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COMPACT STRUCTURES ON $C^* \times C^*$

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0. Introduction.

0.1. By a "surface" we mean a connected 2-dimensional complex manifold. By a 'compact structure' on a surface V we mean an equivalence class of pairs (S, C) where S is a compact surface with $S - C$ as a Zariski-open subset biholomorphic to V; two such pairs (S, C) and (S', C') being 'equivalent' if there is a chain $(S, C) = (S_0, C_0), \cdots, (S_i, C_i), \cdots, (S_n, C_n) = (S, C')$ where each pair (S_{i+1}, C_{i+1}) is obtained from (S_i, C_i) by blowing down an exceptional curve of the first kind or by blowing up a point, in C_i. For a given V if (S, C) exists then V is said to be 'compactifiable'. We prove:

THEOREM A. *Let V be proper homotopy equivalent to $C \times C^*$. Then $(P^2, 2L)$ is the only compact structure on it, where $2L$ denotes the union of two lines on P^2.*

THEOREM B. *Let V be proper homotopy equivalent to $C^* \times C^*$. Then the compact structures on V are one of the following:*

(i) *$(P^2; 3L)$ where $3L$ denotes the union of any three lines in general position.*

(ii) *(X, E), where X is the total space of a P^1-bundle over an elliptic curve and E is a section with $(E^2) = 0$.*

(iii) *(H, E), where H is a Hopf surface and E an elliptic curve on H.*

0.2. If we replace "proper homotopy equivalent" by "biholomorphic", then the above two theorems are contained in [Si], [U] and [Su]. The problem of classifying compact structures on $C^* \times C^*$ was first considered by Simha [Si] where he proved that if S is algebraic and C is irreducible, $S - C$ biholomorphic to $C^* \times C^*$, then S is a P^1-bundle over an elliptic curve with a unique section C and $(C^2) = 0$, i.e., a Serre structure. Note that (ii) in Theorem B includes these along with $E \times P^1$. Ueda [U] classified all irrational structures on $C^* \times C^*$ and then Suzuki [Su] completed the picture by classifying the rational ones. Suzuki heavily depends on the results of Nishino-Suzuki [N-S] on cluster sets of holomorphic mappings. In our context these results are not applicable and this is precisely the point where this paper differs from the earlier works. So our main tool is the Mumford-Ramanujam method of studying the combinatorics of C with the help of the local fundamental group G of C. As everybody else, we also make extensive use of Kodaira's classification of surfaces.

0.3. By using Neumann's results [N], the method of this paper can be employed to prove the following two results:

THEOREM C. *Let V be proper homotopy equivalent to the total space of an affine $\boldsymbol{C}$-bundle of degree n ($\geqq 0$) over an elliptic curve. Then any compact structure on V is one of the following*:

(i) *(X, E), where X is a $\boldsymbol{P}^1$-bundle over an elliptic curve and E is a section with $(E^2) = \pm n$.*

(ii) *$(S_{n,\alpha,t}, D_{n,\alpha,t})$ where $S_{n,\alpha,t}$ denote the Inoue surfaces of class* VII *and $D_{n,\alpha,t}$ is the unique curve with $(D^2_{n,\alpha,t}) = 0$ (see* [E1]).

THEOREM D. *Let V be proper homotopy equivalent to the total space of an affine $\boldsymbol{C}$-bundle of degree n over a smooth curve of genus $g \geqq 2$. Then any compact structure on V is (X, Δ), where X is a $\boldsymbol{P}^1$-bundle of degree n over a smooth curve of genus g and Δ is a section with $(\Delta^2) = \pm n$.*

Of course one has to use the main theorem of Enoki [E1] in proving Theorem C. Here again if V is actually an affine $\boldsymbol{C}$-bundle over an elliptic curve, then Theorem C is due to Enoki [E2]. We shall not give any proof of Theorems C and D here. Note, however, that the case when $g = 0$ in Theorem D remains unsolved.

ADDENDUM. Recently, using Theorem 1.1 of Y. Miyaoka's paper, "The maximal number of quotient singularities on surfaces with given numerical invariant, Math. Ann. 268 (1984), p. 159–171", we have settled the case $g = 0$ also in Theorem D, viz.,

"Let V be proper homotopy equivalent to the total space of an affine $\boldsymbol{C}$-bundle of degree n, $n \in \boldsymbol{Z}$, over $\boldsymbol{P}^1$, and let (X, Δ) be a compact structure on V. Then X is a Hirzebruch surface; if $n \neq 0$, then Δ is a smooth section with $\Delta^2 = -n$".

0.4. The above theorems can also be read as a Ramanujam-type topological characterization of certain compactifiable surfaces, e.g., Theorem A can be read as:

THEOREM A′. *Let V be a compactifiable surface which is proper* *homotopy equivalent to* $\boldsymbol{C}\times\boldsymbol{C}^*$. *Then* V *is biholomorphic to* $\boldsymbol{C}\times\boldsymbol{C}^*$, *and carries a unique algebraic structure.*

Indeed we begin by observing that Ramanujam's theorem can be restated as follows:

THEOREM 1 (Ramanujam). *Let* V *be a compactifiable, topologically contractible smooth surface which is simply connected at infinity. Then* V *is biholomorphic to* $\boldsymbol{C}^2$ *and carries a unique algebraic structure.*

Similarly, the result in [G-S] can be restated as:

THEOREM 2 (Gurjar-Shastri). *Let* V *be a two-dimensional normal variety over* $\boldsymbol{C}$. *Suppose* V *is contractible and has finite fundamental group* G *at infinity. If* V *is compactifiable, then* V *is biholomorphic to* $\boldsymbol{C}^2/G$, *where* $G\subset GL(2, \boldsymbol{C})$ *is a small subgroup. In particular,* V *has at most one singular point.*

0.5. In §1 we set up the notation and recall the relevant materials from [M], [R] and [W]. In §2, we state and prove some propositions which describe the combinatorics of C in terms of its associated graph Γ_C. In §3 the proofs of Theorems A and B are completed.

