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THE CLOSURES OF NILPOTENT ORBITS IN THE CLASSICAL SYMMETRIC PAIRS AND THEIR SINGULARITIES

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0. Introduction. Let G be a complex reductive algebraic group with Lie algebra $\mathfrak{g}$ and θ an involution of G as an algebraic group. We also denote by θ the induced involution of $\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to θ, K_θ the subgroup of G consisting of elements $g\in G$ such that $\theta(g)=g$ and $N(\mathfrak{p})$ the nilpotent subvariety of $\mathfrak{p}$. We call the pair $(\mathfrak{g}, \mathfrak{k})$ the symmetric pair defined by (G, θ).

For the symmetric pairs, Sekiguchi [Se1] tried to construct an analogue of the Brieskorn-Slodowy theory ([B], [Sl]) which gives a correspondence between the simple Lie algebras and the rational double points. In [Se1], he introduced the problem to determine the generic singularities in $N(\mathfrak{p})$. To determine the generic singularities, we need the classification of K_θ-orbits in $N(\mathfrak{p})$ and their closure relation. In the classical cases, the classification of nilpotent orbits is given by means of *ab*-diagrams in [O2]. The first purpose of this paper is to determine the closure relation of K_θ-orbits in $N(\mathfrak{p})$ for the classical symmetric pairs. This is completed in §2 by means of a certain ordering of *ab*-diagrams.

For the classical Lie algebras, the nilpotent orbits are classified by Young diagrams, and their closure relation is described by a certain ordering of Young diagrams. Then Kraft and Procesi ([KP2], [KP3]) showed that the smooth equivalence class (cf. §3) $\mathrm{Sing}(\overline{\mathcal{O}}_\eta, \mathcal{O}_\sigma)$ of the closure $\overline{\mathcal{O}}_\eta$ in $\mathcal{O}_\sigma$, which corresponds to a degeneration $\sigma\leq\eta$ of Young diagrams, is determined only by reduced degeneration $\bar{\sigma}\leq\bar{\eta}$, i.e., the degeneration which we obtain from $\sigma\leq\eta$ by erasing the common columns and rows from the left and the upside.

The second purpose of this paper is to give an analogue of the result of Kraft and Procesi for the classical symmetric pairs. The construction $\overline{C_\eta^{(\varepsilon,\omega)}} \xleftarrow{\rho} N_\eta \xrightarrow{\pi} \overline{C_{\eta'}^{(-\varepsilon,-\omega)}}$ (cf. §3), which we need to prove the "cancelling columns", is also used to give a reduction to determine the closure relation.

On the other hand, there exists a natural correspondence between symmetric pairs and real Lie groups. Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair defined by (G, θ) and let $G_{\boldsymbol{R}}$ be the corresponding real group with Lie algebra $\mathfrak{g}_{\boldsymbol{R}}$. Then it is known by Sekiguchi [Se2] that there is a natural correspondence between the set of nilpotent K_θ-orbits in $\mathfrak{p}$ and that of nilpotent $G_{\boldsymbol{R}}$-orbits in $\mathfrak{g}_{\boldsymbol{R}}$. We call this correspondence Sekiguchi's bijection. Then we are naturally led to the problem whether Sekiguchi's bijection preserves the closure relation.

The third purpose of this paper is to answer this problem affirmatively in the classical cases.

What we call the classical symmetric pairs are the following:

(AI) $(\mathfrak{gl}(n, C), \mathfrak{o}(n, C))$, (AII) $(\mathfrak{gl}(n, C), \mathfrak{sp}(n, C))$,

(AIII) $(\mathfrak{gl}(m+n, C), \mathfrak{gl}(m, C)+\mathfrak{gl}(n, C))$, (BDI) $(\mathfrak{o}(m+n, C), \mathfrak{o}(m, C)+\mathfrak{o}(n, C))$,

(DIII) $(\mathfrak{o}(2n, C), \mathfrak{gl}(n, C))$, (CI) $(\mathfrak{sp}(2n, C), \mathfrak{gl}(n, C))$,

(CII) $(\mathfrak{sp}(m+n, C), \mathfrak{sp}(m, C)+\mathfrak{sp}(n, C))$.

For the symmetric pairs of types (AI) and (AII), the closure relation is determined and the analogue of the result of Kraft and Procesi is given in [O1]. Moreover it is easily verified that Sekiguchi's bijection preserves the closure relation. Therefore we treat the symmetric pairs of types (AIII), (BDI), (DIII), (CI) and (CII) in this paper.

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1. Description of Sekiguchi's bijection. In this section, we give the description of Sekiguchi's bijection in the classical cases.

(1.1) Sekiguchi's bijection. Let G be a complex reductive algebraic group with Lie algebra $\mathfrak{g}$ and θ an involution of the algebraic group G. We also denote by θ the involution of $\mathfrak{g}$ induced by $\theta: G \to G$. Put $K_\theta := \{g \in G; \theta(g)=g\}$, $\mathfrak{k} := \{X \in \mathfrak{g}; \theta(X)=X\}$ and $\mathfrak{p} := \{X \in \mathfrak{g}; \theta(X)=-X\}$. Then we call the pair $(\mathfrak{g}, \mathfrak{k})$ the symmetric pair defined by (G, θ), K_θ the isotropy subgroup, and $\mathfrak{p}$ the associated vector space.

