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# PERMUTABILITY OF ENTIRE FUNCTIONS SATISFYING CERTAIN DIFFERENTIAL EQUATIONS

JIAN-HUA ZHENG AND ZHENG-ZHONG ZHOU

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**1.** Two entire functions $f(z)$ and $g(z)$ are said to be permutable if they satisfy the relation

$$f(g(z)) = g(f(z))$$

for any finite complex number $z$. The permutability of entire functions which we are to consider is determining permutable functions of entire functions. As far as the present authors know, there are several papers that discuss this subject (see [8] and [5]) and contain some good ideas which are used in the proof of results.

T. Kobayashi [5] proved the following:

THEOREM A. *Let $f(z) = z + Ce^{az}$, where $a$ and $C$ are constants with $aC \neq 0$. Let $g(z)$ be a non-constant entire function of finite order which is permutable with $f(z)$. Then either $g(z) = f(z) + D$ or $g(z) = z + D$, where $D$ is a constant with* $\exp(aD) = 1$.

The method he used in the proof of Theorem A is very complicated and not suitable in generalizing Theorem A. The purpose of this paper is to decide permutable functions of a class of entire functions satisfying certain differential equation and to indicate an elementary and simple method by which we can generalize Theorem A.

We assume that the reader is familiar with the fundamental concepts in Nevanlinna's theory of meromorphic functions, in particular, with symbols $m(r, f)$, $T(r, f)$ and $M(r, f)$ etc. (see [4]).

**2.** First we state the results of this paper.

THEOREM 1. *Let $f(z)$ and $g(z)$ be both mutually permutable entire functions of finite order and $f(z)$ of positive lower order. Let $P_i(z)$ $(i = 0, 1, \cdots, n+1$, with $n \geqq 1)$ be polynomials not all zero. If $f(z)$ satisfies a differential equation*

$$(1) \qquad P_0(z)f^{(n)}(z) + \cdots + P_n(z)f(z) + P_{n+1}(z) = 0\,,$$

*then there exist polynomials $Q_0(z), \cdots, Q_{n+1}(z)$ not all zero such that*

$$Q_0(z)g^{(n)}(z) + \cdots + Q_n(z)g(z) + Q_{n+1}(z) = 0 .$$

THEOREM 2. *Let $f(z)$ be an entire function of positive lower order and $g(z)$ an entire function of finite order, permutable with $f(z)$. Then the order, lower order, and type-class of $g(z)$ do not exceed ones of $f(z)$, respectively.*

The following theorems are the main results of this paper and reveal the advantage of our method.

THEOREM 3. *Let $f = Q + He^P$, where $Q$ and $H$ ($\not\equiv 0$) are polynomials and $P$ is a non-constant polynomial. Let $g$ be a non-linear entire function of finite order, permutable with $f$. Then $g = Cf + D$, where $C^n = 1$, $D = a_{n-1}(C-1)/(na_n)$, $n = \deg P$, $a_n$ and $a_{n-1}$ are coefficients of the first and second terms of $P$, respectively.*

THEOREM 4. *Let $f(z) = \sin P(z)$, where $P(z)$ is a non-constant polynomial. Let $g(z)$ be an entire function of finite order, permutable with $f(z)$ and non-linear.* (i) *If $f(z)$ is an odd function, then $g(z) = f(z)$ or $g(z) = -f(z)$;* (ii) *if $f(z)$ is not an odd function, then when $\deg P = 1$, we have $g(z) = f(z)$, while when $\deg P > 1$, either $g(z) = f(z)$ or $g(z) = -\sin(P(z) - c)$, where $c$ satisfies $\exp(inc) = (-1)^{n+1}$, $n = \deg P$ and $P(\sin(-z + c)) = -P(\sin z) + c$.*

We need the following lemmas for the proof of the above theorems.

