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PERIODIC SOLUTIONS OF NONLINEAR INTEGRODIFFERENTIAL EQUATIONS

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For ordinary differential equations and functional differential equations with finite delay, it is well known that uniform boundedness and uniform ultimate boundedness imply the existence of a periodic solution by a fixed point theorem (cf. [2], [5], [10]). In order to obtain a similar existence theorem for a periodic solution of a nonlinear Volterra equation, Burton has extended the boundedness concept to one called the g-boundedness [2].

In this paper, we shall discuss the existence of a periodic solution of an integrodifferential equation by using stability properties of a bounded solution. As will be seen later, our result can be applied to the existence of a strictly positive (componentwise) periodic solution for a model in the dynamics of an n-species system in mathematical ecology discussed by Gopalsamy [4], while the results obtained by Burton [2] and Arino, Burton and Haddock [1] cannot be applied.

We consider a nonlinear integrodifferential equation

$$\dot{x}(t)=f(t, x(t))+\int_{-\infty}^{t} E(t, s, x(s), x(t))ds . \tag{1}$$

This equation can be written as

$$\dot{x}(t)=f(t, x(t))+\int_{-\infty}^{0} E(t, t+s, x(t+s), x(t))ds . \tag{2}$$

In the case where f in (1) contains a term with finite delay, we can obtain the same conclusion, but we shall consider equation (1) here in order to make the statements simple.

Let $x:(-\infty, A)\to R^n$, $-\infty<A\leqq\infty$, be a continuous function. For $t<A$, define x_t by the relation $x_t(s)=x(t+s)$, $s\leqq 0$. We denote by BC the space of all R^n-valued bounded continuous functions on $(-\infty, 0]$.

We impose the following assumptions on (1).

(A) $f: R\times R^n\to R^n$ is a continuous function and $E(t, s, x, y)$ is defined and continuous for $-\infty<s\leqq t<\infty$, $x\in R^n$ and $y\in R^n$.

(B) There is a $T>0$ such that $f(t+T, x)=f(t, x)$ for all $t\in R$, $x\in R^n$ and $E(t+T, s+T, x, y)=E(t, s, x, y)$ for all $t\in R$, $s\leqq t$, $x\in R^n$ and $y\in R^n$.

(C) For any $r>0$, there exists an $L_1(r)>0$ such that

$$\int_{-\infty}^{t} |E(t, s, x(s), x(t))|\, ds \leqq L_1(r)$$

for all $t\in R$, whenever $x(s)$ is continuous and $|x(s)|\leqq r$ for all $s\leqq t$.

(D) For any $\varepsilon>0$ and $r>0$, there exists an $S>0$ such that

$$\int_{-\infty}^{t-S} |E(t, s, x(s), x(t))|\, ds \leqq \varepsilon$$

for all $t\in R$, whenever $x(s)$ is continuous and $|x(s)|\leqq r$ for all $s\leqq t$.

For $E(t, t+s, x(t+s), x(t))$, this is written as

$$\int_{-\infty}^{-S} |E(t, t+s, x(t+s), x(t))|\, ds \leqq \varepsilon$$

for all $t\in R$.

REMARK 1. It follows from conditions (C) and (D) that $\int_{-\infty}^{t} E(t, s, x(s), x(t))ds$ is continuous in t, whenever $x(s)$ is continuous and $|x(s)|\leqq r$ for all $s\leqq t$.

Under the above assumptions, if $t_0\in R$ and $\phi\in \mathrm{BC}$, there exists a solution of (1) which passes through (t_0, ϕ) (cf. [3]). Moreover, a solution $x(t)$ can be continuable up to $t=\infty$ if it remains in a compact set in R^n, because $\dot{x}(t)$ is bounded as long as $x(t)$ remains in a compact set in R^n.

In addition to the conditions (A), (B), (C) and (D), we make the following assumption:

(E) The equation (1) has a bounded solution $u(t)$ defined on $[0, \infty)$ which passes through $(0, \phi)$, $\phi\in \mathrm{BC}$.

REMARK 2. If $u(t)$ is a solution of (1), then $u(t+T)$ is also a solution of (1).