Suppose that there exists a real form $G_{\boldsymbol{R}}$ of G which we obtain by a complex conjugation $\tau: G \to G$ (i.e., $G_{\boldsymbol{R}} = \{g \in G; \tau(g)=g\}$) such that $\theta \circ \tau = \tau \circ \theta$ and that the restriction $\theta|_{G_{\boldsymbol{R}}}$ is a Cartan involution of $G_{\boldsymbol{R}}$. We call the real form $G_{\boldsymbol{R}}$ the real group corresponding to the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. As before we denote by τ the complex conjugation of $\mathfrak{g}$ induced by $\tau: G \to G$ and put $\mathfrak{g}_{\boldsymbol{R}} := \mathrm{Lie}\, G_{\boldsymbol{R}} = \{X \in \mathfrak{g}; \tau(X)=X\}$. Then K_θ (resp. $G_{\boldsymbol{R}}$) acts on $\mathfrak{p}$ (resp. $\mathfrak{g}_{\boldsymbol{R}}$) by the adjoint action. We denote by $N(\mathfrak{p})$ (resp. $N(\mathfrak{g}_{\boldsymbol{R}})$) the set of all nilpotent elements in $\mathfrak{p}$ (resp. $\mathfrak{g}_{\boldsymbol{R}}$) and by $[N(\mathfrak{p})]_{K_\theta}$ (resp. $[N(\mathfrak{g}_{\boldsymbol{R}})]_{G_{\boldsymbol{R}}}$) the set of K_θ-orbits (resp. $G_{\boldsymbol{R}}$-orbits) in $N(\mathfrak{p})$ (resp. $N(\mathfrak{g}_{\boldsymbol{R}})$). Put $\mathfrak{k}_{\boldsymbol{R}} := \mathfrak{k} \cap \mathfrak{g}_{\boldsymbol{R}}$ and $\mathfrak{p}_{\boldsymbol{R}} := \mathfrak{p} \cap \mathfrak{g}_{\boldsymbol{R}}$. Then $\mathfrak{g}_{\boldsymbol{R}} = \mathfrak{k}_{\boldsymbol{R}} + \mathfrak{p}_{\boldsymbol{R}}$ is the Cartan decomposition of $\mathfrak{g}_{\boldsymbol{R}}$ and $\mathfrak{g} = (\mathfrak{k}_{\boldsymbol{R}} + \sqrt{-1}\, \mathfrak{p}_{\boldsymbol{R}}) + (\sqrt{-1}\, \mathfrak{k}_{\boldsymbol{R}} + \mathfrak{p}_{\boldsymbol{R}})$ is that of $\mathfrak{g}$. Let φ be the Cartan involution of $\mathfrak{g}$ corresponding to the above decomposition. Then $\varphi = \tau \circ \theta$; in particular, φ commutes with θ.

A triple (h, x, y) consisting of linearly independent elements of a Lie algebra satisfying the relations $[h, x]=2x$, $[h, y]=-2y$, $[x, y]=h$ is called an S-triple. For a symmetric pair $(\mathfrak{g}, \mathfrak{k})$, an S-triple (h, x, y) in $\mathfrak{g}$ is called a normal S-triple if $h \in \mathfrak{k}$ and $x, y \in \mathfrak{p}$. Sekiguchi introduced the following notion.

DEFINITION (Sekiguchi [Se2]). A normal S-triple (h, x, y) of a symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is called a strictly normal S-triple (with respect to φ) if $\varphi(h)=-h$ and $\varphi(x)=-y$.

REMARK 1. In the above setting, a normal S-triple (h, x, y) is a strictly normal S-triple if and only if $\tau(h) = -h$ (i.e., $h \in \sqrt{-1}\,\mathfrak{k}_{\boldsymbol{R}}$) and $\tau(x) = y$.

THEOREM 1 (Sekiguchi [Se2]). *For any non-zero nilpotent K_θ-orbit $\mathcal{O}_\theta \in [N(\mathfrak{p})]_{K_\theta}$, there exists a strictly normal S-triple (h, x, y) such that $x \in \mathcal{O}_\theta$. Such an (h, x, y) is unique up to conjugation by $K_\theta \cap G_{\boldsymbol{R}}$. If we put*

$$h_{\boldsymbol{R}} := \sqrt{-1}(x - y)\,, \quad x_{\boldsymbol{R}} := (x + y + \sqrt{-1}\,h)/2\,, \quad y_{\boldsymbol{R}} := (x + y - \sqrt{-1}\,h)/2\,,$$

then $(h_{\boldsymbol{R}}, x_{\boldsymbol{R}}, y_{\boldsymbol{R}})$ is an S-triple in $\mathfrak{g}_{\boldsymbol{R}}$. Let $\mathcal{O}_{\boldsymbol{R}}$ be the $G_{\boldsymbol{R}}$-orbit generated by $x_{\boldsymbol{R}}$. Then the map $[N(\mathfrak{p})]_{K_\theta} \to [N(\mathfrak{g}_{\boldsymbol{R}})]_{G_{\boldsymbol{R}}}$, $\mathcal{O}_\theta \mapsto \mathcal{O}_{\boldsymbol{R}}$ is a bijection.