LEMMA 1 (cf. [3]). *Let $F_0(z), \cdots, F_m(z)$ be entire functions not vanishing identically and $h_0(z), \cdots, h_m(z)$ ($m \geqq 1$) be arbitrary meromorphic functions not all zero. Let $g(z)$ be a non-constant entire function, $K$ a positive real number and $\{r_j\}$ an unbounded monotone increasing sequence of positive real numbers such that, for each $j$,*

$$T(r_j, h_i) \leqq KT(r_j, g) \quad (i = 0, 1, \cdots, m) ,$$

$$T(r_j, g') \leqq (1 + o(1))T(r_j, g) .$$

*If $F_i(z)$ and $h_i(z)$ ($i = 0, 1, \cdots, m$) satisfy*

$$F_0(g)h_0 + F_1(g)h_1 + \cdots + F_m(g)h_m = 0 ,$$

*then there exist polynomials $P_0(z), \cdots, P_m(z)$ not all zero such that*

$$F_0(z)P_0(z) + F_1(z)P_1(z) + \cdots + F_m(z)P_m(z) = 0 .$$

LEMMA 2 (cf. [7]). *Let $P(z)$ and $Q(z)$ be polynomials of degree greater than one. If the equation*

$$P(f(z)) = f(Q(z))$$

*admits a meromorphic solution* $f(z)$, *then* $\deg P = \deg Q$ *and the solution* $f(z)$ *is not transcendental.*

LEMMA 3 (Borel's theorem [2]). *Let* $F_1(z), \cdots, F_m(z)$ *be entire functions such that* $F_i(z) - F_j(z)$ *is non-constant for* $i \neq j$. *Let* $h_1(z), \cdots, h_m(z)$ *be meromorphic functions of finite order such that*

$$\rho(h_i) < \min\{\rho(\exp(F_s - F_t));\ s, t = 1, \cdots, m,\ s \neq t\}$$

($i = 1, \cdots, m$), *where* $\rho(h_i)$ *denotes the order of* $h_i$. *If*

$$h_1(z)\exp(F_1(z)) + \cdots + h_m(z)\exp(F_m(z)) = 0\ ,$$

*then* $h_1(z) = \cdots = h_m(z) = 0$.

**3.** We state the proof of the above theorems.

PROOF OF THEOREM 1. Since $f(g) = g(f)$ by Polya's theorem (cf. [6]), we have

$$M(M(r, f), g) \geqq M(r, g(f)) = M(r, f(g)) > M(cM(r/2, g), f)\ ,$$

where $c$ is a positive constant between zero and one. Since $g$ is of finite order and $f$ of positive lower order, there exist positive $K_1$ and $K_2$ such that

$$K_1 \log M(r, f) > \log\log M(cM(r/2, g), f) > K_2 \log M(r/2, g) \geqq K_2 T(r/2, g)\ .$$

By [4, Theorem 1.6], for each positive number $r$ and any real number $R > r$, we get $\log^+ M(r, f) < (R + r)(R - r)^{-1} T(R, f)$. Thus $\log^+ M(r, f) < 3T(2r, f)$. Hence we easily obtain

$$T(r, g) < K_3 T(4r, f)\ , \tag{2}$$

for a positive number $K_3$.

Since $f$ is of finite order, there exists Polya's peak $\{r_j\}$ (cf. [1]) of $f$. Namely there exist three sequence $\{r'_j\}$, $\{r''_j\}$, $\{\varepsilon_j\}$ satisfying $r'_j \to +\infty$, $r_j/r'_j \to +\infty$, $r''_j/r_j \to +\infty$, $\varepsilon_j \to 0$ $(j \to +\infty)$ and when $r'_j \leqq t \leqq r''_j$,

$$T(t, f) < (1 + \varepsilon_j)\cdot(t/r_j)^{\rho(f)} \cdot T(r_j, f)\ . \tag{3}$$

By the condition $\rho(f) < +\infty$, we have

$$m(r, f^{(k)}/f) = O(\log r) \quad (k = 1, 2, \cdots) \quad \text{(cf. [4])}\ .$$

Hence

$$\begin{aligned} T(r, f^{(k)}) &\leqq m(r, f^{(k)}/f) + m(r, f) \\ &= O(\log r) + T(r, f) \leqq (1 + o(1))T(r, f) \quad (k = 1, 2, \cdots)\ . \end{aligned} \tag{4}$$

Combining (2) and (3), we get

$$\begin{aligned} T(r_j, g^{(k)}) &\leqq (1 + o(1))T(r_j, g) \\ &< 2K_3 T(4r_j, f) < 2K_3 4^{\rho(f)} T(r_j, f) \quad (k = 1, 2, \cdots)\ , \end{aligned} \tag{5}$$

since $\rho(g) < +\infty$.

