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AFFINE SURFACES WITH HIGHER ORDER PARALLEL CUBIC FORM

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1. Introduction. The study of affine differential geometry started with the work of Blaschke and his coworkers around 1920 ([B]). Let M be a nondegenerate hypersurface in $\boldsymbol{R}^{n+1}$. Then starting from the usual connection D on $\boldsymbol{R}^{n+1}$ and the standard volume form on $\boldsymbol{R}^{n+1}$ given by the determinant, one can induce in a unique way an affine connection ∇ and a semi-Riemannian metric h on M. This semi-Riemannian metric h on M is called the affine metric. Together with the canonical affine connection ∇, the affine metric together with its Levi Civita connection plays an important role in affine differential geometry (see [B] among others). We say that $C=(\nabla h)$ is the cubic form. It is well-known that the vanishing of the cubic form C implies that M is an open part of a nondegenerate quadric. Let M be a 2-manifold. In this case, K. Nomizu and U. Pinkall investigated the condition that the cubic form is parallel with respect to ∇. This condition gives a characterization of the Cayley surface. The condition that $(\hat{\nabla}C)=0$ was studied in [M-N]. Here, using a method different from that in [M-N], we will study affine surfaces with $\hat{\nabla}$-higher order parallel cubic form, i.e., which satisfy $(\hat{\nabla}^n C)=0$ for some natural number n. In Section 2, we recall some elementary facts about affine differential geometry. In Section 3, we give some examples of affine surfaces with $\hat{\nabla}$-higher order parallel cubic form. Finally in Section 4, we show that these examples are essentially the only ones by proving the following two theorems.

THEOREM 4.1. *Let M be a locally strongly convex Blaschke surface in $\boldsymbol{R}^3$ with $\hat{\nabla}^n C=0$. Then either*

(i) *M is affinely equivalent to an open part of an elliptic paraboloid,*

(ii) *M is affinely equivalent to an open part of a nondegenerate ellipsoid or a two-sheeted hyperboloid, or*

(iii) *M is affinely equivalent to an open part of the affine surface described by $xyz=1$.*

Notice that these surfaces are exactly the locally strongly convex affine spheres with constant curvature Blaschke metric. However, in the non-convex case the situation is quite different, as the following theorem shows:

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THEOREM 4.2. *Let $x: M \to \boldsymbol{R}^3$ be a non-convex Blaschke surface with $\hat{\nabla}$-higher order parallel cubic form. Then, either M is an open part of a one-sheeted hyperboloid or every point p of M has a neighbourhood U of M on which one of the following holds*:

(i) *U is affinely congruent with an open part of the Blaschke surface described by $x_3(x_1^2+x_2^2)=1$.*

(ii) *We can identify U with an open part of $\boldsymbol{R}_1^2$ and there exist two polynomials K and L in one variable on U such that $x(U)$ is affinely congruent to $x_{KL}(U)$, where x_{KL} is as defined in Section* 3.

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2. Preliminaries. Let M^n be a differentiable manifold, $\hat{\nabla}$ a torsion-free affine connection on M and T a $(0, k)$-tensor field on M. Let $\hat{R}$ denote the curvature tensor of $\hat{\nabla}$. Then $\hat{R}$ acts on T as a derivation in the following way:

$$\begin{aligned}(\hat{R}\cdot T)&(X, Y, X_1, \cdots, X_k)\\ &= -T(\hat{R}(X, Y)X_1, X_2, \cdots, X_k)\cdots - T(X_1, \cdots, X_{k-1}, \hat{R}(X, Y)X_h)\,.\end{aligned}$$

From the skew-symmetry of $\hat{R}$ it follows that $\hat{R}\cdot T$ is skew-symmetric in its first and second components. Furthermore, it is clear from the definition that if T is symmetric (resp. skew-symmetric) in its i-th and j-th component, $\hat{R}\cdot T$ is also symmetric (resp. skew-symmetric) in its $(i+2)$-nd and $(j+2)$-nd components. Then, the higher derivatives of T with respect to $\hat{R}$ are defined inductively as follows:

$$\hat{R}^n\cdot T=\hat{R}\cdot(\hat{R}^{n-1}\cdot T)\,.$$

Clearly, $\hat{R}^n\cdot T$ is skew symmetric in its $(2k-1)$-st and $2k$-th components for $k=1, 2, \cdots, n$. Furthermore, we have the following lemma the proof of which is similar to that of Lemma 2 in [V].

LEMMA 2.1. *Let M be a differentiable manifold and $\hat{\nabla}$ a torsion-free affine connection on M with curvature tensor $\hat{R}$. If a $(0, k)$-tensor T on M satisfies*

$$\hat{\nabla}^n T=0\,,$$

then

$$\hat{R}^m\cdot T=0\,,$$

where $m=[(m+1)/2]$, and $[x]$ denotes the integer part of x.