This paper arose out of a coffee-table conversation the author had with R. R. Simha. It was felt that instead of using the results of Nishino-Suzuki, one should be able to stick to the Mumford-Ramanujam method throughout. The author is thankful to Simha for showing and explaining him the materials in [Si], [Su] and [U], and also for many other helpful discussions.

1. Mumford-Ramanujam ingredients. In this section we recall the basic facts about the Mumford-Ramanujam method.

1.1. As in the introduction we consider an equivalence class of pairs (S, C), where S is a surface and C is a connected Zariski-closed subset of codimension one (i.e. a curve). C is said to be with simple normal crossings, (SNC) for short, if

(i) each irreducible component C_i of C is smooth, say $C = \cup_{i=0}^{t} C_i$

(ii) $(C_i.C_j) \leqq 1$, for distinct C_i and C_j, and

(iii) $C_i \cap C_j \cap C_k = \varnothing$, for distinct C_i, C_j, C_k.

1.2. Associated to such a pair (S, C) is its dual weighted graph Γ_C defined as follows: The vertex set of Γ_C is the set of irreducible components $\{C_i\}$ of C. Two vertices C_i and C_j are joined by an edge if and only if $(C_i.C_j) = 1$. Each vertex C_i carries two weights Ω_i and g_i where $\Omega_i = (C_i.C_i)$ and g_i is the genus of C_i. (see for instance [W]). Schematically, at the vertex C_i the graph Γ_C may appear as in Figure 1.

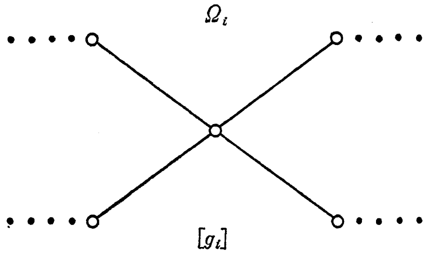

FIGURE 1

Whenever, the genus of a curve is zero this will not be indicated in the schematic presentation (see Figure 2, for instance). We shall denote by $A(n_1, \cdots, n_k)$ the tree dual to a linear chain of rational curves $C_1, \cdots, C_k$ with $C_i^2 = n_i$, $i = 1, \cdots, k$, $(C_i.C_{i+1}) = 1$ and $(C_i.C_j) = 0$ if $|i - j| \neq 1, 0$, as in Figure 2.

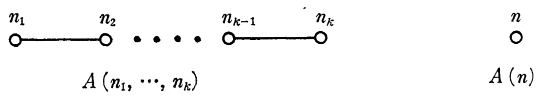

FIGURE 2

The operation of blowing-up and blowing-down on C with the restriction that C always remains (SNC), carry onto the associated graph Γ_C also in an obvious manner. More generally one can take an abstract weighted graph Γ and then define these blowing-up and blowing-down operations of Γ. A graph Γ is said to be *minimal* if we cannot perform any blowing down operation on it, i.e., Γ has no vertex v which is linear in Γ and has weights $\Omega_v = -1$, and $g_v = 0$.

1.3. Given a weighted graph Γ, the intersection matrix I_Γ is defined by

$$(I_\Gamma)_{u,v} = \begin{cases} 1 \text{ if } u \neq v, \text{ and } u \text{ and } v \text{ are joined in } \Gamma \\ \Omega_v \text{ if } u = v \,. \end{cases}$$

I_Γ is symmetric and so defines a symmetric bilinear form over $\boldsymbol{R}$, which will also be denoted by I_Γ. For a curve C with (SNC) we write I_C for I_{Γ_C}. Γ is said to be negative definite, unimodular etc. according as I_Γ is negative definite, unimodular etc. We will often use Γ_C and C interchangeably.

1.4. Given a weighted graph Γ, let $P(\Gamma)$ denote the compact 4-dimensional plumbed manifold with boundary, and let $M_\Gamma = \partial P(\Gamma)$. For a curve C with (SNC) on a surface we can construct a system of tubular neighbourhoods U_C of C so that the boundaries ∂U_C are smooth and have the same homotopy type and a retraction map $\varphi: U_C \to C$ inducing a surjection $\pi_1(U_C) \to \pi_1(C)$. U_C can be idnentified with $P(\Gamma_C)$ (See [M] or [R]). If e_i denotes the homology class represented by a loop that goes around a component C_i exactly once in the positive direction, then the kernel K of $\varphi_*: H_1(M_C) \to H_1(C)$ is generated by $\{e_i\}$ with the relations

$$\sum_{j=0}^{t} (C_i.C_j)e_j = 0 , \quad i = 0, \cdots, t .$$

In particular we have

$$\operatorname{rank} I_C = t + 1 - \operatorname{rank} H_1(M_C) + \operatorname{rank} H_1(C) .$$

