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HADAMARD MANIFOLDS AND THE VISIBILITY AXIOM

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0. Introduction. A complete simply connected Riemannian manifold with non-positive sectional curvature, called an Hadamard manifold and denoted by H, is diffeomorphic to an n-dimensional Euclidean space, where n is the dimension of H. In [5], Eberlein and O'Neill defined points at infinity, denoted by $H(\infty)$, of H and investigated geometry and isometries of the manifold H. They showed that isometries of an Hadamard manifold satisfying the visibility axiom (cf. §1) share many of the properties of linear fractional transformations. If all the sectional curvatures of H are smaller than a negative number, than H satisfies the visibility axiom. On the other hand, using the Gauss-Bonnet formula, Eberlein [4] and Shiohama gave examples of complete two-dimensional Riemannian manifolds of negative Gaussian curvature, not necessarily simply connected, which do not satisfy the visibility axiom. We shall prove a condition expressed in terms of the growth rate of the lengths of Jacobi fields Y such that $Y(0) = 0$, for an Hadamard manifold to satisfy the visibility axiom.

Recently, Gromov introduced the Tits metric on points at infinity $H(\infty)$ and described geometric properties of $H(\infty)$ in terms of the metric. He proved that an Hadamard manifold satisfies the visibility axiom if and only if the Tits distance of any pair of distinct points at infinity is infinite (cf. [1, Lemma 4.14]). Using the result one can get the same result as ours.

Any point O in an Hadamard manifold H is a pole of H. Let (x, r) be the geodesic polar coordinates around O, where x is a point of the unit sphere S in the tangent space at O and r is a positive real number. Then in these polar coordinates $S\times\{r\}$ is the geodesic sphere in H of radius r with center O. Thus the Riemannian metric on the geodesic sphere induces a Riemannian metric on S, denoted by $\psi(\cdot, \cdot; r)$. Then

$$\langle d\exp_{rx}(r\xi), d\exp_{rx}(r\eta)\rangle = \psi(\xi, \eta; r) ,$$

where $\langle\ ,\ \rangle$ is the Riemannian inner product of H and ξ, η are tangent vectors at x. Let TS be the tangent bundle of S. We define a function F on $TS\times(0, \infty)$ by

$$F(\xi, r) = [\psi(\xi, \xi; r)]^{1/2} ,$$

that is, $F(\xi, r)$ is the length of the Jacobi field Y along the geodesic $\exp_O tx$ such that $Y(0) = 0$ and $Y'(0) = \xi$. Thus

$$\lim_{r\to 0} F(\xi, r) = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{\partial F}{\partial r}(\xi, r) = \|\xi\|_O ,$$

where $\| \quad \|_O$ denotes the norm induced by the Riemannian metric of S.

THEOREM. *An Hadamard manifold H satisfies the visibility axiom if and only if there exists a point $O \in H$ such that, for every Lipschitz curve $c: [a, b] \to S$ with non-zero length, we have*

$$\lim_{r\to 0} \int_a^b \frac{\partial F}{\partial r}(\dot{c}(s), r) ds = \infty .$$

Using the theorem, we can produce many Hadamard manifolds which satisfy the visibility axiom and whose sectional curvatures tend to zero as the distance from a point $O \in H$ increases to infinity. For example, if, for every unit tangent vector ξ of the unit sphere in the tangent space at O,

$$\frac{\partial F}{\partial r}(\xi, r) \sim r \quad \text{or} \quad \frac{\partial F}{\partial r}(\xi, r) \sim \log r ,$$

as r tends to infinity, then the Hadamard manifold satisfies the above condition. Greene and Wu proved that the Riemannian manifold of non-positive sectional curvature is flat if the sectional curvature goes to zero rapidly as the distance from a fixed point goes to infinity (cf. [6, Theorems 2 and 4]). Therefore we would like to explain how to get these metrics. Let h_1 and h_2 be non-decreasing C^∞-functions which we obtain by smoothing the following functions $\tilde{h}_1$ and $\tilde{h}_2$ on $(0, \infty)$, respectively:

$$\tilde{h}_1(r) = 1 \text{ for } r \in (0, 1) \quad \text{and} \quad \tilde{h}_1(r) = r \text{ for } r \in [1, \infty)$$

and

$$\tilde{h}_2(r) = 1 \text{ for } r \in (0, e) \quad \text{and} \quad \tilde{h}_2(r) = \log r \text{ for } r \in [e, \infty) .$$

Let $F_1(\xi, r) = \int_0^r h_1(t)dt$ and $F_2(\xi, r) = \int_0^r h_2(t)dt$. Using the equation 7.9, (3) in [2], we see that the metrics satisfy the required properties. Especially, a two-dimensional Hadamard manifold with a Riemannian metric of the form

$$ds^2 = dr^2 + r \log r d\theta^2 \quad \text{for} \quad r \geqq 2$$

is not hyperbolic (cf. [7]) but satisfies the visibility axiom.