Let $f: M^n \to \boldsymbol{R}^{n+1}$ be an immersion of a connected differentiable manifold of dimension n into the affine space $\boldsymbol{R}^{n+1}$ equipped with its usual flat connection D and a parallel volume element ω. Let ξ be an arbitrary local vector field transversal to $f(M^n)$. For any vector fields X and Y on M^n, we write

(2.1) $$D_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)\xi,$$

thus defining an affine connection ∇ and a symmetric tensor of type (0, 2). We call h the second fundamental form. We can also define a volume element θ on M by

(2.2) $$\theta(X_1, X_2, \cdots, X_n) = \omega(f_* X_1, \cdots, f_* X_n, \xi),$$

for any tangent vectors $X_1, \cdots, X_n$ to M^n.

If (M, ∇) is an affine manifold, an immersion f is called an affine immersion if there exists locally a transversal vector field such that (2.1) holds. If (∇, θ) is an equiaffine structure on M, i.e., if ∇ and θ satisfy $\nabla\theta = 0$, then an affine immersion is said to be equiaffine if θ satisfies (2.2).

We say that f is nondegenerate if h is nondegenerate (and this condition is independent of the choice of ξ). In this case, it is known (see [N], [N-P]$_1$) that there is a unique choice of ξ such that the corresponding induced connection ∇, the nondegenerate metric h, and the induced volume element θ satisfy the following conditions:

(i) $\nabla\theta = 0$, thus (∇, θ) is an equiaffine structure on M^n;

(ii) $\theta = \omega_h$ (volume element given by h).

We call ∇ the induced connection and h the affine metric. If h is positive (or negative) definite, the immersion is said to be locally strongly convex. If this is not the case, we say that the immersion is non-convex. Condition (i) implies that $D_X\xi$ is tangent to $f(M^n)$ for any tangent vector X to M^n. We define a tensor field S of type (1, 1) on M^n, called the shape operator, by

(2.3) $$D_X\xi = -f_*(SX).$$

From now on, we will call an affine immersion satisfying (i) and (ii) a *Blaschke immersion*. We have the following fundamental equations.

Equation of Gauss: The curvature tensor R of ∇ is given by

(2.4) $$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY.$$

Equation of Codazzi for h:

(2.5) $$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z).$$

Equation of Codazzi for S:

(2.6) $$(\nabla_X S)(Y) = (\nabla_Y S)(X).$$

Equation of Ricci:

(2.7) $$h(SX, Y) = h(X, SY).$$

Apolarity:

(2.8) $$\nabla\omega_h = 0.$$

We call M an affine sphere if S is a multiple of the identity. Since $\dim(M)\geq 2$, it follows from (2.6) that $S=\lambda I$, where λ is constant on M. If $\lambda=0$, we say that M is an improper affine sphere and if $\lambda\neq 0$, we say that M is a proper affine sphere. From (2.5) it follows that $C(X, Y, Z)=(\nabla h)(X, Y, Z)$ is symmetric in X, Y and Z. We call C the cubic form for the affine immersion f. Apolarity (2.8) can also be expressed by

$$\text{(2.9)} \qquad \operatorname{trace} K_X=0 \qquad \text{for every tangent vector } X,$$

where $K_X Y=\nabla_X Y-\hat{\nabla}_X Y$, $\hat{\nabla}$ denoting the Levi Civita connection for h. Another equivalent condition is

$$\text{(2.10)} \qquad \mathscr{C}(X)=0 \qquad \text{for each } X,$$

where $\mathscr{C}$ is defined by

$$\mathscr{C}(X)=\operatorname{trach}_h\{(Y, Z)\to(\nabla h)(X, Y, Z)\},$$

where trace_h denotes the trace with respect to the affine metric h.

From now on, we will assume that $\dim(M)=2$. Also, we will need the following theorems from [D-N-V] and [M-R].

THEOREM OF RADON ([D-N-V]). *If (M^2, h) is a simply connected, semi-Riemannian 2-manifold and ∇ is a torsion free, strongly compatible connection, i.e., ∇ and h satisfy* (2.5) *and* (2.8), *then there exists a Blaschke immersion f: $M\to\boldsymbol{R}^3$ with induced affine connection ∇ and induced second fundamental form h if and only if $\mathscr{L}=0$, where $\mathscr{L}$ is defined by*

$$\mathscr{L}(Y, Z)=\operatorname{trace}_h\{(X, W)\to(\nabla_X R)(Y, Z)W\}.$$

Furthermore, this immersion is unique up to an affine transformation of $\boldsymbol{R}^3$.