1.5. For a compactifiable surface V which is connected at infinity, we have (S, C), with $S - C$ biholomorphic to V, such that C is connected. By blowing up if necessary, we may assume that C has (SNC). Then $G = \pi_1(M_C)$ is the same as the fundamental group at infinity, $\pi_1^\infty(V)$. On the other hand, $\pi_1(C)$ is a free product

$$\pi_1(C) \simeq \pi_1(\Gamma_C) * \{\pi_1(C_i)\}_{0 \leqq i \leqq t}$$

of $\pi_1(\Gamma_C)$ and the group $\pi_1(C_i)$, and $\pi_1(\Gamma_C)$ is a free group of rank r where r is the number of 'essential' cycles in Γ_C. In this paper we deal with situations where $G \simeq \pi_1(M_C)$ is finite or abelian. So, by the surjection $\pi_1(M_C) \to \pi_1(C)$, we need to consider only the following three cases.

(a) Γ_C is a tree of lines (i.e. each C_i is rational)

(b) Γ_C has only one essential cycle and all C_i are rational

(c) Γ_C is a tree, all C_i are rational except C_0, say, which is an elliptic curve.

1.6. In each of the above cases we have a presentation of G as follows (see [W] for instance): Let $s_{ij} = (C_i.C_j)$.

(a) generators: $C_0, C_1, \cdots, C_t$

relations: $[C_i, C_j] = 1$ whenever $s_{ij} \neq 0$ or $i = j$,

$$\prod_{j=0}^{t} C_j^{s_{ij}} = 1 , \quad \text{for all} \quad i = 0, \cdots, t .$$

(b) In this case let $C_0, \cdots, C_r$ be the vertices on the essential cycle ($r \geqq 2$ by (SNC)). Then

generators: $C_0, \cdots, C_t, u$

relations: $$\begin{cases} C_0^{s_{0,0}} C_1^{s_{0,1}} \cdots C_{r-1}^{s_{0,r-1}} u^{-1} C_r u C_{r+1}^{s_{0,r+1}} \cdots C_t^{s_{0,t}} = 1 , \\ C_0^{s_{i,0}} \cdots C_t^{s_{i,t}} = 1, \text{ for } i \neq 0, r , \end{cases}$$

$$\begin{cases} uC_0u^{-1}C_1^{s_{r,1}} \cdots C_t^{s_{r,t}} = 1\ , \\ [C_i, C_j] = 1 \text{ if } s_{ij} \neq 0 \text{ and } (i, j) \neq (0, r) \text{ or } (r, 0)\ , \\ C_0u^{-1}C_ru = u^{-1}C_ruC_0\ . \end{cases}$$

(c) In this case let C_0 denote the elliptic curve in C.

generators: $C_0, \cdots, C_t, x, y$.

relations:
$$\begin{cases} [x, y]\ C_0^{s_{0,0}} \cdots C_t^{s_{0,t}} = 1\ , \\ \prod_{j=0}^{t} C_i^{s_{ij}} = 1 \text{ for all } i = 1, \cdots, t\ , \\ [x, C_0] = [y, C_0] = [C_i, C_j] = 1 \text{ if } i \neq j \text{ or } s_{ij} \neq 0\ . \end{cases}$$

1.7. Finally, the following group theoretic fact plays a key role in our situation. Let G_i be nontrivial groups and let $g_i \in G_i$. Let H be the quotient of the free product $G_i * \cdots * G_n$ by the single relation $g_1 * \cdots * g_n = 1$;

$$H = (G * \cdots * G_n)/\langle g_1 * \cdots * g_n \rangle\ .$$

Then H is nonabelian for $n \geqq 3$. Indeed, for $n \geqq 4$ we can express H as an appropriate amalgamated free product and check the nonabelianness of H. For $n = 3$, if $H_i = (g_i)$ is the cyclic subgroup of G_i, we may first assume that $H_i \neq (1)$. Then the group $H_0 = (H_1 * H_2 * H_3)/\langle g_1 * g_2 * g_3 \rangle$ becomes a subgroup of H. This group H_0 is well known to be nonabelian.

2. Combinatorics of C.

2.1. In this section S always denotes a (smooth) compact surface and C denotes a (connected) curve with (SNC) on S. We shall need the following easily proved geometric lemmas, before we proceed with the study of the combinatorics of C.

LEMMA 1. *Let L_1 be a line on S with $(L_1^2) = 0$. Then S is algebraic and we have a holomorphic mapping $\psi: S \to \Delta$, where Δ is some non-singular curve; L_1 is a fibre of ψ. Further, if L_2 is a line on S such that $(L_1.L_2) \neq 0$ then $\Delta = \boldsymbol{P}^1$ and L_2 is a section of ψ, i.e., S is a rational ruled surface with L_1 as a good fibre and L_2 as a section. In particular, if D is any (connected) curve on S with $(D.L_1) = 0$, then all components of D_i are rational and Γ_D is a tree, and $(D.L_2) \leqq 1$.*

PROOF. The existence of $\psi: S \to \Delta$ follows from Kodaira and Spencer [K-S] (see 3° of [Su]). The rest of the lemma is common knowledge about rational ruled surfaces.

