

ACTIONS OF COMPACT CONNECTED LIE GROUPS ON SIMPLY CONNECTED RATIONAL COHOMOLOGY 5-SPHERES

Dedicated to the memory of Professor Shigeo Sasaki

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I. Introduction. In [5], Hudson classified simply connected closed 5-manifolds with $SO(3)$ -actions admitting at least one singular orbit up to $SO(3)$ -equivariant diffeomorphisms. In [9], Ōike classified simply connected closed 5-manifolds with effective $U(2)$ -actions up to $U(2)$ -equivariant diffeomorphisms. It is interesting to know what types of Lie groups other than $SO(3)$ and $U(2)$ can act smoothly and effectively on simply connected closed smooth 5-manifolds. In this note we shall particularly consider such actions on simply connected rational cohomology 5-spheres and classify compact connected non-commutative Lie groups other than $SO(3)$ and $U(2)$ acting smoothly and effectively on the spaces up to isomorphisms. By the restriction to non-commutative Lie groups, we avoid the difficulty of classifying 5-manifolds with torus actions. The results which we obtained are Theorems C and D in Sections IV and V respectively, according to the codimension of the principal orbits of actions. Theorem C and its remarks say that all actions of compact connected non-commutative Lie groups on simply connected rational cohomology 5-spheres with codimension two principal orbits are classified completely up to isomorphisms. The classifications of transitive actions and actions with codimension one principal orbits on the same spaces are almost due to Oniščik, Asoh and Ōike and the results are in III. In II, we recall some necessary facts on smooth actions of compact Lie groups for our later use and we shall list up all the compact connected Lie groups which can act smoothly and almost effectively on simply connected closed 5-manifolds. We refer to Bredon [2] for the basic definitions and well known facts on transformation groups.

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II. Preliminaries.

1. First of all, we set notation and recall the well-known results used in this note. Let (G, M, φ) be a smooth and effective action $\varphi: G \times M \rightarrow M$ of a compact connected Lie group G on a simply connected n -dimensional closed smooth manifold M . Let (K) be some isotropy type of this action. We denote the set of points on orbits of the isotropy type (K) by $M_{(K)}$ and the set of fixed points of the restricted K -action on M by

M^K or $F(K, M)$. We always denote the principal isotropy type by (H) and the dimension of the principal orbit by d . Then the orbit space M^* is simply connected and $(n-d)$ -dimensional and if $n-d \leq 2$, then M^* is a manifold with or without boundary ([2, p. 186]). The following results are well-known ([2, p. 184–p. 186 and p. 191]).

(1.1) Let E and B be the sets of the exceptional orbits and of the singular orbits, respectively. Then $\dim(B \cup E) \leq n-1$, $\dim B \leq n-2$, $\dim(B^* \cup E^*) \leq n-d-1$ and $\dim E^* \leq n-d-2$.

(1.2) Let T be a maximal torus of H and let M_o^T denote the union of those components of M^T which intersect $M_{(H)}$ nontrivially. Then the Weyl group $W(T) = N(T)/T$ acts naturally on M_o^T and the natural map $f: M_o^T/W(T) \rightarrow M/G$ is a homeomorphism and the action of the Weyl group $W(T)$ on M_o^T has the type of finite principal isotropy group $(H \cap N(T))/T$.

(1.3) Differentiable slice theorem. Let $G(x)$ be an orbit of type G/K . Then there exists an equivariant tubular neighborhood $G \times_K D^k$ of $G(x)$, where k is the codimension of $G(x)$ in M and K acts orthogonally on the k -disk D^k via the slice representation $\sigma: K \rightarrow O(k)$.

Let (G_1, M_1, φ_1) and (G_2, M_2, φ_2) be two actions. We say that (G_1, M_1, φ_1) is isomorphic to (G_2, M_2, φ_2) , if there exist a Lie group isomorphism $h: G_1 \rightarrow G_2$ and a diffeomorphism $f: M_1 \rightarrow M_2$ satisfying $f(\varphi_1(g, x)) = \varphi_2(h(g), f(x))$, for every $g \in G_1$ and for every $x \in M_1$. If $G_1 = G_2 = G$ and h is the identity map, then (G, M_1, φ_1) is equivariantly diffeomorphic to (G, M_2, φ_2) . Hence if (G, M_1, φ_1) is equivariantly diffeomorphic to (G, M_2, φ_2) , then they are isomorphic.

Let X_1 and X_2 be compact connected G -manifolds with boundaries ∂X_1 and ∂X_2 , respectively. Assume that $M(f) = X_1 \cup_f X_2$ is obtained from X_1 and X_2 by identifying their boundaries under a G -equivariant diffeomorphism $f: \partial X_1 \rightarrow \partial X_2$.

(1.4) Let $f, f': \partial X_1 \rightarrow \partial X_2$ be G -equivariant diffeomorphisms. Then $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G -manifolds, if one of the following conditions is satisfied:

- (1) f is G -diffeotopic to f' ,
- (2) $f^{-1}f'$ is extendable to a G -equivariant diffeomorphism on X_1 ,
- (3) $f'f^{-1}$ is extendable to a G -equivariant diffeomorphism on X_2 (cf. [2, p. 48] or [11, Lemma 5.3.1]).

(1.5) ([2, V 6.2 Corollary]) Let (G, M, φ) have the principal isotropy type (H) and only one singular isotropy type (K) and let the orbit space M^* be diffeomorphic to a contractible manifold Y . Then the set of G -equivariant diffeomorphism classes of proper G -spaces over Y is in one-to-one correspondence with

$$\Gamma = [\partial Y, (N(H) \cap N(K)) \backslash N(H)] / \pi_0(W(H)),$$

where $[\ , \]$ is the set of the homotopy classes and $\pi_0(W(H))$ acts naturally on the set.

REMARK 1.6. (G, M, φ) in (1.5) is proper if $N(H)$ acts transitively on $(G/K)^H$ and this condition always holds if $(G/K)^H$ is connected ([2, V 4.3 Proposition]).

1.7. Let (G, M, φ) be an action. Then there exists a compact connected Lie group G' such that G' is a finite covering group of G and is a direct product of simply connected simple Lie groups and a torus. Let ρ be the projection. Then an action (G', M, φ') is defined by $\varphi'(g', x) = \varphi(\rho(g'), x)$ for $g' \in G'$ and $x \in M$. This action is almost effective (i.e. only the finite subgroup $\rho^{-1}(1)$ acts trivially on M) and G'_x , the isotropy subgroup at x of the G' -action, equals $\rho^{-1}(G_x)$ for every $x \in M$ and hence $G'/G'_x = G/G_x$. Let us define $H' = \rho^{-1}(H)$. Then H' is the principal isotropy subgroup of (G', M, φ') . Since G' also acts almost effectively on the principal orbit G'/H' , we have:

$$(1.8) \quad H' \text{ does not contain any positive dimensional closed normal subgroup of } G'.$$

Conversely, suppose that we have found an almost effective action (G', M, φ') . Then we can construct an effective action (G, M, φ) from the almost effective action. Indeed, denote $Z(G') \cap H'$ by L , where $Z(G')$ is the center of G' , and denote $G'/L, K'/L, H'/L$ by G, K, H , respectively. Furthermore we consider the induced G -action φ defined by $\varphi([g], x) = \varphi'(g, x)$ for $[g] \in G'/L = G$ and $x \in M$. Then (G, M, φ) is an effective action and (K) is a singular isotropy type and (H) is the principal isotropy type.

