such that $|g^{mt(r)}h_{j(r)}(0) - \alpha| + |g^{mt(r)}h_{j(r)}(\zeta) - \beta| < \varepsilon_r$ and $|g^{mt'(r)}\bar{h}_{j(r)}(0) - \alpha| + |g^{mt'(r)}\bar{h}_{j(r)}(\zeta) - \beta| < \varepsilon_r$. Set $g^{mt(r)}h_{j(r)} = f_{hr}$ and $g^{mt'(r)}\bar{h}_{j(r)} = \bar{f}_{hr}$. Similarly it is easy to see that $\{f_{hr}\}_{r=1}^{\infty} \subset G_l$ and $\{\bar{f}_{hr}\}_{r=1}^{\infty} \subset hG_l$. For all $h \in \Gamma_g$, the sequence $\{f_{hr}(R(0, \zeta))\}_{r=1}^{\infty}$ converges to A_g. Note that, for arbitrary s and l $(0 \leqq s, l \leqq S)$, there exist infinitely many elements $k \in \Gamma$ with $G_s = kG_l$. Moreover, since $\langle E_g \rangle$ is an elementary group for $E_g \neq \varnothing$, we obtain

$[\Gamma\colon \langle E_g\rangle] = \infty$. So we can choose k out of Γ_g. Hence if, for all $h \in \Gamma_g$, the sequence $\{f_{hr}\}_{r=1}^{\infty}$ is contained in G_l for a unique l, then the sequence $\{\bar{f}_{kr}\}_{r=1}^{\infty}$ is contained in $kG_l = G_s$, where $k \in \Gamma_g$. Thus there exist at least two cosets each of which contains a convergent sequence with respect to $R(0, \zeta)$ and A_g.

Since each hg^n is a hyperbolic element for $h \in \Gamma_g$ and a sufficiently large n, the above argument shows that there exist at least two cosets each of which contains a convergent sequence with respect to $R(0, \zeta)$ and A_{hg^n}. Lemma 1 shows that A_{hg^n} converges to $L(h(\alpha), \beta)$ as $n \to \infty$. In the same manner as above, from the convergent sequences with respect to $R(0, \zeta)$ and A_{hg^n}, we can choose a convergent subsequence with respect to $R(0, \zeta)$ and $L(h(\alpha), \beta)$, out of which we can make a convergent sequence with respect to $R(0, \zeta)$ and A_g. Hence there exist two cosets G_l and G_s $(l \neq s)$, each of which contains a convergent sequence with respect to $R(0, \zeta)$ and A_g. Each of the two cosets hG_l and hG_s also contains a convergent sequence with respect to $R(0, \zeta)$ and A_g. Considering all $h \in \Gamma_g$ as above, we can conclude that there exist at least three cosets each of which contains a convergent sequence with respect to $R(0, \zeta)$ and A_g.

Applying the above argument to the hyperbolic elements hg^n, we conclude that if there exist p $(\leqq S)$ cosets each of which contains a convergent sequence with respect to $R(0, \zeta)$ and A_{hg^n}, then there exist at least $p+1$ cosets each of which contains a convergent sequence with respect to $R(0, \zeta)$ and A_g. Hence we see that every G_l contains a convergent sequence with respect to $R(0, \zeta)$ and A_g. This completes the proof of Lemma 2.

3. In this section, we prove the Theorem stated in §1. As was stated before Lemma 2, for each $\gamma \in \Gamma$, there exists an n with $\gamma^n \in G$. Since a fixed point of γ is also a fixed point of γ^n, we obtain $H_\Gamma = H_G$ and $P_\Gamma = P_G$. Moreover, for a hyperbolic element γ, the fact $A_\gamma = A_{\gamma^n}$ implies $\{A_\gamma\}_{\gamma \in \Gamma} = \{A_g\}_{g \in G}$.

By definition, $T_\Gamma = \varnothing$ if Γ is of the second kind. Since the index of G in Γ is finite, Γ and G are of the same kind. If one of Γ and G is of the second kind, we obtain $T_\Gamma = T_G = \varnothing$. We thus assume both Γ and G are of the first kind. It is clear from the definition that $T_G \subset T_\Gamma$. So we show $T_G \supset T_\Gamma$. Take a point ζ in T_Γ. Since G is of the first kind, the limit set is identical with ∂D. Hence, for two arbitrary open intervals I_1 and I_2 on ∂D, there exists a hyperbolic element of G such that one fixed point is in I_1 and the other is in I_2. Therefore, in order to show that the point ζ is transitive under G, it is sufficient to show that for an arbitrary hyperbolic element g of G, there exists a convergent sequence with respect to $R(0, \zeta)$ and A_g which is contained in G. For an arbitrary A_γ with $\gamma \in \Gamma$, there exists a sequence $\{g_i\}$ in Γ such that the sequence $\{g_i(R(0, \zeta))\}$ converges to A_γ. By Lemma 2, we can choose such a sequence $\{g_i\}$ in G. Since $\{A_\gamma\}_{\gamma \in \Gamma} = \{A_g\}_{g \in G}$, the point ζ is transitive under G. This completes the proof.

