Tôhoku Math. J.
43 (1991), 123–126

BRILL-NOETHER THEORY FOR VECTOR BUNDLES OF RANK 2

MONTSERRAT TEIXIDOR I BIGAS

(Received January 30, 1990, revised March 23, 1990)

Denote by $U(n, d)$ the moduli space of stable vector bundles of rank n and degree d on a fixed algebraic non-singular curve of genus g. Let $W^r_{n,d}$ be the subscheme of those vector bundles which have at least $r+1$ independent sections. By the theory of determinantal varieties, if $W^r_{n,d}$ is not empty, then every component of it has dimension at least $\rho(n, r, d, g)=n^2(g-1)+1-(r+1)(r+1-d+n(g-1))$ and one expects equality to hold for a generic curve when the right hand side is greater than zero.

Sundaram [S] proved, using a degeneration argument, that $W^1_{2,d}$ is non-empty for any curve for odd d, $g\leq d\leq 2g-2$. In fact, his proof shows that it has the right dimension $\rho(2, 1, d, g)=2d-3$ for a generic curve.

Here we see that the dimension is exactly ρ for a generic curve in the full range of d where it makes sense. We also characterize (in terms of special divisors) the curves for which the dimension is bigger. The proof does not need degeneration arguments but is based instead on the geometric theory of extensions used by Atiyah, Newstead, Lange-Narashimhan et al. (see for instance [L, N]).

For more general results about $W^r_{n,d}$ for generic C see [T].

I would like to thank P. Newstead for bringing this problem to my attention and for useful comments. This work was carried out while the author was working at Liverpool University on a project financed by S.E.R.C.

THEOREM. *Given any non-singular curve C and a d, $3\leq d\leq 2g-1$, $W^1_{2,d}$ has a component of dimension $\rho(2, 1, d, g)=2d-3$, and a generic point on it corresponds to a vector bundle whose space of sections has dimension 2 and the generic section has no zeroes. If C is generic, this is the only component of $W^1_{2,d}$. Moreover, $W^1_{2,d}$ has extra components if and only if $W^1_{1,k}$ is not empty and* $\dim W^1_{1,k}\geq d+2k-2g-1$ *for some k with $2k<d$.*

NOTE. For $d\geq 2g$, any vector bundle of rank two has at least two sections (by Riemann-Roch's Theorem) and therefore $W^1_{2,d}=U(2, d)$.

For $d\leq 2$, $W^1_{2,d}$ is empty. Indeed, if a non-trivial vector bundle E has at least two sections, then a linear combination of them must have a zero (otherwise the map from $\mathcal{O}^2\to E$ given by the sections is an isomorphism because it is bijective at every fiber). Hence, E has a line subbundle of degree at least one and this would contradict stability if $\deg E\leq 2$.

PROOF. Consider a vector bundle E in $W^1_{2,d}$. A section s gives rise to a non-zero map $s: \mathcal{O}\to E$. Let D be the divisor of zeroes of s. By the stability of E, if k is the degree of D, then $2k\leq d$. Write the exact sequence

$$0 \longrightarrow \mathcal{O}(D) \xrightarrow{s} E \xrightarrow{\pi} L \longrightarrow 0 \tag{1}$$

where L is a line bundle. Consider a section t of E independent of s. Then, either $\mathcal{O}(D)$ has at least two sections or the map $t: \mathcal{O}\to E$ does not factor through s and hence gives rise to a section of L.

Assume first $h^0(\mathcal{O}(D))=1$. Then, there is a pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(D) & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \pi t} & & \\ 0 & \longrightarrow & \mathcal{O}(D) & \longrightarrow & E\times_L\mathcal{O} & \longrightarrow & \mathcal{O} & \longrightarrow & 0\,. \end{array} \tag{2}$$

Consider the points $e\in \mathrm{Ext}^1(L, \mathcal{O}(D))=H^1(L^{-1}\otimes\mathcal{O}(D))$ and $e'\in\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(D))= H^1(\mathcal{O}(D))$ respectively corresponding to the two rows of (2). Clearly, the surjection in the second row has a section and therefore $e'=0$.