2. From now on, we assume that M is a simply connected closed smooth 5-manifold. Let (G, M, φ) be an action and let (G', M, φ') be the action defined in 1.7. Then G and G' also act effectively and almost effectively on the principal orbit $G/H = G'/H'$, respectively. Hence by the theorem of Eisenhardt [6, II Theorem 3.1], $\dim G = \dim G' \leq d(d+1)/2$ and if the equality holds, then G/H is a natural sphere S^d or a real projective space $RP(d)$. Let $d(G') = d(G)$ be the smallest dimension of the principal orbits of all G' - (or G -) actions on M . This dimension was computed by Mann [7, Theorem 1], and $\dim G' \leq 15$ since $d \leq 5$. By both conditions, we have the following result.

LEMMA 2.1. *Let G act smoothly and effectively on a simply connected smooth 5-manifold and let $(G', \dim G', d(G'))$ be a triple of $G', \dim G'$ and $d(G')$ defined above. Then $(G', \dim G', d(G'))$ is one of the following.*

$(T, 1, 1), (T^2, 2, 2), (Spin(3), 3, 2), (T^3, 3, 3), (Spin(3) \times T, 4, 3), (Spin(4), 6, 3), (Spin(3) \times T^2, 5, 4), (Spin(4) \times T, 7, 4), (SU(3), 8, 4), (Spin(5), 10, 4), (SU(3) \times T, 9, 5), (Spin(4) \times Spin(3), 9, 5), (Spin(6), 15, 5).$

In the rest of this note, we classify the actions $\{(G, M, \varphi)\}$ such that G is a non-commutative Lie group and M is a rational cohomology 5-sphere.

III. Classification of transitive actions and actions with codimension one principal orbits.

1. Transitive actions. Let (G, M, φ) be an action with the codimension zero principal orbit G/H . Then G acts transitively on M and hence M is equivariantly diffeomorphic to a principal orbit G/H , where G acts effectively on G/H by left translation. For this reason, we consider a pair (G, L) of Lie groups, where L is a closed subgroup of G , G/L is a closed 5-manifold and G acts effectively on G/L by left translation. Two such pairs (G_1, L_1) and (G_2, L_2) are isomorphic if there exists an isomorphism $\sigma: G_1 \rightarrow G_2$ satisfying $\sigma(L_1) = L_2$. Let (G_1, M_1, φ_1) and (G_2, M_2, φ_2) be two actions with the codimension zero principal orbits G_1/H_1 and G_2/H_2 , respectively. Then (G_1, M_1, φ_1) is isomorphic to (G_2, M_2, φ_2) if and only if (G_1, H_1) is isomorphic to (G_2, H_2) .

Now we shall classify the pairs (G, H) such that G/H is a simply connected rational cohomology 5-sphere. The covering group G' of G defined in II.1.7 is a direct product of a simple Lie group G_1 and a rank one Lie group G_2 . Moreover, the restricted G_1 -action on $G'/H' = G/H$ is also transitive (see [8, Theorem I]). Hence $G'/H' = G_1/H_1$, where $H_1 = G_1 \cap H$. Since G_1 is simple and G_1/H_1 is a rank one homogeneous space, (G_1, H_1) is isomorphic to $(SU(3), SU(2))$, $(SU(3), SO(3))$ or $(SO(6), SO(5))$ by [10, Theorem 4]. Hence we have:

THEOREM A. *Let (G, M, φ) be an action with the codimension zero principal orbit G/H . Then the pair (G, H) is isomorphic to one of the following.*

(1) $(SU(3), SU(2))$, (2) $(U(3), U(2))$, (3) $(SO(6), SO(5))$, (4) $(SU(3), SO(3))$,

where H is a natural subgroup of G in all cases and M is the Wu-manifold in (4), while it is the natural sphere in the other cases.

2. Actions with codimension one principal orbits. Let (G, M, φ) be an action with codimension one principal orbits. Then by (II.1.1) E is empty and we get the following decomposition of M (e.g., [11, Section 1]). There exist just two singular orbits $G(x_1) = G/K_1$ and $G(x_2) = G/K_2$, where K_1 and K_2 are some closed subgroups of G such that $H \subset K_1 \cap K_2$. We denote the codimensions of G/K_1 and G/K_2 in M by k_1 and k_2 , respectively. Then $k_i \geq 2$ ($i = 1, 2$) and there exists a G -equivariant decomposition

$$(2.1) \quad M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2 = G/H,$$

where X_i is the mapping cylinder of the projection $p_i: G/H \rightarrow G/K_i$. Moreover X_i is G -equivariantly diffeomorphic to $G \times_{K_i} D^{k_i}$ and there is a G -equivariant diffeomorphism f of G/H such that M is G -equivariantly diffeomorphic to

$$(2.2) \quad M(f, \sigma_1, \sigma_2) = (G \times_{K_1} D^{k_1}) \cup_f (G \times_{K_2} D^{k_2}),$$

where $f = R_{[a]} \in W(H)$ for some $a \in N(H)$, and $R_{[a]}$ is the right translation defined by

$R_{[a]}(gH) = gHa = gaH$ for every $gH \in G/H$. Here K_i acts on D^{k_i} via the slice representation $\sigma_i: K_i \rightarrow O(k_i)$ and this K_i -action is transitive on the $(k_i - 1)$ -sphere ∂D^{k_i} . Hence $\partial D^{k_i} = K_i/H$ and

(2.3) the fiber bundle $K_i/H \rightarrow G/H \rightarrow G/K_i$ is a $(k_i - 1)$ -sphere bundle.

2.4. From (2.1) and (2.3), we get the cohomology groups of two singular orbits and by Lemma II.2.1 G' must be $S^3 \times S^1$, $S^3 \times S^3$, $S^3 \times T^2$, $S^3 \times S^3 \times S^1$ or $Spin(5)$. Hence from (2.2), we have:

THEOREM B. *Let (G, M, φ) be an action with codimension one principal orbits. Then G is $SO(5)$, $S^3 \times S^1$, $SO(3) \times S^1$, $U(2)$, $SO(4) \times S^1$, $SO(3) \times SO(3)$ or $U(2) \times S^1$. If G is not $U(2)$, then the action is isomorphic to one of the following.*

(1) $(SO(5), S^5, \varphi)$, $\varphi(A, \begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} Ax \\ y \end{pmatrix}$, where $({}^T x, y) \in S^5 \subset \mathbf{R}^5 \times \mathbf{R}$.