We call the above bijection Sekiguchi's bijection.

(1.2) Classical symmetric pairs. In this paper, we treat the classical symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ and the corresponding real group $G_{\boldsymbol{R}}$ in Table I.

TABLE I

Type	(ε, ω)	G	$(\mathfrak{g}, \mathfrak{k})$	$G_{\boldsymbol{R}}$
(AIII)	$\varnothing$	$GL(m+n, \boldsymbol{C})$	$(\mathfrak{gl}(m+n, \boldsymbol{C}), \mathfrak{gl}(m, \boldsymbol{C}) + \mathfrak{gl}(n, \boldsymbol{C}))$	$U(m, n)$
(BDI)	$(1, 1)$	$O(m+n, \boldsymbol{C})$	$(\mathfrak{o}(m+n, \boldsymbol{C}), \mathfrak{o}(m, \boldsymbol{C}) + \mathfrak{o}(n, \boldsymbol{C}))$	$O(m, n)$
(DIII)	$(1, -1)$	$O(2n, \boldsymbol{C})$	$(\mathfrak{o}(2n, \boldsymbol{C}), \mathfrak{gl}(n, \boldsymbol{C}))$	$O^*(2n)$
(CII)	$(-1, 1)$	$Sp(m+n, \boldsymbol{C})$	$(\mathfrak{sp}(m+n, \boldsymbol{C}), \mathfrak{sp}(m, \boldsymbol{C}) + \mathfrak{sp}(n, \boldsymbol{C}))$	$Sp(m, n)$
(CI)	$(-1, -1)$	$Sp(2n, \boldsymbol{C})$	$(\mathfrak{sp}(2n, \boldsymbol{C}), \mathfrak{gl}(n, \boldsymbol{C}))$	$Sp(2n, \boldsymbol{R})$

We first give the description of these symmetric pairs. Let V be a finite dimensional vector space over $\boldsymbol{C}$ and $s: V \to V$ a linear involution. We call such a vector space V a vector space with an involution s. Moreover, if V is endowed with a non-degenerate bilinear form $(\ ,\ )$ such that $(u, v) = \varepsilon(v, u)$ and $(su, v) = \omega(u, sv)$ for all $u, v \in V$, we call V an (ε, ω)-space, where $\varepsilon = \pm 1$ and $\omega = \pm 1$.

Let V be a vector space with an involution s and define an involution θ of $GL(V)$ by $\theta(g) = sgs$ ($g \in GL(V)$). Put

$$V_a := \{v \in V;\, sv = v\}\,, \quad V_b := \{v \in V;\, sv = -v\}\,, \quad m := \dim V_a\,, \quad n := \dim V_b\,,$$

$$\tilde{K}(V) := \{g \in GL(V);\, \theta(g) = g\} \simeq GL(V_a) \times GL(V_b)$$

$$\tilde{\mathfrak{k}}(V) := \{X \in \mathfrak{gl}(V);\, \theta(X) = X\}\,, \quad \tilde{\mathfrak{p}}(V) := \{X \in \mathfrak{gl}(V);\, \theta(X) = -X\}\,.$$

Then $(\mathfrak{gl}(V), \tilde{\mathfrak{k}}(V))$ is a symmetric pair isomorphic to $(\mathfrak{gl}(m+n, \boldsymbol{C}), \mathfrak{gl}(m, \boldsymbol{C}) + \mathfrak{gl}(n, \boldsymbol{C}))$ defined by $(GL(V), \theta)$, $\tilde{K}(V)$ the isotropy subgroup, and $\tilde{\mathfrak{p}}(V)$ the associated vector space. We call $(\mathfrak{gl}(V), \tilde{\mathfrak{k}}(V))$ a symmetric pair of type (AIII). We also call it the symmetric pair defined by the vector space V with the involution s.

Next suppose that V is an (ε, ω)-space. For $X \in \mathfrak{gl}(V)$, we denote by $X^* \in \mathfrak{gl}(V)$ the adjoint of X with respect to $(\ ,\ )$. It is easy to see that $\theta(g^*) = (\theta(g))^*$ for $g \in GL(V)$. Then we put $G(V) := \{g \in GL(V);\, g^* = g^{-1}\}$, $\mathfrak{g}(V) := \operatorname{Lie} G(V) = \{X \in \mathfrak{gl}(V);\, X^* = -X\}$,

$K(V) := G(V) \cap \tilde{K}(V) = \{g \in G(V);\ \theta(g) = g\}$, $\mathfrak{k}(V) := \mathrm{Lie}\, K(V) = \{X \in \mathfrak{g}(V);\ \theta(X) = X\}$, $\mathfrak{p}(V) := \mathfrak{g}(V) \cap \tilde{\mathfrak{p}}(V) = \{X \in \mathfrak{g}(V);\ \theta(X) = -X\}$. Then $(\mathfrak{g}(V), \mathfrak{k}(V))$ is the symmetric pair defined by $(G(V), \theta)$, $K(V)$ the isotropy subgroup and $\mathfrak{p}(V)$ the associated vector space. Here we note that $m = n$ if $\omega = -1$ and that m, n are even if $(\varepsilon, \omega) = (-1, 1)$. The symmetric pair $(\mathfrak{g}(V), \mathfrak{k}(V))$ is isomorphic to the symmetric pair in Table I according as $(\varepsilon, \omega) = (1, 1), (1, -1), (-1, 1), (-1, -1)$. We define the type of the symmetric pairs $(\mathfrak{g}(V), \mathfrak{k}(V))$ to be the first column of Table I. We call $(\mathfrak{g}(V), \mathfrak{k}(V))$ the symmetric pair defined by the (ε, ω)-space V.