Let K be the bounded closed subset in R^n such that $\phi(s)\in K$ for all $s\leqq 0$ and $u(t)\in K$ for all $t\geqq 0$. For any $\theta, \psi\in \mathrm{BC}$, we set

$$\rho_j(\theta, \psi)=\sup_{-j\leqq s\leqq 0} |\theta(s)-\psi(s)|\,,$$

$$\rho(\theta, \psi)=\sum_{j=1}^{\infty} \rho_j(\theta, \psi)/[2^j(1+\rho_j(\theta, \psi))]\,.$$

Clearly, $\rho(\theta_n, \theta)\to 0$ as $n\to\infty$ if and only if $\theta_n(s)\to\theta(s)$ uniformly on any compact subset of $(-\infty, 0]$ as $n\to\infty$.

Now we introduce some stability properties with respect to the set K and the metric ρ.

DEFINITION 1. The bounded solution $u(t)$ of (1) is said to be uniformly stable with respect to K and ρ, if for any $\varepsilon>0$ and any $t_0 \geqq 0$, there exists a $\delta(\varepsilon)>0$ such that $\rho(u_{t_0}, x_{t_0})<\delta(\varepsilon)$ implies $\rho(u_t, x_t)<\varepsilon$ for all $t \geqq t_0$, where $x(t)$ is a solution of (1) such that $x_{t_0}(s) \in K$ for all $s \leqq 0$.

Then we have the following theorem.

THEOREM 1. *Under the assumptions* (A), (B), (C), (D) *and* (E), *if the bounded solution $u(t)$ of* (1) *is uniformly stable with respect to K and ρ, then $u(t)$ is an asymptotically almost periodic solution of* (1).

PROOF. Let $\{t_k\}$ be a sequence such that $t_k \to \infty$ as $k \to \infty$. For each t_k, there exists a positive integer N_k such that $N_k T \leqq t_k < (N_k+1)T$. If we set $t_k = N_k T + s_k$, then $0 \leqq s_k < T$. If we set $u^k(t)=u(t+t_k)$, then $u^k(t)$ is a solution of the system

$$\dot{x}(t)=f(t+s_k, x(t))+\int_{-\infty}^{t} E(t+s_k, s+s_k, x(s), x(t))ds\,. \tag{3}$$

Since $u(t)$ is uniformly stable with respect to K and ρ, $u^k(t)$ is also uniformly stable with respect to K and ρ with the same pair $(\varepsilon, \delta(\varepsilon))$ as the one for $u(t)$.

By our assumption (C), there exists an $L>0$ such that $|\dot{u}(t)| \leqq L$ for all $t \geqq 0$. Taking a subsequence, if necessary, we can assume that $s_k \to s_* \in [0, T]$ as $k \to \infty$ and the sequence $\{u(t+t_k)\}$ converges uniformly on any compact set in $(-\infty, \infty)$ as $k \to \infty$.

Since $s_k \to s_*$ and $u(t_k+s)$ converges uniformly on any compact set in $(-\infty, 0]$, for any $\varepsilon>0$ there exists a positive integer $n_1(\varepsilon)$ such that if k, $m \geqq n_1(\varepsilon)$, then

$$\rho(u_0^k, u_0^m)<\delta(\varepsilon)/2 \quad \text{and} \quad |s_k - s_m|<\delta(\varepsilon)/4L\,, \tag{4}$$

where $\delta(\varepsilon)$ is the number for the uniform stability of $u(t)$ with respect to K and ρ.

Moreover, choosing a number $N=N(\varepsilon)>0$ such that $\sum_{j=N+1}^{\infty} 1/2^j<\delta(\varepsilon)/4$, we can find an $n_2(\varepsilon)>0$ such that if k, $m \geqq n_2(\varepsilon)$, then $t_m \geqq N+1$ and

$$\rho_N(u_t^m, u_{t+s_m-s_k}^m)=\sup_{s\in[-N,0]} |u^m(t+s)-u^m(t+s+s_k-s_m)| \leqq L|s_k-s_m|<\delta(\varepsilon)/4$$

for all $t \geqq 0$. Thus we have

$$\rho(u_t^m, u_{t+s_k-s_m}^m)<\delta(\varepsilon)/2 \qquad \text{for all} \quad t \geqq 0\,. \tag{5}$$

This implies that if k, $m \geqq n_2(\varepsilon)$, then $\rho(u_0^m, u_{s_k-s_m}^m)<\delta(\varepsilon)/2$. Thus, if k, $m \geqq n_0(\varepsilon)=\max(n_1(\varepsilon), n_2(\varepsilon))$, we have

$$\rho(u_0^k, u_{s_k-s_m}^m) \leqq \rho(u_0^k, u_0^m)+\rho(u_0^m, u_{s_k-s_m}^m)<\delta(\varepsilon)\,.$$

