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AUTOMORPHISMS AND EQUIVALENCE OF BOUNDED REINHARDT DOMAINS NOT CONTAINING THE ORIGIN

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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Introduction. A domain in $\boldsymbol{C}^n$ stable under rotations in the coordinate axis is called a Reinhardt domain. The equivalence problem for bounded Reinhardt domains containing the origin was solved by Sunada [7]. In that paper, he also determined automorphisms of such domains.

In this paper, we shall give an answer to the equivalence problem for general bounded Reinhardt domains. Further, we try to determine automorphisms of a certain class of bounded Reinhardt domains not containing the origin.

To each Reinhardt domain $D \subset \boldsymbol{C}^n$, there is associated an integer t between 0 and n such that the value $n-t$ measures, in a sense, how many zero coordinates D contains (see Section 2). For example, D contains the origin precisely when $t=0$, while D is contained in $(\boldsymbol{C}^*)^n$ precisely when $t=n$. We shall deal with this extreme case $t=n$ as well as the simplest intermediate case in which $n=2$ and $t=1$.

This paper is organized as follows. In Section 1, we collect notation, terminology and basic results on bounded domains. In Sections 2 and 3, we discuss basic concepts and results on Reinhardt domains. Section 4 deals with the equivalence problem for bounded Reinhardt domains. In Section 5, we study the structure of a certain class of subalgebras of the Lie algebra consisting of all holomorphic vector fields on an n-dimensional Reinhardt domain with $t=n$. The result is used in Section 6 for determining automorphisms of n-dimensional bounded Reinhardt domains with $t=n$. Sections 7 and 8 are devoted to the determination of automorphisms of two-dimensional bounded Reinhardt domains with $t=1$.

Closely related results have been obtained by Bedford [2] and Barrett [1]. Their approach is analytic, while our approach is group-theoretic.

The author would like to thank Professor Sunada who introduced him to the study of automorphisms and equivalence of bounded Reinhardt domains not containing the origin.

1. Preliminaries. In this section, we collect notation, terminology and basic results on bounded domains needed later.

The set of non-zero complex numbers is denoted by C^*. For a subset E of C^n, write $E^* = E \cap (C^*)^n$. As usual, $U(n)$ denotes the unitary group of degree n. In particular, $U(1)$ denotes the multiplicative group of complex numbers of absolute value 1. If $E = \{\cdots\}$ is a subset of a vector space V over a field F, the linear subspace of V spanned by E is denoted by $E_F = \{\cdots\}_F$.

If M is a differentiable manifold and p is a point of M, the tangent space to M at p is denoted by T_pM. If $f: M \to M'$ is a differentiable mapping between two differentiable manifolds M and M' and p is a point of M, the differential of f at p is denoted by $(df)_p$.

An automorphism of a complex manifold M means a biholomorphic mapping of M onto itself. The group of all automorphisms of M is denoted by $\mathrm{Aut}(M)$. A complex manifold M is said to be homogeneous if $\mathrm{Aut}(M)$ acts transitively on M. Two complex manifolds are said to be holomorphically equivalent if there is a biholomorphic mapping between them.

We now recall basic results on bounded domains.

If D is a bounded domain in C^n, then $\mathrm{Aut}(D)$ has the structure of a Lie group with respect to the compact-open topology and acts as a Lie transformation group on D. Moreover, if z is any point of D, then the isotropy subgroup $\mathrm{Aut}(D)_z = \{\varphi \in \mathrm{Aut}(D) \mid \varphi(z) = z\}$ of $\mathrm{Aut}(D)$ at z is compact, and its isotropy representation $\mathrm{Aut}(D)_z \ni \varphi \to (d\varphi)_z \in GL(T_zD)$ is faithful, where $GL(T_zD)$ denotes the group of all complex linear transformations of T_zD viewed as a complex vector space in a canonical manner. The identity component of $\mathrm{Aut}(D)$ is denoted by $G(D)$. For each point z of D, the $G(D)$-orbit $G(D)\cdot z = \{g\cdot z \mid g \in G(D)\}$ of z is a submanifold of D.

To each bounded domain D in C^n, there is associated a Hermitian metric on D which is called the Bergman metric. If $\varphi: D \to D'$ is a biholomorphic mapping between two bounded domains D and D' in C^n, then φ is an isometry with respect to the Bergman metrics of D and D'. In particular, the Bergman metric of a bounded domain D in C^n is invariant under $\mathrm{Aut}(D)$. As a consequence, if $\dim G(D)\cdot z = 2n$ for a point z of D, then D is homogeneous. Indeed, the condition $\dim G(D)\cdot z = 2n$ implies that $G(D)\cdot z$ is an open submanifold of D. Since $G(D)$ is a group of isometries of D with respect to the Bergman metric, Kobayashi and Nomizu [5, I, Corollary 4.8] shows that $G(D)\cdot z$ coincides with D, and hence D is homogeneous.

For a domain D in $\boldsymbol{C}^n$, we denote by $\mathfrak{X}(D)$ the complex Lie algebra of all holomorphic vector fields on D with the Poisson bracket. If D is bounded, then the set of all complete holomorphic vector fields on D is a finite-dimensional real subalgebra of $\mathfrak{X}(D)$, and is denoted by $\mathfrak{g}(D)$. The subalgebra $\mathfrak{g}(D)$ can be canonically identified with the Lie algebra of the Lie group $\mathrm{Aut}(D)$. An application of Liouville's theorem yields that $\mathfrak{g}(D) \cap \sqrt{-1}\mathfrak{g}(D) = \{0\}$.

2. Basic concepts on Reinhardt domains. In this section, we discuss some basic concepts and results on Reinhardt domains.

For each element $\alpha = (\alpha_1, \cdots, \alpha_n)$ of $(\boldsymbol{C}^*)^n$, we define an element π_α of $\mathrm{Aut}(\boldsymbol{C}^n)$ by the coordinatewise multiplication

$$\pi_\alpha(z_1, \cdots, z_n) = (\alpha_1 z_1, \cdots, \alpha_n z_n) .$$

π_α will sometimes be viewed as an element of $\mathrm{Aut}((\boldsymbol{C}^*)^n)$. Write $T = (U(1))^n$. The group T acts as a group of automorphisms on $\boldsymbol{C}^n$ by

$$\alpha \cdot z = \pi_\alpha(z) \quad \text{for} \quad \alpha \in T \text{ and } z \in \boldsymbol{C}^n .$$

For $i = 1, \cdots, n$, let T_i be the subgroup of T defined by

$$T_i = \{(\alpha_1, \cdots, \alpha_n) \in T \mid \alpha_j = 1 \quad \text{for all} \quad j \neq i\} .$$

DEFINITION. A Reinhardt domain in $\boldsymbol{C}^n$ is a domain D in $\boldsymbol{C}^n$ which is stable under T, that is, such that $\alpha \cdot D \subset D$ for every $\alpha \in T$.

Let D be a Reinhardt domain in C^n. The group T then acts as a group of automorphisms on D. The subgroup of $\mathrm{Aut}(D)$ induced by T is denoted by $T(D)$, or simply by T for short. Also, the subgroups of $\mathrm{Aut}(D)$ induced by $T_1, \cdots, T_n$ are denoted, respectively, by $T_1(D), \cdots, T_n(D)$, or simply by $T_1, \cdots, T_n$ for short. For each point z of D, the $T(D)$-orbit $\Pi_z := T(D) \cdot z = \{\alpha \cdot z \mid \alpha \in T(D)\}$ of z is a torus in D. Note that z belongs to $D^* = D \cap (\boldsymbol{C}^*)^n$ precisely when $\dim \Pi_z = n$. Introduce the constant $t = \min_{z \in D} \dim \Pi_z$, the minimal dimension of the tori Π_z for $z \in D$. This constant takes a value between 0 and n, and is a fundamental invariant associated to the Reinhardt domain D. For example, D contains the origin precisely when $t = 0$, while D is contained in $(\boldsymbol{C}^*)^n$ precisely when $t = n$.

DEFINITION. An algebraic automorphism of $(\boldsymbol{C}^*)^n$ is an automorphism of $(\boldsymbol{C}^*)^n$ whose components are given by Laurent monomials, that is, of the form

$$(*) \qquad \begin{aligned} &(\boldsymbol{C}^*)^n \ni (z_1, \cdots, z_n) \to (w_1, \cdots, w_n) \in (\boldsymbol{C}^*)^n , \\ &w_i = \alpha_i z_1^{a_{1i}} \cdots z_n^{a_{ni}} , \quad i = 1, \cdots, n , \end{aligned}$$

where $A: = (a_{ij}) \in GL(n, \boldsymbol{Z})$ and $\alpha := (\alpha_i) \in (\boldsymbol{C}^*)^n$.

The set $\mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$ of all algebraic automorphisms of $(\boldsymbol{C}^*)^n$ is a subgroup of $\mathrm{Aut}((\boldsymbol{C}^*)^n)$. The group $\mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$ is a Lie group of dimension $2n$ with respect to the compact-open topology, and its identity component is given by $\pi_{(\boldsymbol{C}^*)^n} := \{\pi_\alpha | \alpha \in (\boldsymbol{C}^*)^n\}$.

Let φ be an algebraic automorphism of $(\boldsymbol{C}^*)^n$ and write $\varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z))$. In general, the components $\varphi_1, \cdots, \varphi_n$ have zeroes or poles along each coordinate hyperplane. If, for two domains D and D' in $\boldsymbol{C}^n$ not necessarily contained in $(C^*)^n$, they have no poles on D and $\varphi: D \to \boldsymbol{C}^n$ maps D biholomorphically onto D', then we say that φ induces a biholomorphic mapping of D onto D'.

Every algebraic automorphism φ of $(\boldsymbol{C}^*)^n$ has the property that

$$\varphi T \varphi^{-1} = T\,. \tag{1}$$

As a consequence, if φ induces a biholomorphic mapping of a Reinhardt domain $D \subset \boldsymbol{C}^n$ onto a domain $D' \subset \boldsymbol{C}^n$, then D' is also a Reinhardt domain. This type of transformation is a useful tool in studying automorphisms of a Reinhardt domain, as we shall see later. Consider a biholomorphic mapping $\varphi: D \to D'$ between two Reinhardt domains D and D' in $\boldsymbol{C}^n$. Then φ is induced by an algebraic automorphism of $(\boldsymbol{C}^*)^n$ if and only if it is equivariant with respect to the T-actions, or equivalently if and only if it has the property that

$$\varphi T(D) \varphi^{-1} = T(D')\,. \tag{2}$$

Indeed, the "only if" part follows immediately from (1) applied to φ. For the "if" part, see the corollary to Proposition 1 of the next section and the remark after it. Biholomorphic mappings between Reinhardt domains equivariant with respect to the T-actions may be considered as natural isomorphisms in the category of Reinhardt domains. We are led therefore to introducing the following equivalence relation between Reinhardt domains.

DEFINITION. Two Reinhardt domains in $\boldsymbol{C}^n$ are said to be algebraically equivalent if there is a biholomorphic mapping between them induced by an algebraic automorphism of $(\boldsymbol{C}^*)^n$.

The following concepts will be needed later.

DEFINITION. An algebraic (resp. linear) automorphism of a Reinhardt domain D in $\boldsymbol{C}^n$ is an automorphism of D induced by an algebraic automorphism of $(\boldsymbol{C}^*)^n$ (resp. complex linear transformation of $\boldsymbol{C}^n$).

The set $\mathrm{Aut}_{\mathrm{alg}}(D)$ (resp. $GL(D)$) of all algebraic (resp. linear) automorphisms of a Reinhardt domain D in $\boldsymbol{C}^n$ is a subgroup of $\mathrm{Aut}(D)$. The group $\mathrm{Aut}_{\mathrm{alg}}(D)$ (resp. $GL(D)$) may be viewed as a subgroup of $\mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$ (resp. $GL(n, \boldsymbol{C})$). It then follows that $\mathrm{Aut}_{\mathrm{alg}}(D)$ (resp. $GL(D)$) is closed in $\mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$ (resp. $GL(n, \boldsymbol{C})$), and therefore that $\mathrm{Aut}_{\mathrm{alg}}(D)$ (resp. $GL(D)$) is a Lie group with respect to the compact-open topology. Note that $T(D)$ is contained in $\mathrm{Aut}_{\mathrm{alg}}(D)$.

There is a useful correspondence between Reinhardt domains and tube domains. First recall the definition of a tube domain and fix notation. If Ω is a domain in $\boldsymbol{R}^n$, the tube domain $T_\Omega = \Omega + \sqrt{-1}\boldsymbol{R}^n$ over Ω is the domain in $\boldsymbol{C}^n$ consisting of all points $\zeta = \xi + \sqrt{-1}\eta \in \boldsymbol{C}^n = \boldsymbol{R}^n + \sqrt{-1}\boldsymbol{R}^n$ ($\xi, \eta \in \boldsymbol{R}^n$) such that $\xi \in \Omega$. For each element η of $\boldsymbol{R}^n$, we define the translation $\sigma_\eta \in \mathrm{Aut}(T_\Omega)$ by

$$\sigma_\eta(\zeta) = \zeta + \sqrt{-1}\eta .$$

For each point ζ of T_Ω, write $\Sigma_\zeta = \{\sigma_\eta(\zeta) \mid \eta \in \boldsymbol{R}^n\}$. Now, define the mapping $\mathrm{ord}\colon (\boldsymbol{C}^*)^n \to \boldsymbol{R}^n$ by

$$\mathrm{ord}(z_1, \cdots, z_n) = (-(2\pi)^{-1}\log|z_1|, \cdots, -(2\pi)^{-1}\log|z_n|) .$$

Clearly ord is an open mapping. If E is a subset of $\boldsymbol{C}^n$, the image of E^* under ord is called the logarithmic image of E. To each Reinhardt domain D in $(\boldsymbol{C}^*)^n$, there is associated a tube domain T_Ω over the logarithmic image $\Omega := \mathrm{ord}(D)$ of D. T_Ω naturally becomes a covering manifold of D. Indeed, introduce the covering $\varpi\colon \boldsymbol{C}^n \to (\boldsymbol{C}^*)^n$ defined by

$$\varpi(\zeta_1, \cdots, \zeta_n) = (\exp(-2\pi\zeta_1), \cdots, \exp(-2\pi\zeta_n)) .$$

Then we have $T_\Omega = \varpi^{-1}(D)$, and the restriction $\varpi\colon T_\Omega \to D$ is a covering projection. The covering transformation group for ϖ is given by $\sigma_{\boldsymbol{Z}^n} := \{\sigma_\eta \mid \eta \in \boldsymbol{Z}^n\}$. For each point ζ of T_Ω, we have $\Sigma_\zeta = \varpi^{-1}(\Pi_{\varpi(\zeta)})$, and the restriction $\varpi\colon \Sigma_\zeta \to \Pi_{\varpi(\zeta)}$ is a covering projection. The tube domain T_Ω is called the covering tube domain of D and the covering projection $\varpi\colon T_\Omega \to D$ is called the canonical covering projection.