Differentiating both sides of the equality $f(g) = g(f)$ step by step, we have

$$
\begin{aligned}
&f'(g)g' = g'(f)f' , \\
&f''(g)g'^2 + f'(g)g'' = g''(f)f'^2 + g'(f)f'' , \\
&\cdots\cdots\cdots\cdots \\
&f^{(n)}(g)g'^n + f^{(n-1)}(g)g'^{n-1}g'' + \cdots + f'(g)g^{(n)} \\
&\qquad = g^{(n)}(f)f'^n + g^{(n-1)}(f)f'^{n-1}f'' + \cdots + g'(f)f^{(n)} .
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&f'(g) = (f'/g')g'(f) , \\
&f''(g) = (f'/g')^2 g''(f) + (f'' - g''f'/g')g'(f) , \\
&\cdots\cdots\cdots\cdots \\
&f^{(n)}(g) = (f'/g')^n g^{(n)}(f) + \cdots .
\end{aligned}
\tag{6}
$$

We can rewrite (1) as

$$P_0(g)f^{(n)}(g) + \cdots + P_n(g)f(g) + P_{n+1}(g) = 0 . \tag{7}$$

Substituting each equality in (6) into (7), we have

$$h_0 g^{(n)}(f) + \cdots + h_n g(f) + h_{n+1} = 0 ,$$

where all of $h_0, h_1, \cdots, h_{n+1}$ are the differential polynomials in $z, f, f', \cdots, f^{(n)}, g, g', \cdots, g^{(n)}$. Therefore by the above discussion, we can find out a positive number $A$ such that

$$T(r_j, h_i) < AT(r_j, f) \quad (i = 0, 1, \cdots, n+1) ,$$
$$T(r_j, f') \leqq (1 + o(1))T(r_j, f) .$$

Thus by Lemma 1, we complete the proof of Theorem 1.

REMARK. Fromt he proof of Theorem 1, it is clear that Theorem 2 holds.

PROOF OF THEOREM 3. From the expression of $f(z)$, we easily get

$$Hf' - (H' + HP')f = HQ' - Q(H' + HP') = A(z) \quad \text{(say)} . \tag{8}$$

By Theorem 1, there exist three polymomials $Q_0, Q_1, Q_2$ not all zero such that

$$Q_0 g' - Q_1 g = Q_2 . \tag{9}$$

By Lemma 2, we have $Q_0 \not\equiv 0$ and $Q_1 \not\equiv 0$, unless $g$ is a linear function. From (8) and (9), we have

$$Q_0(f)g'(f) - Q_1(f)g(f) = Q_2(f) ,$$
$$H(g)f'(g) - (H'(g) + H(g)P'(g))f(g) = A(g) ,$$
$$f(g) = g(f) , \quad f'(g)g' = g'(f)f' .$$

By simple calculation, we have

$$[H(g)Q_1(f)f' - Q_0(f)(H'(g) + H(g)P'(g))g']f(g)$$
$$= -H(g)Q_2(f)f' + Q_0(f)A(g)g' .$$

Therefore by Lemma 3 we have

$$(Q_1(f)/Q_0(f))f' = [(H'(g) + H(g)P'(g))/H(g)]g' . \tag{10}$$

We separately treat two cases.

(I) The case where $H$ is not a constant. Clearly $(H' + HP')/H$ is not a polynomial. By (9), for any finite complex number $u$, $g - u$ has at most finitely many multiple zeros. In fact, if $Q_2 + uQ_1 \not\equiv 0$, the result clearly holds; if $Q_2 + uQ_1 \equiv 0$, $g - u$ has at most finitely many zeros. We may assume without loss of generality that $g$ can take any finite complex number. In fact, suppose that $g = c_1 + e^X$, where $c_1$ is a constant and $X$ is a polynomial. By Theorem 2, $\deg X$ is not greater than $\deg P$. We easily get $f = c_1 + c_2(z - c_1)^m \exp P$, for some constant $c_2$ and non-negative integer $m$. Thus it follows from $f(g) = g(f)$ that

$$\exp(X(f)) = c_2 \exp(mX + P(g)) , \quad \text{and} \quad X(f) = mX + c_3 + P(g) ,$$

where $c_3$ is a constant. By Lemma 3, we easily obtain $m = 0$ and $(X - c_3)(f) = P(g)$. This is a special case of the following discussion.