Since $u^m(t)=u(t+t_m)$ is a solution of

$$\dot{x}(t)=f(t+s_m, x(t))+\int_{-\infty}^{t} E(t+s_m, s+s_m, x(s), x(t))ds\,,$$

$u^m(t+s_k-s_m)$ is a solution of (3). However $u^k(t)$ is uniformly stable with respect to K and

ρ, $u^m_{s_k-s_m}(s)\in K$ for all $s\leqq 0$ and $\rho(u^k_0, u^m_{s_k-s_m})<\delta(\varepsilon)$ if $k, m\geqq n_0(\varepsilon)$, and hence we have

$$\rho(u^k_t, u^m_{t+s_k-s_m})<\varepsilon \qquad \text{for all} \quad t\geqq 0$$

if k, $m\geqq n_0(\varepsilon)$. On the other hand, (5) implies that if k, $m\geqq n_0(\varepsilon)$,

$$\rho(u^m_t, u^m_{t+s_k-s_m})<\varepsilon \qquad \text{for all} \quad t\geqq 0\,.$$

Thus, if k, $m\geqq n_0(\varepsilon)$, we have

$$\rho(u^k_t, u^m_t)<2\varepsilon \qquad \text{for all} \quad t\geqq 0\,,$$

which implies that if k, $m\geqq n_0(\varepsilon)$,

$$|u(t+t_k)-u(t+t_m)|\leqq \sup_{s\in[-1,0]}|u(t+t_k+s)-u(t+t_m+s)|<8\varepsilon$$

for $\varepsilon\leqq 1/8$ and all $t\geqq 0$. Thus we see that for any sequence $\{t_k\}$ such that $t_k\to\infty$ as $k\to\infty$, there exists a subsequence $\{t_{k_j}\}$ of $\{t_k\}$ for which $u(t+t_{k_j})$ converges uniformly on $[0, \infty)$ as $j\to\infty$. This shows that $u(t)$ is asymptotically almost periodic in t.

This theorem is similar to a result obtained by Yoshizawa for the functional differential equation with finite delay [9]. Hino [7] also has extended Yoshizawa's result to the functional differential equation on the phase space considered axiomatically by Hale and Kato [6].

In what follows, we need the following definitions.

DEFINITION 2. The bounded solution $u(t)$ of (1) is said to be weakly uniformly asymptotically stable with respect to K and ρ, if it is uniformly stable with respect to K and ρ and there is a $\delta_0>0$ such that if $t_0\geqq 0$ and $\rho(u_{t_0}, x_{t_0})<\delta_0$, then $\rho(u_t, x_t)\to 0$ as $t\to\infty$, where $x(t)$ is a solution of (1) such that $x_{t_0}(s)\in K$ for all $s\leqq 0$.

DEFINITION 3. The bounded solution $u(t)$ of (1) is said to be globally weakly uniformly asymptotically stable with respect to K and ρ, if it is uniformly stable with respect to K and ρ and $\rho(u_t, x_t)\to 0$ as $t\to\infty$, whenever $x(t)$ is a solution of (1) such that $x_{t_0}(s)\in K$ for $s\leqq 0$ at some $t_0\geqq 0$.

THEOREM 2. *Under the assumptions* (A), (B), (C), (D) *and* (E), *if the bounded solution $u(t)$ of* (1) *is weakly uniformly asymptotically stable with respect to K and ρ, then the equation* (1) *has an mT-periodic solution $p(t)$ for some integer $m\geqq 1$ such that $p(t)\in K$ for all $t\in R$.*

PROOF. Set $u^k(t)=u(t+kT)$, $k=1, 2, \cdots$. Then there exists a subsequence $\{u^{k_j}(t)\}$ of $\{u^k(t)\}$ which converges to some function $w(t)$ uniformly on any compact set in $(-\infty, 0]$ as $j\to\infty$, where $w(t)$ is a bounded continuous function on $(-\infty, 0]$. Thus $\rho(u^{k_j}_0, w_0)\to 0$ as $j\to\infty$, and therefore there is a positive integer p such that $\rho(u^{k_p}_0, u^{k_{p+1}})<\delta_0$, where δ_0 is the number for weakly uniformly asymptotic stability of $u(t)$ with respect to K and ρ. Set $m=k_{p+1}-k_p$ and consider the solution $u^m(t)=$

$u(t+mT)$ of (1). Then we have

$$\rho(u^m_{k_pT}, u_{k_pT}) = \rho(u_0^{k_p+1}, u_0^{k_p}) < \delta_0$$

and $u^m_{k_pT}(s) \in K$ for all $s \leqq 0$, which implies that

$$\rho(u^m_t, u_t) \to 0 \quad \text{as} \quad t \to \infty\,, \tag{6}$$

because $u(t)$ is weakly uniformly asymptotically stable with respect to K and ρ.