Furthermore, in the proposition in Section 2, we get the following: Let H be an Hadamard manifold and let γ be a geodesic in H. Then the angular length of γ with respect to any point O is at most π.

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1. Notation and Preliminaries. In this paper H will always denote an Hadamard manifold, that is, a complete, connected, simply connected Riemannian manifold of non-positive sectional curvature.

The following two facts are well known (cf. [5]).

FACT 1. The length of a Jacobi field along a geodesic of H is a convex function.

FACT 2. Let γ_1 and γ_2 be geodesics of H. Then $d(\gamma_1(t), \gamma_2(t))$ is a convex function with respect to the variable t, where $d(\gamma_1(t), \gamma_2(t))$ is the distance between $\gamma_1(t)$ and $\gamma_2(t)$.

Thus the function $F(\xi, r)$ is a convex function with respect to the variable r. Let O, P and Q be points in H. The angle subtended by P, Q at O, denoted by $\sphericalangle_O(P, Q)$, is defined as the angle subtended by $\dot{\gamma}_P(0)$, $\dot{\gamma}_Q(0)$ at 0 in the tangent space of H at O, where γ_P and γ_Q are geodesics from O to P and from O to Q, respectively. For a point x in S and for a real number ε with $0<\varepsilon<\pi$, the set $\{P\in H\colon \sphericalangle_O(\gamma_x(1), P)<\varepsilon\}$, denoted by $C(x, \varepsilon)$, is called the cone of vertex O, axis x and angle ε.

DEFINITION 1 (cf. [5]). An Hadamard manifold H is said to satisfy the visibility axiom if, for a point O in H and for a positive number ε, there exists a positive real number R depending only on O and ε with the following property: If τ is a geodesic with $d(\tau, O) \geqq R$, then $\sphericalangle_O(\tau(t_1), \tau(t_2)) < \varepsilon$ for all real numbers t_1, t_2.

PROPOSITION A (cf. [5, Proposition 4.4]). *An Hadamard manifold H satisfies the visibility axiom if and only if, for two different geodesics γ_1 and γ_2 with $\gamma_1(0) = \gamma_2(0)$, there exists a positive real number R such that $d(O, \tau_t) < R$ for all real numbers t, where τ_t is the geodesic segment joining $\gamma_1(t)$ and $\gamma_2(t)$.*

By Proposition A, we have the following fact: An Hadamard manifold H satisfies the visibility axiom if and only if any pair of distinct points at infinity can be joined by a geodesic of H.

DEFINITION 2. A continuous curve in a Riemannian manifold is called a Lipschitz curve if all its local coordinates are locally Lipschitz functions.

Since the local coordinates of a Lipschitz curve are differentiable almost everywhere, we can define the length of the curve.

2. Angular length. Let O be a point in an Hadamard manifold H and S the unit sphere in the tangent space of H at O. Then S is a Riemannian manifold with the induced metric. Let γ be a geodesic of H with unit speed such that $\gamma(0)$ is the point which attains the distance between O and γ. Let $c\colon(-\infty, \infty)\to S$ and $r\colon(-\infty, \infty)\to(0, \infty)$ be curves such that $\gamma(t)=\exp_O r(t)c(t)$. We call the length of c the angular length of γ at O.

PROPOSITION. *Set* $\rho := d(O, \gamma) = d(O, \gamma(0))$. *Then*

(1) $\rho^{-1}\int_{-\infty}^{\infty} F(\dot{c}(s), \rho)ds \leqq \pi$.

(2) *Angular length* $(c) \leqq \pi$.