LEMMA 2. *Suppose Γ_C is as in Figure 3.*

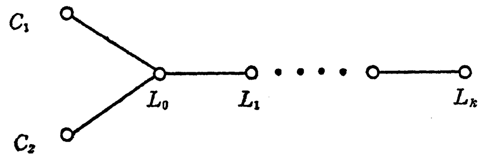

FIGURE 3

where L_i are lines and $(L_0^2) = -1$, $k \geqq 1$. Then the linear subtree T consisting of $\{L_i\}_{i\geqq 2}$ is negative definite or empty.

PROOF. Assume on the contrary, that T is not negative definite. By blowing down (inside T) we may assume that T is minimal. Now it follows that $(L_i^2) \geqq 0$ for some $i \geqq 2$. By blowing up and blowing down, if necessary, (and of course not requiring minimality any more) we may assume $(L_2^2) = 0$. Now by the above lemma S is a rational ruled surface with L_2 as a fibre and L_1 as a section. The curves C_1, L_0 and C_2 are contained in a singular fibre F. By blowing down L_0 we obtain a singular fibre F_0 with $(F_0.L_1) \geqq 2$ which is absurd.

LEMMA 3. *Consider a subtree of lines in Γ_C and its subtree T as in Figure 4.*

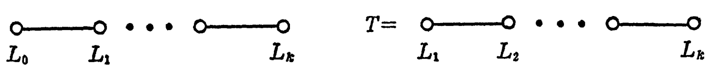

FIGURE 4

Suppose T is not negative definite. Let Γ_0 be any connected subgraph of Γ_C such that no component of Γ_0 meets any component of T. Then $(\Gamma_0.L_0) \leqq 1$ and Γ_0 is a tree of lines. In particular, $\Gamma_0 \cup \{L_0\}$ is also a tree.

PROOF. As in the proof of Lemma 2, we first reduce to the case when $(L_1^2) = 0$. Then Lemma 3 follows from Lemma 1.

2.2. We say a graph Γ satisfies the eigen-value condition (E), if any subspace on which I_Γ is positive semidefinite is of real dimension at most one. We say Γ satisfies (R) if no graph equivalent to Γ has a subtree of the form shown in figure 3, with $k \geqq 2$, where L_0, L_1, L_2 are lines, $(L_0^2) = -1$, and $(L_2^2) \geqq 0$. Lemma 2 implies that if $\Gamma = \Gamma_C$ for a curve C on a compact surface then Γ satisfies (R). The proposition on page 80 of [R] can now be restated as follows:

PROPOSITION 1. *Let Γ be a minimal graph satisfying* (E) *and* (R). *If $\pi_1(M_\Gamma) = (1)$, then Γ is linear.*

For a curve C with (SNC), on a compact surface S, such that I_C is nondegenerate, condition (E) is always satisfied. For, the nondegeneracy of I_C implies that $H_2(C) \to H_2(S)$ is injective and then (E) is an easy consequence of the Hodge Index Theorem. If $H_1(M_C)$ is finite, it follows that I_C is nondegenerate (see 1.4). In the above proposition, once Γ is linear it is easily seen that Γ is equivalent to $A(1)$ or to the empty tree. Thus we have:

PROPOSITION 1′. *Let C be a curve with* (SNC) *on a compact surface S. Suppose $\pi_1(M_C) = (1)$. Then Γ_C is equivalent to $A(1)$ or to the empty tree.*

2.3. In [Sh] we have proved:

PROPOSITION 2. *Let T be a minimal tree satisfying* (E) *and* (R) *with more than one branch points. If all simple branches of T are negative definite, then $\pi_1(M_T)$ is noncyclic and infinite.*

REMARK. Indeed, it suffices to assume that for some free vertex $v \in T$, $T - \{v\}$ satisfies (E), instead of assuming that T itself satisfies (E), since in the proof of this proposition, (E) is used only to say that T does not have two disjoint subtrees isomorphic to $A(-1, -1)$. In order to avoid repetition, we shall use this stronger form of Proposition 2 and prove:

PROPOSITION 2′. *Let C be a curve with* (SNC), *on a compact surface S, such that $T = \Gamma_C$ is a tree. If $\pi_1(M_T) \simeq \boldsymbol{Z}$, then T is equivalent to $A(0)$ or $A(1, 1)$.*

We first prove three more lemmas:

LEMMA 4. *Suppose T is as in the above proposition, except that $\pi_1(M_T)$ is any cyclic group. Suppose T is minimal and not linear. If T_1 is any simple branch of T, then $T_1 \neq A(0)$.*

PROOF. Let T_1 be a simple branch of T at an extremal branch point $u \in T$. Write $T-\{u\}$ as a disjoint union of subtrees, $T-\{u\}= T_1 \amalg \cdots \amalg T_k$, $k \geqq 3$. Then we have

$$\pi_1(M_T)/\langle u\rangle \simeq (\pi_1(M_{T_1}) * \cdots * \pi_1(M_{T_k}))/\langle v_1 * \cdots * v_k\rangle ,$$

where $v_i \in T_i$ are the vertices joined to u in T. By 1.7, we must have $\pi_1(M_{T_i}) = (1)$ for some i. Thus if $T_1 = A(0)$, we have $\pi_1(M_{T_1}) \simeq \boldsymbol{Z}$. Hence we may assume $\pi_1(M_{T_2}) = (1)$, say. It follows that I_{T_2} is nondegenerate. If I_{T_2} is not negative definite, then T_2 will support a divisor D, with $(D^2) > 0$. On the other hand, if C_1 is the curve in T_1, then $(C_1)^2 = 0$ and

$(C_1.D)=0$, contradicting the Hodge Index theorem. Hence I_{T_2} is negative definite. By Proposition 1, it follows that T_2 is not minimal. This means (since T is minimal), that $\Omega_{v_2}=-1$ and v_2 is linear in T_2. Since T is minimal, v_2 is a branch point of T. This implies that T does not satisfy (R), contradicting 2.2. So the proof of Lemma 4 is completed.