On the other hand, $u(t)$ is asymptotically almost periodic in t by Theorem 1, and hence

$$u(t) = p(t) + q(t)\,, \tag{7}$$

where $p(t)$ is almost periodic in t and $q(t)$ is a continuous function such that $q(t) \to 0$ as $t \to \infty$. Since it follows from (6) that

$$u(t+mT) - u(t) \to 0 \quad \text{as} \quad t \to \infty\,,$$

we have

$$p(t) - p(t+mT) \to 0 \quad \text{as} \quad t \to \infty\,.$$

Therefore $p(t) = p(t+mT)$ for all $t \in R$, because $p(t)$ is almost periodic in t.

Now we shall show that $p(t)$ is a solution of (1). If we set $u^k(t) = u(t+kmT)$, $k = 1, 2, \cdots$, then we have

$$u(t+kmT) = p(t) + q(t+kmT)\,,$$

and hence $u^k(t) \to p(t)$ uniformly on any compact set in R as $k \to \infty$. Thus it is clear that $p(t) \in K$ for all $t \in R$. For $t \geqq a > -\infty$, we have

$$u^k(t) = u^k(a) + \int_a^t \left\{ f(s, u^k(s)) + \int_{-\infty}^s E(s, v, u^k(v), u^k(s))dv \right\} ds \tag{8}$$

for sufficiently large k, because $u^k(t)$ is a solution of (1). There exists a $c > 0$ such that $|u^k(t)| \leqq c$ and $|p(t)| \leqq c$ for all $t \in R$. Then, by the condition (D), for this c and any $\varepsilon > 0$, there exists an $S = S(\varepsilon, c) > 0$ such that

$$\int_{-\infty}^{s-S} |E(s, v, u^k(v), u^k(s))|\, dv \leqq \varepsilon$$

and

$$\int_{-\infty}^{s-S} |E(s, v, p(v), p(s))|\, dv \leqq \varepsilon\,.$$

Thus we have

$$\left| \int_{-\infty}^{s} E(s, v, u^k(v), u^k(s))dv - \int_{-\infty}^{s} E(s, v, p(v), p(s))dv \right|$$

$$\leqq \int_{-\infty}^{s-S} | E(s, v, u^k(v), u^k(s)) | dv + \int_{-\infty}^{s-S} | E(s, v, p(v), p(s)) | dv$$

$$+ \int_{s-S}^{s} | E(s, v, u^k(v), u^k(s)) - E(s, v, p(v), p(s)) | dv$$

$$\leqq 2\varepsilon + \int_{s-S}^{s} | E(s, v, u^k(v), u^k(s)) - E(s, v, p(v), p(s)) | dv .$$

Moreover, if $k \geqq k_0(\varepsilon)$ for some $k_0(\varepsilon) > 0$,

$$\int_{s-S}^{s} | E(s, v, u^k(v), u^k(s)) - E(s, v, p(v), p(s)) | dv < \varepsilon ,$$

because E is continuous and $u^k(v)$ converges to $p(v)$ uniformly on $[s-S, s]$ as $k \to \infty$. Thus we have

$$\int_{-\infty}^{s} E(s, v, u^k(v), u^k(s))dv \to \int_{-\infty}^{s} E(s, v, p(v), p(s))dv \tag{9}$$

as $k \to \infty$. Therefore, by our assumption (C) and Lebesgue's convergence theorem, we have

$$\int_{a}^{t} \int_{-\infty}^{s} E(s, v, u^k(v), u^k(s))dvds \to \int_{a}^{t} \int_{-\infty}^{s} E(s, v, p(v), p(s))dvds$$

as $k \to \infty$. Letting $k \to \infty$ in (8), we obtain

$$p(t) = p(a) + \int_{a}^{t} \left\{ f(s, p(s)) + \int_{-\infty}^{s} E(s, v, p(v), p(s))dv \right\} ds$$

for all $t \geqq a$. This shows that $p(t)$ is a solution of (1). Thus the equation (1) has an mT-periodic solution, because $p(t) = p(t + mT)$.

