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PARTIAL THETA FUNCTION EXPANSIONS

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Introduction. In a series of recent papers, Andrews [1], [2] discussed in detail certain groups of formulae, which he found in Ramanujan's "Lost" Notebook (as he preferred to call it) and has given remarkable and ingeneous proofs for some of Ramanujan's tricky and mysterious indentities.

In the second paper of the series he considered the idea of expanding θ-functions in terms of partial theta functions. He considered the four families of trigonometric polynomials:

$$\theta_{1;N}(Z;q)=2q^{1/4}\operatorname{Sin}Z\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1-2q^{2m}\operatorname{Cos}2Z+q^{4m}), \tag{1.1}$$

$$\theta_{2;N}(Z;q)=2q^{1/4}\operatorname{Cos}Z\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1+2q^{2m}\operatorname{Cos}2Z+q^{4m}), \tag{1.2}$$

(introduced by Watson [5, p. 67])

$$\theta_{3;N}(Z;q)=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1+2q^{2m-1}\operatorname{Cos}2Z+q^{4m-2}), \tag{1.3}$$

$$\theta_{4;N}(Z;q)=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1-2q^{2m-1}\operatorname{Cos}2Z+q^{4m-2}). \tag{1.4}$$

These trigonometric polynomials are partial products of the four classical theta functions, first treated extensively by Jacobi. These are

$$\theta_1(Z;q)=2\prod_{n=1}^{\infty}(1-q^{2n})q^{1/4}\operatorname{Sin}Z\prod_{m=1}^{\infty}(1-2q^{2m}\operatorname{Cos}2Z+q^{4m}), \tag{1.5}$$

$$\theta_2(Z;q)=2\prod_{n=1}^{\infty}(1-q^{2n})q^{1/4}\operatorname{Cos}Z\prod_{m=1}^{\infty}(1+2q^{2m}\operatorname{Cos}2Z+q^{4m}), \tag{1.6}$$

$$\theta_3(Z;q)=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{\infty}(1+2q^{2m-1}\operatorname{Cos}2Z+q^{4m-2}), \tag{1.7}$$

$$\theta_4(Z;q)=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{\infty}(1-2q^{2m-1}\operatorname{Cos}2Z+q^{4m-2}). \tag{1.8}$$

These can be alternatively expressed as

$$\theta_1(Z;q)=2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\operatorname{Sin}(2n+1)Z=\lim_{N\to\infty}\theta_{1;N}(Z;q)\,, \tag{1.9}$$

$$\theta_2(Z;q)=2\sum_{n=0}^{\infty}q^{(n+1/2)^2}\operatorname{Cos}(2n+1)Z=\lim_{N\to\infty}\theta_{2;N}(Z;q)\,, \tag{1.10}$$

$$\theta_3(Z;q)=1+2\sum_{n=1}^{\infty}q^{n^2}\operatorname{Cos}2nZ=\lim_{N\to\infty}\theta_{3;N}(Z;q)\,, \tag{1.11}$$

$$\theta_4(Z;q)=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\operatorname{Cos}2nZ=\lim_{N\to\infty}\theta_{4;N}(Z;q)\,. \tag{1.12}$$

Consider the expansion

$$\sum_{N=0}^{\infty}T_N(i,j;q,q_1)\theta_{i;N}(z;q)=\theta_j(z;q_1)\,,\qquad i,j=1,2,3,4\,. \tag{1.13}$$

Now the question is, if such an expansion exists, what is the form of these coefficients $T_N(i,j;q,q_1)$. Ramanujan's wonderful observation was that in a number of cases these coefficients have very elegant closed forms, two of which we give below:

$$T_n(4,4;q,q^3)=\frac{q^{2n^2}}{(1-q^2)(1-q^4)\cdots(1-q^{4n})} \tag{1.14}$$

and

$$T_n(4,4;q,q^2)=\frac{q^{n^2}\prod_{m=0}^{\infty}(1-q^{2m+1})}{(1-q)(1-q^3)\cdots(1-q^{2n-1})(1-q^4)(1-q^8)\cdots(1-q^{4n})}\,. \tag{1.15}$$

Andrews proved Ramanujan's results by the help of a lemma [2, p. 176].