THEOREM 2.1 ([M-R]). *Let M be an affine sphere which is flat with respect to h. Then, up to an affine transformation of $\boldsymbol{R}^3$, M is an open part of one of the following*:

(1) *the elliptic paraboloid $x_3=x_1^2+x_2^2$, if h is locally strongly convex and M is an improper affine sphere,*

(2) *a ruled surface of the form $x_3=x_1x_2+\phi(x_2)$, where ϕ is an arbitrary function depending only on x_2, if h is not convex and M is an improper affine sphere,*

(3) *the surface given by $x_1x_2x_3=1$, if h is convex and M is a proper affine sphere,*

(4) *the surface described by $x_3(x_1^2+x_2^2)=1$, if h is not convex and M is an proper affine sphere.*

3. Some examples of affine surfaces with $\hat{\nabla}$-higher order parallel cubic form. In this section, we will give some examples of affine surfaces which satisfy $(\hat{\nabla}^n C)=0$ for some n. In the next section, we will show that these examples are basically the only ones.

If $n=0$, i.e., if M has vanishing cubic form, it is well-known ([B], [N-P]$_2$) that

M must be affinely equivalent to a nondegenerate quadric. Affine surfaces which satisfy $\hat{\nabla}C=0$ have been classified by Nomizu and Magid in [M-N]. They obtain the following classification result:

Let M be an affine Blaschke surface which satisfies $\hat{\nabla}C=0$. Assume furthermore that C is not identically zero on M. Then, there exists an affine transformation of $\boldsymbol{R}^3$ such that one of the following holds:

(3.1) *M is an open part of the affine Blaschke surface described by $x_1x_2x_3=1$.*

(3.2) *M is an open part of the affine Blaschke surface described by $x_3(x_1^2+x_2^2)=1$.*

(3.3) *M is an open part of the Cayley surface, i.e., the surface described by $x_3=x_1x_2+x_2^3$.*

Other examples of surfaces with $\hat{\nabla}$-higher order parallel cubic form can be found in [M-R], where the authors classify the affine spheres which are flat with respect to h. Apart from open parts of the paraboloids and the surfaces given by (3.1), (3.2) and (3.3), the only other such surfaces can be described as the graph of

$$x_3=x_1x_2+\phi(x_2)\,,$$

where ϕ is an arbitrary function depending only on x_2. A straightforward computation shows that such a surface satisfies $\hat{\nabla}^nC=0$ for some n if and only if ϕ is a polynomial in x_2 of degree at most $n+2$. However, these surfaces are just special cases of the ones which we will consider next.

We take $M=\boldsymbol{R}^2$ and we consider on M the standard Minkowski metric h. Let $\hat{\nabla}$ be the Levi Civita connection of h. Then on $\boldsymbol{R}^2$, we can consider globally defined coordinates $\{y, z\}$ such that ∂_y and ∂_z satisfy

$$h(\partial_y, \partial_y)=h(\partial_z, \partial_z)=0\,, \qquad h(\partial_y, \partial_z)=1\,.$$

Then we can define a torsion-free affine connection ∇ on M, which is strongly compatible with respect to h, by

$$\nabla_{\partial_y}\partial_y=2\sqrt{2}(K(y)+zL(y))\partial_z, \qquad \nabla_{\partial_y}\partial_z=\nabla_{\partial_z}\partial_y=\nabla_{\partial_z}\partial_z=0\,,$$

where K and L are differentiable function of one variable defined on the whole of $\boldsymbol{R}$. If we denote the curvature tensor of ∇ by R, we find after a straightforward computation that

$$R(\partial_y, \partial_z)\partial_z=0\,, \tag{3.4}$$

$$R(\partial_y, \partial_z)\partial_y=-2\sqrt{2}L(y)\partial_z\,. \tag{3.5}$$

From this, we deduce that

$$(\nabla_{\partial_y}R)(\partial_y, \partial_z)\partial_z=0=(\nabla_{\partial_z}R)(\partial_y, \partial_z)\partial_y\,.$$

Therefore, we can apply Radon's theorem to obtain a Blaschke immersion from $(\boldsymbol{R}^2, h)$

into $\boldsymbol{R}^3$, with ∇ as induced affine connection and with h as affine metric. We will denote this immersion by x_{KL}. Notice that by the definition of ∇ and h, we immediately obtain that the immersion is ruled. Further, using (3.4), (3.5) and the Gauss equation, we see that the Blaschke immersion obtained in this way is also affinely minimal. Notice that if we choose the function L to be identically zero, we deduce from (3.4), (3.5) and the Gauss equation that M is an improper affine sphere. Then, it is not too difficult to see that the immersions x_{K0} coincide with the examples found by M. Magid and P. Ryan.

By a straightforward computation, one can check that $x_{KL}(\boldsymbol{R}^2)$ satisfies $(\hat{\nabla}^n C)=0$, for some n, if and only if K (resp. L) is a polynomial of degree at most $n-1$ (resp. $n-2$).

4. Proof of the theorem. We will divide the proof in several lemmas. Let M be a Blaschke surface in $\boldsymbol{R}^3$ such that $\hat{\nabla}^n C=0$ for some n. Then, we know by Lemma 2.1 that $\hat{R}^m\cdot C=0$ for some m. Affine surfaces satisfying that condition are characterized by the following lemma.