THEOREM 3. *Under the assumptions* (A), (B), (C), (D) *and* (E), *if the bounded solution $u(t)$ of* (1) *is globally weakly uniformly asymptotically stable with respect to K and ρ, then the equation* (1) *has a T-periodic solution $p(t)$ such that $p(t) \in K$ for all $t \in R$, which is globally weakly uniformly asymptotically stable with respect to K and ρ.*

PROOF. Since $u(t)$ is uniformly stable with respect to K and ρ, it is asymptotically almost periodic in t by Theorem 1, and hence $u(t) = p(t) + q(t)$, where $p(t)$ is almost periodic in t and $q(t)$ is a continuous function such that $q(t) \to 0$ as $t \to \infty$. Since $u(t + T)$ is also a solution of (1) such that $u_T(s) \in K$ for all $s \leqq 0$ and $u(t)$ is globally weakly uniformly asymptotically stable with respect to K and ρ, we have $\rho(u_t, u_{t+T}) \to 0$ as $t \to \infty$, and hence

$p(t)=p(t+T)$ for all $t\in R$. Thus we can show that $p(t)$ is a T-periodic solution of (1) by the same arguments as in the proof of Theorem 2.

Now we shall see that $p(t)$ is uniformly stable with respect to K and ρ. Set $u^k(t)=u(t+kT)$, $k=1, 2, \cdots$. For any $t_0\in[0, \infty)$, let $x(t)$ be a solution of (1) such that $x_{t_0}(s)\in K$ for all $s\leqq 0$ and $\rho(p_{t_0}, x_{t_0})<\delta(\varepsilon/2)/2$, where $\delta(\varepsilon)$ is the number for uniform stability of $u(t)$. Since $u^k(t)\to p(t)$ uniformly on any compact set in $(-\infty, t_0]$, we have

$$\rho(u^k_{t_0}, p_{t_0})<\delta(\varepsilon/2)/2\ ,$$

if k is sufficiently large. Since $u^k(t)$ is uniformly stable with respect to K and ρ with the same pair $(\varepsilon, \delta(\varepsilon))$ as the one for $u(t)$, we have $\rho(u^k_t, p_t)<\varepsilon/2$ for all $t\geqq t_0$. Moreover, we have

$$\rho(u^k_{t_0}, x_{t_0})\leqq\rho(u^k_{t_0}, p_{t_0})+\rho(p_{t_0}, x_{t_0})<\delta(\varepsilon/2)\ ,$$

which implies that $\rho(u^k_t, x_t)<\varepsilon/2$ for all $t\geqq t_0$. Therefore, if $\rho(p_{t_0}, x_{t_0})<\delta(\varepsilon/2)/2$, then $\rho(p_t, x_t)\leqq\rho(p_t, u^k_t)+\rho(u^k_t, x_t)<\varepsilon$ for all $t\geqq t_0$, which shows that $p(t)$ is uniformly stable with respect to K and ρ.

Since $u(t)$ is globally weakly uniformly asymptotically stable with respect to K and ρ and since $p_{t_0}(s)\in K$, $x_{t_0}(s)\in K$ for all $s\leqq 0$, we have $\rho(u_t, p_t)\to 0$ as $t\to\infty$ and $\rho(u_t, x_t)\to 0$ as $t\to\infty$. Therefore $\rho(p_t, x_t)\to 0$ as $t\to\infty$.

This shows that $p(t)$ is globally weakly uniformly asymptotically stable with respect to K and ρ.

REMARK 3. In particular, consider the case where $E(t, s, x, y)$ has the form

$$E(t, s, x, y)=K(t-s)G(t, x, y)\ . \tag{10}$$

Here $K(\sigma)$ is an $n\times m$ matrix function such that $\int_0^\infty|K(\sigma)|d\sigma<\infty$ and $G(t, x, y)$ is a continuous $m\times 1$ vector function such that $G(t+T, x, y)=G(t, x, y)$ for some $T>0$ and all $t\in R$, $x\in R^n$ and $y\in R^n$. Then for any $\varepsilon>0$, there exists an $S>0$ such that

$$\int_{-\infty}^{t-S}|K(t-s)|ds\leqq\varepsilon\qquad\text{for all}\quad t\in R\ .$$