Here are some observations about the relations between Reinhardt domains and their logarithmic images. First, if we denote by $GL(n, \boldsymbol{Z})\cdot\boldsymbol{R}^n$ the group of all affine transformations of $\boldsymbol{R}^n$ whose linear parts belong to $GL(n, \boldsymbol{Z})$, then there is a group homomorphism of $\mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$ onto $GL(n, \boldsymbol{Z})\cdot\boldsymbol{R}^n$ which associates to each element $\varphi \in \mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$ written in the form $(*)$ the element $\bar\varphi \in GL(n, \boldsymbol{Z})\cdot\boldsymbol{R}^n$ defined by

$$(3) \qquad \bar\varphi(\xi) = \xi A + \mathrm{ord}(\alpha) .$$

It follows that if an algebraic automorphism φ of $(\boldsymbol{C}^*)^n$ induces a biholomorphic mapping $\varphi\colon D \to D'$ between two Reinhardt domains D and D'

in $\boldsymbol{C}^n$, then $\varphi(D^*) = D'^*$ and

$$\bar{\varphi}(\mathrm{ord}(D^*)) = \mathrm{ord}(D'^*) \,. \tag{4}$$

Secondly, if D is a Reinhardt domain in $\boldsymbol{C}^n$, then D is bounded if and only if

$$\mathrm{ord}(D^*) \subset \{(\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n \mid \xi_1 > c_1, \cdots, \xi_n > c_n\}$$

for some constants $c_1, \cdots, c_n$. Thirdly, if D is a Reinhardt domain in $(\boldsymbol{C}^*)^n$, then D is algebraically equivalent to a bounded Reinhardt domain if and only if the logarithmic image $\mathrm{ord}(D)$ of D has the convex hull containing no complete straight lines. This is a consequence of the observations above and the fact that if an open subset Ω of $\boldsymbol{R}^n$ has the convex hull containing no complete straight lines, then there exists an element f of $GL(n, \boldsymbol{Z}) \cdot \boldsymbol{R}^n$ such that

$$f(\Omega) \subset \{(\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n \mid \xi_1 > 0, \cdots, \xi_n > 0\} \,.$$

Finally, if an algebraic automorphism φ of $(\boldsymbol{C}^*)^n$ maps a Reinhardt domain $D \subset \boldsymbol{C}^n$ whose logarithmic image has the convex hull containing no complete straight lines biholomorphically onto a Reinhardt domain $D' \subset \boldsymbol{C}^n$, then (4) applied to φ implies that the logarithmic image of D' also has the convex hull containing no complete straight lines.

LEMMA 1. *Let D be an open subset of $\boldsymbol{C}^n$ whose logarithmic image has the convex hull containing no complete straight lines. If α is an element of $(\boldsymbol{C}^*)^n$ such that $\pi_\alpha(D) = D$, then $\alpha \in T$.*

PROOF. If we write $\Omega = \mathrm{ord}(D^*)$, and if $\bar{\pi}_\alpha$ is the translation of $\boldsymbol{R}^n$ defined by $\bar{\pi}_\alpha(\xi) = \xi + \mathrm{ord}(\alpha)$, then the assumption $\pi_\alpha(D) = D$ implies that $\bar{\pi}_\alpha(\Omega) = \Omega$ (cf. (3), (4)). If $\hat{\Omega}$ is the convex hull of Ω, then $\bar{\pi}_\alpha(\hat{\Omega}) = \hat{\Omega}$. Since $\hat{\Omega}$ contains no complete straight lines, it follows that $\mathrm{ord}(\alpha) = 0$, and hence that $\alpha \in \mathrm{ord}^{-1}(0) = T$. q.e.d.

COROLLARY. *If D is a Reinhardt domain in $\boldsymbol{C}^n$ whose logarithmic image has the convex hull containing no complete straight lines, then the identity component of $\mathrm{Aut}_{\mathrm{alg}}(D)$ coincides with $T(D)$.*

PROOF. This follows from the fact that the identity component of $\mathrm{Aut}_{\mathrm{alg}}(D)$ coincides with that of $\{\pi_\alpha \in \pi_{(\boldsymbol{C}^*)^n} \mid \pi_\alpha(D) = D\}$.

The following result is well-known (cf. Narasimhan [6]).

THEOREM. *If D is a Reinhardt domain in $\boldsymbol{C}^n$, then every holomorphic function f on D can be expanded into a "Laurent series"*

$$f(z) = \sum_{\nu \in \boldsymbol{Z}^n} a_\nu z^\nu$$

which converges absolutely and uniformly on any compact set in D, where $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ for the coordinate $z=(z_1, \cdots, z_n)$ of $\boldsymbol{C}^n$ and $\nu=(\nu_1, \cdots, \nu_n) \in \boldsymbol{Z}^n$. Moreover, the coefficients a_ν's are uniquely determined by f.

COROLLARY. *Suppose that a holomorphic function f on a Reinhardt domain D in $\boldsymbol{C}^n$ satisfies*

$$f(z_1, \cdots, z_{i-1}, \alpha z_i, z_{i+1}, \cdots, z_n) = \alpha^k f(z) \tag{5}$$

for an integer k and for every $\alpha \in U(1)$ and every $z = (z_1, \cdots, z_n) \in D$. Then f has the form

$$f(z) = g(z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n) z_i^k$$

for some holomorphic function g on the Reinhardt domain in $\boldsymbol{C}^{n-1}$ given as the image of the domain D under the projection

$$p\colon \boldsymbol{C}^n \ni z \to (z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n) \in \boldsymbol{C}^{n-1} .$$

PROOF. Let $f(z) = \sum_{\nu \in \boldsymbol{Z}^n} a_\nu z^\nu$ be the Laurent expansion of f. Substituting this into (5) and applying the uniqueness assertion of the above theorem, we see that

$$\alpha^{\nu_i} a_\nu = \alpha^k a_\nu \text{ for every } \alpha \in U(1) \text{ and every } \nu = (\nu_1, \cdots, \nu_n) \in \boldsymbol{Z}^n .$$

Hence, if the coefficient a_ν of the Laurent expansion of f is not equal to 0, then $\alpha^{\nu_i - k} = 1$ for every $\alpha \in U(1)$, so that $\nu_i = k$. We obtain the desired result, since $f(z)/z_i^k$ is then a holomorphic function on D independent of the variable z_i, that is, $f(z)/z_i^k$ has the form $f(z)/z_i^k = g(p(z))$ for some holomorphic function g on $p(D)$. q.e.d.

LEMMA 2. *Let G be a group of automorphisms of a Reinhardt domain D in $\boldsymbol{C}^n$. If the center of G contains the subgroup*

$$T_{k+1}(D) \cdots T_n(D) = \{\alpha^{(k+1)} \cdots \alpha^{(n)} \mid \alpha^{(k+1)} \in T_{k+1}(D), \cdots, \alpha^{(n)} \in T_n(D)\}$$

of $T(D)$, and if Δ is the Reinhardt domain in $\boldsymbol{C}^k$ given as the image of the domain D under the projection

$$p\colon \boldsymbol{C}^n \ni (z_1, \cdots, z_n) \to (z_1, \cdots, z_k) \in \boldsymbol{C}^k ,$$

then, to each element φ of G, there correspond an automorphism $\tau(\varphi)$ of Δ and a holomorphic mapping $\gamma(\varphi)$ of Δ into $(\boldsymbol{C}^*)^{n-k}$ for which φ has the form

$$\varphi\colon D \ni (z', z'') \to (w', w'') \in D ,$$
$$\begin{cases} w' = \tau(\varphi)(z') , \\ w'' = \pi_{\gamma(\varphi)(z')}(z'') , \end{cases}$$

where $z', w' \in \boldsymbol{C}^k$ and $z'', w'' \in \boldsymbol{C}^{n-k}$. Further, the map $\tau\colon G \to \mathrm{Aut}(\Delta)$ sending

φ to $\tau(\varphi)$ is a group homomorphism. If the logarithmic image of D has the convex hull containing no complete straight lines, then the kernel $\ker \tau$ *of τ coincides with $T_{k+1}(D) \cdots T_n(D)$.*

PROOF. Let φ be any element of G. In terms of the coordinates in $\boldsymbol{C}^n$, write $\varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z))$, where $\varphi_1, \cdots, \varphi_n$ are holomorphic functions on D. Since φ commutes with every element of $T_{k+1}(D) \cdots T_n(D)$, the functions $\varphi_1, \cdots, \varphi_n$ satisfy

$$\varphi_i(z_1, \cdots, z_k, \alpha_{k+1}z_{k+1}, \cdots, \alpha_n z_n) = \varphi_i(z) , \qquad i = 1, \cdots, k ,$$
$$\varphi_i(z_1, \cdots, z_k, \alpha_{k+1}z_{k+1}, \cdots, \alpha_n z_n = \alpha_i \varphi_i(z) , \qquad i = k+1, \cdots, n ,$$

for every $(\alpha_{k+1}, \cdots, \alpha_n) \in (U(1))^{n-k}$ and every $z = (z_1, \cdots, z_n) \in D$. Hence, by a successive use of the above corollary, we see that they have the form

$$\varphi_i(z) = g_i(z_1, \cdots, z_k) , \qquad i = 1, \cdots, k ,$$
$$\varphi_i(z) = g_i(z_1, \cdots, z_k)z_i , \qquad i = k+1, \cdots, n ,$$

for some holomorphic functions $g_1, \cdots, g_n$ on Δ. Write

$$\tau(\varphi) = (g_1, \cdots, g_k) \quad \text{and} \quad \gamma(\varphi) = (g_{k+1}, \cdots, g_n) .$$

To prove the first assertion, it suffices to prove that $\tau(\varphi)$ gives an automorphism of Δ, while $\gamma(\varphi)$ gives a mapping into $(\boldsymbol{C}^*)^{n-k}$. The fact that $\gamma(\varphi)$ gives a mapping into $(\boldsymbol{C}^*)^{n-k}$ is immediate from the fact that φ is an automorphism of D. To prove that $\tau(\varphi)$ gives an automorphism of Δ, note first that $\tau(\varphi)(\Delta) \subset \Delta$. This follows from the relation $p \circ \varphi = \tau(\varphi) \circ p$. Now, let φ' be any element of G and consider $\tau(\varphi')$ and $\tau(\varphi\varphi')$. Then we have $\tau(\varphi')(\Delta) \subset \Delta$ and $\tau(\varphi\varphi')(\Delta) \subset \Delta$, as noted above, and it is readily verified that

$$\tau(\varphi\varphi') = \tau(\varphi)\tau(\varphi') \tag{6}$$

on Δ. If we take φ^{-1} as φ', then (6) implies that

$$\tau(\varphi)\tau(\varphi^{-1}) = \tau(\varphi\varphi^{-1}) = \tau(\mathrm{id}) = \mathrm{id}$$

on Δ, where id denotes the identity mapping. Since a similar argument shows that $\tau(\varphi^{-1})\tau(\varphi) = \mathrm{id}$ on Δ, it follows that $\tau(\varphi)$ gives an automorphism of Δ, which proves the first assertion.

The first half of the second assertion is an immediate consequence of (6). It remains to prove the latter half of the second assertion. It is obvious that $T_{k+1}(D) \cdots T_n(D)$ is contained in $\ker \tau$. To show the reverse inclusion, let φ be any element of $\ker \tau$. Then φ has the form

$$\varphi: D \ni (z', z'') \to (w', w'') \in D\ ,\tag{7}$$
$$\begin{cases} w' = z'\ , \\ w'' = \pi_{\gamma(\varphi)(z')}(z'')\ , \end{cases}$$

for the holomorphic mapping $\gamma(\varphi) = (g_{k+1}, \cdots, g_n)$ of Δ into $(\boldsymbol{C}^*)^{n-k}$, where $z', w' \in \boldsymbol{C}^k$ and $z'', w'' \in \boldsymbol{C}^{n-k}$, and $g_{k+1}, \cdots, g_n$ are nowhere-vanishing holomorphic functions on Δ. Fix any point z' of Δ^* and define an open subset $D_{z'}$ of $\boldsymbol{C}^{n-k}$ by

$$D_{z'} = \{z'' \in \boldsymbol{C}^{n-k} \,|\, (z', z'') \in D\}\ .$$

As a consequence of (7), we have $\pi_{\gamma(\varphi)(z')}(D_{z'}) = D_{z'}$. Since the assumption that the logarithmic image of D has the convex hull containing no complete straight lines implies that the logarithmic image of $D_{z'}$ has the convex hull containing no complete straight lines, it follows from Lemma 1 that $\gamma(\varphi)(z') \in T = (U(1))^{n-k}$, that is, that $|g_i(z')| = 1$, $i = k+1, \cdots, n$. Since this holds for every point z' of the non-empty open subset Δ^* of Δ, we see that $g_{k+1}, \cdots, g_n$ are constants of absolute value 1. Therefore, in view of (7), φ belongs to $T_{k+1}(D) \cdots T_n(D)$, so that $\ker \tau$ is contained in $T_{k+1}(D) \cdots T_n(D)$, which proves the latter half of the second assertion. q.e.d.

As a special case of the above lemma, we obtain the following corollary.

COROLLARY. *Let G be a group of automorphisms of a Reinhardt domain D in $\boldsymbol{C}^n$. If the center of G contains the group $T(D)$, then $G \subset \pi_{(\boldsymbol{C}^*)^n}$.*

3. Two propositions. Continuing our study in the preceding section, we establish two basic results on Reinhardt domains. The first result gives a criterion for a biholomorphic mapping between two bounded Reinhardt domains to be induced by an algebraic automorphism of $(\boldsymbol{C}^*)^n$. The second result is about the structure of the group of linear automorphisms of a bounded Reinhardt domain.

PROPOSITION 1. *Let $\varphi: D \to D'$ be a biholomorphic mapping between two Reinhardt domains D and D' in $\boldsymbol{C}^n$. If D or D' is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^n$, and if there exists a point z_0 of D^* such that $\varphi(\Pi_{z_0}) = \Pi_{\varphi(z_0)}$, then φ is induced by an algebraic automorphism of $(\boldsymbol{C}^*)^n$.*

PROOF. Since it follows from the relation $\dim \Pi_{\varphi(z_0)} = \dim \varphi(\Pi_{z_0}) = \dim \Pi_{z_0} = n$ that $\varphi(\Pi_{z_0}) = \Pi_{\varphi(z_0)} \subset D'^*$, we can find a Reinhardt domain D_0 in $\boldsymbol{C}^n$ such that $\Pi_{z_0} \subset D_0 \subset D$, $\varphi(D_0) \subset D'^*$ and $\mathrm{ord}(D_0)$ is simply connected.

If T_Ω is the covering tube domain of D_0 and ϖ is the canonical covering projection, then, because of the simple connectedness of $\Omega = \mathrm{ord}(D_0)$, the covering $\varpi: T_\Omega \to D_0$ is the universal covering of D_0. If $T_{\Omega'}$ is the covering tube domain of D'^* and ϖ' is the canonical covering projection, and if $\tilde{\varphi}: T_\Omega \to T_{\Omega'}$ is a lifting of $\varphi: D_0 \to D'^*$ and ζ_0 is a point of T_Ω such that $\varpi(\zeta_0) = z_0$, then the assumption $\varphi(\Pi_{z_0}) = \Pi_{\varphi(z_0)}$ implies that

$$\tilde{\varphi}(\Sigma_{\zeta_0}) = \Sigma_{\tilde{\varphi}(\zeta_0)} , \tag{8}$$

and we have the following commutative diagram:

$$\begin{array}{ccc} \Sigma_{\zeta_0} & \xrightarrow{\tilde{\varphi}} & \Sigma_{\tilde{\varphi}(\zeta_0)} \\ \varpi \downarrow & & \downarrow \varpi' \\ \Pi_{z_0} & \xrightarrow[\varphi]{} & \Pi_{\varphi(z_0)} \end{array} \tag{9}$$

Clearly $\varpi: \Sigma_{\zeta_0} \to \Pi_{z_0}$ and $\varpi': \Sigma_{\tilde{\varphi}(\zeta_0)} \to \Pi_{\varphi(z_0)}$ are the universal coverings of the tori Π_{z_0} and $\Pi_{\varphi(z_0)}$, respectively, and $\tilde{\varphi}: \Sigma_{\zeta_0} \to \Sigma_{\tilde{\varphi}(\zeta_0)}$ is a lifting of $\varphi: \Pi_{z_0} \to \Pi_{\varphi(z_0)}$.