(2) $(S^3 \times S^1, S^5, \varphi_{m,n})$, $\varphi_{m,n}((q, t), (q', t')) = (qq't^{-m}, t^n t')$,

where $(q', t') \in S^5 \subset \mathbf{H} \times \mathbf{C}$ and $(m, n) = 1$.

(3) $(U(2) \times S^1, S^5, \varphi)$, $\varphi((A, t), {}^T(z_1, z_2, z_3)) = \begin{pmatrix} A \cdot {}^T(z_1, z_2) \\ tz_3 \end{pmatrix}$

(4) $(SO(4) \times S^1, S^5, \varphi)$, $\varphi((A, t), \begin{pmatrix} w \\ z \end{pmatrix}) = \begin{pmatrix} Aw \\ tz \end{pmatrix}$, where $({}^T w, z) \in S^5 \subset \mathbf{R}^4 \times \mathbf{C}$.

(5) $(SO(3) \times SO(3), S^5, \varphi)$, $\varphi((A, B), \begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} Ax \\ By \end{pmatrix}$,

where $({}^T x, {}^T y) \in S^5 \subset \mathbf{R}^3 \times \mathbf{R}^3$.

(6) $(SO(3) \times S^1, W^5(k), \varphi_k)$, $\varphi_k((A, t), {}^T(z_0, z_1, z_2, z_3)) = \begin{pmatrix} t^2 z_0 \\ A \cdot {}^T(z_1 t^k, z_2 t^k, z_3 t^k) \end{pmatrix}$,

where k is an odd integer and $W^5(k) = \{(z_0, z_1, z_2, z_3) \in \mathbf{C}^4 \mid \sum |z_i|^2 = 1, z_0^k + z_1^2 + z_2^2 + z_3^2 = 0\}$.

REMARK 2.5. In (2) if a pair (m, n) is not equal to (m', n') , then $\varphi_{m,n}$ is not isomorphic to $\varphi_{m',n'}$. In (6) if $k \neq k'$, then φ_k is not isomorphic to $\varphi_{k'}$.

REMARK 2.6. In any case, M is the natural sphere. Hence this result is the same as the 5-dimensional case of the classification theorem of actions on simply connected \mathbf{Z}_2 -cohomology spheres by Asoh [1]. Hence we omit the proof.

REMARK 2.7. When G is $U(2)$, by [9, Theorem] the action $(U(2), M, \psi)$ with the same conditions as in Theorem B is isomorphic to the following (1) or (2):

(1) $(U(2), S^5, \varphi)$, $\varphi(A, {}^T(z_1, z_2, z_3)) = \begin{pmatrix} z_1(\det A)^k \\ A \cdot {}^T(z_2(\det A)^{-m}, z_3(\det A)^{-m}) \end{pmatrix}$,

where $k \neq 0$ and $(k, 2m - 1) = 1$.

(2) $(U(2), M_k^5, \varphi_k)$, where $M_k^5 = (U(2) \times_{O(2)} D^2) \cup_{f_k} (S^3 \times D^2)$ is diffeomorphic to the Wu-manifold $SU(3)/SO(3)$ which is not a \mathbf{Z}_2 -sphere. (for detailed definitions of f_k ,

M_k^5 and φ_k , see Ōike [9].)

IV. Classification of actions with codimension two principal orbits.

1. In this section we shall show the following result.

THEOREM C. *Let (G, M, φ) be an action with codimension two principal orbits. Then G is S^3 , $SO(3)$, $SO(3) \times S^1$, $U(2)$ or $SO(4)$. If G is neither $SO(3)$ nor $U(2)$, then the action is isomorphic to one of the following.*

- (1) $(SU(2), SU(3)/SO(3), \varphi), \varphi(A, [B]) = [\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} B]$.
- (2) $(Sp(1), S^5, \varphi), \varphi(q, (q', z)) = (qq', z)$ where $(q', z) \in S^5 \subset \mathbf{H} \times \mathbf{C}$.
- (3) $(SO(3) \times S^1, S^5, \varphi), \varphi((A, t), {}^T(Tx, y, z)) = {}^T(T(Ax), ty, z)$ where $({}^Tx, y, z) \in S^5 \subset \mathbf{R}^3 \times \mathbf{C} \times \mathbf{R}$.
- (4) $(SO(4), S^5, \varphi), \varphi(A, \begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} Ax \\ y \end{pmatrix}$ where $({}^Tx, {}^Ty) \in S^5 \subset \mathbf{R}^4 \times \mathbf{R}^2$.

REMARK 1.1. When G is $SO(3)$, $F(SO(3), M)$ is empty or consists of k points (see 3.2 and 3.3). Hudson classified all $SO(3)$ -actions on simply connected closed 5-manifolds. Especially, if M is a simply connected rational cohomology 5-sphere, then the result is as follows.

(1) If $F(SO(3), M)$ is empty, then $(SO(3), M, \psi)$ with the same conditions as in Theorem C is isomorphic to

$$(SO(3), S^5, \varphi), \varphi(A, \begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} Ax \\ Ay \end{pmatrix} \text{ where } ({}^Tx, {}^Ty) \in S^5 \subset \mathbf{R}^3 \times \mathbf{R}^3.$$

(2) If $F(SO(3), M)$ consists of k points, then $k \geq 2$ and $(SO(3), M, \psi)$ with the same conditions as in Theorem C is isomorphic to $(SO(3), M^{j_1 j_2 \dots j_k}, \varphi_{j_1 j_2 \dots j_k})$ (for detailed definitions of $M^{j_1 j_2 \dots j_k}$ and $\varphi_{j_1 j_2 \dots j_k}$, see Hudson [5, II, III and IV]).

REMARK 1.2. When G is $U(2)$, $F(U(2), M) = S^1$ (see 4.3) and the action $(U(2), M, \psi)$ with the same conditions as in Theorem C is isomorphic to

$$(U(2), S^5, \varphi), \varphi(A, {}^T(z_1, z_2, z_3)) = \begin{pmatrix} z_1 \\ A \cdot {}^T(z_2, z_3) \end{pmatrix} \text{ (see Ōike [9, Theorem]).}$$

In the rest of this section, we shall prove Theorem C.

2. Since M is compact connected simply connected and $d=3$, M^* is a compact connected simply connected 2-manifold with or without boundary. Hence M^* is a 2-disk D^2 or a 2-sphere S^2 . By (II.1.1), $\dim(B^* \cup E^*) \leq 1$ and $\dim E^* \leq 0$. Then we have:

LEMMA 2.1. $M^* = D^2, B^* = \partial M^* = S^1$ and $E^* = \emptyset$.