In this paper we shall show that the lemma of Andrews can be easily extended in a more general basic hypergeometric setting and thus giving the possibility of studying more general expansions. Indeed if we divide by d and then make $d\to\infty$ in the expansion, subsequently proved in this paper, we get Ramanujan's expansions.

THEOREM 1.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n(-q/a;q^2)_n((\alpha_r);q^2)_n(tq^\lambda)^n}{(q^2;q^2)_{2n}((\beta_s);q^2)_n}&={}_r\phi_{s+1}\left[\begin{matrix}(\alpha_r);\,q^2;\,tq^\lambda\\ q^2,(\beta_s);\end{matrix}\right]\\&+\sum_{N=1}^{\infty}\frac{(a^N+a^{-N})q^{N^2}((\alpha_r);q^2)_N(tq^\lambda)^N}{(q^2;q^2)_{2N}((\beta_s);q^2)_N}\times\sum_{n=0}^{\infty}\frac{((\alpha_r)q^{2N};q^2)_n(tq^\lambda)^n}{((\beta_s)q^{2N};q^2)_n(q^2;q^2)_n(q^{4N+2};q^2)}\end{aligned} \tag{1.16}$$

where (α_r) denotes the sequences $\alpha_1,\alpha_2,\cdots,\alpha_r$ and λ is a suitable constant.

PROOF.

$$\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n(-q/a;q^2)_n((\alpha_r);q^2)_n(tq^\lambda)^n}{(q^2;q^2)_{2n}((\beta_s);q^2)_n}=\sum_{n=0}^{\infty}\frac{q^{n^2+\lambda n}a^{-n}t^n(-aq^{-2n+1};q^2)_{2n}((\alpha_r);q^2)_n}{(q^2;q^2)_{2n}((\beta_s);q^2)_n}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a^{-n+m}q^{n^2-2nm+m^2+\lambda n}((\alpha_r);q^2)_n t^n}{(q^2;q^2)_m(q^2;q^2)_{2n-m}((\beta_s);q^2)_n}$$

$$=\sum_{N=-\infty}^{\infty}a^Nq^{N^2}\sum_{n=0}^{\infty}\frac{((\alpha_r);q^2)_n(tq^\lambda)^n}{(q^2;q^2)_{n+N}(q^2;q^2)_{n-N}((\beta_s);q^2)_n}$$

$$={}_r\phi_{s+1}\left[\begin{matrix}(\alpha_r);q^2;tq^\lambda\\ q^2,(\beta_s);\end{matrix}\right]+\sum_{N=1}^{\infty}(a^N+a^{-N})q^{N^2}\sum_{n=0}^{\infty}\frac{((\alpha_r);q^2)_{n+N}(tq^\lambda)^{n+N}}{(q^2;q^2)_n(q^2;q^2)_{n+2N}((\beta_s);q^2)_{n+N}}$$

$$={}_r\phi_{s+1}\left[\begin{matrix}(\alpha_r);q^2;tq^\lambda\\ q^2,(\beta_s);\end{matrix}\right]+\sum_{N=1}^{\infty}\frac{(a^N+a^{-N})q^{N^2}((\alpha_r);q^2)_N(tq^\lambda)^N}{(q^2;q^2)_{2N}((\beta_s);q^2)_N}$$

$$\times\sum_{n=0}^{\infty}\frac{((\alpha_rq^{2N});q^2)_n(tq^\lambda)^n}{(q^2;q^2)_n(q^{4N+2};q^2)_n((\beta_sq^{2N});q^2)_n}.$$