Therefore the functions of both sides of the equality (10) are meromorphic. Furthermore, $Q_0/(Q_0, Q_1)$ and $H/(H' + HP', H)$ are not constant, which we denote by $F(z)$ and $E(z)$, respectively, where $(Q_0, Q_1)$ denotes the greatest common factor of $Q_0$ and $Q_1$. From (10), we know that $E(g)$ and $F(f)$ have zeros at the same points. By Theorem 2, $\rho(g) \leqq \rho(f)$. $f - u$ and $g - u$ have at most finitely many multiple zeros for any complex number $u$. Hence by (10) and the above discussion, we have

$$B(z)E(g) = C(z)F(f)e^{W(z)} ,$$

where $B$, $C$ and $W$ are polynomials and $\deg W \leqq \rho(f)$. Obviously

$$B(f)E(f(g)) = C(f)F(f(f))\exp(W(f)) ,$$
$$B(f)E(Q(g) + H(g)\exp(P(g))) = C(f)F(Q(f) + H(f)\exp(P(f)))\exp(W(f)) .$$

By Lemma 3, we get without difficulty

$$P(g) = aP(f) + bW(f) + G(z) = (aP + bW)(f) + G(z) = L(f) + G \quad (\text{say}) ,$$

where $a$ and $b$ are constants, $G(z)$ is a polynomial. Thus we obtain

$$P(Q(g)+H(g)\exp(P(g)))=L(Q(f)+H(f)\exp(P(f)))+G(f)\,.$$

By Lemma 3 again, we have $P(g)=dP(f)+S(z)$, where $d$ is a constant and $S(z)$ a polynomial. We have $d=1$ by the same method as in the above. Then it follows that

$$P(g)=P(f)+S\,, \tag{11}$$

$$P(Q(g)+H(g)\exp(S+P(f)))=P(Q(f)+H(f)\exp(P(f)))+S(f)\,. \tag{11'}$$

Comparing the coefficients of the first and second terms of both sides of the equality (11′), we see that $H^n(g)\exp(nS)=H^n(f)$ and

$$\begin{aligned} &na_nQ(g)H^{n-1}(g)\exp((n-1)S)+a_{n-1}H^{n-1}(g)\exp((n-1)S)\\ &\quad =na_nQ(f)H^{n-1}(f)+a_{n-1}H^{n-1}(f)\,. \end{aligned}$$

Therefore

$$cH(g)e^S=H(f)\,, \tag{12}$$

$$c(na_nQ(g)+a_{n-1})=na_nQ(f)+a_{n-1}\,,\quad \text{namely}\,,$$

$$Q(g)=c^{-1}Q(f)+a_{n-1}(1-c)/(na_nc)\,, \tag{13}$$

where $c$ is a constant with $c^n=1$. Combining (11), (12) and (13), we have

$$\begin{aligned} g(f)&=f(g)=Q(g)+H(g)\exp(P(g))\\ &=c^{-1}Q(f)+c^{-1}H(f)e^{-S}\exp(P(f)+S)+q=c^{-1}f(f)+q\,, \end{aligned}$$

hence $g=c^{-1}f+q$, where $q=a_{n-1}(1-c)/(na_nc)$.

(II) The case where $H$ is a constant. We may assume that $H=1$. From (10), we get immediately $P(g)=L(f)$, where $L$ is an integral of $(Q_1/Q_0)$. By the same method as in (I), we may assume that $f$ takes any finite complex number. Thus it follows that $L$ is a polynomial. Therefore by the same method as in (I), we have the theorem.

PROOF OF THEOREM 4. Obviously, $f$ satisfies the equation

$$f'^2+P'^2f^2=P'^2\,. \tag{14}$$

Since $f'(g)g'=g'(f)f'$ and $f^2(g)=g^2(f)$, we have

$$(f'/g')^2g'^2(f)+P'^2(g)g^2(f)=P'^2(g)\,.$$

By the same method as in the proof of Theorem 1, we can find out three polynomials $Q_0$, $Q_1$, $Q_2$ not all zero such that

$$Q_0g'^2+Q_1g^2=Q_2\,. \tag{15}$$

By Lemma 2, we have $Q_0\not\equiv 0$ and $Q_1\not\equiv 0$, unless $g$ is a linear function.