(1.3) Realization of classical symmetric pairs and the corresponding real groups. Here let us give the realization of the symmetric pairs and the real groups in Table I in terms of matrix algebra as follows.

(AIII) Put $V := \boldsymbol{C}^{m+n}$ and define a linear involution s by

$$s := \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix} \in GL(V),$$

where I_n is the identity matrix of size n. Define a hermitian form f on V by

$$f(u, v) := {}^t\bar{u}sv \qquad (u, v \in V),$$

where $\bar{u}$ is the ordinary complex conjugation of $u \in V$. Then f is positive definite on V_a and negative definite on V_b. Denote by $(X)_f^*$ the adjoint of $X \in \mathfrak{gl}(V)$ with respect to f and put $\tau(g) := \{(g)_f^*\}^{-1}$ $(g \in GL(V))$, $G_{\boldsymbol{R}} := \{g \in GL(V);\ \tau(g) = g\}$. Then $G_{\boldsymbol{R}} = U(m, n)$ is the real Lie group corresponding to the symmetric pair $(\mathfrak{gl}(V), \tilde{\mathfrak{k}}(V))$ defined by the vector space V with the involution s.

(BDI), (DIII), (CII), (CI) Put $(\varepsilon, \omega) = (\pm 1, \pm 1)$ and $V = \boldsymbol{C}^{m+n}$. We suppose that $m = n$ if $\omega = -1$ and m, n are even if $(\varepsilon, \omega) = (-1, 1)$. Put

$$s = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$$

and define a bilinear form $(\ ,\ )$ on V by $(u, v) = {}^t uJv$ $(u, v \in V)$, where, for each (ε, ω), we define the matrix J as follows:

$$(\varepsilon, \omega) = (1,\ 1) \qquad J = I_{m+n},$$

$$(\varepsilon, \omega) = (1, -1) \qquad J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

$$(\varepsilon, \omega) = (-1,\ 1) \qquad J = \left(\begin{array}{cc|cc} 0 & I_{m/2} & & \\ -I_{m/2} & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & I_{n/2} \\ \multicolumn{2}{c|}{0} & -I_{n/2} & 0 \end{array}\right)$$

$$(\varepsilon, \omega)=(-1, -1) \qquad J=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then V is an (ε, ω)-space with respect to s and $(\ ,\ )$. We define an anti-linear map $\tau: V \to V$ by $\tau(v)=\sqrt{\omega}\, sJ\bar{v}$ $(v \in V)$ (i.e., $\tau(\alpha u+\beta v)=\bar{\alpha}\tau(u)+\bar{\beta}\tau(v)$ $(\alpha, \beta \in \boldsymbol{C}, u, v \in V)$). Then we have the following (cf. [BC]):

$$\tau^2(v)=\varepsilon\omega v\,, \quad (\tau(u), \tau(v))=\overline{(u, v)} \qquad (u, v \in V)\,.$$

Define a complex conjugation τ of the group $G(V)$ by $\tau(g)=\tau \circ g \circ \tau^{-1}$ $(g \in G(V))$ and put $G_{\boldsymbol{R}}=\{g \in G(V);\ \tau(g)=g\}$. Then $G_{\boldsymbol{R}}$ is the real Lie group corresponding to the symmetric pair $(\mathfrak{g}(V), \mathfrak{k}(V))$ defined by the (ε, ω)-space V. Moreover $G_{\boldsymbol{R}}$ is isomorphic to the real group in Table I corresponding to each (ε, ω) (cf. [BC]). For the simplicity of expression, we attach $(\varepsilon, \omega)=\varnothing$ to the symmetric pair of type (AIII) and the corresponding real group.

REMARK 2 (cf. [D]). (1) Suppose that $\tau^2=-\mathrm{id}_V$ (i.é., $\varepsilon\omega=-1$). Let $\boldsymbol{H}=\{\alpha+j\beta;\ \alpha, \beta \in \boldsymbol{C}\}$ $(\alpha j=j\bar{\alpha})$ be the quaternion algebra with the conjugation $\overline{(\alpha+j\beta)}=\bar{\alpha}-j\beta$ $(\alpha, \beta \in \boldsymbol{C})$. Define the right action of $\boldsymbol{H}$ on V by $v(\alpha+j\beta)=v\alpha+\tau(v)\beta$ $(v \in V, \alpha, \beta \in \boldsymbol{C})$. Then V is a right $\boldsymbol{H}$-vector space such that $\dim_{\boldsymbol{H}} V=(1/2)\dim_{\boldsymbol{C}} V$. Define $f_-: V \times V \to \boldsymbol{H}$ by

$$f_-(u, v):=-\overline{(u, \tau(v))}-\overline{(u, v)}j \qquad (u, v \in V)\,.$$

Then we have the following:

$$f_-(v, u)=-\varepsilon\overline{f_-(u, v)}\,, \quad f_-(up, vq)=\bar{p}f_-(u, v)q \qquad (u, v \in V, p, q \in \boldsymbol{H})\,.$$

By using f_-, we can write $G_{\boldsymbol{R}}$ as $G_{\boldsymbol{R}}=\{g \in GL(V);\ f_-(gu, gv)=f_-(u, v) \text{ for all } u, v \in V\}$.