4. First, we give a corollary to the Theorem. This is a generalization of the Theorem in [3].

COROLLARY. *Let G be a finitely generated Fuchsian group of the first kind and let F be a convex fundamental polygon for G. Let $\mathscr{N}$ be a tesselation of F under G, that is, $\mathscr{N} = \{g(F) \mid g \in G\}$. Suppose that there exists an element γ of* Möb(D) *with $\gamma(\mathscr{N}) = \mathscr{N}$. If ζ is a transitive point under G, then so is $\gamma(\zeta)$. If ζ is a hyperbolic (or parabolic) fixed point of G, then so is $\gamma(\zeta)$.*

PROOF. We consider the group $\Gamma = \langle G, \gamma \rangle$. By the assumption $\gamma(\mathscr{N}) = \mathscr{N}$ and by the fact that F has finitely many sides, Γ has finitely many elliptic fixed points in F and has at most one elliptic fixed point on each side of F. Hence there are finitely many fixed points of elliptic elements of Γ on each $g(\mathrm{Cl}\, F)$, where g is an element of G. So the elliptic fixed points do not accumulate in D. Hence Γ is a discrete group (see [1, p. 201]) and $[\Gamma : G]$ is finite. If ζ is a transitive point under G, then ζ is a transitive point under Γ. So $\gamma(\zeta)$ is a transitive point under Γ. By the Theorem, $\gamma(\zeta)$ is a transitive point under G. For hyperbolic and parabolic fixed points, we can prove the assertion similarly.

Next, we give an example.

EXAMPLE. Let G be the Fuchsian group treated in [3] which acts on the unit disk D: namely, the Dirichlet fundamental region F of G with the center at the origin 0 is a non-Euclidean regular $4g$-sided polygon ($g \geqq 2$). We label the sides of F as $\{s_i\}_{i=0}^{4g-1}$ counterclockwise from a certain side of F. The identification of $\{s_i\}_{i=0}^{4g-1}$ is given by $\alpha_i(s_{4i-2}) = s_{4i-4}$ and $\beta_i(s_{4i-3}) = s_{4i-1}$ for $i = 1, 2, \cdots, g$. By u_i we denote the non-Euclidean middle point of s_i. We denote by v_0 the vertex of F which lies between s_0 and s_{4g-1} and by v_i the vertex of F which lies between s_i and s_{i-1} for $i = 1, 2, \cdots, 4g-1$. By w_i we denote the non-Euclidean middle point between the origin 0 and v_i for $i = 0, 1, \cdots, 4g-1$. Let f_1 be the elliptic element of order $4g$ which fixes the origin 0 and let f_2 be the elliptic element of order two in Möb(D) which fixes w_0. We consider the group $\Gamma = \langle G, f_1, f_2 \rangle$. The set of the elliptic fixed points of Γ in $\mathrm{Cl}\, F$ is $\{0, u_i, v_i, w_i\}_{i=0}^{4g-1}$. By the argument in the proof of the Corollary, Γ is a Fuchsian group. Each point of $\{u_i, w_i\}_{i=0}^{4g-1}$ is an elliptic fixed point of order two of Γ. Suppose that $L(\zeta_1, \zeta_2)$ passes through u_i (or w_i). Our Theorem shows that, if the point ζ_1 is transitive under G, then so is ζ_2 while, if the point ζ_1 is a hyperbolic fixed point of G, then so is ζ_2. By $\tilde{u}_i$ (or $\tilde{w}_i$) we denote the projection of u_i (or w_i) on the Riemann surface D/G. Let $\varphi_t(\tilde{z}, \theta)$ ($\tilde{z} \in D/G$ and $\theta \in [0, 2\pi)$) be the geodesic flow starting at $\tilde{z}$ in the direction θ. The above implies that if $\varphi_t(\tilde{u}_i, \theta)$ (or $\varphi_t(\tilde{w}_i, \theta)$) is dense in $T_1(D/G)$ then so is $\varphi_t(\tilde{u}_i, \theta + \pi)$ (or $\varphi_t(\tilde{w}_i, \theta + \pi)$).

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