We show that $\tilde{\varphi}: T_\Omega \to T_{\Omega'}$ is a complex affine mapping. Note first that the restriction $\tilde{\varphi}: \Sigma_{\zeta_0} \to \Sigma_{\tilde{\varphi}(\zeta_0)}$ is an affine mapping. Indeed, consider the Bergman metrics of D and D'. Since the Bergman metric of D is invariant under $\mathrm{Aut}(D)$, therefore under $T(D)$, the submanifold Π_{z_0} of D with the induced Riemannian structure is a Euclidean torus. Similarly, by means of the Bergman metric of D', the submanifold $\Pi_{\varphi(z_0)}$ of D' has the structure of a Euclidean torus. Since the biholomorphic mapping $\varphi: D \to D'$ is an isometry with respect to the Bergman metrics, we see that $\varphi: \Pi_{z_0} \to \Pi_{\varphi(z_0)}$ is an isometry between the Euclidean tori Π_{z_0} and $\Pi_{\varphi(z_0)}$, and our assertion follows from the fact that $\tilde{\varphi}: \Sigma_{\zeta_0} \to \Sigma_{\tilde{\varphi}(\zeta_0)}$ is a lifting of $\varphi: \Pi_{z_0} \to \Pi_{\varphi(z_0)}$. Now, if Re and Im are the projections of $T_{\Omega'}$ into $\boldsymbol{R}^n$ defined by

$$\mathrm{Re}\,\zeta = \xi \text{ and } \mathrm{Im}\,\zeta = \eta \text{ for } \zeta = \xi + \sqrt{-1}\eta \in T_{\Omega'} = \Omega' + \sqrt{-1}\boldsymbol{R}^n \quad (\xi \in \Omega', \eta \in \boldsymbol{R}^n) ,$$

then, by (8), the mapping

$$\boldsymbol{R}^n \ni \eta \to \mathrm{Re}\,\tilde{\varphi}(\zeta_0 + \sqrt{-1}\eta) \in \boldsymbol{R}^n$$

is constant, while, by what we have noted above, the mapping

$$\boldsymbol{R}^n \ni \eta \to \mathrm{Im}\,\tilde{\varphi}(\zeta_0 + \sqrt{-1}\eta) \in \boldsymbol{R}^n$$

is affine. This implies that if, in terms of the coordinates in $\boldsymbol{C}^n$, we write $\tilde{\varphi}(\zeta_1, \cdots, \zeta_n) = (\tilde{\varphi}_1(\zeta), \cdots, \tilde{\varphi}_n(\zeta))$, where $\zeta = (\zeta_1, \cdots, \zeta_n)$ and $\tilde{\varphi}_1, \cdots, \tilde{\varphi}_n$ are holomorphic functions on T_Ω, and if $\zeta_i = \xi_i + \sqrt{-1}\eta_i$ $(\xi_i, \eta_i \in \boldsymbol{R})$,

$i = 1, \cdots, n$, then the functions $\partial^2\tilde{\varphi}_i(\zeta)/\partial\eta_j\partial\eta_k$, $i, j, k = 1, \cdots, n$, vanish identically on Σ_{ζ_0}. Σ_{ζ_0} is a totally real submanifold of T_Ω of dimension n. Thus $\partial^2\tilde{\varphi}_i(\zeta)/\partial\zeta_j\partial\zeta_k$, $i, j, k = 1, \cdots, n$, vanish identically on T_Ω, and hence $\tilde{\varphi}: T_\Omega \to T_{\Omega'}$ is a complex affine mapping.

Because of the result of the preceding paragraph, φ has the form $\tilde{\varphi}(\zeta) = \zeta A + \beta$ for some $A \in GL(n, \boldsymbol{C})$ and some $\beta \in \boldsymbol{C}^n$. We show that A belongs to $GL(n, \boldsymbol{Z})$. It is enough to show that $\boldsymbol{Z}^n A = \boldsymbol{Z}^n$. Write $A = B + \sqrt{-1}C$, where B and C are real square matrices of degree n. Then, by (8), we have

$$\operatorname{Re}\tilde{\varphi}(\zeta_0) = \operatorname{Re}\tilde{\varphi}(\zeta_0 + \sqrt{-1}\eta) = -\eta C + \operatorname{Re}\tilde{\varphi}(\zeta_0)$$

for every $\eta \in \boldsymbol{R}^n$, so that $\eta C = 0$ for every $\eta \in \boldsymbol{R}^n$, and hence $C = 0$. Since this implies that, for $\eta \in \boldsymbol{R}^n$, $\eta A \in \boldsymbol{R}^n$ and $\tilde{\varphi}\sigma_\eta = \sigma_{\eta A}\tilde{\varphi}$, the desired result follows from the fact that, in (9), the covering transformation groups of the coverings $\varpi: \Sigma_{\zeta_0} \to \Pi_{z_0}$ and $\varpi': \Sigma_{\tilde{\varphi}(\zeta_0)} \to \Pi_{\varphi(z_0)}$ are given by the restrictions of $\sigma_{\boldsymbol{Z}^n}$ to Σ_{ζ_0} and $\Sigma_{\tilde{\varphi}(\zeta_0)}$, respectively.

If $A = (a_{ij})$ and $\beta = (\beta_i)$, then, in view of the definition of the covering projections ϖ and ϖ', it follows that $\varphi: D_0 \to D'^*$ has the form

$$\varphi: D_0 \ni (z_1, \cdots, z_n) \to (w_1, \cdots, w_n) \in D'^* ,$$
$$w_i = \exp(-2\pi\beta_i) z_1^{a_{1i}} \cdots z_n^{a_{ni}} , \qquad i = 1, \cdots, n ,$$

and therefore, by analytic continuation, that φ is induced by an algebraic automorphism of $(\boldsymbol{C}^*)^n$. q.e.d.

COROLLARY. *Let $\varphi: D \to D'$ be a biholomorphic mapping between two Reinhardt domains D and D' in $\boldsymbol{C}^n$. If D or D' is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^n$, and if $\varphi T(D)\varphi^{-1} = T(D')$, then φ is induced by an algebraic automorphism of $(\boldsymbol{C}^*)^n$.*

PROOF. It follows from the assumption $\varphi T(D)\varphi^{-1} = T(D')$ that $\varphi(\Pi_z) = \Pi_{\varphi(z)}$ for every $z \in D$. Therefore, for any point z_0 of D^*, the mapping φ satisfies the condition of the proposition. q.e.d.

REMARK. The conclusion of the above corollary holds even if the boundedness assumption on D, D' is removed. This more general result can be shown by using the corollary to the theorem of the preceding section.

PROPOSITION 2. *If D is a bounded Reinhardt domain in $\boldsymbol{C}^n$, then, by a change of coordinates*

$$(z_1, \cdots, z_n) \to (r_1 z_1, \cdots, r_n z_n)$$

for some positive constants $r_1, \cdots, r_n$, the group $GL(D)$ is contained in $U(n)$.

PROOF. We show first that $GL(D)$ is compact. It is enough to show that if $\{\varphi^{(m)} | m = 1, 2, \cdots\}$ is any sequence of elements of $GL(D)$, then there exists a subsequence of $\{\varphi^{(m)} | m = 1, 2, \cdots\}$ which converges to an element of $GL(D)$. Select a basis $u_1, \cdots, u_n$ for the complex vector space $\boldsymbol{C}^n$ consisting of points of D. Since every element of $GL(D)$ maps $u_1, \cdots, u_n$ into the fixed bounded domain D, the set of entries of elements of $GL(D)$ is bounded, and therefore there exists a subsequence of $\{\varphi^{(m)} | m = 1, 2, \cdots\}$, denoted again by $\{\varphi^{(m)} | m = 1, 2, \cdots\}$, which converges to some complex square matrix φ of degree n. If φ does not belong to $GL(n, \boldsymbol{C})$, then $\varphi(D)$ is contained in a proper vector subspace of $\boldsymbol{C}^n$. Since the boundedness of D implies that the set $\varphi^{(m)}(D)$ approaches uniformly to the set $\varphi(D)$ as m goes to ∞, it follows that $\varphi^{(m)}(D) \neq D$ for sufficiently large m. This contradicts the assumption $\varphi^{(m)} \in GL(D)$. We thus conclude that $\varphi \in GL(n, \boldsymbol{C})$. Since $GL(D)$ is closed in $GL(n, \boldsymbol{C})$, we obtain $\varphi \in GL(D)$, which proves our assertion.

Because of the result of the preceding paragraph, there exists a $GL(D)$-invariant Hermitian inner product $\langle \sharp, \sharp \rangle$ on $\boldsymbol{C}^n$. Since $GL(D)$ contains T, we see that

$$\langle z, w \rangle = \sum_{i=1}^{n} a_i z_i \bar{w}_i \tag{10}$$

for some positive constants $a_1, \cdots, a_n$, where $z = (z_1, \cdots, z_n)$, $w = (w_1, \cdots, w_n) \in \boldsymbol{C}^n$. By a change of coordinates

$$(z_1, \cdots, z_n) \rightarrow (\sqrt{a_1} z_1, \cdots, \sqrt{a_n} z_n) ,$$

we can take $a_1 = \cdots = a_n = 1$ in (10), and then $GL(D)$ is contained in $U(n)$. q.e.d.

4. Equivalence of Reinhardt domains. This section deals with the equivalence problem for bounded Reinhardt domains.

We first present a group-theoretic characterization of $T(D)$ and a result from the theory of Lie groups.

PROPOSITION 1. *If D is a bounded Reinhardt domain in $\boldsymbol{C}^n$, then $T(D)$ is a maximal torus in $G(D)$.*

PROOF. It is obvious that $T(D)$ is a torus in $G(D)$, that is, a connected compact abelian subgroup of $G(D)$. Let T' be any torus in $G(D)$ containing $T(D)$. By the corollary to Lemma 2 of Section 2, T' is a torus in $\pi_{(\boldsymbol{C}^*)^n}$. Since $T(D)$ is clearly a maximal torus in $\pi_{(\boldsymbol{C}^*)^n}$, we see that T' coincides with $T(D)$. q.e.d.

THE CONJUGACY THEOREM (cf. Hochschild [4]). *If T and T' are two* *maximal tori in a connected Lie group* G, *then there exists an element* $g \in G$ *such that* $gTg^{-1} = T'$.

We shall now give an answer to the equivalence problem for bounded Reinhardt domains.

THEOREM 1. *If two bounded Reinhardt domains in* $\boldsymbol{C}^n$ *are holomorphically equivalent, then they are algebraically equivalent.*

Note that if an algebraic automorphism of $(\boldsymbol{C}^*)^n$ induces a biholomorphic mapping between two Reinhardt domains in $\boldsymbol{C}^n$ containing the origin, then it must be of the form

$$(**) \qquad \begin{gathered} (\boldsymbol{C}^*)^n \ni (z_1, \cdots, z_n) \to (w_1, \cdots, w_n) \in (\boldsymbol{C}^*)^n , \\ w_i = \alpha_i z_{\sigma(i)} , \qquad i = 1, \cdots, n , \end{gathered}$$

where σ is a permutation of $\{1, \cdots, n\}$ and $(\alpha_1, \cdots, \alpha_n) \in (\boldsymbol{C}^*)^n$. Hence, from the above theorem, we obtain the following consequence.

COROLLARY (Sunada [7]). *If two bounded Reinhardt domains in* $\boldsymbol{C}^n$ *containing the origin are holomorphically equivalent, then there is a biholomorphic mapping between them induced by an algebraic automorphism of* $(\boldsymbol{C}^*)^n$ *of the form* $(**)$.

To prove Theorem 1, it suffices to prove the following proposition.

PROPOSITION 2. *If* $\varphi: D \to D'$ *is a biholomorphic mapping between two bounded Reinhardt domains* D *and* D' *in* $\boldsymbol{C}^n$, *then* φ *can be written in the form*

$$\varphi = \varphi'\varphi'' ,$$

where $\varphi' \in G(D')$ *and* $\varphi'' \in \mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$.

PROOF. By Proposition 1, $T(D)$ is a maximal torus in $G(D)$. Since $\varphi G(D) \varphi^{-1} = G(D')$, we see that $\varphi T(D) \varphi^{-1}$ is a maximal torus in $G(D')$, while, again by Proposition 1, $T(D')$ is a maximal torus in $G(D')$. Therefore the conjugacy theorem shows that there exists an element $\psi \in G(D')$ such that $\psi(\varphi T(D) \varphi^{-1})\psi^{-1} = (\psi\varphi) T(D) (\psi\varphi)^{-1}$ coincides with $T(D')$. It follows from the corollary to Proposition 1 of the preceding section that $\psi\varphi \in \mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$. Putting $\varphi' = \psi^{-1}$ and $\varphi'' = \psi\varphi$, we obtain the desired result. q.e.d.

COROLLARY. *If* D *is a bounded Reinhardt domain in* $\boldsymbol{C}^n$, *then* $\mathrm{Aut}(D)$ *coincides with* $G(D) \cdot \mathrm{Aut}_{\mathrm{alg}}(D)$.

PROOF. Apply the above proposition to the case of $D = D'$. q.e.d.

REMARK. All the results of this section remain true for Reinhardt domains D such that $\mathrm{Aut}(D)$ has the structure of a Lie group with respect to the compact-open topology.

We conclude this section by discussing the equivalence of annuli as an illustration of Theorem 1.

EXAMPLE. For a real number r with $0 < r < 1$, let $A(r)$ be the annulus in $\boldsymbol{C}$ defined by

$$A(r) = \{z \in \boldsymbol{C} \mid r < |z| < r^{-1}\} .$$

Consider two annuli $A(r)$ and $A(r')$. Theorem 1 asserts that $A(r)$ and $A(r')$ are holomorphically equivalent if and only if there exists an element of $\mathrm{Aut}_{\mathrm{alg}}(\boldsymbol{C}^*)$ which maps $A(r)$ onto $A(r')$. Since the group $\mathrm{Aut}_{\mathrm{alg}}(\boldsymbol{C}^*)$ is given by

$$\mathrm{Aut}_{\mathrm{alg}}(\boldsymbol{C}^*) = \{\boldsymbol{C}^* \ni z \to \alpha z^a \in \boldsymbol{C}^* \mid \alpha \in \boldsymbol{C}^*,\ a = \pm 1\} , \tag{11}$$

the latter condition is equivalent to $r = r'$. Thus we obtain a classically known result on the equivalence of annuli.