LEMMA 4.1. *Let M be a Blaschke surface in $\boldsymbol{R}^3$ such that $\hat{R}^m\cdot C=0$ for some m. Then either*

(i) *M is an open part of a nondegenerate ellipsoid or hyperboloid,*

or

(ii) *$\hat{R}=0$, i.e., M is flat with respect to h on the whole of M.*

PROOF. Since $\dim M=2$ and $\hat{\nabla}$ is the Levi Civita connection of the semi-Riemannian metric h, the curvature tensor $\hat{R}$ of $\hat{\nabla}$ is given by

$$\hat{R}(X, Y)Z=\mu(h(Y, Z)X-h(X, Z)Y),$$

where μ is a differentiable function on M. Then, we will first show by induction that the following formulas hold at every point p of M:

$$(4.1)\quad \begin{aligned}
&(\hat{R}^{2n+1}\cdot C)(e_1, e_2, e_1, e_2, \cdots, e_1, e_1, e_1)=(-1)^{n+1}\varepsilon^n 3^{2n+1}\mu^{2n+1}C(e_2, e_2, e_2)\\
&(\hat{R}^{2n+1}\cdot C)(e_1, e_2, e_1, e_2, \cdots, e_1, e_2, e_2)=(-1)^{n}\varepsilon^{n+1} 3^{2n+1}\mu^{2n+1}C(e_2, e_2, e_2)\\
&(\hat{R}^{2n+1}\cdot C)(e_1, e_2, e_1, e_2, \cdots, e_2, e_2, e_2)=(-1)^{n}\varepsilon^{n+1} 3^{2n+1}\mu^{2n+1}C(e_1, e_1, e_1)\\
&(\hat{R}^{2n+1}\cdot C)(e_1, e_2, e_1, e_2, \cdots, e_1, e_1, e_2)=(-1)^{n+1}\varepsilon^n 3^{2n+1}\mu^{2n+1}C(e_1, e_1, e_1),
\end{aligned}$$

where $\{e_1, e_2\}$ is a basis of T_pM such that $h(e_1, e_1)=\varepsilon$, $\varepsilon\in\{-1, 1\}$, $h(e_2, e_2)=1$ and $h(e_1, e_2)=0$. First, by applying the definition of $\hat{R}\cdot C$ and the apolarity condition, we obtain that

$$\begin{aligned}
\hat{R}\cdot C(e_1, e_2, e_1, e_1, e_1)&=-3C(\hat{R}(e_1, e_2)e_1, e_1, e_1)\\
&=3\mu\varepsilon C(e_2, e_1, e_1)=-3\mu C(e_2, e_2, e_2).
\end{aligned}$$

Similarly, we find that

$$\hat{R}\cdot C(e_1, e_2, e_2, e_2, e_2)=3\mu\varepsilon C(e_1, e_1, e_1),$$

$$\hat{R}\cdot C(e_1, e_2, e_1, e_1, e_2) = -3\mu C(e_1, e_1, e_1),$$
$$\hat{R}\cdot C(e_1, e_2, e_1, e_2, e_2) = 3\mu\varepsilon C(e_2, e_2, e_2).$$

Hence (4.1) holds for $n=0$. Therefore let us now assume that (4.1) holds for a natural number n and prove that (4.1) also holds for $n+1$. Using the skew symmetry of $\hat{R}^{2n+1}\cdot C$ in its $(4k+1)$-st and $(4k+2)$-nd components for $k=1, 2, \cdots, n$, and the induction hypothesis, we obtain that

$$\begin{aligned}(\hat{R}^{2n+2}\cdot C)(e_1, e_2, \cdots, e_1, e_1, e_1) &= 3\mu\varepsilon(\hat{R}^{2n+1}\cdot C)(e_1, e_2, \cdots, e_2, e_1, e_1)\\ &= \varepsilon^{n+1}(-1)^{n+1}3^{2n+2}\mu^{2n+2}C(e_1, e_1, e_1).\end{aligned}$$

Similarly, we also obtain that

$$(\hat{R}^{2n+2}\cdot C)(e_1, e_2, \cdots, e_1, e_1, e_2) = \varepsilon^{n+2}(-1)^{n+2}3^{2n+2}\mu^{2n+2}C(e_2, e_2, e_2).$$
$$(\hat{R}^{2n+2}\cdot C)(e_1, e_2, \cdots, e_1, e_2, e_2) = \varepsilon^{n+2}(-1)^{n+2}3^{2n+2}\mu^{2n+2}C(e_1, e_1, e_1).$$
$$(\hat{R}^{2n+2}\cdot C)(e_1, e_2, \cdots, e_2, e_2, e_2) = \varepsilon^{n+1}(-1)^{n+1}3^{2n+2}\mu^{2n+2}C(e_2, e_2, e_2).$$
$$(\hat{R}^{2n+3}\cdot C)(e_1, e_2, \cdots, e_1, e_1, e_1) = \varepsilon^{n+1}(-1)^{n+2}3^{2n+3}\mu^{2n+3}C(e_2, e_2, e_2).$$
$$(\hat{R}^{2n+3}\cdot C)(e_1, e_2, \cdots, e_1, e_1, e_2) = \varepsilon^{n+1}(-1)^{n+2}3^{2n+3}\mu^{2n+3}C(e_1, e_1, e_1).$$
$$(\hat{R}^{2n+3}\cdot C)(e_1, e_2, \cdots, e_1, e_2, e_2) = \varepsilon^{n+2}(-1)^{n+1}3^{2n+3}\mu^{2n+3}C(e_2, e_2, e_2).$$
$$(\hat{R}^{2n+3}\cdot C)(e_1, e_2, \cdots, e_2, e_2, e_2) = \varepsilon^{n+2}(-1)^{n+1}3^{2n+3}\mu^{2n+3}C(e_1, e_1, e_1).$$