LEMMA 5. *Let T be as in Proposition 2′ and T_1 be any simple branch of T. Suppose T is minimal and not linear. Then all weights on T_1 are* $\leqq -2$.

PROOF. Suppose T_1 has some nonnegative weights. By the above lemma, $T_1 \neq A(0)$. So, as in Lemma 1, by blowing-up and blowing-down, if necessary, within T_1, so as not to disturb the nonlinearity of T, we can assume that T_1 contains $A(0, 0)$ in such a way that $T_1 - A(0, 0)$ is either empty or a simple branch of $T'=T-A(0, 0)$. As in Lemma 1, we now obtain a $\boldsymbol{P}^1$-fibration $\psi\colon S \to \boldsymbol{P}^1$ such that $T' = T - A(0, 0)$ is contained in a single fibre of ψ. In particular, T' is linear. Moreover, $\pi_1(M_T) \simeq \pi_1(M_{T'}) \simeq \boldsymbol{Z}$. Hence it easily follows that T' is the full fibre. By (R), it follows that T' is minimal, and hence $T' = A(0)$. Thus T is linear which is a contradiction. This proves Lemma 5.

LEMMA 6. *Let T be as in Proposition 2′, T' be any proper subtree (i.e., $T' \subsetneqq T$). Then T' satisfies* (E).

PROOF. Since $\pi_1(M_T) \cong \boldsymbol{Z}$, rank $H_1(M_T)=1$. Thus, if $K=\operatorname{Ker}(i_*\colon H_2(C) \to H_2(S))$, then $K \cong \boldsymbol{Z}$. Let $D=\sum_{i=0}^{t} \lambda_i C_i$ be the divisor such that the homology class (D) generates K. We claim that $\lambda_i \neq 0$, for all i. Assuming on the contrary, let us say, $\lambda_0 = 0$. Write $T - \{C_0\}$ as a disjoint union of trees, $T - \{C_0\} = T_1 \amalg \cdots \amalg T_k$, $k \geqq 1$, and let $D = \sum D_i$ with $\operatorname{supp}(D_i) \subset T_i$. We claim first that $D_i = 0$ except for one i. If not, suppose $D_1 \neq 0$ and $D_2 \neq 0$. It follows that I_{T_1} and I_{T_2} are nondegenerate. Also since $(D)^2=0$, one easily sees that $(D_1)^2 = (D_2)^2 = 0$. Hence both I_{T_1} and I_{T_2} have positive eigenvalues which contradicts the Hodge Index theorem. Thus we may assume that $D = D_1$ say. Now suppose C_1 is the vertex in T_1 joined to C_0 in T. Then $0=(C_1.D)=\lambda_1$. Repeating the above argument we finally arrive at the conclusion that $\lambda_i = 0$ for all i which is absurd. Thus $\lambda_i \neq 0$ for any i. Hence it follows that if T' is any proper subtree of T then $I_{T'}$ is nondegenerate, and hence T' satisfies (E).

PROOF OF PROPOSITION 2′. We shall assume that T is minimal, and show that it is linear, and then the conclusion of the Proposition 2′ follows easily.

So, suppose T is not linear. By Lemma 5, all simple branches of T have weights $\leqq -2$. In particular, for any simple branch Γ of T, we have $\pi_1(M_\Gamma)$ is nontrivial finite cyclic group. By **1.7**, it follows that T has more than one branch points. By Lemma 6, it follows that T cannot have two disjoint subtrees isomorphic $A(-1, -1)$. Hence by Proposition 2 and the remark below that, $\pi_1(M_T)$ is noncyclic. This contradiction completes the proof of Proposition 2′.

2.4. Finally we come to the central result of this paper.

PROPOSITION 3. *Let C be a curve with* (SNC) *on a compact surface S. Suppose $S - C = V$ is proper homotopy equivalent to $\boldsymbol{C}^* \times \boldsymbol{C}^*$. Then Γ_C is equivalent to one of the trees shown in Figure* 5.

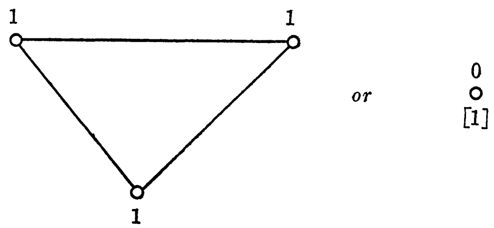

FIGURE 5

PROOF. Note that $G = \pi_1^\infty(V) \simeq \pi_1^\infty(\boldsymbol{C}^* \times \boldsymbol{C}^*) \simeq \boldsymbol{Z}^3 \simeq \pi_1(M_C)$. Thus as observed in 1.5, Γ_C may have three distinct features. In the first two cases below we closely follow [U].

(i) We first show that Γ_C is not a tree of lines or equivalently $H_1(C) \neq (0)$.