5. Holomorphic vector fields on a Reinhardt domain. Let D be a Reinhardt domain in $\boldsymbol{C}^n$. Then $T(D)$ acts as a Lie transformation group on D. The subalgebra of $\mathfrak{X}(D)$ corresponding to $T(D)$ is denoted by $\mathfrak{t}(D)$. Since the group $T(D)$ is abelian, the subalgebra $\mathfrak{t}(D)$ is abelian. For $i = 1, \cdots, n$, let H_i be the infinitesimal transformation of the one-parameter subgroup

$$\{D \ni (z_1, \cdots, z_n) \to (z_1, \cdots, z_{i-1}, \exp(\sqrt{-1}\theta)z_i, z_{i+1}, \cdots, z_n) \in D \mid \theta \in \boldsymbol{R}\}$$

of $\mathrm{Aut}(D)$. Then $\mathfrak{t}(D)$ is given by $\mathfrak{t}(D) = \{H_1, \cdots, H_n\}_{\boldsymbol{R}}$. Also, the subalgebras $\mathfrak{t}_1(D), \cdots, \mathfrak{t}_n(D)$ of $\mathfrak{t}(D)$ corresponding, respectively, to $T_1(D), \cdots, T_n(D)$ are given by $\mathfrak{t}_i(D) = \{H_i\}_{\boldsymbol{R}}$, $i = 1, \cdots, n$. Note that the holomorphic vector fields $H_1, \cdots, H_n$ have the form

$$H_i = \sqrt{-1}z_i(\partial/\partial z_i) , \qquad i = 1, \cdots, n . \tag{12}$$

Suppose that D is contained in $(\boldsymbol{C}^*)^n$ and consider a finite-dimensional real subalgebra $\mathfrak{g}$ of $\mathfrak{X}(D)$ containing $\mathfrak{t}(D)$. In this section, we prove some fundamental lemmas concerning the structure of $\mathfrak{g}$ which we need in the next section.

We begin with preliminary observations. Since D is contained in $(\boldsymbol{C}^*)^n$, it follows from (12) that the holomorphic vector fields $H_1, \cdots, H_n$ form a basis of $\mathfrak{X}(D)$, that is, every element X of $\mathfrak{X}(D)$ can be written in the form

$$X = \sum_{i=1}^{n} f_i H_i ,$$

where $f_1, \cdots, f_n$ are holomorphic functions on D. For $i = 1, \cdots, n$, let $f_i(z) = \sum_{\nu \in \boldsymbol{Z}^n} a_\nu^{(i)} z^\nu$ be the Laurent expansion of f_i, and write $X_\nu = z^\nu(\sum_{i=1}^n a_\nu^{(i)} H_i)$. Then X has the expression

$$X = \sum_{\nu \in \boldsymbol{Z}^n} X_\nu . \tag{13}$$

In what follows, every element of $\mathfrak{X}(D)$ will be expressed as above. Note that the sum in (13) converges absolutely and uniformly on any compact set in D, and that $X = 0$ precisely when $X_\nu = 0$ for all $\nu \in \boldsymbol{Z}^n$. If $H = \sum_{i=1}^n c_i H_i \in \mathfrak{t}(D)_C$ and $X = \sum_{\nu \in \boldsymbol{Z}^n} X_\nu \in \mathfrak{X}(D)$, then it follows that

$$(\mathrm{ad}\, H)X = [H, X] = \sum_{\nu \in \boldsymbol{Z}^n} (\sqrt{-1} \sum_{i=1}^n c_i \nu_i) X_\nu ,$$

where $\nu = (\nu_1, \cdots, \nu_n)$, and hence that, for a polynomial P with complex coefficients, we have

$$P(\mathrm{ad}\, H)X = \sum_{\nu \in \boldsymbol{Z}^n} P(\sqrt{-1} \sum_{i=1}^n c_i \nu_i) X_\nu .$$

LEMMA 1. *There exists a positive integer N such that if $X = \sum_{\nu \in \boldsymbol{Z}^n} X_\nu$ is any element of $\mathfrak{g}$, then $X_\nu = 0$ for all $\nu = (\nu_1, \cdots, \nu_n) \in \boldsymbol{Z}^n$ with $\max_{1 \leq i \leq n} |\nu_i| > N$.*

PROOF. For $i = 1, \cdots, n$, let P_i be the minimal polynomial of the endomorphism $\mathrm{ad}\, H_i$ of $\mathfrak{g}$, and take a positive integer N_i such that $P_i(\sqrt{-1}k) \neq 0$ for all $k \in \boldsymbol{Z}$ with $|k| > N_i$. Then a desired integer N is given by $N = \max_{1 \leq i \leq n} N_i$. Indeed, let $X = \sum_{\nu \in \boldsymbol{Z}^n} X_\nu$ be any element of $\mathfrak{g}$. Then

$$0 = P_i(\mathrm{ad}\, H_i)X = \sum_{\nu \in \boldsymbol{Z}^n} P_i(\sqrt{-1}\nu_i) X_\nu \quad \text{for every} \quad i = 1, \cdots, n ,$$

where ν_i is the i-th component of ν. Thus

$$P_i(\sqrt{-1}\nu_i) X_\nu = 0 \text{ for every } i = 1, \cdots, n \text{ and every } \nu = (\nu_1, \cdots, \nu_n) \in \boldsymbol{Z}^n .$$

Hence, if $\nu = (\nu_1, \cdots, \nu_n) \in \boldsymbol{Z}^n$, and if $\max_{1 \leq i \leq n} |\nu_i| > N$, so that $|\nu_i| > N_i$ for some i, then, since $P_i(\sqrt{-1}\nu_i) \neq 0$, we have $X_\nu = 0$, as asserted. q.e.d.

LEMMA 2. *If $X = \sum_{\nu \in \boldsymbol{Z}^n} X_\nu$ is any element of $\mathfrak{g}$, then $X_\nu + X_{-\nu} \in \mathfrak{g}$ for every $\nu \in \boldsymbol{Z}^n$.*

PROOF. Let N be a positive integer as in Lemma 1. For $i = 0, 1, \cdots, N$, we define a polynomial $P_i(x)$ by

$$P_0(x) = \prod_{\lambda=1}^{N} (x^2 + \lambda^2) ,$$

$$P_i(x) = \prod_{\lambda=0}^{i-1} (x^2 + \lambda^2) \prod_{\lambda=i+1}^{N} (x^2 + \lambda^2) , \quad i \neq 0 .$$

Consider first the case where $\nu=0$. From Lemma 1 and the relations $P_0(\pm\sqrt{-1}\lambda)=0$, $\lambda=1,\cdots,N$, it follows that

$$P_0(\operatorname{ad} H_1)\cdots P_0(\operatorname{ad} H_n)X=\sum_{\mu\in Z^n}P_0(\sqrt{-1}\mu_1)\cdots P_0(\sqrt{-1}\mu_n)X_\mu=P_0(0)^nX_0\ ,$$

where $\mu=(\mu_1,\cdots,\mu_n)$. Since $P_0(x)$ has real coefficients and $P_0(0)\neq 0\in \boldsymbol{R}$, we see that

$$X_0=P_0(0)^{-n}P_0(\operatorname{ad} H_1)\cdots P_0(\operatorname{ad} H_n)X\in \mathfrak{g}\ .$$

Consider next the case where $\nu=(\nu_1,\cdots,\nu_n)\neq 0$. If $\max_{1\leqq i\leqq n}|\nu_i|>N$, then, by Lemma 1, $X_\nu+X_{-\nu}=0\in\mathfrak{g}$. Suppose that $\max_{1\leqq i\leqq n}|\nu_i|\leqq N$. Without loss of generality, we may assume that $\nu_1\neq 0$. For $k=1,\cdots,n$, write

$$X^{(k)}=\sum_{\mu\in Z^{n-k}}X_{(\nu^{(k)},\mu)}+\sum_{\mu\in Z^{n-k}}X_{(-\nu^{(k)},\mu)}\ ,$$

where $\nu^{(k)}=(\nu_1,\cdots,\nu_k)$. We show by induction on k that $X^{(k)}\in\mathfrak{g}$ for every $k=1,\cdots,n$. For this, it is enough to prove the following two assertions:

(i) $X^{(1)}\in\mathfrak{g}$;

(ii) If $X^{(k)}\in\mathfrak{g}$, then $X^{(k+1)}\in\mathfrak{g}$.

First we prove (i). From Lemma 1 and the relations $P_{|\nu_1|}(\sqrt{-1}\nu_1)=P_{|\nu_1|}(-\sqrt{-1}\nu_1)$ and $P_{|\nu_1|}(\pm\sqrt{-1}\lambda)=0$, $\lambda=0,\cdots,|\nu_1|-1, |\nu_1|+1,\cdots,N$, it follows that

$$\begin{aligned}P_{|\nu_1|}(\operatorname{ad} H_1)X&=\sum_{\mu\in Z^n}P_{|\nu_1|}(\sqrt{-1}\mu_1)X_\mu\\&=\sum_{\mu\in Z^{n-1}}P_{|\nu_1|}(\sqrt{-1}\nu_1)X_{(\nu_1,\mu)}+\sum_{\mu\in Z^{n-1}}P_{|\nu_1|}(-\sqrt{-1}\nu_1)X_{(-\nu_1,\mu)}\\&=P_{|\nu_1|}(\sqrt{-1}\nu_1)\Big(\sum_{\mu\in Z^{n-1}}X_{(\nu_1,\mu)}+\sum_{\mu\in Z^{n-1}}X_{(-\nu_1,\mu)}\Big)=P_{|\nu_1|}(\sqrt{-1}\nu_1)X^{(1)}\ .\end{aligned}$$

Since $P_{|\nu_1|}(x)$ has real coefficients and $P_{|\nu_1|}(\sqrt{-1}\nu_1)\neq 0\in\boldsymbol{R}$, we see that

$$X^{(1)}=P_{|\nu_1|}(\sqrt{-1}\nu_1)^{-1}P_{|\nu_1|}(\operatorname{ad} H_1)X\in\mathfrak{g}\ ,$$

and the proof of (i) is complete.

Next we prove (ii). If $\nu_{k+1}=0$, then, by an argument similar to the case where $\nu=0$ and the induction hypothesis,

$$X^{(k+1)}=P_0(0)^{-1}P_0(\operatorname{ad} H_{k+1})X^{(k)}\in\mathfrak{g}\ .$$

Suppose that $\nu_{k+1}\neq 0$. Arguing as in the proof of (i), we see that

$$\begin{aligned}P_{|\nu_{k+1}|}(\operatorname{ad} H_{k+1})X^{(k)}=P_{|\nu_{k+1}|}(\sqrt{-1}\nu_{k+1})\Big(&\sum_{\mu\in Z^{n-k-1}}X_{(\nu^{(k)},\nu_{k+1},\mu)}\\&+\sum_{\mu\in Z^{n-k-1}}X_{(\nu^{(k)},-\nu_{k+1},\mu)}+\sum_{\mu\in Z^{n-k-1}}X_{(-\nu^{(k)},\nu_{k+1},\mu)}+\sum_{\mu\in Z^{n-k-1}}X_{(-\nu^{(k)},-\nu_{k+1},\mu)}\Big)\ ,\end{aligned}$$

and hence, by the induction hypothesis, that if we write as Y the right hand side of the above equality multiplied by $P_{|\nu_{k+1}|}(\sqrt{-1}\nu_{k+1})^{-1}$, then

$$Y = P_{|\nu_{k+1}|}(\sqrt{-1}\nu_{k+1})^{-1}P_{|\nu_{k+1}|}(\mathrm{ad}\, H_{k+1})X^{(k)} \in \mathfrak{g} .$$

Let H be the element of $\mathfrak{t}(D)$ defined by

$$H = \sum_{i=1}^{k} \nu_i H_i + H_{k+1}$$

and let Q be the polynomial defined by

$$Q(x) = x^2 + \left(\sum_{i=1}^{k} \nu_i^2 - \nu_{k+1}\right)^2 .$$

From Lemma 1 and the relations

$$Q\left(\sqrt{-1}\left(\sum_{i=1}^{k} \nu_i^2 + \nu_{k+1}\right)\right) = Q\left(\sqrt{-1}\left(-\sum_{i=1}^{k} \nu_i^2 - \nu_{k+1}\right)\right)$$

and

$$Q\left(\sqrt{-1}\left(\sum_{i=1}^{k} \nu_i^2 - \nu_{k+1}\right)\right) = Q\left(\sqrt{-1}\left(-\sum_{i=1}^{k} \nu_i^2 + \nu_{k+1}\right)\right) = 0 ,$$

it follows that

$$\begin{aligned} Q(\mathrm{ad}\, H)Y = & \sum_{\mu \in Z^{n-k-1}} Q\left(\sqrt{-1}\left(\sum_{i=1}^{k} \nu_i^2 + \nu_{k+1}\right)\right)X_{(\nu^{(k)},\nu_{k+1},\mu)} \\ & + \sum_{\mu \in Z^{n-k-1}} Q\left(\sqrt{-1}\left(\sum_{i=1}^{k} \nu_i^2 - \nu_{k+1}\right)\right)X_{(\nu^{(k)},-\nu_{k+1},\mu)} \\ & + \sum_{\mu \in Z^{n-k-1}} Q\left(\sqrt{-1}\left(-\sum_{i=1}^{k} \nu_i^2 + \nu_{k+1}\right)\right)X_{(-\nu^{(k)},\nu_{k+1},\mu)} \\ & + \sum_{\mu \in Z^{n-k-1}} Q\left(\sqrt{-1}\left(-\sum_{i=1}^{k} \nu_i^2 - \nu_{k+1}\right)\right)X_{(-\nu^{(k)},-\nu_{k+1},\mu)} \\ = & \, Q\left(\sqrt{-1}\left(\sum_{i=1}^{k} \nu_i^2 + \nu_{k+1}\right)\right)\left(\sum_{\mu \in Z^{n-k-1}} X_{(\nu^{(k+1)},\mu)} + \sum_{\mu \in Z^{n-k-1}} X_{(-\nu^{(k+1)},\mu)}\right) . \end{aligned}$$

Since $Q(x)$ has real coefficients and since, by the assumptions $\nu_1 \neq 0$ and $\nu_{k+1} \neq 0$,

$$Q\left(\sqrt{-1}\left(\sum_{i=1}^{k} \nu_i^2 + \nu_{k+1}\right)\right) = -4\left(\sum_{i=1}^{k} \nu_i^2\right)\nu_{k+1} \neq 0 \in \boldsymbol{R} ,$$

we see that

$$X^{(k+1)} = Q\left(\sqrt{-1}\left(\sum_{i=1}^{k} \nu_i^2 + \nu_{k+1}\right)\right)^{-1} Q(\mathrm{ad}\, H)Y \in \mathfrak{g} ,$$

and the proof of (ii) is complete.

We have thus shown that $X^{(k)} \in \mathfrak{g}$ for every $k = 1, \cdots, n$. In particular, $X_\nu + X_{-\nu} = X^{(n)}$ belongs to $\mathfrak{g}$. q.e.d.

Note that $\{X \in \mathfrak{g} \mid X = X_0\}$, $\{X \in \mathfrak{g} \mid X = X_\nu + X_{-\nu}\}$ ($\nu \neq 0 \in \boldsymbol{Z}^n$) are vector subspaces of $\mathfrak{g}$. Lemma 2 asserts that $\mathfrak{g}$ is the direct sum of these subspaces.