PROOF. Suppose that $\partial M^* = \emptyset$. Then $M^* = S^2$ and all orbits are of the same dimension. Hence from Conner's result [4], G has rank one and acts almost freely (i.e.,

all isotropy subgroups are discrete) on M . Hence G is $SO(3)$ or S^3 . Let $p: M \rightarrow M^*$ be the projection. Then $\tilde{H}^k(p^{-1}(x); \mathbf{Q}) \cong \tilde{H}^k(G/G_x; \mathbf{Q}) \cong [\tilde{H}^k(G; \mathbf{Q})]^{G_x} = 0$ ($k=0, 1, 2$) for every $x \in M$. Hence from the Vietoris-Begle mapping theorem $p^*: H^2(M^*; \mathbf{Q}) \cong H^2(M; \mathbf{Q})$. This is a contradiction. Therefore $\partial M^* \neq \emptyset$. Hence $M^* = D^2$. Since $B^* = \partial D^2$, $E^* = \emptyset$ ([2, IV Theorem 8.6]). q.e.d.

Let $G(x)$ be a singular orbit of type G/K and let N be the boundary of a G -equivariant tubular neighborhood of $G(x)$. Then G acts effectively on the 4-manifold N and its orbit space N^* is an interval. Hence G acts on N with the codimension one principal orbit G/H and two singular orbits $G/K_1, G/K_2$, where $H \subset K_1 \cap K_2$ and $K_i \subseteq K$. Hence from III. 2, we get:

(2.2) N is G -equivariantly diffeomorphic to $(G \times_{K_1} D^{k_1}) \cup_f (G \times_{K_2} D^{k_2})$ and $\partial D^{k_i} = K_i/H$, where k_i is the codimension of the singular orbit G/K_i in N ($i=1, 2$).

Since $F(G, M) \subset B^* = S^1$, we have:

(2.3) $F(G, M)$ is empty, consists of a finite number of points or is S^1 .

From the list in Lemma II.2.1, G' is $Spin(3) = S^3$, $Spin(3) \times T$ or $Spin(4) = S^3 \times S^3$. We shall prove Theorem C in each of these cases as well as the cases in (2.3).

3. The case where G' is S^3 . In this case, G is S^3 or $SO(3)$ and H^0 is trivial. Let T be a maximal torus of G and let (K) be a singular isotropy type. Since $\dim K \geq 1$, K^0 is T or G . Then K is $T, N(T)$ or G , since $K \subset N(K^0) = N(T)$.

3.1. Suppose that $F(G, M) = \emptyset$. We first show the following.

LEMMA 3.1.1. *Singular isotropy types are unique and the type is (T) .*

PROOF. We first find that $F(T, M) = S^1$, because the singular isotropy type is (T) or $(N(T))$, $F(T, M)$ is a rational cohomology sphere and $F(T, M) \cap G(x) = F(T, M_{(K)}) \cap G/K$ consists of one point or two points if K is $N(T)$ or T , respectively. Next suppose that singular isotropy types are not unique. Then the types are (T) and $(N(T))$. Hence the orbit of type $G/N(T)$ is the isolated singular orbit, i.e., its slice representation $\sigma: N(T) \rightarrow O(3)$ has no trivial representations as a direct summand (see [12] or [2, p. 213]). Hence the induced $N(T)/T \cong \mathbf{Z}_2$ -action on $F(T, M)$ has finite fixed points. This contradicts that $F(T, M)/\mathbf{Z}_2 = M_{(K)}^* = S^1$. Finally suppose the unique singular isotropy type is $(N(T))$. Then $F(T, M) \cap G(x)$ consists of one point for any singular orbit $G(x)$. We consider a fiber bundle $G/K = \mathbf{R}P(2) \rightarrow M_{(K)} \xrightarrow{p_1} B^* = S^1$ and define a map $c: M_{(K)}^* \rightarrow M_{(K)}$ by $c(x^*) = G(x) \cap F(T, M)$ for every $x^* \in M_{(K)}^*$. Then c is a cross-section and hence $M_{(K)}$ is isomorphic to $G/K \times S^1$. The equivariant diffeomorphism $f_1: G/K \times S^1 = G/K \times F(T, M_{(K)}) \rightarrow M_{(K)}$ is defined by $f_1(gK, x) = \varphi(g, x)$. We identify B^* with $F(T, M)$. Let U be a closed invariant tubular neighborhood of $M_{(K)}$ in M and let $q: \partial U \rightarrow M_{(K)}$ be the projection of the induced bundle with the fiber $K/H = S^1$. Then q

induces a diffeomorphism $\bar{q}: (\partial U)^* \rightarrow M_{(k)}^*$. Set $W = M - \text{int } U$. Then W is a G -manifold with only one isotropy type (H) and W^* is a 2-disk. Hence W is equivariantly diffeomorphic to $G/H \times D^2$. Thus $\partial U = \partial W$ is equivariantly diffeomorphic to $G/H \times S^1$. We denote the diffeomorphism by f . Let $\bar{S} = f^{-1}(1 \times S^1) \subset \partial U$. Then \bar{S} is a 1-sphere and $G(x) \cap \bar{S}$ consists of one point for any $x \in \partial U$. We have $q(\bar{S}) = F(T, M)$, since $\varphi(T, q(\bar{x})) = q(\bar{x})$ for any $\bar{x} \in \bar{S}$. We consider the following commutative diagram:

$$\begin{array}{ccccc}
 \partial U & \xrightarrow{q} & & & M_{(k)} \\
 & \swarrow f_2 & & \searrow f_1 & \\
 & G/H \times \bar{S} & \xrightarrow{p \times q'} & G/K \times S^1 & \\
 p_2 \downarrow & \downarrow & & \downarrow & \downarrow p_1 \\
 (\partial U)^* = \bar{S} & \xrightarrow{\bar{q}} & & S^1 = M_{(k)}^* &
 \end{array}$$

where $p: G/H \rightarrow G/K$ is the projection, q' is the restriction of q , f_2 is the equivariant diffeomorphism defined by $f_2(gH, x) = \varphi(g, x)$ and we identify $(\partial U)^*$ with \bar{S} . Then q' is a diffeomorphism and hence U is equivariantly diffeomorphic to $N \times \bar{S}$, where N is the total space of the D^2 -bundle associated with p and $M = U \cup W$ is diffeomorphic to $(N \times \bar{S}) \cup_{f \circ f_2} (G/H \times D^2)$. Hence $\pi_1(M) = \mathbb{Z}_2$ by van Kampen's theorem. This is a contradiction. q.e.d.

Now we construct an example.