In the homology exact sequence

$$H_3(S) \to H_3(S, C) \to H_2(C) \xrightarrow{i_*} H_2(S) \to H_2(S, C) \to H_1(C) \to H_1(S) \to H_1(S, C)$$

we have $H_1(S, C) \simeq H^3(V) = (0)$ and $H_3(S, C) \simeq H^1(V) \simeq \boldsymbol{Z} \oplus \boldsymbol{Z}$. Thus if $H_1(C) = (0)$, then $H_1(S) = 0$, $H_3(S) \simeq H^1(S) = 0$. It follows that $\operatorname{rank} \operatorname{Im}(i_*) = \operatorname{rank} H_2(C) - 2 = t - 1$. But $\operatorname{rank} \operatorname{Im}(i_*) = \operatorname{rank} I_C = t + 1 - \operatorname{rank} H_1(M_C) + \operatorname{rank} H_1(C) = t - 2$. Hence $H_1(C) \neq 0$.

(ii) Now consider the case when Γ_C is a tree having one vertex, say C_0, with $g_0 = 1$. Of course other vertices are of genus zero. We may assume Γ_C is minimal and then we should show that Γ_C is a single vertex v with $\Omega_v = 0$ and $g_v = 1$.

Assuming the contrary, suppose $\Gamma_C - \{C_0\} = \cup_{i=1}^k T_i$, where T_i is a nonempty tree of rational curves. Let $C_i \in T_i$ be the vertex joined to

C_0. Ueda ([U] page 87) has shown that $\pi_1(M_{T_i}) = (1)$ for each $i \geqq 1$. His proof of the fact that each T_i is negative definite holds here also. By virtue of Proposition 1, T_i are not minimal. Since Γ_C is minimal, this means that $(C_i^2) = -1$ and T_i has exactly two branches at C_i, say γ_{i1} and γ_{i2}. Using the presentation of G in (b) of 1.5 we have

$$G/\langle C_i\rangle \simeq (\pi_1(M_{\Gamma_0}) * \pi_1(M_{\gamma_{i1}}) * \pi_1(M_{\gamma_{i2}}))/\langle C_0 * D_{i1} * D_{i2}\rangle$$

where Γ_0 is the branch of $\Gamma - \{C_i\}$ containing C_0, and where D_{ij} are the vertices in γ_{ij} joined to C_i. Hence $\pi_1(M_{\Gamma_0}) \neq (1)$. On the other hand, $G/\langle C_i\rangle$ is abelian and hence one of the groups $\pi_1(M_{\gamma_{ij}}) = (1)$, say $j = 1$. Repeat this argument with γ_{i1} in place of T_i and conclude that $(D_{i1}^2) = -1$. Thus T_i has a subtree isomorphic to $A(-1, -1)$ which is indefinite, contradicting the observation that T_i is negative define. Thus $\Gamma_C = \{C_0\}$. That $(C_0^2) = 0$ follows from the fact that if $(C_0^2) = \pm n$, then $\pi_1(M_{C_0})$ is isomorphic to $\{x, y, z\,|\,[x, y] = [x, y], [y, z] = x^n\}$ which is abelian if and only if $n = 0$.

(iii) Finally let Γ_C be a graph of lines with one essential cycle. Let $C_0, \cdots, C_r$, be the vertices on this cycle. ($r \geqq 2$ by (SNC)). We assume that Γ_C is minimal and then show that Γ_C has no other vertices ($t = r$). Then by Proposition 3 of [U], it follows that Γ_C is one of the trees shown in Figure 6.

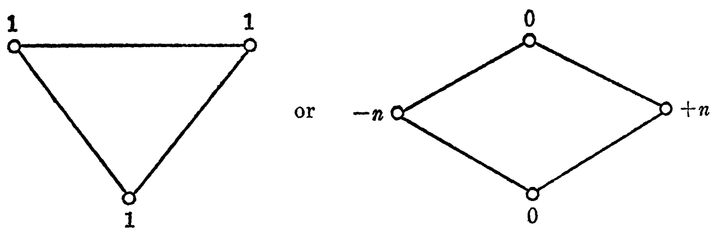

FIGURE 6

Since the latter is easily seen to be equivalent to the former, this would complete the proof of Proposition 3.

So let $T_0 = C_0 \cup \cdots \cup C_r$ be the cycle in Γ_C. First we show that there are no branch points of Γ_C in $\Gamma_C - T_0$. If possible let $D_0 \in \Gamma_C - T_0$ be a branch points of Γ_C. Write $\Gamma_C - \{D_0\} = \Gamma_0 \cup \Gamma_1 \cdots \cup \Gamma_k$, say, $k \geqq 2$. Here Γ_0 denotes the component containing T_0. Let $E_i \in \Gamma_i$ be the vertices of Γ_i joined to D_0. Then we have

$$G/\langle D_0\rangle \simeq (G_0 * G_1 * \cdots * G_k)/\langle E_0 * \cdots * E_k\rangle$$

where $G_i = \pi_1(M_{\Gamma_i})$. $G_0 \neq (1)$ since Γ_0 contains T_0. Since G is abelian, we may assume that $G_2 = \cdots = G_k = (1)$. By Lemma 3, Γ_2 is negative definite also. Proceed now exactly as in the paragraph (ii) to produce a subtree of Γ_2 isomorphic to $A(-1, -1)$, and thereby a contradiction. The branch points of Γ, if any, are on the cycle T_0.