The subalgebra $\mathfrak{t}(D)$ is contained in $\{X \in \mathfrak{g} \mid X = X_0\}$. Under an additional assumption on $\mathfrak{g}$, they coincide:

LEMMA 3. *If* $\mathfrak{g} \cap \sqrt{-1}\mathfrak{g} = \{0\}$, *then* $\mathfrak{t}(D)$ *coincides with* $\{X \in \mathfrak{g} \mid X = X_0\}$.

PROOF. If X is any element of $\mathfrak{g}$ such that $X = X_0$, then X can be written in the form $X = Y + \sqrt{-1}Z$ with $Y \in \mathfrak{t}(D)$ and $Z \in \mathfrak{t}(D)$. By assumption, we get

$$Z = \sqrt{-1}(Y - X) \in \mathfrak{g} \cap \sqrt{-1}\mathfrak{g} = \{0\},$$

hence $X = Y \in \mathfrak{t}(D)$. q.e.d.

6. Automorphisms of n-dimensional Reinhardt domains with $t = n$. In the study of automorphisms of bounded Reinhardt domains in $\boldsymbol{C}^n$ with $t = n$, or bounded Reinhardt domains in $(\boldsymbol{C}^*)^n$, it is more natural to include the domains which are algebraically equivalent to such domains in our consideration. In view of the observations in Section 2 concerning the relations between Reinhardt domains and their logarithmic images, we shall deal with Reinhardt domains in $(\boldsymbol{C}^*)^n$ whose logarithmic images have the convex hulls containing no complete straight lines.

The purpose of this section is to prove the following:

THEOREM 2. *If D is a Reinhardt domain in $(\boldsymbol{C}^*)^n$ whose logarithmic image has the convex hull containing no complete straight lines, then $G(D)$ coincides with $T(D)$.*

Combining the above theorem with the corollary to Proposition 2 of Section 4, we obtain the following consequence.

COROLLARY. *If D is a Reinhardt domain in $(\boldsymbol{C}^*)^n$ whose logarithmic image has the convex hull containing no complete straight lines, then* $\mathrm{Aut}(D)$ *coincides with* $\mathrm{Aut}_{\mathrm{alg}}(D)$.

Before starting the proof of the theorem, we discuss an illustrative example.

EXAMPLE. Let $A(r)$ be the annulus defined in the example in Section 4. Then Theorem 2 and (11) show that

$$\mathrm{Aut}(A(r)) = \mathrm{Aut}_{\mathrm{alg}}(A(r)) = \{A(r) \ni z \to \alpha z^a \in A(r) \mid \alpha \in U(1),\ a = \pm 1\}.$$

This is a classically known result on automorphisms of annuli.

We turn to the proof of Theorem 2. When $n = 1$, the assertion of the theorem is well-known. Hence we assume that $n \geqq 2$. It is enough to prove that $\mathfrak{g}(D)$ coincides with $\mathfrak{t}(D)$. Suppose on the contrary that $\mathfrak{g}(D)$ does not coincide with $\mathfrak{t}(D)$. Since $\mathfrak{g}(D)$ is a finite-dimensional real subalgebra of $\mathfrak{X}(D)$ containing $\mathfrak{t}(D)$, and since $\mathfrak{g}(D) \cap \sqrt{-1}\mathfrak{g}(D) = \{0\}$, the lemmas of the preceding section apply to $\mathfrak{g}(D)$. By Lemmas 2 and 3, there exists a non-zero element ν of $\boldsymbol{Z}^n$ such that

$$\mathfrak{p} := \{X \in \mathfrak{g}(D) \mid X = X_\nu + X_{-\nu}\} \neq \{0\} .$$

Let φ be an element of $\mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$ of the form $(*)$ for which the i-th component of $\nu^t A^{-1} \in \boldsymbol{Z}^n$ is positive for $i=1$ and equal to 0 for $i=2, \cdots, n$, where ${}^tA$ denotes the transpose of A. In view of (2) and the observations in Section 2, by a change of coordinates $\varphi: (z_1, \cdots, z_n) \to (w_1, \cdots, w_n)$, we may assume that $\nu = (\nu_1, 0, \cdots, 0)$ and $\nu_1 > 0$. Moreover, by Lemma 1, we can take ν to be maximal in the sense that every element $Y = Y_\mu + Y_{-\mu} \in \mathfrak{g}(D)$ with $\mu = (\mu_1, 0, \cdots, 0)$ and $\mu_1 > \nu_1$ is equal to 0. Write

$$\mathfrak{g} = \mathfrak{t}(D) + \mathfrak{p} .$$

Then $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}(D)$. In fact, the following relations hold:

(a) $[\mathfrak{t}(D), \mathfrak{t}(D)] \subset \mathfrak{t}(D)$;

(b) $[\mathfrak{t}(D), \mathfrak{p}] \subset \mathfrak{p}$;

(c) $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t}(D)$.

(a) is obvious. To show (b), it is enough to prove that $(\mathrm{ad}\, H_i)\mathfrak{p} \subset \mathfrak{p}$ for every $i = 1, \cdots, n$. If $X = X_\nu + X_{-\nu}$ is any element of $\mathfrak{p}$, then

$$(14) \qquad (\mathrm{ad}\, H_i)X = \sqrt{-1}\nu_i(X_\nu - X_{-\nu}) \quad \text{for every} \quad i = 1, \cdots, n ,$$

where ν_i is the i-th component of ν. By the definition of $\mathfrak{p}$, the right hand side of (14) belongs to $\mathfrak{p}$, which proves our assertion. It remains to show (c). If X and Y are any elements of $\mathfrak{p}$, and if we write $Z = [X, Y]$, then a straightforward computation yields that Z has the form $Z = Z_{2\nu} + Z_0 + Z_{-2\nu}$. Since, by Lemma 2, $Z_{2\nu} + Z_{-2\nu} \in \mathfrak{g}(D)$, it follows from the maximality of ν that $Z_{2\nu} + Z_{-2\nu} = 0$, so that $Z = Z_0$. Therefore, by Lemma 3, $Z \in \mathfrak{t}(D)$, as desired. Note that $\{H_2, \cdots, H_n\}_{\boldsymbol{R}}$ is contained in the center of $\mathfrak{g}$. Indeed, because of (14) and the fact that the subalgebra $\mathfrak{t}(D)$ is abelian, we have $(\mathrm{ad}\, H_i)\mathfrak{g} = (\mathrm{ad}\, H_i)\mathfrak{t}(D) + (\mathrm{ad}\, H_i)\mathfrak{p} = \{0\}$ for every $i = 2, \cdots, n$.

Consider a connected Lie subgroup G of $G(D)$ corresponding to $\mathfrak{g}$. Since the center of G contains the subgroup $T_2(D) \cdots T_n(D)$ of $T(D)$, and since, by assumption, the logarithmic image of D has the convex hull containing no complete straight lines, it follows from Lemma 2 of Section 2 that if Δ is the Reinhardt domain in $\boldsymbol{C}^*$ given as the image of the domain D under the projection $\boldsymbol{C}^n \ni (z_1, \cdots, z_n) \to z_1 \in \boldsymbol{C}$, then there exists a group homomorphism $\tau: G \to \mathrm{Aut}(\Delta)$ between the groups G and $\mathrm{Aut}(\Delta)$ with $\ker \tau = T_2(D) \cdots T_n(D)$. The group $\mathrm{Aut}(\Delta)$ has the structure of a Lie group with respect to the compact-open topology, and it is readily verified that $\tau: G \to \mathrm{Aut}(\Delta)$ is a Lie group homomorphism. We observe that

$$\dim \mathrm{Aut}(\Delta) \leqq 2 \tag{15}$$

and

$$\dim \ker \tau = \dim T_2(D) \cdots T_n(D) = n - 1 . \tag{16}$$

If we write $J = \mathrm{ad}\,\nu_1^{-1} H_1$, then (14) implies that $J\mathfrak{p} \subset \mathfrak{p}$ and $J^2 = -\mathrm{id}$ on $\mathfrak{p}$. Consequently, the dimension of $\mathfrak{p}$ is greater than or equal to 2. Since the sum $\mathfrak{g} = \mathfrak{t}(D) + \mathfrak{p}$ is direct, we see that

$$\dim G = \dim \mathfrak{g} = \dim \mathfrak{t}(D) + \dim \mathfrak{p} \geqq n + 2 . \tag{17}$$

On the other hand, since

$$\dim G - \dim \ker \tau = \dim \tau(G) \leqq \dim \mathrm{Aut}(\Delta) ,$$

it follows from (15) and (16) that

$$\dim G \leqq \dim \ker \tau + \dim \mathrm{Aut}(\Delta) \leqq n - 1 + 2 = n + 1 .$$

This contradicts (17). We thus conclude that $G(D) = T(D)$.

7. Automorphisms of two-dimensional Reinhardt domains with $t = 1$. The remainder or this paper is devoted to the determination of automorphisms of bounded Reinhardt domains in $\boldsymbol{C}^2$ with $t = 1$.

Let D be a bounded Reinhardt domain in $\boldsymbol{C}^2$ with $t = 1$. Without loss of generality, we may assume that $D \cap \{z_1 = 0\} \neq \varnothing$. For simplicity, we shall write $G(D) = G$, $\mathfrak{g}(D) = \mathfrak{g}$, etc. We begin by noting that D is inhomogeneous. Indeed, otherwise, D is holomorphically equivalent to the ball $\{(z_1, z_2) \in \boldsymbol{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$ or the polydisk $\{(z_1, z_2) \in \boldsymbol{C}^2 \mid |z_1| < 1, |z_2| < 1\}$ (see E. Cartan [3]), so that D is homeomorphic to a cell. But it is readily verified that $H_1(D, \boldsymbol{R}) \neq 0$ when $D \cap \{z_2 = 0\} = \varnothing$, while $H_3(D, \boldsymbol{R}) \neq 0$ when $D \cap \{z_2 = 0\} \neq \varnothing$. Hence D is inhomogeneous. Fix any point $o = (0, b)$ $(b \neq 0)$ of $D \cap \{z_1 = 0\}$ and let M be the G-orbit of o.

LEMMA 1. *The dimension of M is equal to either* 1 or 3.

PROOF. The fact that M contains the one-dimensional torus Π_o implies that $\dim M \geqq 1$. Since D is inhomogeneous, it follows (see Section 1) that each G-orbit has dimension less than or equal to 3, so that $\dim M \leqq 3$. To complete the proof of the lemma, it suffices to prove that $\dim M \neq 2$. Suppose on the contrary that $\dim M = 2$. Note first that, for each point

z of D^*, the dimension of the G-orbit $G\cdot z$ of z is greater than or equal to 2, and the equality holds precisely when $G\cdot z$ coincides with Π_z. Now, if $M\cap D^* \neq \emptyset$, say $z\in M\cap D^*$, then, by what we have noted above, $M = G\cdot z = \Pi_z \subset D^*$. Since this contradicts the assumption that M contains the point $o\in D - D^*$, it follows that $M\subset D - D^*$, and therefore that M is a connected open subset of $D\cap\{z_1 = 0\}$. As a consequence, M is a complex submanifold of D. The fact that M is the union of T-orbits implies that

$$M = \{(0, z_2)\in \boldsymbol{C}^2 \mid r < |z_2| < R\}$$

for some constants r and R with $0 \leqq r < R < \infty$, so that M is holomorphically equivalent to an annulus or a punctured disk. Since every element of G induces an automorphism of the complex manifold M, this contradicts the fact that G acts transitively on M. q.e.d.

Let K be the isotropy subgroup of G at the point o and let $\rho\colon K\to GL(T_oD)$ be the linear isotropy representation. Then K contains the one-dimensional torus T_1, so that $\dim K \geqq 1$, and M can be expressed as the homogeneous space $M = G/K$.

LEMMA 2. *The dimension of K is equal to* 1, *and consequently the identity component of K coincides with T_1.*

PROOF. Since ρ is faithful, it suffices to prove that $\dim\rho(K) = 1$. Since K is compact and has dimension greater than or equal to 1, the identity component $\rho(K)^\circ$ of $\rho(K)$ is a connected compact subgroup of $GL(T_oD)\simeq GL(2, \boldsymbol{C})$ and has dimension greater than or equal to 1. Therefore $\rho(K)^\circ$ is isomorphic to $U(1)$, $(U(1))^2$, $SU(2)$ or $U(2)$. Note that $\rho(K)$, hence $\rho(K)^\circ$, leaves the subspace T_oM of T_oD stable, and that, by the above lemma, the dimension of T_oM is equal to either 1 or 3. If $\rho(K)^\circ$ is isomorphic to $SU(2)$ or $U(2)$, then it acts irreducibly on T_oD. This contradicts what we have noted above. Suppose that $\rho(K)^\circ$ is isomorphic to $(U(1))^2$. Then $\rho(K)^\circ$ contains the canonical complex structure of T_oD. Indeed, when $GL(T_oD)$ is viewed as $GL(2, \boldsymbol{C})$, we see that $\rho(K)^\circ$ is conjugate to the subgroup

$$\left\{\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\in GL(2, \boldsymbol{C})\,\middle|\, \alpha_1, \alpha_2\in U(1)\right\}$$

which contains the complex structure. Consequently every proper $\rho(K)^\circ$-stable subspace of T_oD is of dimension 2. Again, this contradicts what we have noted above. We thus conclude that $\rho(K)^\circ$ is isomorphic to $U(1)$, so that $\dim\rho(K) = \dim\rho(K)^\circ = \dim U(1) = 1$. q.e.d.

We now look into the group G according to the dimension of M.

PROPOSITION 1. *If* $\dim M = 1$, *then* G *coincides with* T.

PROOF. It follows from Lemma 2 and the assumption $\dim M = 1$ that $\dim G = \dim M + \dim K = 2 = \dim T$. Since G is connected, we see that G coincides with T. q.e.d.

PROPOSITION 2. *If* $\dim M = 3$, *then the dimension of* G *is equal to* 4. *Moreover, the center* $\mathfrak{z}$ *of* $\mathfrak{g}$ *is a one-dimensional subalgebra of* $\mathfrak{t}$ *which does not coincide with* $\mathfrak{t}_1$.

PROOF. By the same argument as in the proof of the above proposition, we have $\dim G = 4$, which proves the first assertion.