Hence (4.1) holds for every n. Using the apolarity condition, we find from (4.1) and the assumption of the lemma that for each point p of M the following holds:

$$\mu(p)=0 \quad \text{or} \quad C_p=0. \tag{4.2}$$

Let $U=\{p\in M \mid \mu(p)\neq 0\}$. Then, U is an open part of M. If $U\neq\emptyset$, then $\mu=0$ on the whole of M and we obtain the lemma. Therefore, we may assume that U is not empty. Then, from (4.2) it follows that C vanishes identically on U. Thus from Berwald's theorem it follows that every connected component of U is an open part of a quadric. Since μ is different from zero on U, these quadrics must be ellipsoids or hyperboloids. But for an ellipsoid or an hyperboloid μ is a constant different from zero. Hence, since μ is differentiable on M, $U=M$ and thus connected. Therefore M is an open part of a nondegenerate ellipsoid or a nondegenerate hyperboloid. This completes the proof. ■

From now on, we will assume that M is not a part of a quadric. Then, it follows from Lemma 4.1 that M is flat with respect to h. Let $p\in M$. Since M is flat with respect to h, we know that there exist coordinates $\{u, v\}$ defined on a neighbourhood U of p such that

$$h(x_u, x_u)=\varepsilon\,, \qquad h(x_v, x_v)=1\,, \qquad h(x_u, x_v)=0\,,$$

where $\varepsilon=1$ or -1 on U. So $\hat{\nabla}_{x_u}x_u=\nabla_{x_u}x_v=\hat{\nabla}_{x_v}x_u=\hat{\nabla}_{x_v}x_v=0$. Therefore, it follows from $\hat{\nabla}^n C=0$ and the apolarity condition that there exist polynomials P and Q of two variables u and v, which are of degree at most $n-1$ such that

$$C(x_u, x_u, x_u)=-2P(u, v)\,, \qquad C(x_v, x_v, x_v)=-2Q(u, v)\,,$$

$$C(x_u, x_v, x_v)=2\varepsilon P(u, v)\,, \qquad C(x_v, x_u, x_u)=2\varepsilon Q(u, v)\,.$$

Then, the components of the induced connection ∇ and of the curvature tensor R are computed in the following two lemmas.

LEMMA 4.2. *On U we have*

$$\nabla_{x_u}x_u=P\varepsilon x_u-\varepsilon Q x_v\,, \qquad \nabla_{x_v}x_v=-Px_u+Qx_v\,, \qquad \nabla_{x_u}x_v=\nabla_{x_v}x_u=-Qx_u-P\varepsilon x_v\,.$$

PROOF. On U, we define functions a_1 up to a_6 by

$$\nabla_{x_u}x_u=a_1x_u+a_2x_v\,, \qquad \nabla_{x_v}x_v=a_3x_u+a_4x_v\,, \qquad \nabla_{x_u}x_v=\nabla_{x_v}x_u=a_5x_u+a_6x_v\,.$$

Then, we know that $-2P=C(x_u, x_u, x_u)=-2h(\nabla_{x_u}x_u, x_u)=-2\varepsilon a_1$. Hence $a_1=\varepsilon P$. The other equations are then obtained in a similar way. ■

LEMMA 4.3. *The curvature tensor R of ∇ is given by*

$$R(x_u, x_v)x_v=(-P_u+Q_v-2(P^2\varepsilon+Q^2))x_u+(Q_u+P_v\varepsilon)x_v\,,$$

$$R(x_u, x_v)x_u=(-Q_u-P_v\varepsilon)x_u+\varepsilon(-P_u+Q_v+2(P^2\varepsilon+Q^2))x_v\,.$$

PROOF. Since $\nabla_{x_u}x_v-\nabla_{x_v}x_u=[x_u, x_v]=0$, we have

$$\begin{aligned}R(x_u, x_v)x_v&=\nabla_{x_u}\nabla_{x_v}x_v-\nabla_{x_v}\nabla_{x_u}x_v\\&=-P_ux_u+Q_ux_v+Q_vx_u+P_v\varepsilon x_v-2P\varepsilon(Px_u-Qx_v)+2Q(-Qx_u-P\varepsilon x_v)\\&=(-P_u+Q_v-2(P^2\varepsilon+Q^2))x_u+(Q_u+P_v\varepsilon)x_v\,.\end{aligned}$$