(3.1.2) Let k be any positive integer and let $p_k: G/\mathbb{Z}_k \rightarrow G/T$ be the natural projection. Then this is a fiber bundle with the fiber $T/\mathbb{Z}_k = S^1$. Let N_k be the total space of the associated D^2 -bundle. Then we define a G -manifold $M(k)$ by $M(k) = (G/\mathbb{Z}_k \times D^2) \cup_{\text{id}} (N_k \times S^1)$, where $\text{id}: \partial(G/\mathbb{Z}_k \times D^2) = G/\mathbb{Z}_k \times S^1 \rightarrow G/\mathbb{Z}_k \times S^1 = \partial(N_k \times S^1)$ and G acts naturally on G/\mathbb{Z}_k and N_k and acts trivially on D^2 and S^1 .

PROPOSITION 3.1.3. *Let G be S^3 or $SO(3)$ and let (G, M, φ) be an action with two isotropy types (T) and (\mathbb{Z}_k) .*

- (1) *If $k = 1$, then the action is equivariantly diffeomorphic to the following (i) or (ii):*
 - (i) (3.1.2) for $k = 1$.
 - (ii) *The case (1) in Theorem C if G is S^3 , while the case (1) in Remark 1.1 if G is $SO(3)$.*
- (2) *If $k \geq 2$, then the action is equivariantly diffeomorphic to (3.1.2) for the same k .*

PROOF. If $k = 1$, then $(G/K)^H = G/T$ is connected. If $k \geq 2$, then $(G/K)^H = (G/T)^{\mathbb{Z}_k} = N(T)/T$ and $N(H) = N(T)$. Hence $N(H)$ acts transitively on $(G/K)^H$. Hence every action is proper by Remark II.1.6. By (II.1.5), it turns out that the number of G -equivariant diffeomorphism classes of G -manifolds is two if $k = 1$ and is one if $k \geq 2$. q.e.d.

Since we have $H^2(M(k); \mathcal{Q}) \cong \mathcal{Q}$ by the Mayer-Vietoris exact sequence, $M(k)$ is not a rational cohomology sphere. Therefore if $F(G, M) = \emptyset$, then the action (G, M, φ) with the same conditions as in Theorem C is equivariantly diffeomorphic to the case (1) (ii) in Proposition 3.1.3.

3.2. Suppose that $F(G, M)$ consists of k points. Then G is $SO(3)$. Indeed, suppose that G is S^3 . Let D^5 be a slice at $x \in F(G, M)$ such that $D^5 \cap F(G, M) = \{x\}$. Then S^3 acts on ∂D^5 via the slice representation $\sigma: S^3 \rightarrow SO(5)$ and the action has no fixed points. Hence the representation σ is irreducible. Therefore σ must be the weight 2 representation: $S^3 \rightarrow SO(3) \rightarrow SO(5)$ (cf. [2, p. 44]). Hence this S^3 -action on D^5 is not effective. This is a contradiction. Thus G is $SO(3)$ and such $SO(3)$ -actions on simply connected closed 5-manifolds were classified by Hudson [5]. In this case, $(SO(3), M, \varphi)$ is equivariantly diffeomorphic to the case (2) in Remark 1.1.

3.3. Suppose that $F(G, M) = S^1$. Let N be a G -equivariant tubular neighborhood of $F(G, M)$ and let $q: N \rightarrow F(G, M)$ be the projection. Then G acts effectively and transitively on $\partial q^{-1}(x) = S^3$ for every $x \in F(G, M)$. Hence $G/H = S^3$. Thus $G = S^3$ and $H = \{1\}$. Since $(G/K)^H$ is connected, any action is proper by Remark II.1.6 and the actions are unique up to G -equivariant diffeomorphisms by (II.1.5) and are G -equivariantly diffeomorphic to the case (2) in Theorem C.

4. The case where G' is $S^3 \times S^1$. Let $L_1 = (1 \times S^1) \cap H' \subset L = Z(G') \cap H' \subset Z(G') = Z_2 \times S^1$. Then G'/L_1 is isomorphic to $S^3 \times S^1$ and acts almost effectively on M . Hence we may assume that L_1 is trivial. Then $G = G'/L$ is $S^3 \times S^1$, $SO(3) \times S^1$ or $U(2)$. Since $\dim H' = 1$, H'^o is a one-dimensional torus.

4.1. Suppose that $F(G, M) = \emptyset$. We shall show that this case cannot occur. Let K' be a singular isotropy subgroup. Then $1 < \dim K' < 4$. Hence K'^o is T^2 or $S^3 \times 1$. We first show that the action has unique singular isotropy type. If there exist at least two singular isotropy types, then the action has one isolated singular orbit. Let G/K be the isolated singular orbit. Then K acts orthogonally on a slice D^k and $\partial D^k/K$ is an interval, where k is 3 or 4 if K'^o is T^2 or $S^3 \times 1$, respectively. Hence the K -action on ∂D^k has the codimension one principal orbit K/H and two singular orbits K/K_1 and K/K_2 , where K'^o , $K_1'^o$ and $K_2'^o$ are either T^2 or $S^3 \times 1$ and $K' \supseteq K_i'$ ($i = 1, 2$) (cf. 2. and II.2). Hence T^2 and $S^3 \times 1$ cannot be the identity components of the singular isotropy subgroups at the same time. Hence all singular isotropy subgroups are of the same dimension. Moreover $\partial D^k = S^{k-1}$ is K -equivariantly diffeomorphic to

$$M(f, \sigma_1, \sigma_2) = (K \times_{K_1} D^{k_1}) \cup_f (K \times_{K_2} D^{k_2}),$$

where $k_1 = k_2 = k - 1$ and $f \in W(H)$ is right translation on K/H . Since $D^{k_i} \rightarrow K \times_{K_i} D^{k_i} \rightarrow K/K_i$ is a fiber bundle and the base space consists of finite points, $K \times_{K_i} D^{k_i}$ is equivariantly

diffeomorphic to $K/K_i \times D^{k_i}$. Hence we have $H^{k-1}(S^{k-1}; \mathbf{Z}) \cong H^{k-2}(K/K_i \times S^{k-1}; \mathbf{Z})$ by the Mayer-Vietoris exact sequence. This is a contradiction. Now let (K) be the unique singular isotropy type.