To prove the second assertion, we begin with preliminary observations. Since the subspace T_oM of T_oD is $\rho(T_1)$-stable and of dimension 3, it follows that T_oM has the form

$$T_oM = T_oP + T_o\Pi_o \quad \text{(direct sum)}$$

for the complex submanifold $P = \{(z_1, z_2) \in D \mid z_2 = b\}$ of D and the one-dimensional torus $\Pi_o = \{(0, \alpha b) \in D \mid \alpha \in U(1)\}$ in D. Let $q: G \to M$ be the orbit map defined by $q(g) = g \cdot o$. Note that, by Lemma 2, the kernel of the differential $(dq)_e: \mathfrak{g} \to T_oM$ of q at the unit element e of G coincides with $\mathfrak{t}_1$. If we write $\mathfrak{g}' = (dq)_e^{-1}(T_oP)$, then $\mathfrak{g}'$ is a three-dimensional subspace of $\mathfrak{g}$ containing $\mathfrak{t}_1$. Moreover, $\mathfrak{g}'$ is $\mathrm{Ad}(T_1)$-stable, where Ad denotes the adjoint representation of G. Indeed, since $(dq)_e\mathrm{Ad}(\alpha) = \rho(\alpha)(dq)_e$ for all $\alpha \in T_1 \subset K$, the fact that T_oP is $\rho(T_1)$-stable implies that $\mathfrak{g}'$ is $\mathrm{Ad}(T_1)$-stable. Since T_1 is compact, so that $\mathrm{Ad}(T_1)$ is compact, we can find an $\mathrm{Ad}(T_1)$-stable subspace $\mathfrak{p}$ of $\mathfrak{g}'$ complementary to $\mathfrak{t}_1$. It follows that $[\mathfrak{t}_1, \mathfrak{p}] \subset \mathfrak{p}$. Also, $\mathrm{Ad}(T_1)$ acts as $SO(2)$ on the two-dimensional subspace $\mathfrak{p}$, and consequently there exists an element H_1 of $\mathfrak{t}_1$ such that, on the subspace $\mathfrak{p}$, the endomorphism $\mathrm{ad}\, H_1$ belongs to $\mathrm{Ad}(T_1)$ and

$$(\mathrm{ad}\, H_1)^2 = -\mathrm{id}\,. \tag{18}$$

By the definition of the subspace $\mathfrak{p}$, we have the following direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} + \mathfrak{p}\,. \tag{19}$$

Turning to the proof of the second assertion, let Z be any element of $\mathfrak{z}$ and write $Z = X + Y$, where $X \in \mathfrak{t}$ and $Y \in \mathfrak{p}$. It follows that $0 = (\mathrm{ad}\, H_1)Z = (\mathrm{ad}\, H_1)X + (\mathrm{ad}\, H_1)Y = (\mathrm{ad}\, H_1)Y$. This and (18) imply that $Y = 0$, so that $Z = X \in \mathfrak{t}$. Thus $\mathfrak{z}$ is contained in $\mathfrak{t}$. Further, $\mathfrak{z}$ is at most one-dimensional and does not coincide with $\mathfrak{t}_1$, since, by the existence of the element H_1, we have $\mathfrak{z} \cap \mathfrak{t}_1 = \{0\}$. In view of (19) and these facts, to prove the second assertion, it suffices to prove that there exists a non-zero element H of $\mathfrak{t}$ such that $[H, \mathfrak{p}] = \{0\}$. If $[\mathfrak{t}_2, \mathfrak{p}] = \{0\}$, then the above assertion is obvious. Hence we assume that $[\mathfrak{t}_2, \mathfrak{p}] \neq \{0\}$. We prove our assertion in several steps.

We show that $[\mathfrak{t}_2, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t}$. Since $\mathrm{Ad}(T_1)$ acts trivially on $\mathfrak{t}$, while it acts as $SO(2)$ on $\mathfrak{p}$, there exists an element s of $\mathrm{Ad}(T_1)$ such that

$$s(X+Y) = X - Y \quad \text{for} \quad X \in \mathfrak{t} \text{ and } Y \in \mathfrak{p} .$$

Clearly s is a Lie algebra involution of $\mathfrak{g}$ and the eigenspaces for the eigenvalues 1 and -1 of s are given by $\mathfrak{t}$ and $\mathfrak{p}$, respectively. If $X \in \mathfrak{t}_2$ and $Y \in \mathfrak{p}$, then

$$s([X, Y]) = [s(X), s(Y)] = [X, -Y] = -[X, Y] ,$$

which proves $[\mathfrak{t}_2, \mathfrak{p}] \subset \mathfrak{p}$. If $X \in \mathfrak{p}$ and $Y \in \mathfrak{p}$, then

$$s([X, Y]) = [s(X), s(Y)] = [-X, -Y] = [X, Y] ,$$

which proves $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t}$. Note that the triple $(\mathfrak{g}, \mathfrak{t}, s)$ is a symmetric Lie algebra (cf. Kobayashi and Nomizu [5, II, Chapter XI]).

The relation $[\mathfrak{t}_2, \mathfrak{p}] \subset \mathfrak{p}$ implies that if $X \in \mathfrak{t}_2$, then $\mathrm{ad}\, X$ induces an endomorphism of $\mathfrak{p}$. We show that there exists an element H_2 of $\mathfrak{t}_2$ such that, on the subspace $\mathfrak{p}$, the endomorphism $\mathrm{ad}\, H_2$ coincides with an endomorphism $\mathrm{Ad}(\alpha_0)$ for some element α_0 of T_1. Take a non-zero element X of $\mathfrak{t}_2$. If α is any element of T_1, then we have

$$\mathrm{Ad}(\alpha)[X, Y] = [\mathrm{Ad}(\alpha)X, \mathrm{Ad}(\alpha)Y] = [X, \mathrm{Ad}(\alpha)Y] \quad \text{for all} \quad Y \in \mathfrak{p} .$$

This means that the restriction of $\mathrm{ad}\, X$ to $\mathfrak{p}$ commutes with the action of $\mathrm{Ad}(T_1)$ on $\mathfrak{p}$. Since $\mathrm{Ad}(T_1)$ acts as $SO(2)$ on $\mathfrak{p}$, it follows that, on the subspace $\mathfrak{p}$, the endomorphism $\mathrm{ad}\, X$ coincides with an endomorphism $c \cdot \mathrm{Ad}(\alpha_0)$ for an element α_0 of T_1 and a non-negative constant c. By the assumption $[\mathfrak{t}_2, \mathfrak{p}] \neq \{0\}$, the constant c is not equal to 0. Putting $H_2 = c^{-1}X$, we obtain a desired element.

We show that $[\mathfrak{p}, \mathfrak{p}] \neq \{0\}$. Suppose $[\mathfrak{p}, \mathfrak{p}] = \{0\}$. Let J denote the canonical complex structure of D and J_z the value of J at a point $z \in D$. If u is a non-zero element of $T_o P$, then $T_o P = \{u, J_o u\}_{\boldsymbol{R}}$. Take elements X and Y of $\mathfrak{p}$ for which $(dq)_e(X) = u$ and $(dq)_e(Y) = J_o u$, and define a mapping $f: \boldsymbol{C} \to D$ by

$$f(\xi + \sqrt{-1}\eta) = q(\exp \xi X \exp \eta Y) \quad \text{for} \quad \xi \in \boldsymbol{R} \text{ and } \eta \in \boldsymbol{R} ,$$

where exp denotes the exponential mapping of $\mathfrak{g}$ into G. Then f is holomorphic on $\boldsymbol{C}$. To prove this, let I denote the canonical complex structure of $\boldsymbol{C}$ and I_ζ the value of I at a point $\zeta \in \boldsymbol{C}$. Note first that

$$(20) \qquad (df)_0 I_0 = J_o(df)_0 .$$

This is a consequence of the relations $(df)_0((\partial/\partial\xi)_0) = u$, $(df)_0((\partial/\partial\eta)_0) = J_o u$ and $I_0((\partial/\partial\xi)_0) = (\partial/\partial\eta)_0$, where $(\partial/\partial\xi)_0$ and $(\partial/\partial\eta)_0$ are the values of $\partial/\partial\xi$ and $\partial/\partial\eta$ at 0. Now, let $\zeta_0 = \xi_0 + \sqrt{-1}\eta_0$ $(\xi_0, \eta_0 \in \boldsymbol{R})$ be any point of $\boldsymbol{C}$. Since the assumption $[\mathfrak{p}, \mathfrak{p}] = \{0\}$ implies that $[X, Y] = 0$, and hence that $\exp\xi X \exp\eta Y = \exp\eta Y \exp\xi X$ for all $\xi \in \boldsymbol{R}$ and all $\eta \in \boldsymbol{R}$, we see that if $\varphi \in \mathrm{Aut}(D)$ and $\psi \in \mathrm{Aut}(\boldsymbol{C})$ are defined to be $\varphi(z) = (\exp\xi_0 X \exp\eta_0 Y)\cdot z$ and $\psi(\zeta) = \zeta - \zeta_0$, then f can be written in the form $f = \varphi f \psi$, and therefore

$$(df)_{\zeta_0} = (d\varphi)_o(df)_0(d\psi)_{\zeta_0} .$$

It follows from this relation and (20) that

$$\begin{aligned}(df)_{\zeta_0} I_{\zeta_0} &= (d\varphi)_o(df)_0(d\psi)_{\zeta_0} I_{\zeta_0} = (d\varphi)_o(df)_0 I_0(d\psi)_{\zeta_0} \\ &= (d\varphi)_o J_o(df)_0(d\psi)_{\zeta_0} = J_{f(\zeta_0)}(d\varphi)_o(df)_0(d\psi)_{\zeta_0} = J_{f(\zeta_0)}(df)_{\zeta_0} ,\end{aligned}$$

so that f is holomorphic on $\boldsymbol{C}$. Since f is obviously non-constant, this contradicts the boundedness of D. We thus conclude that $[\mathfrak{p}, \mathfrak{p}] \neq \{0\}$.

We show that $(\mathrm{ad}\, H_2)^2 = -\mathrm{id}$ on $\mathfrak{p}$. Let X be any element of $\mathfrak{p}$. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t}$ and $\mathrm{ad}\, H_2 = \mathrm{Ad}(\alpha_0) \in \mathrm{Ad}(T_1)$ on $\mathfrak{p}$, it follows that

$$\begin{aligned}[X, Y] &= \mathrm{Ad}(\alpha_0)[X, Y] = [\mathrm{Ad}(\alpha_0)X, \mathrm{Ad}(\alpha_0)Y] = [[H_2, X], [H_2, Y]] \\ &= [[[H_2, X], H_2], Y] + [H_2, [[H_2, X], Y]] \\ &= -[[H_2, [H_2, X]], Y] = -[(\mathrm{ad}\, H_2)^2 X, Y]\end{aligned}$$

for every $Y \in \mathfrak{p}$, so that $[(\mathrm{ad}\, H_2)^2 X + X, Y] = 0$ for every $Y \in \mathfrak{p}$. Since $\mathfrak{p}$ is two-dimensional and $[\mathfrak{p}, \mathfrak{p}] \neq \{0\}$, we see that $(\mathrm{ad}\, H_2)^2 X + X = 0$, which proves our assertion.

We now prove that there exists a non-zero element H of $\mathfrak{t}$ such that $[H, \mathfrak{p}] = \{0\}$. We have seen that there exist an element H_1 of $\mathfrak{t}_1$ and an element H_2 of $\mathfrak{t}_2$ such that, on the subspace $\mathfrak{p}$, both endomorphisms $\mathrm{ad}\, H_1$ and $\mathrm{ad}\, H_2$ belong to $\mathrm{Ad}(T_1) \simeq SO(2)$ and $(\mathrm{ad}\, H_1)^2 = (\mathrm{ad}\, H_2)^2 = -\mathrm{id}$. It is readily verified that $\mathrm{ad}\, H_1 = \pm(\mathrm{ad}\, H_2)$ on $\mathfrak{p}$. Changing the sign of H_2 suitably, we may assume that $\mathrm{ad}\, H_1 = \mathrm{ad}\, H_2$ on $\mathfrak{p}$. Then we have $(\mathrm{ad}(H_1 - H_2))\mathfrak{p} = \{0\}$. Since $H_1 - H_2 \neq 0 \in \mathfrak{t}$, a desired element H is given by $H = H_1 - H_2$. q.e.d.

8. Automorphisms of two-dimensional Reinhardt domains with $t = 1$ (continued). To each bounded domain D, there is associated the constant $h = \min_{z \in D} \dim G(D)\cdot z$, the minimal dimension of the orbits $G(D)\cdot z$ for

$z \in D$. Note that if $\varphi: D \to D'$ is a biholomorphic mapping between two bounded domains D and D' in C^n, then, since $\varphi G(D)\varphi^{-1} = G(D')$, we have

$$\dim G(D')\cdot\varphi(z) = \dim \varphi(G(D)\cdot z) = \dim G(D)\cdot z \tag{21}$$

for every $z \in D$. As a consequence, the constant h is a biholomorphic invariant.

We begin with a result concerning the invariant h.

THEOREM 3. *If D is a bounded Reinhardt domain in C^2 with $t = 1$, then the invariant h is equal to either* 1 *or* 3.

PROOF. $t = 1$ implies that $\dim G(D)\cdot z \geqq \dim \Pi_z \geqq 1$ for every $z \in D$, so that $h \geqq 1$. Since D is inhomogeneous, it follows (see Section 1) that if we select any point z of D, then $\dim G(D)\cdot z \leqq 3$, so that $h \leqq 3$. To complete the proof of the theorem, it suffices to prove that $h \neq 2$. Suppose on the contrary that $h = 2$. Then there exists a point z_0 of D such that $\dim G(D)\cdot z_0 = 2$. We see that $z_0 \in D^*$, since, for each point z of $D - D^*$, the dimension of the $G(D)$-orbit of z is equal to either 1 or 3 by Lemma 1 of Section 7. It follows from the relation $\dim G(D)\cdot z_0 = 2 = \dim \Pi_{z_0}$ that

$$G(D)\cdot z_0 = \Pi_{z_0} . \tag{22}$$

We show that $\mathrm{Aut}(D) = \mathrm{Aut}_{\mathrm{alg}}(D)$. If φ is any element of $\mathrm{Aut}(D)$, then, by (21) applied to φ, we have $\dim G(D)\cdot\varphi(z_0) = \dim G(D)\cdot z_0 = 2$, and therefore, by the same argument as above, $G(D)\cdot\varphi(z_0) = \Pi_{\varphi(z_0)}$. Since $\varphi G(D)\varphi^{-1} = G(D)$, it follows from (22) and this relation that

$$\varphi(\Pi_{z_0}) = \varphi(G(D)\cdot z_0) = G(D)\cdot\varphi(z_0) = \Pi_{\varphi(z_0)} .$$

Since $z_0 \in D^*$, Proposition 1 of Section 3 shows that φ is induced by an algebraic automorphism of $(C^*)^2$, so that $\mathrm{Aut}(D) = \mathrm{Aut}_{\mathrm{alg}}(D)$.

Consequently, the identity component of $\mathrm{Aut}_{\mathrm{alg}}(D)$ coincides with $G(D)$. On the other hand, because of our observations in Section 2, the identity component of $\mathrm{Aut}_{\mathrm{alg}}(D)$ is given by $T(D)$. We thus conclude that $G(D) = T(D)$, so that $t = h = 2$, a contradiction. q.e.d.

THEOREM 4. *If D is a bounded Reinhardt domain in C^2 with $t = 1$ and $h = 1$, then $G(D)$ coincides with $T(D)$.*

PROOF. By the assumption $h = 1$, there exists a point o of D such that $\dim G(D)\cdot o = 1$. It is obvious that $o \in D - D^*$. Without loss of generality, we may assume that $o \in D \cap \{z_1 = 0\}$. $G(D)$ coincides with $T(D)$ by Proposition 1 of Section 7 applied to the orbit $M := G(D)\cdot o$. q.e.d.

Combining the above theorem with the corollary to Proposition 2 of Section 4, we obtain the following consequence.

COROLLARY. *If D is a bounded Reinhardt domain in $\boldsymbol{C}^2$ with $t=1$ and $h=1$, then* $\mathrm{Aut}(D)$ *coincides with* $\mathrm{Aut}_{\mathrm{alg}}(D)$.