(2) Suppose that $\tau^2=\mathrm{id}_V$ (i.e., $\varepsilon\omega=1$). If we write $V_{\boldsymbol{R}}:=\{v \in V;\ \tau(v)=v\}$, $V_{\boldsymbol{R}}$ is a real vector space of dimension $\dim_{\boldsymbol{R}} V_{\boldsymbol{R}}=\dim_{\boldsymbol{C}} V$ and $G_{\boldsymbol{R}}$ is naturally identified as

$$G_{\boldsymbol{R}} \simeq \{g \in GL(V_{\boldsymbol{R}});\ (gu, gv)=(u, v) \text{ for all } u, v \in V_{\boldsymbol{R}}\}\,.$$

RERMARK 3. In the cases $(\varepsilon, \omega)=(\pm 1, \pm 1)$, we have

$$(v, \tau(v))={}^t vJ\sqrt{\omega}\, sJ\bar{v}=\omega\sqrt{\omega}\,{}^t vJJs\bar{v}=\varepsilon\omega\sqrt{\omega}\,{}^t vs\bar{v} \ (v \in V)\,.$$

In particular, if $v \in V_a \cup V_b \setminus \{0\}$, we have $(v, \tau(v)) \neq 0$.

(1.4) Classification of nilpotent orbits of the symmetric pairs. Here we give the classification of nilpotent K_θ-orbits in $\mathfrak{p}$.

Let $(\mathfrak{gl}(V), \tilde{\mathfrak{k}}(V))$ be the symmetric pair defined by a vector space V with an involution s. For any nilpotent element $X \in \tilde{\mathfrak{p}}(V)=\{X \in \mathfrak{gl}(V);\ XV_a \subset V_b,\ XV_b \subset V_a\}$, we can take a Jordan basis

$$\{X^p a_i;\ 1 \le i \le r_a,\ 0 \le p < \lambda_i\} \cup \{X^q b_j;\ 1 \le j \le r_b,\ 0 \le q < \mu_j\}$$

of V such that $a_i \in V_a$, $b_j \in V_b$ and $X^{\lambda_i}a_i=0$, $X^{\mu_j}b_j=0$. By letting a string

$$\overbrace{abab\cdots\cdots}^{\lambda_i} \quad \text{(resp. } \overbrace{baba\cdots\cdots}^{\mu_j} \text{)}$$

correspond to $\{X^p a_i;\ 0 \le p < \lambda_i\}$ (resp. $\{X^q b_j;\ 0 \le q < \mu_j\}$), we get a diagram η_X which is the sum of such strings. Here we always put the longer string above the shorter one. Such a diagram is called an *ab*-diagram. It is easy to see that the *ab*-diagram η_X is independent of the choice of a Jordan basis. Therefore we call η_X the *ab*-diagram of X. For two nilpotent elements X and Y of $\tilde{\mathfrak{p}}(V)$, we see that $\eta_X=\eta_Y$ if and only if X and Y are conjugate under $\tilde{K}(V)$. Thus we have a one-to-one correspondence between the set of nilpotent $\tilde{K}(V)$-orbits in $\tilde{\mathfrak{p}}(V)$ and the set $D(m, n)$ of *ab*-diagrams η such that $n_a(\eta)=\dim V_a=m$ and $n_b(\eta)=\dim V_b=n$, where $n_a(\eta)$ (resp. $n_b(\eta)$) is the number of the a's (resp. the b's) in η:

$$[N(\tilde{\mathfrak{p}}(V))]_{\tilde{K}(V)} \simeq D(m, n)\,.$$

Next let us give the classification of nilpotent orbits of the symmetric pair $(\mathfrak{g}(V), \mathfrak{k}(V))$. For a fixed $(\varepsilon, \omega)=(\pm 1, \pm 1)$, let us call the *ab*-diagrams in Table II primitive (ε, ω)-diagrams. We call an *ab*-diagram, which is a sum of primitive (ε, ω)-diagrams, an (ε, ω)-diagram.

TABLE II

Type	(ε, ω)	*ab*-diagrams		
(BDI)	(1, 1)	$ab\cdots ba$,	$ba\cdots ab$,	$ba\cdots ba$ $ab\cdots ab$,
(DIII)	(1, −1)	$ba\cdots ba$ $ba\cdots ba$,	$ab\cdots ab$ $ab\cdots ab$,	$ab\cdots ba$ $ba\cdots ab$,
(CII)	(−1, 1)	$ab\cdots ba$ $ab\cdots ba$,	$ba\cdots ab$ $ba\cdots ab$,	$ba\cdots ba$ $ab\cdots ab$,
(CI)	(−1, −1)	$ba\cdots ba$,	$ab\cdots ab$,	$ab\cdots ba$ $ba\cdots ab$,

We denote by $D^{(\varepsilon,\omega)}(m, n)$ the set of (ε, ω)-diagrams η such that $n_a(\eta)=m$ and $n_b(\eta)=n$.