PROOF. Let β be the angle as in Figure 1. We have

$$F(\dot{c}(s), r(s)) = \sin\beta\ .$$

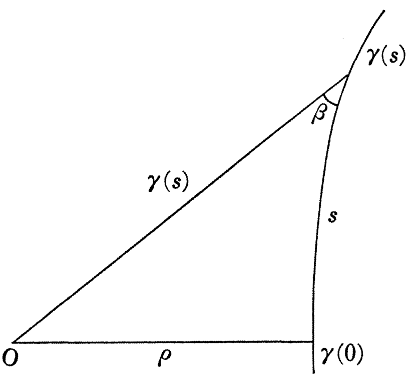

FIGURE 1

Compare the triangle in Figure 1 with the triangle in the Euclidean plane whose sides have the same length, see Figure 2. The angle $\tilde{\beta}$ in Figure 2 satisfies $\beta \leqq \tilde{\beta} \leqq \pi/2$ (cf. [3, Corollary 6.4.3]) and has

$$\sin\tilde{\beta} \leqq \rho/r(s) \qquad \text{(see Figure 2)}\ .$$

Thus

$$\frac{F(\dot{c}(s), r(s))}{r(s)} \leqq \frac{\rho}{(r(s))^2}\ .$$

Since the function $F(\dot{c}(s), r)/r$ is non-decreasing in r,

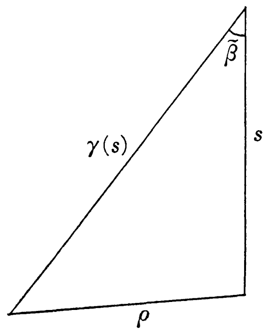

FIGURE 2

$$\frac{F(\dot{c}(s), \rho)}{\rho} \leqq \frac{F(\dot{c}(s), r(s))}{r(s)} .$$

The cosine inequality applied to the triangle in Figure 1 gives

$$\rho^2 + s^2 \leqq (r(s))^2 .$$

Thus

$$\frac{F(\dot{c}(s), \rho)}{\rho} \leqq \frac{\rho}{\rho^2 + s^2} .$$

Integrating with respect to s, we have (1). Using the equality

$$\text{Length } (c) = \lim_{r\to 0} r^{-1}\int_{-\infty}^{\infty} F(\dot{c}(s), r) ds ,$$

we have (2). q.e.d.

3. Proof of the theorem. We use the following lemma which we can prove easily using the Arzela-Ascoli theorem.

LEMMA. *Let $c_k\colon [0, 1] \to M$, $k = 1, 2, \cdots$, be C^1-curves in a compact Riemannian manifold M with the following properties.*

(1) The parameter s of c_k is proportional to the arc length of c_k from $c_k(0)$.

(2) The lengths of c_k's are bounded by a positive number L. Then there exist a Lipschitz curve c and a subsequence $\{c_{k_i}\}$ of $\{c_k\}$ which converges to the curve c pointwise.

Now we prove the theorem. Suppose an Hadamard manifold H satisfies the visibility axiom. Let $c\colon [a, b] \to S$ be as in the statement of the theorem. We shall show the inequality

$$\int_a^b \frac{\partial F}{\partial r}(\dot{c}(s), r) ds \geqq 1 ,$$

for large enough r. The theorem can then be deduced by partitioning c into an arbitrarily large number of pieces. By considering only part of c if necessary, one may assume that $\measuredangle_O(c(a), c(b)) > 0$. Since H satisfies the visibility axiom, there is $R > 0$ such that for any $r > 0$, the geodesic σ from $\exp_O rc(a)$ to $\exp_O rc(b)$ must pass within distance R of O. It follows that if $r > R$, then

$$f(r) := \int_a^b F(\dot{c}(s), r)ds \geqq \mathrm{dist}(\exp_O rc(a), \exp_O rc(b)) \geqq 2(r - R)$$

(see Figure 3). Since $f(r)$ is a convex function of r, it follows from the above inequality that

$$\frac{df}{dr}(r) \geqq 1$$

for large enough r.

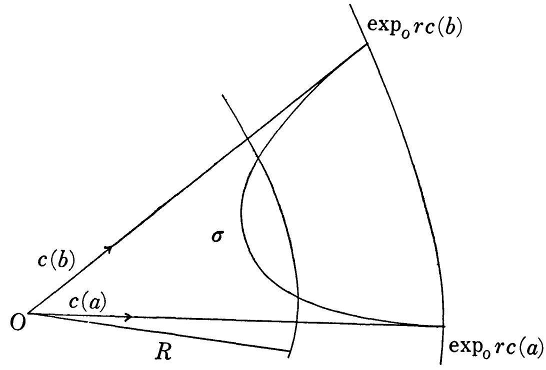

FIGURE 3

Conversely, assume that for every Lipschitz curve $c: [0, 1] \to S$ with non-zero length we have

$$\lim_{r\to\infty} \int_a^b \frac{\partial F}{\partial r}(\dot{c}(s), r)ds = \infty .$$