Consider the case where $h=3$. Let D be a bounded Reinhardt domain in $\boldsymbol{C}^2$ with $t=1$ and $h=3$. As in the preceding section, we assume that $D\cap\{z_1=0\}\neq\varnothing$ and fix a point $o=(0, b)$ $(b\neq 0)$ of $D\cap\{z_1=0\}$. In what follows, we use the notation of the preceding section. Note first that $\dim G\cdot z=3$ for every $z\in D$. In particular, we have $\dim G\cdot o=3$. Now, applying Proposition 2 of Section 7 to the orbit $M:=G\cdot o$, we see that the center of $\mathfrak{g}$ is a one-dimensional subalgebra of $\mathfrak{t}$ which does not coincide with $\mathfrak{t}_1$. This implies that the identity component Z of the center of G is a one-dimensional subtorus of T which does not coincide with T_1, so that we may write

$$Z=\{(\alpha^k, \alpha^l)\in T\,|\,\alpha\in U(1)\}\ , \tag{23}$$

where k and l are relatively prime integers and l is positive.

Let φ be any element of G. In terms of the coordinates in $\boldsymbol{C}^2$, write $\varphi(z_1, z_2)=(\varphi_1(z_1, z_2), \varphi_2(z_1, z_2))$, where φ_1 and φ_2 are holomorphic functions on D. Since φ commutes with every element of Z, the functions φ_1 and φ_2 satisfy

$$\begin{aligned}\varphi_1(\alpha^k z_1, \alpha^l z_2)&=\alpha^k\varphi_1(z_1, z_2)\ ,\\ \varphi_2(\alpha^k z_1, \alpha^l z_2)&=\alpha^l\varphi_2(z_1, z_2)\ ,\end{aligned} \tag{24}$$

for every $\alpha\in U(1)$ and every $(z_1, z_2)\in D$. For $i=1, 2$, let

$$\varphi_i(z_1, z_2)=\sum_{(\nu_1,\nu_2)\in\boldsymbol{Z}^2} a^{(i)}_{(\nu_1,\nu_2)} z_1^{\nu_1} z_2^{\nu_2} \tag{25}$$

be the Laurent expansion of φ_i. Note that, since $D\cap\{z_1=0\}\neq\varnothing$, we have $a^{(1)}_{(\nu_1,\nu_2)}=a^{(2)}_{(\nu_1,\nu_2)}=0$ for every $(\nu_1, \nu_2)\in\boldsymbol{Z}^2$ with $\nu_1<0$. Substituting (25) into (24) and applying the uniqueness assertion of the theorem of Section 2, we get

$$\begin{aligned}a^{(1)}_{(\nu_1,\nu_2)}\alpha^{k\nu_1+l\nu_2}&=a^{(1)}_{(\nu_1,\nu_2)}\alpha^k\ ,\\ a^{(2)}_{(\nu_1,\nu_2)}\alpha^{k\nu_1+l\nu_2}&=a^{(2)}_{(\nu_1,\nu_2)}\alpha^l\ ,\end{aligned} \tag{26}$$

for every $\alpha\in U(1)$ and every $(\nu_1, \nu_2)\in\boldsymbol{Z}^2$.

Using the above observation, we prove the following lemma.

LEMMA 1. *The integer l is equal to* 1. *Moreover, if $D\cap\{z_2=0\}\neq\varnothing$, then k is also equal to* 1.

PROOF. Since $\dim G\cdot o=3$, the orbit $G\cdot o$ contains a point $a=(a_1, a_2)$

of D for which $a_1 \neq 0$. Take an element φ of G which maps the point o onto the point a and, in terms of the coordinates in C^2, write $\varphi(z_1, z_2) = (\varphi_1(z_1, z_2), \varphi_2(z_1, z_2))$, where φ_1 and φ_2 are holomorphic functions on D. By the observation above to φ, the coefficients of the Laurent expansions of φ_1 and φ_2 satisfy (26). Now the function $z_2 \to \varphi_1(0, z_2)$ is not identically 0, because $\varphi_1(0, b) = a_1 \neq 0$. Therefore there exists an integer ν_2 such that $a^{(1)}_{(0,\nu_2)} \neq 0$. It follows from (26) that $\alpha^{l\nu_2 - k} = 1$ for every $\alpha \in U(1)$, so that $l\nu_2 = k$. Since the integers k and l are relatively prime and l is positive, we see that $l = 1$. To prove the second assertion, suppose that $D \cap \{z_2 = 0\} \neq \emptyset$ and select a point o' of $D \cap \{z_2 = 0\}$. Since $\dim G \cdot o' = 3$, the orbit $G \cdot o'$ contains a point $a' = (a'_1, a'_2)$ of D for which $a'_2 \neq 0$. Taking an element φ of G which maps the point o' onto the point a' and repeating an argument similar to the above, we have $k\nu_1 = l = 1$ for some integer ν_1. Since the assumption $D \cap \{z_1 = 0\} \neq \emptyset$ implies that $\nu_1 \geqq 0$, it follows that $k = 1$. q.e.d.

We now look into the group G according as $D \cap \{z_2 = 0\}$ is empty or not.

Consider first the case where $D \cap \{z_2 = 0\} = \emptyset$. Let ψ be the algebraic automorphism of $(C^*)^2$ defined by

$$\psi(z_1, z_2) = (z_1 z_2^{-k}, z_2) ,$$

where k is the integer given in (23). Then ψ induces an automorphism of $C \times C^*$, and, by Lemma 1, we have $\psi Z \psi^{-1} = T_2$. Therefore, replacing D by $\psi(D)$, we may assume that $Z = T_2$. Note that $\psi(D)$ is not necessarily bounded. But, in view of the observations in Section 2, the logarithmic image of $\psi(D)$ has the convex hull containing no complete straight lines. Further, $\psi(D)$ is bounded in the z_2-direction, that is, $\psi(D)$ is contained in a set $\{(z_1, z_2) \in C^2 \mid |z_2| < M\}$ for some constant M. Note also that $\psi(0, z_2) = (0, z_2)$ implies that $\psi(D) \cap \{z_1 = 0\}$ is not empty and contains the point o.

Since the center of G contains the subgroup T_2 of T, and since the logarithmic image of D has the convex hull containing no complete straight lines, Lemma 2 of Section 2 shows that if Δ is the Reinhardt domain in C containing the origin given as the image of the domain D under the projection $p: C^2 \ni (z_1, z_2) \to z_1 \in C$, then, to each element φ of G, there correspond an automorphism $\tau(\varphi)$ of Δ and a holomorphic mapping $\gamma(\varphi)$ of Δ into C^* for which φ has the form

$$\varphi: D \ni (z_1, z_2) \to (w_1, w_2) \in D ,$$
$$\begin{cases} w_1 = \tau(\varphi)(z_1) , \\ w_2 = \gamma(\varphi)(z_1) z_2 , \end{cases}$$

and the map $\tau: G \to \mathrm{Aut}(\Delta)$ is a group homomorphism with $\ker \tau = T_2$. Applying, if necessary, a change of coordinates $(z_1, z_2) \to (rz_1, z_2)$ for some positive constant r, we may assume that Δ is the unit disk B_1 or the complex plane $\boldsymbol{C}$. The group $\mathrm{Aut}(\Delta)$ has the structure of a Lie group with respect to the compact-open topology, and it is readily verified that τ is a Lie group homomorphism. Since $\dim \tau(G) = \dim G - \dim \ker \tau = \dim G - \dim T_2 = 4 - 1 = 3$, it follows that $\tau(G) = \mathrm{Aut}(B_1)$ when $\Delta = B_1$, while $\tau(G) = U(1)\cdot\boldsymbol{C}$ when $\Delta = \boldsymbol{C}$, where $U(1)\cdot\boldsymbol{C}$ denotes the group of all complex affine transformations of $\boldsymbol{C}$ whose linear parts belong to $U(1)$. As a consequence, the center $\mathrm{Cent}\, G$ of G itself coincides with T_2. Indeed, $\tau(\mathrm{Cent}\, G)$ is contained in the center of $\tau(G)$, which reduces to the identity element, so $\mathrm{Cent}\, G \subset \ker \tau = T_2$.

We examine the structure of the domain D. First we define a real-analytic homeomorphism ι of Δ into $\tau(G)$ as follows: In each of the cases $\Delta = B_1$ and $\Delta = \boldsymbol{C}$, there exists a Lie subgroup K of $\tau(G)$ which acts freely and transitively on Δ. The mapping ι associates to each point z_1 of Δ the element f of K such that $f(0) = z_1$. Next we define a real-valued function s on G by $s(\varphi) = |\gamma(\varphi)(0)|$. Since the mapping $G \ni \varphi \to \varphi(o) = \varphi(0, b) = (\tau(\varphi)(0), \gamma(\varphi)(0)b) \in D$ is real-analytic and $\gamma(\varphi)(0) \neq 0$, the function s is real-analytic, while it is invariant under the right action of T_2 on G. Therefore s defines a real-analytic function on $\tau(G) = G/T_2$, which we denote also by s. Now consider the projection $p: D \to \Delta$. We observe that

$$p^{-1}(0) = \{(0, z_2) \in \boldsymbol{C}^2 \,|\, r_i < |z_2| < R_i,\ i \in I\}$$

for some constants r_i, R_i, $i \in I$, such that $0 \leqq r_i < R_i < \infty$, $i \in I$, and

$$\{r_i < |z_2| < R_i\} \cap \{r_{i'} < |z_2| < R_{i'}\} = \varnothing$$

if $i \neq i'$, where I is a non-empty, at most countable index set. It is readily verified that, for each point z_1 of Δ, the set $p^{-1}(z_1)$ is given by

$$p^{-1}(z_1) = \{(z_1, z_2) \in \boldsymbol{C}^2 \,|\, s\circ\iota(z_1) r_i < |z_2| < s\circ\iota(z_1) R_i,\ i \in I\}\ ,$$

hence $D = \bigcup_{i \in I} D_i$ (disjoint union), where D_i, $i \in I$, are the domains in $\boldsymbol{C}^2$ defined by

$$D_i = \{(z_1, z_2) \in \boldsymbol{C}^2 \,|\, z_1 \in \Delta,\ s\circ\iota(z_1) r_i < |z_2| < s\circ\iota(z_1) R_i\}\ , \quad i \in I\ .$$

Since D is connected, I must consist of only one element. Hence, if we write $r = r_i$ and $R = R_i$, then D has the form

$$D = \{(z_1, z_2) \in \boldsymbol{C}^2 \,|\, z_1 \in \Delta,\ s\circ\iota(z_1) r < |z_2| < s\circ\iota(z_1) R\}\ .$$

By a change of coordinates $(z_1, z_2) \to (z_1, R^{-1} z_2)$, we may assume that $R = 1$.

Write $R(z_1) = s \circ \iota(z_1)$. Then the positive-valued function $R(z_1)$ on $\varDelta$ is real-analytic, and satisfies the condition that $R(0) = 1$ and

(27) $\quad R(\tau(\varphi)(z_1)) = |\gamma(\varphi)(z_1)|R(z_1)$ for every $\varphi \in G$ and every $z_1 \in \varDelta$.

Hence we have

(28) $\quad R(\alpha z_1) = R(z_1)$ for every $\alpha \in U(1)$ and every $z_1 \in \varDelta$.

To proceed further, it is necessary to discuss the cases of $\varDelta = B_1$ and $\varDelta = \boldsymbol{C}$ separately.

When $\varDelta = B_1$, following Thullen [8], we can show that D has the form

$$D = \{(z_1, z_2) \in \boldsymbol{C}^2 \,|\, |z_1| < 1,\ r(1 - |z_1|^2)^{p/2} < |z_2| < (1 - |z_1|^2)^{p/2}\}$$

for a non-negative constant p, and then G consists of all transformations of the form

$$D \ni (z_1, z_2) \to (w_1, w_2) \in D \,,$$
$$\begin{cases} w_1 = \alpha(z_1 + \beta)(1 + \bar{\beta}z_1)^{-1} \,, \\ w_2 = \gamma(1 - |\beta|^2)^{p/2}(1 + \bar{\beta}z_1)^{-p}z_2 \,, \end{cases}$$

where $\alpha, \gamma \in U(1)$ and $\beta \in B_1$.

Suppose that $\varDelta = \boldsymbol{C}$. We first determine the domain D. Let φ be any element of G. Then we have $\tau(\varphi) \in \tau(G) = U(1)\cdot\boldsymbol{C}$. If we write $\tau(\varphi)(z_1) = \alpha z_1 + \beta$, where $\alpha \in U(1)$ and $\beta \in \boldsymbol{C}$, then φ has the form

$$\varphi\colon D \ni (z_1, z_2) \to (w_1, w_2) \in D \,,$$
(29)
$$\begin{cases} w_1 = \alpha z_1 + \beta \,, \\ w_2 = \gamma(\varphi)(z_1)z_2 \,, \end{cases}$$

and (27) applied to φ means that

(30) $\quad R(\alpha z_1 + \beta) = |\gamma(\varphi)(z_1)|R(z_1)$ for every $z_1 \in \varDelta$.

Hence we see that the function $\log R(z_1)$ on $\varDelta$ satisfies the functional equation

$$\log R(\alpha z_1 + \beta) = \log|\gamma(\varphi)(z_1)| + \log R(z_1) \,.$$

To derive a differential equation $\log R(z_1)$ satisfies, apply the Laplacian $L = \partial^2/\partial x_1^2 + \partial^2/\partial y_1^2$ on $\varDelta$ to both sides of the above equation, where $z_1 = x_1 + \sqrt{-1}y_1$ $(x_1, y_1 \in \boldsymbol{R})$. Then we get

(31) $\quad L(\log R)(\alpha z_1 + \beta) = L(\log R)(z_1)$,

because $\log|\gamma(\varphi)(z_1)|$ is a harmonic function on $\varDelta$ and the differential operator L is invariant under $U(1)\cdot\boldsymbol{C}$. The fact that $\tau(G) = U(1)\cdot\boldsymbol{C}$ implies that (31) holds for every $\alpha \in U(1)$ and every $\beta \in \boldsymbol{C}$. Hence, fixing $z_1 \in \Delta$ and putting $\alpha = 1$, we find that $L(\log R)(z_1 + \beta) = L(\log R)(z_1)$ for every $\beta \in \boldsymbol{C}$, which leads to the differential equation

$$(32) \qquad L(\log R)(z_1) = \text{const.}$$

in Δ. By (28), the function $\log R(z_1)$ is invariant under $U(1)$. Consequently, in terms of polar coordinates $z_1 = \rho \exp(\sqrt{-1}\theta)$, we have $\log R(z_1) = Q(\rho)$ for some real-analytic function Q on $\boldsymbol{R}$, and (32) becomes

$$(33) \qquad L(Q)(\rho) = \partial^2 Q(\rho)/\partial\rho^2 + \rho^{-1}\partial Q(\rho)/\partial\rho = \text{const.}$$

Note that $Q(0) = \log R(0) = 0$. Let $Q(\rho) = \sum_{\nu=1}^{\infty} a_\nu \rho^\nu$ be the Taylor expansion of Q. Substituting this into the differential equation (33), we get

$$\sum_{\nu=1}^{\infty} \nu^2 a_\nu \rho^{\nu-2} = \text{const.}$$

Therefore we see that $a_\nu \neq 0$ only if $\nu = 2$, so that $Q(\rho) = c\rho^2$ for some real constant c. In view of the definition of $Q(\rho)$, this implies that $\log R(z_1) = c\,|z_1|^2$, or $R(z_1) = \exp(c\,|z_1|^2)$, and we conclude that D has the form

$$D = \{(z_1, z_2) \in \boldsymbol{C}^2 \,|\, r\exp(c|z_1|^2) < |z_2| < \exp(c|z_1|^2)\} .$$

Since D must be holomorphically equivalent to a bounded domain, it is necessary that $c \neq 0$. Conversely, each domain of the above form with $c \neq 0$ is algebraically equivalent to a bounded domain under the transformation $(z_1, z_2) \to (z_1 z_2^a, z_2^a)$, where $a = 1$ when $c < 0$ and $a = -1$ when $c > 0$. In our situation, D is bounded in the z_2-direction, so $c < 0$. By a change of coordinates $(z_1, z_2) \to (\sqrt{-c}\,z_1, z_2)$, we may assume that $c = -1$.