(i) If K^o is T^2 , then K^o is also T^2 . Since $F(T^2, M)$ is a positive dimensional cohomology sphere, G is $S^3 \times S^1$ or $SO(3) \times S^1$. (If G is $U(2)$, then it is known that $F(T^2, M)$ is $S^1 \times S^0$ ([9, (3.4)]).) Since $\text{rank } G = \text{rank } K$, we may assume that $K^o = T^2 = T_0 \times S^1$, where T_0 is a maximal torus of S^3 or $SO(3)$. Since $K \subset N(K^o) = N(T^2) = N(T_0) \times S^1$, K is T^2 or $N(T^2)$. Assume that K is T^2 . Then T^2 acts effectively and orthogonally on a slice D^3 via the slice representation $\sigma: T^2 \rightarrow O(3)$ and the T^2 -action on $\partial D^3 = S^2$ has two fixed points and the principal orbit $K/H = S^1$. Since (H^o) is (T) , we have $\sigma = \sigma_{m,n}$ defined by

$$\sigma_{m,n}(t_1, t_2) = \begin{pmatrix} & & 0 \\ t_1^m & t_2^n & \\ 0 & 0 & 1 \end{pmatrix} \in SO(2) \subset O(3),$$

where $(t_1, t_2) \in T^2 = T_0 \times S^1$ and m and n are some positive integers with $(m, n) = 1$ or $m = 0$ and $n = 1$ by (II.1.8). We get $n = 1$ since $L_1 = (1 \times S^1) \cap H$ is trivial. Hence $H = \{(t, t^{-m}), | t \in S^1\}$. Therefore the action is proper and unique up to G -equivariant diffeomorphisms by (II.1.5) and [2, V 4.3 Proposition]. Hence M is G -equivariantly diffeomorphic to $G \times_{T^2} S^3$, where T^2 acts on $S^3 = D^3 \cup D^3$ by $\sigma_{m,1}$. Hence M is not a cohomology 5-sphere. Next assume that K is $N(T^2)$. Then $N(T^2)$ acts effectively and orthogonally on a slice D^3 via the slice representation $\sigma: N(T^2) \rightarrow O(3)$. Since the singular isotropy types are unique, σ has a one-dimensional trivial representation θ as a direct summand. Hence we may assume that $\sigma = \sigma_1 \oplus \theta$ where $\sigma_1(N(T^2)) \subset O(2) = \left\{ \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset O(3)$ and $\sigma_1|_{T^2} = \sigma_{m,1}$. Let $[\alpha] \in N(T_0)/T_0 \cong \mathbf{Z}_2$ be a generator. Then we have $\sigma_1(\alpha, 1) \in SO(2)$, since $\sigma_1(\alpha, 1)$ commutes with each element of $\sigma_1(1 \times S^1) = SO(2)$. Hence we get $m = 0$ and $\sigma_1(\alpha, 1) = \pm I_2$ from the relations between α and T_0 in $N(T_0)$. Therefore G is $SO(3) \times S^1$ and we can show in the same way as above that M is equivariantly diffeomorphic to $G \times_{N(T^2)} S^3$. Hence M is not a cohomology 5-sphere.

(ii) If K^o is $S^3 \times 1$, then K' is $S^3 \times \mathbf{Z}_k$ (k is a positive integer). Then we have $H' = T_0 \times \mathbf{Z}_k$, because K'/H' is S^2 . Hence k must be one since $L_1 = (1 \times S^1) \cap H'$ is trivial. Hence G is $SO(3) \times S^1$, K is $SO(3) \times 1$ and H is $T_0 \times 1$. Therefore the action is proper and unique by (II.1.5) and Remark II.1.6. Thus the action is $SO(3) \times S^1$ -equivariantly diffeomorphic to $(SO(3) \times S^1, S^4 \times S^1, \psi)$ given by $\psi((A, t), \left(\begin{pmatrix} x \\ y \end{pmatrix}, z \right)) = \left(\begin{pmatrix} Ax \\ y \end{pmatrix}, tz \right)$, where $(T_x, T_y) \in S^4 \subset \mathbf{R}^3 \times \mathbf{R}^2$. Hence M is not a cohomology 5-sphere.

4.2. Suppose that $F(G, M)$ consists of k points. We first show the following lemma.

LEMMA 4.2.1. G is $SO(3) \times S^1$ and $F(G, M)$ consists of two points.

PROOF. Let $F(G, M) = \{x_i \mid i = 1, 2, \dots, k\}$ and let D_i^5 be the slice at x_i such that $D_i^5 \cap F(G, M) = \{x_i\}$. Then G acts effectively and orthogonally on $\partial D_i^5 = S^4$ via the slice representation $\sigma_i: G \rightarrow SO(5)$ and its G -action on S^4 has the principal orbit of type G/H and two singular orbits G/K_1 and G/K_2 , where $H \subset K_1 \cap K_2$ and $K_s \neq G$ ($s = 1, 2$). We investigate all the faithful representations with this condition and find out that σ_i is $\gamma \oplus \rho$ for every i and G is $SO(3) \times S^1$, where $(\gamma \oplus \rho)(A, t) = \begin{pmatrix} A & 0 \\ 0 & t \end{pmatrix} \in SO(5)$, for every $(A, t) \in SO(3) \times S^1$. Therefore, $(H) = (SO(2) \times 1)$, $(K_1) = (SO(2) \times S^1)$ and $(K_2) = (SO(3) \times 1)$. Next we shall show that $k = 2$. Let T_1 be a maximal torus of H (i.e., $T_1 = H$). Then from (II.1.2) there exists a homeomorphism $f: M_o^{T_1}/W(T_1) \rightarrow M^* = D^2$, where $W(T_1) = W(SO(2)) \times S^1 = \mathbf{Z}_2 \times S^1$. Since $\dim M_o^{T_1} \geq 2$, $M_o^{T_1} = M^{T_1}$ is a cohomology 3-sphere and isotropy types of this $W(T_1)$ -action are (1×1) , $(1 \times S^1)$, $(\mathbf{Z}_2 \times 1)$ and $(W(T_1))$ (see Figure 1). If we consider reduced $W(T_1)^o = 1 \times S^1$ -action on M^{T_1} , then

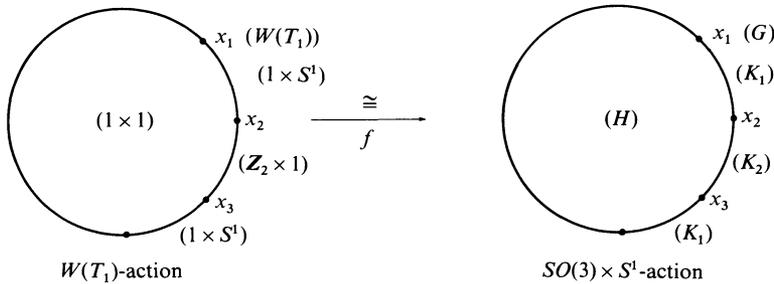


FIGURE 1

$F(W(T_1)^o, M^{T_1})$ is not empty and hence $F(W(T_1)^o, M^{T_1})$ is a cohomology 1-sphere. Hence

$$M_{(W(T_1)^o)}^{T_1}/W(T_1) = F(W(T_1)^o, M^{T_1})/W(T_1) = F(W(T_1)^o, M^{T_1})/(\mathbf{Z}_2 \times 1),$$

where $\mathbf{Z}_2 \times 1 = W(T_1)/W(T_1)^o$, is connected and hence $k = 2$. q.e.d.

Let $F(G, M) = \{x_1, x_2\}$. Since the number of isolated singular orbits is two, M is G -equivariantly diffeomorphic to $M(f) = D_1 \cup_f D_2$. Here D_i is a G -equivariant tubular neighborhood of the fixed point x_i , i.e., D_i is a 5-dimensional disk and $f: \partial D_1 \rightarrow \partial D_2$ is a G -equivariant diffeomorphism. Then we have the following lemma.