Write now $\Gamma_C = T_0 \cup T'_1 \cup \cdots \cup T'_m$ where each T'_i is linear. Again by Lemma 3 each T'_i is negative definite, $i \geqq 1$. Now if $\Gamma_C \neq T_0$ then at least one of the $C_0, \cdots, C_r$ is a branch point, say C_0. Write $\Gamma_C - \{C_0\} = \gamma_0 \cup T_1 \cup \cdots \cup T_l$, where γ_0 is the branch that contains $C_1, C_2, \cdots, C_r$ and $l \geqq 1$. Let $D_i \in T_i$ be the vertex that is joined to C_0. Let $G_0 = \pi_1(M_{\gamma_0})$, $G_i = \pi_1(M_{T_i})$. Since there are no branch points of Γ_C on T_i and Γ_C is minimal, it follows that T_i are also minimal. Hence G_i are finite non-trivial cyclic groups of order m_i, $i \geqq 1$. Using the presentation (c) of 1.6 for G, we have

$$G/\langle C_0 \rangle \simeq (G_0 * \boldsymbol{Z}(u) * G_1 * \cdots * G_l)/\langle C_1 u C_r u^{-1} * D_1 * \cdots * D_l \rangle \,.$$

First of all it follows that $l \leqq 1$. If C_1 and C_r are nontrivial elements of G_0 then $x = C_1 u C_r u^{-1} \in G_0 * \boldsymbol{Z}(u)$ is an element of infinite order and hence we have $G/\langle C_0 \rangle \simeq C_0 * \boldsymbol{Z}(u)/\langle x^{m_1} \rangle$. Since $m_1 \neq 1$, this latter group is not abelian and so we conclude that $C_1 = 1$ or $C_r = 1$ in C_0. But then it is easily seen that $\boldsymbol{Z}(u)$ is a free factor of $G/\langle C_0 \rangle$ and hence $G/\langle C_0 \rangle$, being abelian, is isomorphic to $\boldsymbol{Z}(u)$. This is absurd, since $G \simeq \boldsymbol{Z}^3$. This completes the proof of Proposition 3.

2.5. We remark that in Proposition 3, the assumption on $V = S - C$ is unnecessarily strong. It suffices to assume that $\pi_1(M_C) \simeq \boldsymbol{Z}^3$. Then except for (i) the rest of the argument still holds, and (i) can be proved in a different way.

3. Proofs of Theorems A and B.

3.1. Let $S - C = V$ be proper homotopy equivalent to $\boldsymbol{C} \times \boldsymbol{C}^*$. Then $G = \pi_1^\infty(S - C) \simeq \pi_1^\infty(\boldsymbol{C} \times \boldsymbol{C}^*) \simeq \boldsymbol{Z}$. As observed in 1.5 all components of C are rational and Γ_C has at most one essential cycle (case (c) cannot occur). We shall first show that Γ_C has no cycles. If not, we then have $H_1(C) \simeq \boldsymbol{Z}$. So $K = \operatorname{Ker} \varphi_* = (0)$. Hence I_C is unimodular. If I_C has positive eigenvalues then we get an effective divisor D on S with $(D^2) > 0$. This implies S is algebraic. On the other hand, in the homology exact sequence

$$H_2(S, C) \to H_1(C) \to H_1(S) \to H_1(S, C)$$

we have $H_2(S, C) \simeq H^2(S - C) = 0$ and $H_1(S, C) \simeq H^3(S - C) = 0$. So the Betti number $\beta_1(S) = 1$. Since algebraic surfaces have even first Betti numbers this proves that I_C has no positive eigen values. Hence I_C is negative definite. Now by 2.5 and 2.3 of [W] it follows that $\pi_1(M_C)$

cannot be infinite cyclic which is a contradiction.

3.2. Thus Γ_C is a tree of lines. By Proposition 2′ we can assume that it is $A(0)$ or $A(1, 1)$. In the former case by Lemma 1 we have a surjective holomorphic mapping $\psi: S \to \Delta$ with C as a good fibre. It follows that $\pi_1(S - C) \simeq \pi_1(\Delta - \psi(C))$ which is a free group of rank $\neq 1$. This contradiction shows that Γ_C is equivalent to $A(1, 1)$. Thus we have shown that any compactification of V is equivalent to (S, C) where $\Gamma_C = A(1, 1)$. It is not difficult to see that $S \simeq \boldsymbol{P}^2$ as required, in Theorem A.

3.3. We now consider Theorem B. So let $S - C = V$ be proper homotopy equivalent $\boldsymbol{C}^* \times \boldsymbol{C}^*$. By Proposition 3 we may assume that Γ_C is a cycle of three lines as described, or consists of a single elliptic curve E with $(E^2) = 0$. In the former case it is easily seen that $S \simeq \boldsymbol{P}^2$ as required. The latter case is essentially contained in [Si] (S, algebraic) and [U] (S, nonalgebraic) except that these authors work under somewhat stronger hypothesis. Below we shall prove a lemma which along with completing the proof of Theorem B will also give the proof of the corresponding (easier) portion of Theorems C and D.