2. Applications of Theorem 1. The summation formula given in Andrews [3] is

$${}_3\phi_2\left[\begin{matrix}a,b,cq;q;d/abq\\ c,d;\end{matrix}\right]=\prod\left[\begin{matrix}d/a,d/b\\ d/abq,d\end{matrix}\right]\left\{1-\frac{d}{aq}-\frac{d}{bq}+\frac{d}{q}+\frac{dc(1-a)(1-b)}{abq(1-c)}\right\}$$

By writing b for a, c for b, d for c, q^2 for d, and q^2 for q in the above Andrews' summation formula we get

$$(2.1)\quad {}_3\phi_2\left[\begin{matrix}b,c,dq^2;q^2;1/bc\\ d,q^2\end{matrix}\right]=\prod\left[\begin{matrix}q^2/b,q^2/c;q^2\\ 1/bc,q^2\end{matrix}\right]\times\left\{2-\frac{1}{b}-\frac{1}{c}+\frac{d(1-b)(1-c)}{bc(1-d)}\right\}.$$

Using (2.1) to sum the n-series on the right of (1.16) by taking $\alpha_1=b$, $\alpha_2=c$, $\alpha_3=dq^2$, $\beta_1=d$, $\lambda=2$ and $t=1/bcq^2$, we get after some simplification,

$$(2.2)\quad \sum_{n=0}^{\infty}\frac{(-aq;q^2)_n(-q/a;q^2)_n(b;q^2)_n(c;q^2)_n(dq^2;q^2)_n(1/bc)^n}{(q^2;q^2)_{2n}(d;q^2)_n}$$

$$=\prod\left[\begin{matrix}q^2/b,q^2/c;q^2\\ q^2,1/bc\end{matrix}\right]\times\left[\left\{1-\frac{1}{b}-\frac{1}{c}+\frac{d(1-b)(1-c)}{bc(1-d)}\right\}\right.$$

$$+\sum_{N=1}^{\infty}\frac{(a^N+a^{-N})q^{N^2}(b;q^2)_N(c;q^2)_N(dq^2;q^2)_N(1/bc)^N}{(d;q^2)_N(q^2/b;q^2)_N(q^2/c;q^2)_N}$$

$$\left.\times\left\{1-\left(\frac{1}{b}+\frac{1}{c}\right)q^{2N}+q^{4N}+dq^{2N}\frac{(1-bq^{2N})(1-cq^{2N})}{bc(1-dq^{2N})}\right\}\right].$$

Letting $d\to\infty$ in (2.2) we get Andrew's Lemma [2, p. 176].

3. Putting $a=-e^{2iz}$ and letting $b\to\infty$, $c\to\infty$ in (2.2) and using (1.4) we get

$$(3.1)\quad \sum_{n=0}^{\infty}\frac{q^{2n^2-2n}(1-dq^{2n})}{(q^2;q^2)_{2n}}\theta_{4;n}(z;q)$$

$$=-d-2d\sum_{N=1}^{\infty}(-1)^N q^{3N^2}\operatorname{Cos}2NZ+2\sum_{N=-\infty}^{\infty}(-1)^N q^{3N^2+2N}\operatorname{Cos}2NZ$$

$$=-d-d[\theta_4(z;q^3)-1]+\sum_{N=-\infty}^{\infty}(-1)^N q^{3N^2+2N}e^{2iNZ}+\sum_{N=-\infty}^{\infty}(-1)^N q^{3N^2+2N}e^{-2iNZ}$$

$$=-d\theta_4(z;q^3)+\sum_{N=-\infty}^{\infty}(-1)^N q^{3N^2}e^{2iZ(z+\pi\tau)}+\sum_{N=-\infty}^{\infty}(-1)^N q^{3N^2}e^{2iN(-z+\pi\tau)}$$

$$=-d\theta_4(z;q^3)+\theta_4(z+\pi\tau;q^3)+\theta_4(-z+\pi\tau;q^3)+\theta_4(-z+\pi\tau;q^3)$$

$$=-d\theta_4(z;q^3)-q^{-3}e^{2iZ}\theta_4(z;q^3)-q^{-3}e^{2iZ}\theta_4(z;q^3)$$

$$=-d\theta_4(z;q^3)-2q^{-3}\operatorname{Cos}2Z\theta_4(z;q^3)=-(d+2q^{-3}\operatorname{Cos}2Z)\theta_4(z;q^3)\,.$$

We have used equations (4), (8), (17) of Rainville [4, pp. 316–317] to put the expansion in an elegant form.