LEMMA 4.2.2. Let F be a $SO(3) \times SO(2)$ -equivariant diffeomorphism on S^4 such that the following diagram is commutative:

$$(4.2.3) \quad \begin{array}{ccc} S^4 & \xrightarrow{F} & S^4 \\ \gamma \oplus \rho \downarrow & & \downarrow \gamma \oplus \rho \\ S^4 & \xrightarrow{F} & S^4 \end{array}$$

Then F is $SO(3) \times SO(2)$ -diffeotopic to F' defined by

$$F' \left(\begin{pmatrix} u \\ v \end{pmatrix} \right) = \begin{pmatrix} \varepsilon I_3 & 0 \\ 0 & s_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{for every } ({}^T u, {}^T v) \in S^4 \subset \mathbf{R}^3 \times \mathbf{R}^2,$$

where s_0 is some element of $SO(2)$ and $\varepsilon = \pm 1$.

PROOF. Let $S^4 = \{({}^T u, {}^T v) \in \mathbf{R}^3 \times \mathbf{R}^2 \mid |u|^2 + |v|^2 = 1\}$ and $I = \{({}^T \hat{u}, {}^T \hat{v}) \in S^4 \mid \hat{u} = {}^T(|u|, 0, 0), \hat{v} = {}^T(|v|, 0), |u|^2 + |v|^2 = 1\}$. Then we identify I with the orbit space S^4/G and denote the orbit map $p: S^4 \rightarrow I$ by $p(G(x)) = G(x) \cap I$ since S^4/G is an interval and $G(x) \cap I$ consists of one point for any $x \in S^4$, where G is $SO(3) \times SO(2)$. Let $F \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, where $F_1 = F_1 \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbf{R}^3$, $F_2 = F_2 \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbf{R}^2$ and $|F_1|^2 + |F_2|^2 = 1$. Then $F_1 = F_1(u)$ and $F_2 = F_2(v)$ by the commutativity of (4.2.3). Hence for any $\begin{pmatrix} u \\ v \end{pmatrix} \in S^4$, there exists $(A(u), s(v)) \in G$ such that $u = A(u) \cdot \hat{u}$, $v = s(v) \cdot \hat{v}$ and

$$F \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F_1(u) \\ F_2(v) \end{pmatrix} = \begin{pmatrix} A(u) & 0 \\ 0 & s(v) \end{pmatrix} \begin{pmatrix} F_1(\hat{u}) \\ F_2(\hat{v}) \end{pmatrix}.$$

Moreover there exists $(A(F_1), s(F_2)) \in G$ such that $F_1(\hat{u}) = A(F_1) \cdot \hat{F}_1$ and $F_2(\hat{v}) = s(F_2) \cdot \hat{F}_2$, where $\hat{F}_1 = \hat{F}_1(\hat{u}) = {}^T(|F_1(\hat{u})|, 0, 0)$ and $\hat{F}_2 = \hat{F}_2(\hat{v}) = {}^T(|F_2(\hat{v})|, 0) = {}^T((1 - |F_1(\hat{u})|^2)^{1/2}, 0)$. Hence we have

$$F \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F_1(\hat{u}) \\ F_2(\hat{v}) \end{pmatrix} = \begin{pmatrix} A(u) & 0 \\ 0 & s(v) \end{pmatrix} \begin{pmatrix} A(F_1) & 0 \\ 0 & s(F_2) \end{pmatrix} \begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix}.$$

This presentation of F is dependent on F alone. We denote $\hat{F}: I \rightarrow I$ by $\hat{F} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix}$. Then \hat{F} is a diffeomorphism with $\hat{F} \begin{pmatrix} \hat{0} \\ \hat{1} \end{pmatrix} = \begin{pmatrix} \hat{0} \\ \hat{1} \end{pmatrix}$, $\hat{F} \begin{pmatrix} \hat{1} \\ \hat{0} \end{pmatrix} = \begin{pmatrix} \hat{1} \\ \hat{0} \end{pmatrix}$ and $p \circ F = \hat{F} \circ p$ (cf. [2, VI.5]). Let $J: I \rightarrow I$ be the identity map defined by $J \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} J_1(\hat{u}) \\ J_2(\hat{v}) \end{pmatrix} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$. Then the map $h: [0, 1] \times I \rightarrow I$ given by

$$h \left(t, \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \right) = \begin{pmatrix} (1-t)\hat{F}_1(\hat{u}) + t \cdot J_1(\hat{u}) \\ \hat{v}(t) \end{pmatrix},$$

where $0 \leq t \leq 1$ and $\hat{v}(t) = {}^T((1 - |(1-t)\hat{F}_1(\hat{u}) + t \cdot J_1(\hat{u})|^2)^{1/2}, 0)$, define a diffeotopy between \hat{F} and J . Let $H: [0, 1] \times S^4 \rightarrow S^4$ be a homotopy defined by

$$H \left(t, \begin{pmatrix} u \\ v \end{pmatrix} \right) = \begin{pmatrix} A(u) & 0 \\ 0 & s(v) \end{pmatrix} \begin{pmatrix} A(F_1) & 0 \\ 0 & s(F_2) \end{pmatrix} h \left(t, \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \right).$$

Then H is dependent on F alone and hence H is a well-defined G -diffeotopy between $F \begin{pmatrix} u \\ v \end{pmatrix} = H(0, \begin{pmatrix} u \\ v \end{pmatrix})$ and $H(1, \begin{pmatrix} u \\ v \end{pmatrix})$. We set $H(1, \begin{pmatrix} u \\ v \end{pmatrix}) = F' \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F'_1(u) \\ F'_2(v) \end{pmatrix}$. Then by the

commutativity of (4.2.3) F'_1 is an $SO(3)$ -equivariant diffeomorphism on S^2 and F'_2 is an $SO(2)$ -equivariant diffeomorphism on S^1 . Hence $F'_1 \in N(SO(2))/SO(2) \cong \mathbf{Z}_2$ and $F'_2 \in N(1)/1 \cong SO(2)$. Thus we get F' defined above. q.e.d.

By this lemma and (II.1.4), we have $M(f) = M(\text{id}) = S^5$. Thus the case (3) in Theorem C is proved.

4.3. Suppose that $F(G, M) = S^1$. In this case, we can show that G is $U(2)$ and H is $U(1)$ like 3.3. Hence by (II.1.5), the $U(2)$ -action of this type is $U(2)$ -equivariantly diffeomorphic to the case in Remark 1.2.

5. The case where G' is $S^3 \times S^3$. Since $d=3$, we have $\dim H' = 3$. Hence H'^o is isomorphic to S^3 .

LEMMA 5.1. $F(G', M) = S^1$.