3.4. LEMMA. *Let S be a compact surface, $C \subset S$ be an irreducible smooth curve of genus $g \geqq 1$ with $S - C = V$ having the proper homotopy type of the total space of an affine $\boldsymbol{C}$-bundle over a smooth curve of genus g. Then either S is a smooth $\boldsymbol{P}^1$-bundle with C as a section or S is a Hopf surface, with $(C)^2 = 0$ and $g = 1$.*

PROOF. In the homology exact sequence

$$0 \to H_3(S) \to H_3(S, C) \to H_2(C) \to H_2(S) \to H_2(S, C)$$
$$\to H_1(C) \to H_1(S) \to H_1(S, C)\ ,$$

we have $H_i(S, C) \simeq H^{4-i}(V)$ for each i. Since V is homotopy equivalent to a curve of genus g, we have $H_1(S, C) = 0$, $H_2(S, C) \simeq \boldsymbol{Z}$ and $H_3(S, C) \simeq \boldsymbol{Z}^{2g}$. It follows that $\beta_1(C) - 1 \leqq \beta_1(S) \leqq \beta_1(C) = 2g$ and $\beta_3(S) \leqq 2$. Thus if $\beta_1(S)$ is even then $\beta_1(S) = \beta_1(C) = 2g$ and the topological Euler characteristic is $\chi(S) = 2\chi(C)$. On the other hand if $\beta_1(S)$ is odd then $\beta_2(S) = 0$; hence $(C^2) = 0$. Also in this case we have $\chi(S) = 4 - 4g = 2\chi(C)$. We shall now examine these two cases separately.

Case when $\beta_1(S)$ is even. By Theorem 3 of [K4], $h^{1,0} = q(S) = g$. Consider the Albanese map $\alpha: S \to \mathrm{Alb}(S)$. We claim that α cannot have two dimensional image. For if it does, then $\alpha^*: H^4(\mathrm{Alb}(S)) \to H^4(S)$ is nontrivial. It follows easily that $\alpha^*: H^2(\mathrm{Alb}(S)) \to H^2(S)$ will have an image of rank at least six. Since $\beta_2(S) \leqq 2$ we conclude $\mathrm{Im}\, \alpha$ has dimension $\leqq 1$. Since $g \geqq 1$, α cannot be a constant map and so $\mathrm{Im}\, \alpha$ is a curve Δ.

Then, it is well-known α is a surjective morphism with connected fibres and Δ is a smooth curve of genus g. Since the composite $C \to S \xrightarrow{\alpha} \Delta$ induces an isomorphism in homology it follows that C is a smooth section of α.

We claim that $\alpha: S \to \Delta$ is a $\boldsymbol{P}^1$-bundle morphism with smooth fibres. If not let F_i, $1 \leqq i \leqq r$, be the singular fibres of α. Since α has a section, none of these is a multiple fibre. Hence $\chi(F_i) > \chi(F)$ where F denotes a smooth fibre. Now we have

$$\chi(S) = \chi(F)\chi(\Delta) + \sum_{i=1}^{r} \chi(F_i) = 2\chi(\Delta) \leqq 0 .$$

This shows that $\chi(F) > 0$ unless $\chi(\Delta) = 0$ and in any case $r = 0$. Hence α is smooth.

If $\chi(\Delta) \neq 0$ then $\chi(F) > 0$, i.e., $F \simeq \boldsymbol{P}^1$. Restriction of α to $S - C$ yields an exact sequence of fundamental groups

$$1 \to \pi_1(F - \text{a point}) \to \pi_1(S - C) \to \pi_1(\Delta) \to 1 .$$

Thus if $\chi(\Delta) = 0$, i.e., $g = 1$, then $\pi_1(S - C) \simeq \boldsymbol{Z} + \boldsymbol{Z} \simeq \pi_1(\Delta)$ and hence $\pi_1(F - \text{a point}) = (1)$, i.e., $F \simeq \boldsymbol{P}^1$. Thus in any case $\alpha: S \to \Delta$ is a smooth $\boldsymbol{P}^1$-fibration as required.

Case when $\beta_1(S)$ is odd. Clearly S is nonalgebraic and since $\beta_2(S) = 0$ as observed above, we have $p_g = 0$. Hence by Theorem 2.1 of Kadaira [K2] S is a surface of class VII_0. In particular, $\beta_1(S) = 1$ and hence C is an elliptic curve. Here again there are two subcases. If S has nonconstant meromorphic functions then, by Theorem 4.1 and Theorem 4.3 of [K1], S is an elliptic surface, $\varphi: S \to \Delta$, over a curve Δ with C as a fibre. Since $\chi(S) = 2\chi(C) = 0$, singular fibres of φ, if any, will be of type mI_0. Let F_i, $i = 1, \cdots, k$ be the singular fibres of $\varphi | V$, of type $m_i I_0$, respetively. The local analysis of the fundamental group of a tubular neighbourhood of these fibres as done in §5 of [K4], together with a simple application of Van Kampen's theorem yield a surjection of the fundamental group $\pi_1(V)$ onto a free product $(\boldsymbol{Z}/(m_1)) * \cdots * (\boldsymbol{Z}/(m_k))$. On the other hand, since $g = 1$, it follows from the hypothesis that $\pi_1(V) \simeq \boldsymbol{Z} \oplus \boldsymbol{Z}$. Hence $k \leqq 1$. Thus $\varphi: S \to \Delta$ is an elliptic fibre space with at most two singular fibres, and these singular fibres are of type mI_0. By Lemma 8 of [K3] S is a Hopf surface. Finally if S has no nonconstant meromorphic functions then we appeal to Theorem 34 of [K3] to conclude that S is a Hopf surface.

This completes the proof of the lemma and thereby the proof of Theorem B.

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