The proof of the second formula is completely similar. ■

LAMMA 4.4. *The polynomials P and Q, defined on U, satisfy the following system of differential equations*:

(i) $$(Q_{vv}-\varepsilon Q_{uu}-2P_{uv})-6(PP_v\varepsilon+QQ_v)+2(QP_u-PQ_u)=0\,,$$

(ii) $$(P_{vv}\varepsilon-P_{uu}+2Q_{uv})+6(PP_u\varepsilon+QQ_u)+2\varepsilon(QP_v-PQ_v)=0\,.$$

PROOF. By Radon's theorem, we know that ∇ and R satisfy

$$\varepsilon(\nabla_{x_u}R)(x_u, x_v)x_u+(\nabla_{x_v}R)(x_u, x_v)x_v=0\,. \tag{4.2}$$

On the other hand, by using Lemma 4.2 and Lemma 4.3, we obtain that

$$
\begin{aligned}
(4.3)\quad &(\nabla_{x_u}R)(x_u, x_v)x_u \\
&=\nabla_{x_u}(R(x_u, x_v)x_u)-R(x_u, x_v)\nabla_{x_u}x_u-R(\nabla_{x_u}x_u, x_v)x_u-R(x_u, \nabla_{x_u}x_v)x_u \\
&=(-Q_{uu}-P_{uv}\varepsilon)x_u+(-P_{uu}\varepsilon+Q_{uv}\varepsilon+4\varepsilon(PP_u\varepsilon+QQ_u))x_v \\
&\quad+(-Q_u-P_v\varepsilon)(P\varepsilon x_u-Q\varepsilon x_v)+(-P_u\varepsilon+Q_v\varepsilon+2\varepsilon(P^2\varepsilon+Q^2))(-Qx_u-P\varepsilon x_v) \\
&\quad-P\varepsilon((-Q_u-P_v\varepsilon)x_u+\varepsilon(-P_u+Q_v+2(P^2\varepsilon+Q^2))x_v) \\
&\quad+Q\varepsilon((-P_u+Q_v-2(P^2\varepsilon+Q^2))x_u+(Q_u+P_v\varepsilon)x_v) \\
&=(-Q_{uu}-P_{uv}\varepsilon-4Q\varepsilon(P^2\varepsilon+Q^2))x_u+(-P_{uu}\varepsilon+Q_{uv}\varepsilon+6\varepsilon(PP_u\varepsilon+QQ_u) \\
&\quad+2(QP_v-PQ_v)-4P(P^2\varepsilon+Q^2))x_v\,.
\end{aligned}
$$

Similarly, we obtain that

$$
\begin{aligned}
(4.4)\qquad (\nabla_{x_v}R)(x_u, x_v)x_v=&(-P_{uv}+Q_{vv}+4Q(P^2\varepsilon+Q^2)-6(PP_v\varepsilon+QQ_v) \\
&+2(P_uQ-PQ_u))x_u+(Q_{uv}+P_{vv}\varepsilon+4\varepsilon P(P^2\varepsilon+Q^2))x_v\,.
\end{aligned}
$$

Then, substituting (4.3) and (4.4) into (4.2) completes the proof. ■

PROOF OF THEOREM 4.1. Let us assume that M is not affinely equivalent to an open part of a nondegenerate ellipsoid or a two-sheeted hyperboloid. Then, it follows from Lemma 4.1 that M is flat with respect to h. Let $p\in M$. Then, we know from Lemma 4.2 and Lemma 4.3 that there exist polynomials P and Q in two variables u and v, on a neighbourhood U of p, which satisfy the system of differential equations described in Lemma 4.4. Since M is locally strongly convex, we have $\varepsilon=1$ in Lemma 4.4. First, we will prove that P and Q must be constants. Let us assume that $\deg P\neq\deg Q$. If $\deg P>\deg Q$, we denote by P_1 the terms of the highest degree of P. Then it follows from Lemma 4.4 (i) and (ii) that

$$((P_1)^2)_v=0\,,\qquad ((P_1)^2)_u=0\,.$$

Hence P_1 is a constant. Therefore, P is a constant and Q is zero. Similarly, if $\deg(Q)>\deg(P)$, we find that Q is a constant and that P is zero. Therefore, we may assume that $\deg(P)=\deg(Q)$. We will assume that P and Q are not both constants and derive a contradiction. Let P_1 (resp. Q_1) denote the terms of highest degree of P (resp. Q). Then $\deg P_1=\deg Q_1=\deg P=\deg Q>0$. Therefore, by looking at the terms of highest degree in Lemma 4.4 (i) and (ii), we find that P_1 and Q_1 must satisfy the system of equations