We now determine the group G. If we write any element φ of G in the form (29), then, by (30),

$$|\gamma(\varphi)(z_1)| = R(\alpha z_1 + \beta)/R(z_1) .$$

Since $R(z_1) = \exp(-|z_1|^2)$, it follows from this relation that

$$\begin{aligned} |\gamma(\varphi)(z_1)| &= \exp\{-(|\alpha z_1 + \beta|^2 - |z_1|^2)\} \\ &= \exp\{-(2\operatorname{Re}\alpha\bar{\beta}z_1 + |\beta|^2)\} = \exp\operatorname{Re}\{-(2\alpha\bar{\beta}z_1 + |\beta|^2)\} , \end{aligned}$$

where $\operatorname{Re} c$ denotes the real part of a complex number c. Therefore the function $\gamma(\varphi)(z_1)$ has the form

$$\gamma(\varphi)(z_1) = \gamma\exp\{-(2\alpha\bar{\beta}z_1 + |\beta|^2)\}$$

for some element γ of $U(1)$, and we conclude that G consists of all transformations of the form

$$D \ni (z_1, z_2) \to (w_1, w_2) \in D\ ,$$
$$\begin{cases} w_1 = \alpha z_1 + \beta\ , \\ w_2 = \gamma[\exp\{-(2\alpha\bar{\beta}z_1 + |\beta|^2)\}]z_2\ , \end{cases}$$

where $\alpha, \gamma \in U(1)$ and $\beta \in \boldsymbol{C}$.

Consider next the case where $D \cap \{z_2 = 0\} \neq \emptyset$. To determine the group G, we show first that every element of G is induced by a complex linear transformation of $\boldsymbol{C}^2$. Let φ be any element of G. In terms of the coordinates in $\boldsymbol{C}^2$, write $\varphi(z_1, z_2) = (\varphi_1(z_1, z_2), \varphi_2(z_1, z_2))$, where φ_1 and φ_2 are holomorphic functions on D. Apply the observation before Lemma 1 to φ. Then, because of Lemma 1, we see that, for $k = l = 1$, the coefficients of the Laurent expansions of φ_1 and φ_2 satisfy (26). Thus, if $a^{(i)}_{(\nu_1,\nu_2)} \neq 0$, then $\nu_1 + \nu_2 = 1$. On the other hand, since $D \cap \{z_1 = 0\} \neq \emptyset$ and $D \cap \{z_2 = 0\} \neq \emptyset$, we get $\nu_1 \geqq 0$ and $\nu_2 \geqq 0$ if $a^{(i)}_{(\nu_1,\nu_2)} \neq 0$. Therefore we have $a^{(i)}_{(\nu_1,\nu_2)} \neq 0$ only if $(\nu_1, \nu_2) = (1, 0)$ or $(\nu_1, \nu_2) = (0, 1)$, so that φ is induced by a complex linear transformation of $\boldsymbol{C}^2$.

It follows from what we have shown above that G coincides with the identity component of $GL(D)$. Proposition 2 of Section 3 shows that, by a change of coordinates $(z_1, z_2) \to (r_1 z_1, r_2 z_2)$ for some positive constants r_1 and r_2, the group G is contained in $U(2)$. Since $\dim G = 4 = \dim U(2)$, we see that G coincides with $U(2)$. As a consequence, we have $\mathrm{Cent}\, G = Z = \{(\alpha, \alpha) \in T \mid \alpha \in U(1)\}$.

In view of the connectedness of D, the fact that $G = U(2)$ implies that D has the form

$$D = \{(z_1, z_2) \in \boldsymbol{C}^2 \mid r < |z_1|^2 + |z_2|^2 < R\}$$

for some constants r and R with $0 \leqq r < R < \infty$. By a change of coordinates $(z_1, z_2) \to (R^{-1/2} z_1, R^{-1/2} z_2)$, we may assume that $R = 1$, and then G still coincides with $U(2)$.

Summarizing our results, we obtain the following theorem.

THEOREM 5. *Every bounded Reinhardt domain in* $\boldsymbol{C}^2$ *with* $t = 1$ *and* $h = 3$ *is algebraically equivalent to one of the domains listed in* (i)-(iii) *below.*

(i) $\{(z_1, z_2) \in \boldsymbol{C}^2 \mid |z_1| < 1,\ r(1 - |z_1|^2)^{p/2} < |z_2| < (1 - |z_1|^2)^{p/2}\}$ $(p \geqq 0,\ 0 \leqq r < 1)$.

(ii) $\{(z_1, z_2) \in \boldsymbol{C}^2 \mid r \exp(-|z_1|^2) < |z_2| < \exp(-|z_1|^2)\}$ $(0 \leqq r < 1)$.

(iii) $\{(z_1, z_2) \in \boldsymbol{C}^2 \mid r < |z_1|^2 + |z_2|^2 < 1\}$ $(0 \leqq r < 1)$.

Further, for the domains D above, the groups G(D) are given as follows:

For a domain D of type (i), *G(D) consists of all transformations of the form*

$$D \ni (z_1, z_2) \to (w_1, w_2) \in D\ ,$$
$$\begin{cases} w_1 = \alpha(z_1 + \beta)(1 + \bar{\beta}z_1)^{-1}\ , \\ w_2 = \gamma(1 - |\beta|^2)^{p/2}(1 + \bar{\beta}z_1)^{-p}z_2\ , \end{cases}$$

where α, $\gamma \in U(1)$ *and* $\beta \in B_1$.

For a domain D *of type* (ii), $G(D)$ *consists of all transformations of the form*

$$D \ni (z_1, z_2) \to (w_1, w_2) \in D ,$$
$$\begin{cases} w_1 = \alpha z_1 + \beta , \\ w_2 = \gamma[\exp\{-(2\alpha\bar{\beta}z_1 + |\beta|^2)\}]z_2 , \end{cases}$$

where α, $\gamma \in U(1)$ *and* $\beta \in \boldsymbol{C}$.

For a domain D *of type* (iii), $G(D)$ *coincides with* $U(2)$.

COROLLARY. *Let* D *be a bounded Reinhardt domain in* $\boldsymbol{C}^2$ *with* $t=1$ *and* $h=3$. *If* D *is a domain of type* (i) *with* $p=0$ *and* $r \neq 0$, *then* $\mathrm{Aut}(D) = G(D) \cup \theta G(D)$, *where* $\theta\colon D \ni (z_1, z_2) \to (z_1, rz_2^{-1}) \in D$. *If* D *is not holomorphically equivalent to any such domain, then* $\mathrm{Aut}(D) = G(D)$.

This is an immediate consequence of the above theorem, the corollary to Proposition 2 of Section 4 and the following proposition.

PROPOSITION. *For the domains* D *of types* (i)-(iii) *defined in Theorem* 5, *the groups* $\mathrm{Aut}_{\mathrm{alg}}(D)$ *are given as follows*:

If D *is a domain of type* (i), *then* $\mathrm{Aut}_{\mathrm{alg}}(D) = T(D)$ *when* $p \neq 0$ *or* $p=0$ *and* $r=0$, *while* $\mathrm{Aut}_{\mathrm{alg}}(D) = T(D) \cup \theta T(D)$ *when* $p=0$ *and* $r \neq 0$, *where* θ *is as in the corollary above.*

If D *is a domain of type* (ii), *then* $\mathrm{Aut}_{\mathrm{alg}}(D) = T(D)$.

If D *is a domain of type* (iii), *then* $\mathrm{Aut}_{\mathrm{alg}}(D) = T(D) \cup \sigma T(D)$, *where* $\sigma\colon D \ni (z_1, z_2) \to (z_2, z_1) \in D$.

To prove the proposition, we begin with two lemmas. The proof of Lemma 2 below is straightforward, and is omitted.

LEMMA 2. *Let* D *be a Reinhardt domain in* $\boldsymbol{C}^2$ *and let* $\varphi\colon D \ni (z_1, z_2) \to (w_1, w_2) \in D$ *be an element of* $\mathrm{Aut}_{\mathrm{alg}}(D)$. *If* $D \cap \{z_1=0\} \neq \emptyset$ *and* $D \cap \{z_2=0\} = \emptyset$, *then* φ *has the form*

$$\begin{cases} w_1 = \alpha_1 z_1 z_2^b , \\ w_2 = \alpha_2 z_2^d , \end{cases} \tag{34}$$

where $b \in \boldsymbol{Z}$, $d = \pm 1$ *and* $(\alpha_1, \alpha_2) \in (\boldsymbol{C}^*)^2$. *If* $D \cap \{z_1=0\} \neq \emptyset$ *and* $D \cap \{z_2=0\} \neq \emptyset$, *then* φ *has the form*

$$\begin{cases} w_1 = \alpha_1 z_{\tau(1)} , \\ w_2 = \alpha_2 z_{\tau(2)} , \end{cases} \tag{35}$$

where τ *is a permutation of* $\{1, 2\}$ *and* $(\alpha_1, \alpha_2) \in (\boldsymbol{C}^*)^2$.

LEMMA 3. *Let* D *be a Reinhardt domain in* $\boldsymbol{C}^2$ *such that* $D \cap \{z_1 = 0\} \neq \emptyset$ *and* $D \cap \{z_2 = 0\} = \emptyset$, *that* $D \cap \{z_1 = 0\}$ *is bounded, and that, for* *every* $c \in \varDelta$, *the set* $D \cap \{z_2 = c\}$ *is bounded, where* $\varDelta$ *is the domain in* $\boldsymbol{C}$ *given as the image of the domain* D *under the projection* $\boldsymbol{C}^2 \ni (z_1, z_2) \to z_2 \in \boldsymbol{C}$. *If* φ *is an element of* $\mathrm{Aut}_{\mathrm{alg}}(D)$ *written in the form* (34), *and if* $d = 1$, *then* $\varphi \in T(D)$.

PROOF. It suffices to prove that $b = 0$ and $|\alpha_1| = |\alpha_2| = 1$. If we write $D_0 = \{z_2 \in \boldsymbol{C} \mid (0, z_2) \in D\}$, then, by assumption, D_0 is a bounded open subset of $\boldsymbol{C}$. Since the restriction $\varphi: D \cap \{z_1 = 0\} \to D \cap \{z_1 = 0\}$ induces an automorphism $D_0 \ni z_2 \to \alpha_2 z_2 \in D_0$ of D_0, it follows that $|\alpha_2| = 1$. If, for every $c \in \varDelta$, we write $D_c = \{z_1 \in \boldsymbol{C} \mid (z_1, c) \in D\}$, then, by assumption, D_c is a bounded open subset of $\boldsymbol{C}$. Since the relation $|\alpha_2| = 1$ implies that the restriction $\varphi: D \cap \{z_2 = c\} \to D \cap \{z_2 = \alpha_2 c\}$ induces an automorphism $D_c \ni z_1 \to \alpha_1 c^b z_1 \in D_c$ of D_c, it follows that $|\alpha_1 c^b| = 1$. Since this holds for every $c \in \varDelta$, we see that $b = 0$ and $|\alpha_1| = 1$. q.e.d.

Turning to the proof of the proposition, let D be as in the proposition.

Suppose first that D is of type (i) with $p \neq 0$, of type (i) with $p = 0$ and $r = 0$, or of type (ii). We observe that D satisfies the conditions of Lemma 3. Let φ be any element of $\mathrm{Aut}_{\mathrm{alg}}(D)$. By Lemma 2, φ has the form (34). Since, for every $\varepsilon > 0$, there exists a point (z_1, z_2) of D such that $|z_2| < \varepsilon$, it follows that $d = 1$, and hence, by Lemma 3, $\varphi \in T(D)$, so that $\mathrm{Aut}_{\mathrm{alg}}(D) = T(D)$.

Suppose next that D is of type (i) with $p = 0$ and $r \neq 0$. We observe that D satisfies the conditions of Lemma 3. Let φ be any element of $\mathrm{Aut}_{\mathrm{alg}}(D)$. By Lemma 2, φ has the form (34). If $d = 1$, then, by Lemma 3, $\varphi \in T(D)$. Assume that $d = -1$. If we write $\varDelta = \{z_2 \in \boldsymbol{C} \mid r < |z_2| < 1\}$, then the restriction $\varphi: D \cap \{z_1 = 0\} \to D \cap \{z_1 = 0\}$ induces an automorphism $\varDelta \ni z_2 \to \alpha_2 z_2^{-1} \in \varDelta$ of the annulus $\varDelta$. This implies that $|\alpha_2| = r$. Since, for every $c \in \varDelta$, the restriction $\varphi: D \cap \{z_2 = c\} \to D \cap \{z_2 = \alpha_2 c^{-1}\}$ induces an automorphism $B_1 \ni z_1 \to \alpha_1 c^b z_1 \in B_1$ of the unit disk B_1, and therefore $|\alpha_1 c^b| = 1$, it follows that $b = 0$ and $|\alpha_1| = 1$. We thus conclude that $\mathrm{Aut}_{\mathrm{alg}}(D) = T(D) \cup \theta T(D)$.

Suppose finally that D is of type (iii). We observe that $D \cap \{z_1 = 0\} \neq \varnothing$ and $D \cap \{z_2 = 0\} \neq \varnothing$. Let φ be any element of $\mathrm{Aut}_{\mathrm{alg}}(D)$. By Lemma 2, φ has the form (35). If $\tau = \mathrm{id}$, then it follows from the boundedness of D that $\varphi \in T(D)$. If $\tau \neq \mathrm{id}$, then the consideration of $\sigma\varphi$ yields that $\varphi \in \sigma T(D)$, where σ is as in the proposition. We thus conclude that $\mathrm{Aut}_{\mathrm{alg}}(D) = T(D) \cup \sigma T(D)$, and the proof of the proposition is complete.

REFERENCES

[1] D. E. BARRETT, Holomorphic equivalence and proper mapping of bounded Reinhardt domains not containing the origin, Comment. Math. Helv. 59 (1984), 550–564.

[2] E. BEDFORD, Holomorphic mapping of products of annuli in C^n, Pacific J. Math. 87 (1980), 271–281.
[3] E. CARTAN, Sur les domaines bornés homogènes de l'espace de n variables complexes, Abh. Math. Sem. Hamburg 11 (1935), 116–162.
[4] G. HOCHSCHILD, The Structure of Lie Groups, Holden-Day, San Francisco, 1965.
[5] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, I, 1963; II, 1968, Wiley-Interscience, New York.
[6] R. NARASIMHAN, Several Complex Variables, Univ. of Chicago Press, 1971.
[7] T. SUNADA, Holomorphic equivalence problem for bounded Reinhardt domains, Math. Ann. 235 (1978), 111–128.
[8] P. THULLEN, Zu den Abbildungen durch analytische Funktionen mehrerer Veräuderlichen, Math. Ann. 104 (1931), 244–259.

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