Therefore, without the continuity of $K(\sigma)$, we can show that $p(t)$ in the proof of Theorems 2 and 3 is a solution, because (9) in the proof of Theorem 2 can be prove in the following way: For any $\varepsilon>0$, there exists a $k_0(\varepsilon)>0$ such that if $k\geqq k_0(\varepsilon)$,

$$\begin{aligned}&\left|\int_{-\infty}^{s}K(s-v)G(s, u^k(v), u^k(s))dv-\int_{-\infty}^{s}K(s-v)G(s, p(v), p(s))dv\right|\\&\quad\leqq\int_{-\infty}^{s-S}|K(s-v)||G(s, u^k(v), u^k(s))|dv+\int_{-\infty}^{s-S}|K(s-v)||G(s, p(v), p(s))|dv\\&\qquad+\int_{s-S}^{s}|K(s-v)||G(s, u^k(v), u^k(s))-G(s, p(v), p(s))|dv\leqq 2\varepsilon G^*+K^*\varepsilon\ ,\end{aligned}$$

where $G^* = \max\{|G(t, x, y)|: t \in R, |x| \leqq c, |y| \leqq c\}$ and $K^* = \int_0^\infty |K(\sigma)| d\sigma$. Thus we can see that

$$\int_{-\infty}^{s} K(s-v)G(s, u^k(v), u^k(s))dv \to \int_{-\infty}^{s} K(s-v)G(s, p(v), p(s))dv$$

as $k \to \infty$. Therefore, Theorems 2 and 3 hold without the continuity assumption on $K(\sigma)$.

EXAMPLE (cf. [4]). Consider a system of integrodifferential equations

$$\dot{x}_i(t) = x_i(t)\left\{b_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}(t) \int_{-\infty}^{t} K_{ij}(t-v)x_j(v)dv\right\}, \tag{11}$$

where $b_i(t)$ and $a_{ij}(t)$ $(i, j = 1, 2, \cdots, n)$ are continuous and positive T-periodic functions on R and K_{ij}: $[0, \infty) \to [0, \infty)$ $(i, j = 1, 2, \cdots, n;\ i \neq j)$ denote delay kernels such that

$$\int_0^\infty K_{ij}(s)ds = 1 \quad \text{and} \quad \int_0^\infty sK_{ij}(s)ds < \infty \qquad (i, j = 1, 2, \cdots, n;\ i \neq j).$$

We assume that

$$b_i^l > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u(b_j^u/a_{jj}^l), \qquad i = 1, 2, \cdots, n,$$

where

$$b_i^l = \inf_{t \in R} b_i(t), \qquad b_i^u = \sup_{t \in R} b_i(t),$$

$$a_{ij}^l = \inf_{t \in R} a_{ij}(t), \qquad a_{ij}^u = \sup_{t \in R} a_{ij}(t), \qquad (i, j = 1, 2, \cdots, n).$$

If we set

$$\alpha_i = b_i^u/a_{ii}^l \quad \text{and} \quad \beta_i = \left[b_i^l - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u(b_j^u/a_{jj}^l)\right]\Big/ a_{ii}^u,$$

then $0 < \beta_i < \alpha_i$ for each i. Then if $u(t)$ is a solution of (11) through $(0, \phi)$ such that $0 < \beta_i \leqq \phi_i(s) \leqq \alpha_i$ for all $s \leqq 0$, then we have $\beta_i \leqq u_i(t) \leqq \alpha_i$ for all $t \geqq 0$. Let K be a bounded closed set in R^n such that

$$K = \{(x_1, x_2, \cdots, x_n) \in R^n:\ \beta_i \leqq x_i \leqq \alpha_i \text{ for each } i\}.$$

Moreover, if we assume that there exists a positive constant m such that

$$a_{ii}^l > m + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ji}^u, \qquad i = 1, 2, \cdots, n,$$

we can see that the bounded solution $u(t)$ of (11) is globally weakly uniformly asymptotically stable with respect to K and ρ by using a Liapunov functional

$$V(t, u(\cdot), x(\cdot))$$
$$= \sum_{i=1}^{n} \left[|\log u_i(t) - \log x_i(t)| + \sum_{\substack{j=1 \\ j\neq i}}^{n} \int_0^\infty K_{ij}(s) \left\{ \int_{t-s}^{t} a_{ij}(s+v) |u_j(v) - x_j(v)| dv \right\} ds \right],$$

because we have

$$\dot{V}(t, u(\cdot), x(\cdot)) \leqq \sum_{i=1}^{n} |u_i(t) - x_i(t)| \left\{ -a_{ii}^l + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ji}^u \right\} \leqq -m \sum_{i=1}^{n} |u_i(t) - x_i(t)|.$$

Therefore, by Theorem 3, the system (11) has a T-periodic solution $p(t)$ such that $\beta_i \leqq p_i(t) \leqq \alpha_i$ for all $t \in R$.