PROOF. First suppose that $F(G', M) = \emptyset$. Then all singular isotropy subgroups are 4-dimensional. Let (K') be a singular isotropy type. Then we may assume that $K'^o = S^1 \times S^3$. Then $H'^o = 1 \times S^3$. This contradicts (II.1.8). Next suppose that $F(G', M)$ consists of a finite number of points. Let $x \in F(G', M)$ and let D^5 be the slice at x such that $D^5 \cap F(G', M) = \{x\}$. Then G' acts almost effectively and orthogonally on D^5 via the slice representation $\sigma: G' \rightarrow SO(5)$ and the action has two singular orbits G'/K'_1 and G'/K'_2 . Since $3 < \dim K'_i < 6$, we have $\dim K'_i = 4$. Therefore $K'_i{}^o$ is $S^1 \times S^3$ or $S^3 \times S^1$. This is a contradiction as in the first case. q.e.d.

Since $F(G, M)$ is also S^1 , let U be a G -equivariant tubular neighborhood of $F(G, M) = S^1$ and let $q: U \rightarrow S^1$ be the projection. Then $\partial q^{-1}(x) = S^3$ and G acts effectively and transitively on S^3 . Therefore G is $SO(4)$ and H is $SO(3)$. Let $W = M - \text{int } U$. Then $p_1: W \rightarrow W^*$ is the projection of the fiber bundle with the fiber $G/H = S^3$ and $W^* = D^2$. Hence W is G -equivariantly diffeomorphic to $G/H \times D^2 = S^3 \times D^2$. On the other hand, the D^4 -bundle $q: U \rightarrow S^1$ is orientable since the structure group is $SO(4)$. Hence U is G -equivariantly diffeomorphic to $D^4 \times S^1$. Therefore M is G -equivariantly diffeomorphic to $M(f) = (D^4 \times S^1) \cup_f (S^3 \times D^2)$, where $G = SO(4)$ acts naturally on D^4 and S^3 and trivially on S^1 and D^2 while $f: S^3 \times S^1 \rightarrow S^3 \times S^1$ is a G -equivariant diffeomorphism. Now for such f there exist a smooth map $\alpha: S^1 \rightarrow \mathbf{Z}_2 (\cong N(H)/H)$ and a diffeomorphism $\beta: S^1 \rightarrow S^1$ such that $f(q, t) = (q\alpha(t), \beta(t))$ for $(q, t) \in S^3 \times S^1$. We can extend f to the $SO(4)$ -equivariant diffeomorphism $F: D^4 \times S^1 \rightarrow D^4 \times S^1$ defined by $F(sq, t) = (sq\alpha(t), \beta(t))$, $0 \leq s \leq 1$. Then by (II.1.4), $M(f) = M(\text{id}) = S^5$ and this $SO(4)$ -action is $SO(4)$ -equivariantly diffeomorphic to the case (4) in Theorem C.

This complete the proof of Theorem C.

V. Classification of actions with codimension three principal orbits.

1. In this section we shall prove the following result.

THEOREM D. *Let (G, M, φ) be an action with codimension three principal orbits. Then G must be $SO(3)$.*

REMARK 1.1. By the proof of this theorem, M^* is a simply connected 3-manifold with $\partial M^* = F(SO(3), M)$. Hence M^* is a homotopy 3-disk and $\partial M^* = S^2$. Hence if the Poincaré conjecture is true, $M^* = D^3$ and by (II.1.5) the number of $SO(3)$ -equivariant diffeomorphism classes of such actions is unique (see [5, Appendix]).

2. PROOF OF THEOREM D. Since $d=2$, G' is $Spin(3) = S^3$ by the list in Lemma II.2.1. Hence G is S^3 or $SO(3)$ and H^o is a maximal torus T of G . Since $H^o \subset H \subset N(H^o) = N(T)$, H is T or $N(T)$. Then we have:

LEMMA 2.1. *$H = T$, $E = \emptyset$ and $B \neq \emptyset$.*

PROOF. Since M is simply connected and G is connected, its principal orbit and its exceptional orbit are both orientable ([2, p. 188]). Hence $H = T$ and $E = \emptyset$, because $G/N(T) = RP(2)$ is non-orientable. Next suppose that $B = \emptyset$. Then all orbits are principal and M^* is a connected simply connected 3-manifold. By (II.1.2), there exists a homeomorphism $f: M_o^T/W(T) \rightarrow M^*$, where $M_o^T = M^T$ is also a rational cohomology sphere, $W(T) = \mathbf{Z}_2$ acts on M^T and the $W(T)$ -action has a unique isotropy subgroup $(N(T) \cap H)/T = \{1\}$. Hence M^T is a rational cohomology 3-sphere and the projection $q: M^T \rightarrow M^T/\mathbf{Z}_2 = M^*$ is a principal \mathbf{Z}_2 -bundle. This contradicts the fact that M^* is simply connected. q.e.d.

Since G has no 2-dimensional subgroups, its singular isotropy subgroup is G . Hence all singular orbits are fixed points and $B = B^* = F(G, M)$ is a submanifold of M .

LEMMA 2.2. *B is 2-dimensional and G is isomorphic to $SO(3)$.*

PROOF. Let B_k ($k=0, 1, 2$) be the k -dimensional components of B . Then $B = B_0 \cup B_1 \cup B_2$ (disjoint union). Suppose that $B_0 \neq \emptyset$. Let $y \in B_0$ and let D^5 be an equivariant tubular neighborhood of y . We may assume $D^5 \cap B = \{y\}$. Then G acts orthogonally on $\partial D^5 = S^4$ via the slice representation $\sigma: G \rightarrow SO(5)$ and each orbit of the action is $G/H = S^2$. Hence S^4/G is a simply connected 2-manifold. Adapt (II.1.2) to this action on S^4 , and a contradiction results as in the proof of Lemma 2.1. Next suppose that $B_1 \neq \emptyset$. Let $y \in B_1$ and let B_1^o be the connected component containing y . Let N be an equivariant tubular neighborhood of B_1^o such that $N \cap B = B_1^o$ and let $p: N \rightarrow B_1^o$ be the projection. Then G acts effectively on $\partial p^{-1}(y) = S^3$ and each orbit of this action is $G/H = S^2$ and the orbit space S^3/G is a simply connected closed 1-manifold. This is a contradiction. Hence $B = B_2$. Let N be an equivariant tubular neighborhood of B and let $p: N \rightarrow B$ be the projection. Then G acts orthogonally on $\partial p^{-1}(y) = S^2$ via the restriction to $p^{-1}(y)$ of the slice representation at $y \in B$. We denote the restricted representation by σ . Then $\sigma: G \rightarrow SO(3)$ is the faithful representation since G acts effectively and transitively on $S^2 = G/H$. Hence G is isomorphic to $SO(3)$, because G and $SO(3)$ are closed manifolds of

the same dimension.

q.e.d.

Thus Theorem D has been proved.

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