$$-3(P_1(P_1)_v+Q_1(Q_1)_v)+(Q_1(P_1)_u-P_1(Q_1)_u)=0\,,\tag{4.5}$$

$$3(P_1(P_1)_u+Q_1(Q_1)_u)+(Q_1(P_1)_v-P_1(Q_1)_v)=0\,.\tag{4.6}$$

Then, if we put $K=P_1+iQ_1$ and $L=P_1-iQ_1$, the equations (4.5) and (4.6) respectively become

$$-3(KL_v+LK_v)-i(KL_u-LK_u)=0\,,\tag{4.7}$$

$$(4.8) \qquad 3(KL_u+LK_u)-i(KL_v-LK_v)=0\,.$$

Now, let α be an irreducible factor of K with multiplicity k. Since K is homogeneous and $\deg(K)>0$, so K_u and K_v are not simultaneously zero and α is then an irreducible factor of K_u or K_v with multiplicity exactly $(k-1)$. Further, it follows from (4.7) and (4.8) that

$$\alpha^k\,|\,(-3LK_v+iLK_u)\,,$$
$$\alpha^k\,|\,(3LK_u+iLK_v)\,.$$

Hence $\alpha^k\,|\,LK_v$ and $\alpha^k\,|\,LK_u$. So $\alpha\,|\,L$. Therefore, by applying the same argument with K and L interchanged, we see that K and L have the same irreducible factors. So, we can decompose K and L over $\boldsymbol{C}$ in the following way:

$$(4.9) \qquad K=c_1\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_n^{k_n}\,,$$

$$(4.10) \qquad L=c_2\alpha_1^{l_1}\alpha_2^{l_2}\cdots\alpha_n^{l_n}\,,$$

where $c_1\neq 0\neq c_2$, $n>1$, $k_i\neq 0\neq l_1$, α_i and α_j are distinct mutually irreducible factors of degree 1. Then, if we substitute this expression in (4.7) and (4.8), and compute the result modulo $\alpha_1^{k_1+l_1}$, we find that

$$-3c_1c_2(k_1+l_1)(\alpha_1)_v+ic_1c_2(k_1-l_1)(\alpha_1)_u\equiv 0(\mathrm{mod}\,\alpha_1^{k_1+l_1})\,,$$
$$3c_1c_2(k_1+l_1)(\alpha_1)_u+ic_1c_2(k_1-l_1)(\alpha_1)_v\equiv 0(\mathrm{mod}\,\alpha_1^{k_1+l_1})\,.$$

However, since $\deg(\alpha_1)=1$, we know that $(\alpha_1)_u$ and $(\alpha_1)_v$ are not both zero. Hence

$$c_1c_2(9(k_1+l_1)^2-(k_1-l_1)^2)=0, \qquad \text{or} \quad c_1c_2(8k_1^2+8l_1^2+20k_1l_1)=0\,.$$

Hence, we obtain a contradiction. Therefore, P and Q must be constant on U. But then it follows from Lemma 4.3 and the Gauss equation that

$$Sx_u=-2(P^2+Q^2)x_u, \qquad Sx_v=-2(P^2+Q^2)x_v\,.$$

Hence S_p is a multiple of the identity for every point p of M. Thus M is an affine sphere which is flat with respect to h. Applying then Theorem 2.1, we are done. ∎

So in the last part of this section, we will assume that M is not locally strongly convex, i.e., we will assume that $\varepsilon=-1$. Then, the solutions of the differential equations are given by the following lemma.

LEMMA 4.5. *Let $\varepsilon=-1$. Then, the polynomials P and Q defined on U satisfy the system of differential equations given in Lemma* 4.4 *if and only if one of the following holds*:

(a) *P and Q are constant on U*,

(b) *$P=Q$ and $P_{vv}+P_{uu}=2P_{uv}$*,

(c) *$P+Q=0$ and $P_{vv}+P_{uu}=-2P_{uv}$*.

PROOF. By Lemma 4.4, we know that the polynomials P and Q satisfy the following system of differential equations:

$$(Q_{vv}+Q_{uu}-2P_{uv})-6(-PP_v+QQ_v)+2(QP_u-PQ_u)=0\,,$$
$$(-P_{vv}-P_{uu}+2Q_{uv})+6(-PP_u+QQ_u)-2(QP_v-PQ_v)=0\,.$$

If we put $K=P+Q$, $L=P-Q$, these equations become

$$\frac{1}{2}(K_{vv}+K_{uu}-L_{vv}-L_{uu}-2(K_{uv}+L_{uv}))+3(KL_v+LK_v)+(KL_u-LK_u)=0\,, \tag{4.11}$$

$$\frac{1}{2}(K_{vv}+K_{uu}+L_{vv}+L_{uu}-2(K_{uv}-L_{uv}))+3(KL_u+LK_u)+(KL_v-LK_v)=0\,. \tag{4.12}$$