REMARK 4. In the case where $a_{ij}(t)$ and $b_i(t)$ are almost periodic in t, Murakami [8] has shown that the system (11) has an almost periodic solution under the same conditions.

Finally, we shall discuss the existence of an almost periodic solution of the equation (1).

We define almost periodic functions with parameters in the usual way (cf. [11]). Let $R^* = R^- \times R^n \times R^n$, where $R^- = (-\infty, 0]$.

DEFINITION 4. $E(t, t+s, x, y)$ is said to be almost periodic in t uniformly for $(s, x, y) \in R^*$, if for any $\varepsilon > 0$ and any compact set K^* in R^*, there exists a positive number $L(\varepsilon, K^*)$ such that any interval of length $L(\varepsilon, K^*)$ contains a τ for which

$$(12) \qquad |E(t+\tau, t+s+\tau, x, y) - E(t, t+s, x, y)| \leqq \varepsilon$$

for all $t \in R$ and all $(s, x, y) \in K^*$.

For the properties of an almost periodic function with parameters, see [11].

We assume the conditions (A), (C), (D), (E) and

(B′) $f(t, x)$ is almost periodic in t uniformly for $x \in R^n$ and $E(t, t+s, x, y)$ is almost periodic in t uniformly for $(s, x, y) \in R^*$.

Let K be the bounded closed subset in R^n such that $\phi(s) \in K$ for all $s \leqq 0$ and $u(t) \in K$ for all $t \geqq 0$, where $u(t)$ is the bounded solution of (1) under the condition (E).

DEFINITION 5. The bounded solution $u(t)$ of (1) is said to be totally stable with respect to K and ρ, if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $t_0 \geqq 0$, $\rho(u_{t_0}, x_{t_0}) < \delta(\varepsilon)$ and if $h(t)$ is a continuous function such that $|h(t)| < \delta(\varepsilon)$ for $t \geqq t_0$, then $\rho(u_t, x_t) < \varepsilon$ for all $t \geqq t_0$, where $x(t)$ is a solution of

$$(13) \qquad \dot{x}(t) = f(t, x(t)) + \int_{-\infty}^{t} E(t, s, x(s), x(t)) ds + h(t)$$

such that $x_{t_0}(s) \in K$ for all $s \leqq 0$.

Then we have the following theorem.

THEOREM 4. *Under the assumptions* (A), (B′), (C), (D) *and* (E), *if the bounded solution* $u(t)$ *of* (1) *is totally stable with respect to* K *and* ρ, *then* $u(t)$ *is an asymptotically almost periodic solution of* (1), *and the equation* (1) *has an almost periodic solution.*

PROOF. Let $\{t_k\}$ be a sequence such that $t_k \to \infty$ as $k \to \infty$. If we set $u^k(t) = u(t + t_k)$, then $u^k(t)$ is a solution of the system

$$\dot{x}(t) = f(t + t_k, x(t)) + \int_{-\infty}^{t} E(t + t_k, s + t_k, x(s), x(t))ds . \tag{14}$$

Since $u(t)$ is totally stable with respect to K and ρ, $u^k(t)$ is also totally stable with respect to K and ρ with the same pair $(\varepsilon, \delta(\varepsilon))$ as the one for $u(t)$. For each positive integers k and m, we defined a continuous function $h_{k,m}: R \to R^n$ by

$$\begin{aligned} h_{k,m}(t) = {} & f(t + t_m, u^m(t)) - f(t + t_k, u^m(t)) \\ & + \int_{-\infty}^{0} E(t + t_m, t + s + t_m, u^m(t + s), u^m(t))ds \\ & - \int_{-\infty}^{0} E(t + t_k, t + s + t_k, u^m(t + s), u^m(t))ds . \end{aligned} \tag{15}$$

Then, clearly $u^m(t) = u(t + t_m)$ is a solution of the system

$$\dot{x}(t) = f(t + t_k, x(t)) + \int_{-\infty}^{t} E(t + t_k, s + t_k, x(s), x(t))ds + h_{k,m}(t) . \tag{16}$$

Taking a subsequence of $\{t_k\}$, if necessary, we may assume that $u(s + t_k)$ converges uniformly on any compact interval in $(-\infty, 0]$. Therefore there exists a positive integer $n_1(\varepsilon)$ such that

$$\rho(u_0^k, u_0^m) < \delta(\varepsilon) \qquad \text{if} \quad k, m \geqq n_1(\varepsilon) ,$$

where $\delta(\varepsilon)$ is the number for the total stability of $u(t)$ with respect to K and ρ.