PROPOSITION 1 ([O1, Proposition 4], [O2, Proposition 2]). *Let V be an (ε, ω)-space such that* $\dim V_a=m$ *and* $\dim V_b=n$. *We consider the symmetric pair* $(\mathfrak{gl}(V), \tilde{\mathfrak{k}}(V))$ *of type* (AIII) *and the ones* $(\mathfrak{g}(V), \mathfrak{k}(V))$ *of types* (BDI), (DIII), (CII) and (CI). *Then we have the following*:

(1) *Two elements X, $Y \in \mathfrak{p}(V)$ are conjugate under $K(V)$ if and only if they are con-*

jugate under $\tilde{K}(V)$. In particular, we have a natural inclusion

$$[N(\mathfrak{p}(V))]_{K(V)} \subsetneq [N(\tilde{\mathfrak{p}}(V))]_{\tilde{K}(V)} \simeq D(m, n) .$$

(2) The image of the above inclusion is precisely $D^{(\varepsilon,\omega)}(m, n)$. Therefore we have a natural bijection $[N(\mathfrak{p}(V))]_{K(V)} \simeq D^{(\varepsilon,\omega)}(m, n)$.

(1.5) Classification of nilpotent orbits of the real Lie algebras. We recall the classification of nilpotent orbits of the real Lie algebras due to Bourgoyne and Cushman [BC] and Djoković [D].

For $(\varepsilon, \omega)=(\pm 1, \pm 1)$ or $\varnothing$, let $G_{\boldsymbol{R}}$ be the real reductive group in (1.3). We use the notation of (1.3). Only in the remaining part of this section, let us also denote by f the bilinear form (,) on V in the cases $(\varepsilon, \omega)=(\pm 1, \pm 1)$ just as in the case $(\varepsilon, \omega)=\varnothing$. Since we do not consider the anti-linear map $\tau: V \to V$ in the case $(\varepsilon, \omega)=\varnothing$, we disregard the conditions on $\tau: V \to V$ in our discussion below.

In the above setting, let $\mathcal{O}$ be a $G_{\boldsymbol{R}}$-orbit in the Lie algebra $\mathfrak{g}_{\boldsymbol{R}}$ and $x \in \mathcal{O}$. Then there exists a direct sum decomposition $V=V_1 \oplus V_2 \oplus \cdots \oplus V_r$ $(V_i \neq 0)$ into complex subspaces V_i with the following properties:

(1) Each V_i is x-stable and τ-stable.

(2) $f(V_i, V_j)=\{0\}$ if $i \neq j$.

(3) Each V_i is indecomposable in the sense of (1) and (2).

Let Δ be the type of (x, V) and Δ_i that of $(x|_{V_i}, V_i)$ (for the definition of types, see [BC]). Then we have $\Delta=\Delta_1+\Delta_2+\cdots+\Delta_r$. Thus each type is a sum of indecomposable types and, as shown in [BC], this decomposition is unique. Therefore the set $[N(\mathfrak{g}_{\boldsymbol{R}})]_{G_{\boldsymbol{R}}}$ of nilpotent $G_{\boldsymbol{R}}$-orbits is classified by sums of indecomposable nilpotent types.

For a nilpotent element $x \in \mathfrak{g}_{\boldsymbol{R}}$, the indecomposable nilpotent type Δ_i of $(x|_{V_i}, V_i)$ is one of the types in Table III.

TABLE III

(ε, ω)	Indecomposable nilpotent types	
$\varnothing$	$\Delta_k^\delta(0)$	
$(1, 1)$	$\Delta_k^\delta(0)$ (k: even),	$\Delta_k(0, 0)$ (k: odd)
$(1, -1)$	$\Delta_k(0, 0)$ (k: even),	$\Delta_k^\delta(0, 0)$ (k: odd)
$(-1, 1)$	$\Delta_k^\delta(0, 0)$ (k: even),	$\Delta_k(0, 0)$ (k: odd)
$(-1, -1)$	$\Delta_k(0, 0)$ (k: even),	$\Delta_k^\delta(0)$ (k: odd)

In Table III, $\delta=\pm$ and $k \geq 0$ is an integer. The above types are defined as in [D] as follows:

The case $(\varepsilon, \omega)=\varnothing$. If $\dim V_i=k+1$ and there exists $v \in V_i$ such that

$$(\sqrt{-1})^k \delta f(v, x^k v)>0 ,$$

then Δ_i is denoted by $\Delta_k^\delta(0)$.