Dividing by d and then taking limit as $d\to\infty$, we have

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_{2n}}\theta_{4;n}(z;q)=\theta_{4,n}(z;q^3)\,,$$

which is the identity that follows from Ramanujan's result (1.14), Andrews [2, (1.10), p. 174].

Putting $a=e^{2iz}$ and letting $b\to\infty$, $c\to\infty$ in (2.2) and using (1.3), we get

$$(3.2)\quad \sum_{n=0}^{\infty}\frac{q^{2n^2-2n}(1-dq^{2n})}{(q^2;q^2)_{2n}}\theta_{3;n}(z;q)=-(d-2q^{-3}\operatorname{Cos}2z)\theta_3(z;q^3)\,.$$

Putting $a=e^{-2iz}$, $b=-q$ and letting $c\to\infty$ in (2.2) and using (1.4), we get

$$(3.3)\quad \frac{1}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-2n}(-q;q^2)_n(1-dq^{2n})}{(q^2;q^2)_{2n}}\theta_{4;n}(z;q)=-\left(d-\frac{1}{q}2q^{-2}\operatorname{Cos}2z\right)\theta_4(z;q^2)\,.$$

Putting $a=e^{2iz}$, $b=-q$ and letting $c\to\infty$ in (2.2) and using (1.3), we get

$$(3.4)\quad \frac{1}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-2n}(-q;q^2)_n(1-dq^{2n})}{(q^2;q^2)_{2n}}\theta_{3,n}(z;q)=-\left(d-\frac{1}{q}+2q^{-2}\operatorname{Cos}2z\right)\theta_3(z;q^2)\,.$$

Putting $a=e^{2iz}$, $b=-q$, $c=q$ in (2.2) and using (1.3), we get

$$(3.5)\quad \frac{(-q^{-2};q^2)_\infty}{(-q;q^2)_\infty(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-2n}(-q;q^2)_n(q;q^2)_n(1-dq^{2n})}{(q^2;q^2)_{2n}}\theta_{3;n}(z;q)$$

$$= -\left(d + \frac{d}{q^2} + \frac{2}{q}\operatorname{Cos} 2z\right)\theta_4(z; q).$$

Putting $a = -e^{2iz}$, $b = -q$, $c = q$ in (2.2) and using (1.4), we get

(3.6)
$$\frac{(-q^{-2}; q^2)_\infty}{(-q; q^2)_\infty(-q; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-2n}(-q; q^2)_n (q; q^2)_n (1 - dq^{2n})}{(q^2; q^2)_{2n}} \theta_{4;n}(z; q)$$
$$= -\left(d + \frac{d}{q^2} + \frac{2}{q}\operatorname{Cos} 2z\right)\theta_3(z; q).$$

By putting $d = 2q^{-3}$, $-2q^{-3}$, $\to\infty$, 0, -1, $-q$ in (3.1), $d = 2q^{-3}$, $\to\infty$, 0, -1 in (3.2), $d = 0$, -1 in (3.3), $d = 1/q, 2q^{-2}$ in (3.4), $d = 2/q$ in (3.5), $d = 0$, $-2/q$ in (3.6) we get interesting results.