We shall show that the visibility axiom holds at O. If not, there are $\varepsilon > 0$ and a sequence $\{\gamma_k\}$ of geodesics which subtend an angle greater than ε at O such that $\mathrm{dist}(\gamma_k, O)$ goes to infinity as k tends to infinity. Let $\tilde{c}_k: (-\infty, \infty) \to S$ and $\tilde{r}_k: (-\infty, \infty) \to (0, \infty)$ be the curves in the unit tangent sphere and in $(0, \infty)$, respectively, such that

$$\gamma_k(t) = \exp_O \tilde{r}_k(t)\tilde{c}_k(t) .$$

By the proposition in §2, $\varepsilon \leqq \mathrm{length}(\tilde{c}_k) \leqq \pi$ for all k. Let $c_k: [0, 1] \to S$ be the curve which we obtain by parametrizing $\tilde{c}_k$ proportionally to arc length. As in the above lemma, there is a Lipschitz curve $c\colon [0, 1] \to S$ which is a pointwise limit of a subsequence of $\{c_k\}$, which we also denote by $\{c_k\}$ for simplicity. For any fixed r the curve $\sigma_r(s) := \exp_O rc(s)$ is a pointwise limit of $\sigma_{r,k}(s) := \exp_O rc_k(s)$. Thus

$$\operatorname{length}(\sigma_r) \leqq \lim_{k\to\infty} \operatorname{length}(\sigma_{r,k}) \quad \text{for every} \quad r > 0 .$$

Set

$$f_k(r) := \int_0^1 F(\dot{c}_k(s), r)ds = \operatorname{length}(\sigma_{r,k})$$

and

$$f(r) := \int_0^1 F(\dot{c}(s), r)ds = \operatorname{length}(\sigma_r) .$$

Since $f(r)$ is a convex function with $f(0) = 0$ and $\lim_{r\to 0} f'(r) > 0$, it is increasing for $r > 0$. It follows from the assumption made above that there is $r > 0$ with $f(R) \geqq 4R$. Hence $f_k(R) \geqq 3R$ for any large enough k. Since each f_k is convex and $f_k(0) = 0$, it follows that

$$f_k'(R) = \int_0^1 \frac{\partial F}{\partial r}(\dot{c}_k(s), R)ds \geqq 3 \quad \text{for large enough } k .$$

Also we may assume that for all large enough k we have $\operatorname{dist}(\gamma_k, O) > R$. We will now show that for such k there is a length-decreasing variation of γ_k, which will be a contradiction, since any geodesic in H is length minimizing. Let $r_k(s)$ be the distance from O to γ_k along the geodesic defined by the vector $c_k(s)$. Define $\gamma_{k,\varepsilon}\colon [-\varepsilon, 1 + \varepsilon] \to H$ as follows (see Figure 4):

$$\gamma_{k,\varepsilon}(s) = \exp_O((r_k(s) - \varepsilon - s)c_k(0)) \quad \text{if} \quad -\varepsilon \leqq s \leqq 0 ,$$
$$\gamma_{k,\varepsilon}(s) = \exp_O((r_k(s) - \varepsilon)c_k(s)) \quad \text{if} \quad 0 \leqq s \leqq 1 ,$$
$$\gamma_{k,\varepsilon}(s) = \exp_O((r_k(s) - \varepsilon + s - 1)c_k(1)) \quad \text{if} \quad 1 \leqq s \leqq 1 + \varepsilon .$$

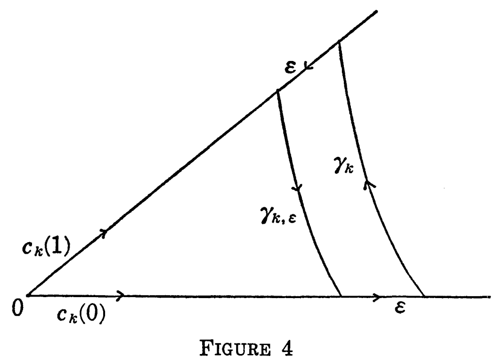

FIGURE 4

Since $\partial F/\partial r(\xi, r)$ is increasing in r,

$$\begin{aligned}\frac{d}{d\varepsilon}(\text{length } \gamma_{k,\varepsilon})|_{\varepsilon=0} &= 2 - \int_0^1 \frac{\partial F}{\partial r}(\dot{c}_k(s), r_k(s))ds \\ &\leqq 2 - \int_0^1 \frac{\partial F}{\partial r}(\dot{c}_k(s), R)ds \leqq -1 .\end{aligned}$$

This proves the theorem.

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