Hence from (14) and (15), we have

$$f'^2(g) + P'^2(g)f^2(g) = P'^2(g)\ ,$$

$$Q_0(f)(g'/f')^2 f'^2(g) + Q_1(f)f^2(g) = Q_2(f)\ .$$

Eliminating $f'^2(g)$ from the above equalities, we have

$$[Q_1(f) - P'^2(g)Q_0(f)(g'/f')^2]\cdot f^2(g) = Q_2(f) - P'^2(g)Q_0(f)(g'/f')^2\ .$$

Thus

$$P'^2(g)Q_0(f)(g'/f')^2 = Q_1(f) = Q_2(f)\ .$$

Obviously, $Q_1 = Q_2$ and

$$P'^2(g)g'^2 = (Q_1(f)/Q_0(f))f'^2\ . \tag{16}$$

Put $B = (Q_1/Q_0)$, which is a polynomial, since $f$ takes any finite complex number. Hence from (14), (15) and (16), we obtain $P'^2(g)(1 - g^2)B = B(f)\cdot(1 - f^2)P'^2$, namely,

$$C(g)B = D(f)P'^2\ ,$$

where $C = P'^2(z)(1 - z^2)$ and $D = B(z)(1 - z^2)$. Furthermore, we have $B(f)C(f(g)) = P'^2(f)D(f(f))$, namely,

$$\begin{aligned} &B(f)C((\exp(iP(g)) - \exp(-iP(g)))/(2i)) \\ &\quad = P'^2(f)D((\exp(iP(f)) - \exp(-iP(f)))/(2i))\ . \end{aligned}$$

Then it follows from Lemma 3 that

$$P(g) = dP(f) + G(z)\ , \tag{17}$$

where $G$ is a polynomial and $d$ a constant. Hence we have immediately

$$\begin{aligned} &P((\exp(idP(f) + iG) - \exp(-idP(f) - iG))/(2i)) \\ &\quad = dP((\exp(iP(f)) - \exp(-iP(f)))/(2i)) + G(f)\ . \end{aligned}$$

By Lemma 3 again, $d = 1$ or $d = -1$; furthermore, when $d = 1$, $\exp(inG) = 1$; when $d = -1$, $\exp(inG) = (-1)^{n+1}$; $n = \deg P$. Hence $G$ is a constant. From (17), we have

$$P(\sin(\pm w + G)) = \pm P(\sin w) + G\ . \tag{18}$$

Now we separately treat two cases.

(I) The case where $P = az + b$, where both $a$ $(\neq 0)$ and $b$ are constants. From (17), we have $g = \pm f + c$, where $c = (G \pm b - b)/a$. Since

$$g(f) = f(g) = \sin P(g) = \pm\sin(P(f) \pm G)\ ,$$

we get $g = \pm\sin(P(z) \pm G) = \pm\sin P(z) + c$, further, $\sin(z \pm G) = \sin z \pm c$. By Lemma 3, we have immediately $c = 0$. It follows that $g = \pm f$.

When $f$ is not an odd function, $g \neq -f$. Indeed, suppose that $g = -f$.

We have $-f(f) = g(f) = f(g) = f(-f)$, namely, $-f(z) = f(-z)$. This is a contradiction.

(II) The case where $\deg P > 1$. We consider two subcases.

(1) The subcase where the equality (18) holds for the positive sign. We are to prove that $G = 0$. Indeed, suppose that $G \neq 0$. By (18) we get without difficulty

$$P(\sin(mG))/(mG) = P(0)/(mG) + 1 .$$

As $m \to +\infty$, the right side of the above equality converges to 1, but since $|\exp(iG)| = 1$, the left side converges to 0. This is a contradiction. Thus we have $G = 0$. Therefore, we have $g = f$.

(2) The subcase where the equality (18) holds for the negative sign. From (17), we have

$$g(f) = f(g) = \sin P(g) = -\sin(P(f) - G) ,$$

namely, $g = -\sin(P(z) - G)$.

When $f$ is an odd function, by Lemma 3, we have $P(-z) = -P(z) + 2k\pi$, where $k$ is an integer. Then it follows from (18) that $P(\sin(z - G)) = P(\sin z) + (2k\pi - G)$. Consequently,

$$P(0) = P(\sin(G - G)) = P(\sin(mG)) + m(2k\pi - G) .$$

By the same method as in the subcase (1), we can prove that $G = 2k\pi$. Thus we have $g = -f$.

Hence we complete the proof of the theorem.

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Department of Mathematics and
Anhui University
HeFei
People's Republic of China

Department of Mathematics
Anhui Normal University
WuHu
People's Republic of China