Denote by D' the (effective) divisor of zeroes of πt. Then $L=\mathcal{O}(D')$ and the map $H^1(\mathcal{O}(D-D'))\to H^1(\mathcal{O}(D))$ corresponding to pull-back of extensions is the dual of $(H^1(\mathcal{O}(D)))^*\cong H^0(\mathcal{O}(K-D))\to H^0(\mathcal{O}(K+D'-D))=(H^1(\mathcal{O}(D-D')))^*$ given by multiplication by t. Clearly, the image of this map is the set of divisors in $H^0(\mathcal{O}(K+D'-D))$ containing the divisor D'. So, the extension e which corresponds to a point in the orthogonal to this image may be identified with a point in the span $\langle D'\rangle$ of D' when thinking of the curve as its image by the linear system $|K+D'-D|$. Using Riemann-Roch's theorem and the fact that $\deg D'=d-k>\deg D=k$, we get $\dim\langle D'\rangle= h^0(K+D'-D)-h^0(K-D)-1=d-k-2$.

Let us now count the dimension of the set of vector bundles E obtained in this way. Every such vector bundle, if stable, admits a different representation of this type, up to scalar multiple, for each one of its sections. Hence the dimension of the set of vector bundles so obtained is

$$\begin{aligned} &\dim\{D\}+\dim\{D'\}+\dim\langle D'\rangle-(h^0(E)-1)\\ &\qquad=\dim C^{(k)}+\dim C^{(d-k)}+d-k-2-(h^0(E)-1)\leq 2d-k-3\leq 2d-3 \end{aligned}$$

with equality only if $k=0$ and a generic E has no more than two independent sections.

Claim 1. For $k=\deg D=0$, the generic E as in (1) satisfies $h^0(E)=2$.

Assume this were not the case. Then, $h^0(E)>2$ implies $h^0(L)>1$ and a second section of L can be lifted to a section of E. Reasoning as above, one gets that the extension e corresponding to a point in $\langle D'\rangle$ is also in the span of some other divisor D'' with D'' in $|L|$. Hence

$$\langle D'\rangle \subset \bigcup_{D''}(\langle D'\rangle \cap \langle D''\rangle)\,. \tag{3}$$

Now, $\dim(\langle D'\rangle \cap \langle D''\rangle) = \dim\langle D'\rangle + \dim\langle D''\rangle - \dim\langle D'+D''\rangle = \dim\langle D'\rangle + \dim\langle D''\rangle - [h^0(K\otimes L) - h^0(K\otimes L - D' - D'') - 1] = h^0(L) - 3$, where the last equality follows by Riemann-Roch theorem using the fact that D' and D'' are divisors in the linear series $|L|$.

From (3), one obtains

$$d-2=\dim\langle D'\rangle \le h^0(L)-1+\dim(\langle D'\rangle \cap \langle D''\rangle)=2h^0(L)-4\,. \tag{4}$$

Therefore $h^0(L)\ge (d/2)+1$. By Clifford's theorem this implies either that C is hyperelliptic and L is of the form ag^1_2 or that L is non-special. In the former case, L and therefore E is not generic. In the latter, by Riemann-Roch's theorem, (4) reads as $d-2\le 2(d+1-g)-4$. Hence $d\ge 2g$ against the hypothesis. This proves Claim 1.

Claim 2. The generic extension (1) with $D=0$, $h^0(E)=2$ gives rise to a stable E.

Assume this were not the case and let F be a line subbundle of degree at least $d/2$.

As $\mathcal{O}$ does not have subsheaves of positive degree, the map from F to E gives rise to a non-trivial map from F to L. Hence $F^{-1}\otimes L$ has a section, i.e., $F^{-1}\otimes L=\mathcal{O}(D'')$ for an effective divisor D'' with $\deg D''=\deg L-\deg F\le d/2$. Then, reasoning as above, we would get that $\langle D'\rangle$ is contained in the union of the spans of D'', where D'' varies on the set of divisors of degree at most $d/2$. Let us prove that this is not possible.

The span of D' has dimension $d-2>d/2-1\ge \dim\langle D''\rangle$. So $\langle D'\rangle$ cannot be contained in the spans of the finite number of subsets of $d/2$ points that it contains. Therefore we can assume that $\dim\langle D'\rangle \cap \langle D''\rangle \le d/2-2$.

As the set of divisors D'' has at most dimension $d/2$

$$\dim\bigcup_{D''}(\langle D'\rangle \cap \langle D''\rangle)\le d/2+d/2-2=d-2=\dim\langle D'\rangle\,.$$

So, if a generic point, say e in $\langle D'\rangle$, is in the span of a divisor $\langle D''\rangle$, then D'' must be generic too. Then projecting from e the points of D'' one obtains linearly dependent points. This happens a fortiori when projecting from $\langle D'\rangle$. But projection from $\langle D'\rangle$ is the canonical map and this is impossible as $\deg D''=d/2\le g$. This proves the claim.