The case $(\varepsilon, \omega)=(1, 1)$. If $\dim V_i=k+1$ with k even and there exists $v\in(V_i)^\tau:=\{v\in V_i;\ \tau(v)=v\}$ such that

$$(\sqrt{-1})^k\delta f(v, x^k v)>0\,,$$

then Δ_i is denoted by $\Delta_k^\delta(0)$. In this case, the signature δ equals $+$ (resp. $-$) if and only if the signature of the symmetric bilinear form $f|_{(V_i)^\tau}$ is $(k/2+1, k/2)$ (resp. $(k/2, k/2+1)$). On the other hand, if $\dim V_i=2(k+1)$ with k odd, then Δ_i is denoted by $\Delta_k(0, 0)$.

The case $(\varepsilon, \omega)=(1, -1)$. If $\dim V_i=2(k+1)$ with k odd and there exists $v\in V_i$ such that

$$(\sqrt{-1})^{k-1}\delta f_-(v, x^k v)>0 \quad \text{(cf. Remark 2)}\,,$$

then Δ_i is denoted by $\Delta_k^\delta(0, 0)$. On the other hand, if $\dim V_i=2(k+1)$ with k even, then Δ_i is denoted by $\Delta_k(0, 0)$.

The case $(\varepsilon, \omega)=(-1, 1)$. If $\dim V_i=2(k+1)$ with k even and there exists $v\in V_i$ such that

$$(\sqrt{-1})^k\delta f_-(v, x^k v)>0\,,$$

then Δ_i is denoted by $\Delta_k^\delta(0, 0)$. If $\dim V_i=2(k+1)$ with k odd, then Δ_i is denoted by $\Delta_k(0, 0)$.

The case $(\varepsilon, \omega)=(-1, -1)$. Suppose that $\dim V_i=k+1$ with k odd, that V_i does not have a non-trivial x-stable decomposition, and that there exists $v\in V_i$ such that

$$(\sqrt{-1})^{k-1}\delta f(v, x^k v)>0\,.$$

Then Δ_i is denoted by $\Delta_k^\delta(0)$. On the other hand, if $\dim V_i=2(k+1)$ with k even and V_i is decomposed into two x-stable subspaces of dimension $k+1$, then Δ_i is denoted by $\Delta_k(0, 0)$.

(1.6) Description of Sekiguchi's bijection. Let V be a vector space with an involution, or an (ε, ω)-space. We consider the symmetric pair $(\mathfrak{g}, \mathfrak{k})=(\mathfrak{gl}(V), \tilde{\mathfrak{k}}(V))$ corresponding to $(\varepsilon, \omega)=\varnothing$, or $(\mathfrak{g}, \mathfrak{k})=(\mathfrak{g}(V), \mathfrak{k}(V))$ corresponding to $(\varepsilon, \omega)=(\pm 1, \pm 1)$. Let $G_{\boldsymbol{R}}$ be the real group corresponding to $(\mathfrak{g}, \mathfrak{k})$ as in (1.3).

Proposition 2. *Let $\mathcal{O}_\theta$ be a nilpotent K_θ-orbit in $\mathfrak{p}$ and $\mathcal{O}_{\boldsymbol{R}}$ the nilpotent $G_{\boldsymbol{R}}$-orbit in $\mathfrak{g}_{\boldsymbol{R}}$ which corresponds to $\mathcal{O}_\theta$ by Sekiguchi's bijection. Let $\eta=\sum_{i=1}^r \eta_i$ be the ab-diagram (resp. (ε, ω)-diagram) corresponding to $\mathcal{O}_\theta$, where η_i is an ab-diagram with a single row (resp. primitive (ε, ω)-diagram) if $(\varepsilon, \omega)=\varnothing$ (resp. $(\varepsilon, \omega)=(\pm 1, \pm 1)$). Let η_i correspond to the type Δ_i as in Table* IV. *Then the type Δ of $\mathcal{O}_{\boldsymbol{R}}$ is $\Delta=\Delta_1+\Delta_2+\cdots+\Delta_r$.*

TABLE IV

$(\varepsilon, \omega)=\varnothing$	$\overbrace{\cdots\cdots baba}^{k+1}$, $\Delta_k^+(0)$	$\overbrace{\cdots\cdots abab}^{k+1}$, $\Delta_k^-(0)$	
$(\varepsilon, \omega)=(\ \ 1, \ \ 1)$	$\overbrace{ab\cdots\cdots ba}^{k+1}$, $\Delta_k^+(0)$	$\overbrace{ba\cdots\cdots ab}^{k+1}$, $\Delta_k^-(0)$	$\overbrace{ba\cdots\cdots ba}^{k+1}$ $ab\cdots\cdots ab$, $\Delta_k(0, 0)$
$(\varepsilon, \omega)=(\ \ 1, -1)$	$\overbrace{ba\cdots\cdots ba}^{k+1}$ $ba\cdots\cdots ba$, $\Delta_k^+(0, 0)$	$\overbrace{ab\cdots\cdots ab}^{k+1}$ $ab\cdots\cdots ab$, $\Delta_k^-(0, 0)$	$\overbrace{ab\cdots\cdots ba}^{k+1}$ $ba\cdots\cdots ab$, $\Delta_k(0, 0)$
$(\varepsilon, \omega)=(-1, \ \ 1)$	$\overbrace{ab\cdots\cdots ba}^{k+1}$ $ab\cdots\cdots ba$, $\Delta_k^+(0, 0)$	$\overbrace{ba\cdots\cdots ab}^{k+1}$ $ba\cdots\cdots ab$, $\Delta_k^-(0, 0)$	$\overbrace{ba\cdots\cdots ba}^{k+1}$ $ab\cdots\cdots ab$, $\Delta_k(0, 0)$
$(\varepsilon, \omega)=(-1, -1)$	$\overbrace{ba\cdots\cdots ba}^{k+1}$, $\Delta_k^+(0)$	$\overbrace{ab\cdots\cdots ab}^{k+1}$, $\Delta_k^-(0)$	$\overbrace{ab\cdots\cdots ba}^{k+1}$ $ba\cdots\cdots ab$. $\Delta_k(0, 0)$