Now we shall show that there is a positive integer $n_2(\varepsilon)$ such that

$$|h_{k,m}(t)| < \delta(\varepsilon) \qquad \text{if} \quad k, m \geqq n_2(\varepsilon) .$$

There exists a $c > 0$ such that $|u^k(t)| \leqq c$ and $|u^m(t)| \leqq c$ for all $t \in R$. Then by the condition (D), for this c and $\delta(\varepsilon) > 0$, there exists an $S = S(\delta, c) > 0$ such that

$$\int_{-\infty}^{-S} |E(t + t_m, t + s + t_m, u^m(t + s), u^m(t))| ds \leqq \delta(\varepsilon)/4$$

and

$$\int_{-\infty}^{-S} |E(t+t_k, t+s+t_k, u^m(t+s), u^m(t))| \, ds \leqq \delta(\varepsilon)/4 .$$

Thus we have

$$|h_{k,m}(t)| \leqq |f(t+t_m, u^m(t)) - f(t+t_k, u^m(t))| + \delta(\varepsilon)/2$$
$$+ \int_{-S}^{0} |E(t+t_m, t+s+t_m, u^m(t+s), u^m(t)) - E(t+t_k, t+s+t_k, u^m(t+s), u^m(t))| \, ds ,$$

and hence if $k, m \geqq n_2(\varepsilon)$ for some $n_2(\varepsilon) > 0$,

$$|f(t+t_m, u^m(t)) - f(t+t_k, u^m(t))| < \delta(\varepsilon)/4$$

and

$$\int_{-S}^{0} |E(t+t_m, t+s+t_m, u^m(t+s), u^m(t)) - E(t+t_k, t+s+t_k, u^m(t+s), u^m(t))| \, ds < \delta(\varepsilon)/4 ,$$

because f and E are almost periodic, which implies that

$$|h_{k,m}(t)| < \delta(\varepsilon) \qquad \text{if} \quad k, m \geqq n_2(\varepsilon) .$$

Thus, if $k, m \geqq n_0(\varepsilon) = \max(n_1(\varepsilon), n_2(\varepsilon))$, we have

$$\rho(u_0^k, u_0^m) < \delta(\varepsilon) \quad \text{and} \quad |h_{k,m}(t)| < \delta(\varepsilon) .$$

Therefore, if $k, m \geqq n_0(\varepsilon)$, we have

$$\rho(u_t^k, u_t^m) < \varepsilon \qquad \text{for all} \quad t \geqq 0 ,$$

since $u^k(t)$ is a solution of (14) which is totally stable with respect to K and ρ and $u_0^m(s) \in K$ for all $s \leqq 0$. This implies that if $k, m \geqq n_0(\varepsilon)$,

$$|u(t+t_k) - u(t+t_m)| \leqq \sup_{s \in [-1, 0]} |u(t+s+t_k) - u(t+s+t_m)| < 4\varepsilon$$

for all $\varepsilon \leqq 1/4$ and all $t \geqq 0$. Thus we see that for any sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$, there exists a subsequence $\{t_{k_j}\}$ of $\{t_k\}$ for which $u(t+t_{k_j})$ converges uniformly on $[0, \infty)$ as $j \to \infty$. This shows that $u(t)$ is an asymptotically almost periodic solution of (1).

Now we have

$$u(t) = p(t) + q(t) ,$$

where $p(t)$ is almost periodic in t and $q(t)$ is a continuous function such that $q(t) \to 0$ as $t \to \infty$. There exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$, such that $p(t+t_k) \to p(t)$ uniformly on R, $f(t+t_k, x) \to f(t, x)$ uniformly on $R \times S$ for any compact set S in R^n and $E(t+t_k, t+s+t_k, x, y) \to E(t, t+s, x, y)$ uniformly on $R \times K^*$ for any compact set K^* in R^*.

Now we set $u^k(t) = u(t+t_k)$. Then $u^k(t)$ converges to $p(t)$ uniformly on any compact set in R as $k \to \infty$, and $u^k(t)$ is a solution of (14). Thus we can show that $p(t)$ is a solution of (1) by the same arguments as in the proof of Theorem 2.

This shows that the equation (1) has an almost periodic solution.

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