If $K=0$ or if $L=0$, we see that (4.11) and (4.12) reduce to Case (b) and Case (c). If K is a non-zero constant, we find from (4.11) and (4.12) that

$$3\tilde{L}_v+\tilde{L}_u=0, \qquad 3\tilde{L}_u+\tilde{L}_v=0\,,$$

where $\tilde{L}$ denotes the terms of highest degree of L. From this it follows that $\tilde{L}$, and hence L, is also a constant. Similarly, we can prove that if L is a non-zero constant, then K is also a constant. Therefore, we may now assume that $\deg(K)>0$ and $\deg(L)>0$. Let $\tilde{K}$ (resp. $\tilde{L}$) denote the terms of highest degree of K (resp. L). Then it follows from (4.11) and (4.12) that $\tilde{K}$ and $\tilde{L}$ satisfy the following system of differential equations:

$$3(\tilde{K}\tilde{L}_v+\tilde{L}\tilde{K}_v)+(\tilde{K}\tilde{L}_u-\tilde{L}\tilde{K}_u)=0\,, \tag{4.13}$$

$$3(\tilde{K}\tilde{L}_u+\tilde{L}\tilde{K}_u)+(\tilde{K}\tilde{L}_v-\tilde{L}\tilde{K}_v)=0\,. \tag{4.14}$$

But now, just as in the proof of Theorem 4.1, we can deduce from this that $\tilde{K}$ and $\tilde{L}$ must have the same irreducible factors over $\boldsymbol{R}$. So we can write

$$\tilde{K}=c_1\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_n^{k_n}\,, \qquad \tilde{L}=c_2\alpha_1^{l_1}\alpha_2^{l_2}\cdots\alpha_n^{l_n}\,,$$

where $\deg(\alpha_i)\in\{1,2\}$, $k_i, l_i\geq 1$, $c_1\neq 0\neq c_2$, $n\geq 1$ and for distinct indices i and j, α_i and α_j are mutually irreducible. Substituting these expression in (4.13) and (4.14) and computing the result modulo $\alpha_1^{k_1+l_1}$, we get

$$3c_1c_2(k_1+l_1)(\alpha_1)_v-c_1c_2(k_1-l_1)(\alpha_1)_u\equiv 0 \pmod{\alpha_1^{k_1+l_1}}\,,$$
$$3c_1c_2(k_1+l_1)(\alpha_1)_u-c_1c_2(k_1-l_1)(\alpha_1)_v\equiv 0 \pmod{\alpha_1^{k_1+l_1}}\,.$$

Since $(\alpha_1)_u$ and $(\alpha_1)_v$ are not both zero modulo $\alpha_1^{k_1+l_1}$, we deduce that

$$c_1^2c_2^2(9(k_1+l_1)^2+(k_1-l_1)^2)=0\,.$$

This is a contradiction. ∎

PROOF OF THEOREM 4.2. Let us assume that M is not a part of a nondegenerate hyperboloid. Then, we know by Lemma 4.1 that M is flat with respect to h. So, we can apply Lemmas 4.2, 4.3, 4.4 and 4.5. Let $p \in M$ and let U be a neighbourhood of p. Since M is flat with respect to h, by taking U sufficiently small, we can identify U with an open part of $\boldsymbol{R}_1^2$. By Lemma 4.5, we have to consider three different cases on U.

Case 1. *P and Q are constant on U and $P^2 \neq Q^2$.* In this case, it follows from Lemma 4.3 and the Gauss equation that $x(U)$ is a proper affine sphere. So U is flat with respect to h and U is a proper affine sphere. Therefore, by the theorem of M. Magid and P. Ryan, we obtain that $x(U)$ is affine congruent to an open part of the surface $x_3(x_1^2+x_2^2)=1$.

Case 2. *$P=Q$ on U.* In this case, we know that P also satisfies the differential equation $P_{uu}+P_{vv}=2P_{uv}$. Then, we make the following change of coordinates on U:

$$y=(1/\sqrt{2})(u+v), \qquad z=(1/\sqrt{2})(-u+v)\,.$$

Then, a straightforward computation shows that $P_{zz}=0$. Hence there exist polynomials of one variable K and L on U such that

$$P(y,z)=K(y)+zL(y)\,.$$

Furthermore, we find that

$$\nabla_{x_z}x_z=\nabla_{x_y}x_z=\nabla_{x_z}x_y=0\,, \qquad \nabla_{x_y}x_y=2\sqrt{2}(K(y)+zL(y))x_z\,,$$

$$h(x_y,x_y)=h(x_z,x_z)=0\,, \qquad h(x_y,x_z)=1\,.$$

Hence, we have two Blaschke immersions from (U,h) into $\boldsymbol{R}^3$ with the same induced connection, namely x and the immersion x_{KL} defined in Section 3. By Radon's theorem there exist an affine transformation A of $\boldsymbol{R}^3$ such that $A(x(U))=x_{KL}(U)$. This completes the proof of the theorem in this case.

Case 3. *$P=-Q$ on U.* The proof of this case is completely similar to the proof of the previous case. ∎

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