4. Theorem 2. We now prove

(4.1)
$$\sum_{n=0}^{\infty} \frac{(-\alpha)_{n+1}(-q/\alpha)_n((\alpha_r))_n(tq^\lambda)^n}{(q)_{2n+1}((\beta_s))_n} = \frac{1-\alpha}{1-q}\, {}_r\phi_{s+1}\begin{bmatrix}(\alpha_r); tq^\lambda \\ (\beta_s), q^2;\end{bmatrix}$$
$$+ \frac{(\alpha^{1+N} + \alpha^{-N})q^{(N^2+N)/2}((\alpha_r))_N(tq^\lambda)^N}{(q)_{2N+1}((\beta_s))_N} \times \sum_{n=0}^{\infty} \frac{((\alpha_r)q^N)_n(tq^\lambda)^n}{(q)_n(q^{2N+2})_n((\beta_s)q^N)_n}.$$

PROOF.

$$\sum_{n=0}^{\infty} \frac{(-\alpha)_{n+1}(-q/\alpha)_n((\alpha_r))_n(tq^\lambda)^n}{(q)_{2n+1}((\beta_s))_n} = \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2+\lambda n}\alpha^{-n}(-\alpha q^{-n})_{2n+1}((\alpha_r))_n t^n}{(q)_{2n+1}((\beta_s))_n}$$
$$= \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} \frac{\alpha^{-n+m} q^{(n^2+n)/2+\lambda m} q^{-nm+(m^2-m)/2}((\alpha_r))_n t^n}{(q)_m(q)_{2n+1-m}((\beta_s))_n}$$
$$= \sum_{n=-\infty}^{\infty} \alpha^N q^{(N^2-N)/2} \sum_{n=0}^{\infty} \frac{((\alpha_r))_n(tq^\lambda)^n}{(q)_{n+N}(q)_{n+1-N}((\beta_s))_n}$$
$$= \sum_{N=1}^{\infty} (\alpha^N + \alpha^{1-N})q^{(N^2-N)/2} \sum_{n=0}^{\infty} \frac{((\alpha_r))_{n+N-1}(tq^\lambda)^{n+N-1}}{(q)_n(q)_{n+2N-1}((\beta_s))_{n+N-1}}$$
$$= \sum_{N=1}^{\infty} (\alpha^N + \alpha^{1-N})q^{(N^2-N)/2} = \sum_{n=0}^{\infty} \frac{((\alpha_r))_{n+N-1}(tq^\lambda)^{n+N-1}}{(q)_n(q)_{n+2N-1}((\beta_s))_{n+N-1}} = \frac{1+\alpha}{1-q}\, {}_r\phi_{s+1}\begin{bmatrix}(\alpha_r); tq^\lambda \\ (\beta_s), q^2;\end{bmatrix}$$
$$+ \sum_{N=1}^{\infty} \frac{(\alpha^{1+N} + \alpha^{-N})q^{(N^2+N)/2}((\alpha_r))_N(tq^\lambda)^N}{(q)_{2N+1}((\beta_s))_N} \times \sum_{n=0}^{\infty} \frac{((\alpha_r)q^N)_n(tq^\lambda)^n}{(q)_n(q^{2N+2})_n((\beta_s)q^N)_n}.$$

5. Application of Theorem 2. Using the summation formula (2.2) after putting $\alpha_1 = b$, $\alpha_2 = c$, $\alpha_3 = dq$, $\beta_1 = d$, $t = 1/bcq$, we have

$$\sum_{n=0}^{\infty} \frac{(-\alpha)_{n+1}(-q/\alpha)_n(b)_n(c)_n(dq)_n(q/bc)^n}{(q)_{2n+1}(d)_n}$$

$$+(1+\alpha)\prod\begin{bmatrix} q^2/b, q^2/c \\ q, \quad q/bc \end{bmatrix}\left\{1-\frac{q}{b}-\frac{q}{c}+q+\frac{dq(1-b)(1-c)}{bc(1-d)}\right\}$$

$$=\sum_{N=1}^{\infty} \frac{(\alpha^{1+n}+\alpha^{-N})q^{(N^2+N)/2}(b)_N(c)_N(dq)_N(q/bc)^N}{(d)_N(q^2/b)_N(q^2/c)_N}$$

$$\times\prod\begin{bmatrix} q^2/b, q^2/c \\ q, \quad q/bc \end{bmatrix}\left\{1-\frac{q^{N+1}}{b}-\frac{q^{N+1}}{c}+q^{2N+1}+\frac{dq^{N+1}(1-bq^N)(1-cq^N)}{bc(1-dq^N)}\right\}.$$