We have now shown that the closure in $W^1_{2,d}$ of the subset given by the extensions (1) with $h^0(\mathcal{O}(D))=1$ has precisely one component W_0 of dimension $2d-3$, all others being of smaller dimension. The generic point of W_0 corresponds in (1) to $D=0$. From this and Claim 1, it is as described in the statement of the theorem. Also from Claim 1 and the assumption $h^0(\mathcal{O}(D))=1$, the bundle corresponding to this generic point has no line subbundle with two independent sections, so W_0 is a component of $W^1_{2,d}$. Since every component of $W^1_{2,d}$ has dimension at least $2d-3$, it follows that all other components must have generic points corresponding to extensions (1) with $h^0(\mathcal{O}(D))\ge 2$ and that there exist such components if and only if the set of bundles corresponding to such extensions has dimension at least $2d-3$.

In order to conclude the proof of the theorem, it remains only to see what happens with the extensions (1) obtained with an $\mathcal{O}(D)$ which has two independent sections.

If we want E to be stable, then $k=\deg D<(\deg E)/2=d/2$.

If $W^1_{1,k}$ is empty, then so is the set of such extensions. Otherwise its dimension is at most (with equality if different pairs $(\mathcal{O}(D), L)$ give rise to different extensions) $\dim W^1_{1,k}+\dim \mathrm{Pic}^{d-k}+h^1(L^{-1}(D))-1=\dim W^1_{1,k}+2g-2+d-2k$ and this number is at least $2d-3$ if and only if $\dim W^1_{1,k}\geq d+2k-2g-1$. Moreover, if $2k<d$, the generic extension obtained in this way is stable. This is proved as before by using the fact that the dimension of the $(d/2-k)$-chordal variety is at most (in fact equal to, cf. [L]) $d-2k-1<d-2k+g-2=\dim|K\otimes L(-D)|$.

Now, for a generic curve, $W^1_{1,k}$ is empty for $2k<g+2$. Moreover, $\dim W^1_{1,k}=\rho(1,1,k,g)=2k-g-2$ for $2k\geq g+2$ and this number is smaller than $d+2k-2g-1$ because, by stability, $2k<d$. This proves the assertion for generic C.

Assume that every extension could be obtained by means of another pair $(\bar{D}, \bar{L})\in W^1_{1,k}\times \mathrm{Pic}^{d-k}$. Then reasoning as before, e belongs to the span of an effective divisor $\tilde{D}$ where $L=\mathcal{O}(\bar{D}+\tilde{D})$. Now, letting $\tilde{D}$ vary in the set of effective divisors of degree $d-2k$ such that $L=\mathcal{O}(\bar{D}+\tilde{D})$ for a $\bar{D}$ in $W^1_{1,k}$, we obtain (for generic L)

$$\dim \bigcup\langle\tilde{D}\rangle=\dim\langle\tilde{D}\rangle+d-2k+\dim W^1_{1,k}-g=2d-4k-1-g+\dim W^1_{1,k}\,.$$

We are assuming this number to be at least $h^1(L^{-1}(D))-1$. Hence, $\dim W^1_{1,k}\geq 2g-d+2k-1$. On the other hand, by Martens Theorem (cf. [A, C, G, H]) $\dim W^1_{1,k}\leq k-2$. Hence, $d-2\geq d-k\geq 2g-1$, contradicting the hypothesis.

References

[A, C, G, H] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, Geometry of algebraic curves I, Springer Verlag, Berlin Heidelberg, New York, 1984.

[L] H. Lange, Higher secant varieties and the theorem of Nagata on ruled surfaces, Manuscripta Math. 47 (1984), 263–269.

[L, N] H. Lange and M. S. Narasimhan, Maximal subbundles of rank two vector bundles on curves, Math. Ann. 266 (1983), 55–72.

[S] N. Sundaram, Special divisors and vector bundles, Tôhoku Math. J. 39 (1987), 175–213.

[T] M. Teixidor, Brill-Noether Theory for stable vector bundles, Duke Math. J. (1991), to appear.

Mathematics Department
Tufts University
Medford, M.A. 02155
U.S.A.