Replacing q by q^2, we have

$$\text{(5.1)} \quad \sum_{n=0}^{\infty} \frac{(-\alpha; q^2)_{n+1}(-q^2/\alpha; q^2)_n(b; q^2)_n(c; q^2)_n(dq^2; q^2)_n(q^2/bc)^n}{(q^2; q^2)_{2n+1}(d; q^2)_n}$$

$$=(1+\alpha)\prod\begin{bmatrix} q^4/b, q^4/c; q^2 \\ q^2, q^2/bc \end{bmatrix}\left\{1-\frac{q^2}{b}-\frac{q^2}{c}+q^2+\frac{dq^2+(1-b)(1-c)}{bc(1-d)}\right\}$$

$$+\sum_{N=1}^{\infty} \frac{(\alpha^{1+n}+\alpha^{-N})q^{N^2+N}(b; q^2)_N(c; q^2)_N(dq^2; q^2)_N(q^2/bc)^N}{(d; q^2)_N(q^4/b; q^2)_N(q^4/c; q^2)_N}$$

$$\times\prod\begin{bmatrix} q^4/b, q^4/c; q^2 \\ q^2, q^2/bc \end{bmatrix}\left\{1-\frac{q^{2N+2}}{b}-\frac{q^{2N+2}}{c}+q^{4N+2}+\frac{dq^{2N+2}(1-bq^{2N})(1-cq^{2N})}{bc(1-dq^{2N})}\right\}.$$

Putting $\alpha=e^{2iz}$, $b\to\infty$, $c=-q^2$ in (5.1), we have

$$\text{(5.2)} \quad \frac{q^{1/4}}{(-q^2; q^2)_\infty}\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-q^2; q^2)_n(1-dq^{2n})}{(q^2; q^2)_{2n+1}}\theta_{2;n}(z; q)$$

$$=(1-d)\theta_2(z; q^2)+2q^{1/2}\operatorname{Cos} z\,\theta_3(z; q^2).$$

Putting $\alpha=e^{2iz}$, $b=q^2$ and letting $c\to\infty$ in (5.1), we have

$$\text{(5.3)} \quad \frac{1}{2q^{1/4}\operatorname{Cos} z(q^2; q^2)_\infty}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2-n}(1-dq^{2n})}{(q^{2n+2}; q^2)_{n+1}}\theta_{2;n}(z; q)$$

$$=1-\frac{(1+d)}{2\operatorname{Cos} z}q^{-1/2}\theta_{2;f}(z; q^2)-\frac{2\operatorname{Sin} z}{\operatorname{Cos} z}\sum_{N=1}^{\infty}(-1)^N q^{2N^2}\operatorname{Sin} 2Nz.$$

Putting $\alpha=-e^{2iz}$, $b=q^2$ and letting $c\to\infty$ in (5.1), we have

$$\text{(5.4)} \quad \frac{1}{2q^{1/4}\operatorname{Sin} z(q^2; q^2)_\infty}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2-n}(1-dq^{2n})}{(q^{2n+2}; q^2)_{n+1}}\theta_{1;n}(z; q)$$

$$=1-\frac{q^{1/2}(1+d)}{2\operatorname{Sin} z}\theta_{1;f}(z; q^2)+\frac{2\operatorname{Cos} z}{\operatorname{Sin} z}\sum_{N=1}^{\infty} q^{2N^2}\operatorname{Sin} 2Nz.$$

Putting $\alpha=-e^{2iz}$, and letting $b\to\infty$, $c\to\infty$ in (5.1), we have

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}(1-dq^{2n})}{(q^2;q^2)_{2n+1}}\theta_{1;n}(z;q)=-(dq^{-1/2}+2q^{-5/2}\operatorname{Cos}2z)\theta_1(z;q^3)\,. \tag{5.5}$$

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