In order to prove Proposition 2, take a strictly normal S-triple (h, x, y) of $(\mathfrak{g}, \mathfrak{k})$ with respect to $\varphi=\tau\circ\theta$ such that $x\in\mathcal{O}_\theta$ (cf. (1.1)). Let $(h_{\boldsymbol{R}}, x_{\boldsymbol{R}}, y_{\boldsymbol{R}})$ be the corresponding S-triple in $\mathfrak{g}_{\boldsymbol{R}}$ (cf. Theorem 1) and S the three-dimensional subalgebra isomorphic to $\mathfrak{sl}(2, \boldsymbol{C})$ spanned by the S-triple (h, x, y):

$$S:=\boldsymbol{C}h+\boldsymbol{C}x+\boldsymbol{C}y=\boldsymbol{C}h_{\boldsymbol{R}}+\boldsymbol{C}x_{\boldsymbol{R}}+\boldsymbol{C}y_{\boldsymbol{R}}\,.$$

Then Proposition 2 is an immediate consequence of the followin two lemmas.

LEMMA 1. *Take the vector space V, the involution s of V, the hermitian form f on V and the complex conjugation τ of $GL(V)$ as in* (1.3, (AIII)). *We consider the symmetric pair* $(\mathfrak{gl}(V), \mathfrak{k}(V))$ *defined by the vector space V with the involution and the corresponding real group $G_{\boldsymbol{R}}=U(m, n)$. Then for the above three-dimensional subalgebra S, we have the following*:

(1) *V has an f-orthogonal direct sum decomposition*

$$V=V_1\oplus V_2\oplus\cdots\oplus V_r$$

into s-stable irreducible S-submodules V_i such that each V_i does not have a non-trivial $x_{\boldsymbol{R}}$-stable and f-orthogonal decomposition.

(2) *Let η_i be the ab-diagram of the nilpotent element $x|_{V_i}$ and Δ_i the type of $(x_{\boldsymbol{R}}|_{V_i}, V_i)$. Then η_i and Δ_i are contained in Table* IV *and the correspondence $\eta_i \leftrightarrow \Delta_i$ is given in Table* IV.

LEMMA 2. *Take the vector space V, the involution s of V, the bilinear form $f=(\ ,\ )$ on V and the anti-linear map $\tau: V \to V$ as in* (1.3, BDI$\sim$CI). *We consider the symmetric pair $(\mathfrak{g}(V), \mathfrak{k}(V))$ defined by the (ε, ω)-space V and the corresponding real group $G_{\boldsymbol{R}}=\{g \in G(V); \tau(g)=\tau \circ g \circ \tau^{-1}=g\}$. Then for the above three-dimensional subalgebra S, we have the following*:

(1) *V has an f-orthogonal direct sum decomposition*

$$V=V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

into s-stable and τ-stable S-submodules V_i such that each V_i does not have a non-trivial $x_{\boldsymbol{R}}$-stable, τ-stable and f-orthogonal decomposition.

(2) *Let η_i be the ab-diagram of $x|_{V_i}$ and Δ_i the type of $(x_{\boldsymbol{R}}|_{V_i}, V_i)$. Then η_i and Δ_i are contained in Table* IV *and the correspondence $\eta_i \leftrightarrow \Delta_i$ is given in Table* IV.

(1.7) Proof of Lemma 1. In order to prove Lemma 1 and Lemm 2, we need the following lemma.

LEMMA 3. *Let S be a Lie algebra spanned by an S-triple (h, x, y). Let W be an irreducible S-module of dimension $k+1$ $(k \geq 0)$ and v a lowest weight vector of W. Put $z:=x+y+\sqrt{-1}h$. Then if we express $z^k v$ as a linear combination of the basis $\{v, xv, x^2v, \cdots, x^kv\}$ of W, the coefficient of v equals $k!(-\sqrt{-1})^k$.*

PROOF. By the representation theory of $\mathfrak{sl}_2$ (for example, [H, Lemma 7.2]), there exists a basis $\{v_0, v_1, \cdots, v_k\}$ such that $hv_i=(k-2i)v_i$ $(0 \leq i \leq k)$, $yv_i=(i+1)v_{i+1}$ $(0 \leq i \leq k-1)$, $xv_i=(k-i+1)v_{i-1}$ $(1 \leq i \leq k)$ and $v_k=v$. If we express z by a matrix with respect to the basis $\{v=v_k, v_{k-1}, \cdots, v_1, v_0\}$, we have

$$z=\begin{pmatrix} -\sqrt{-1}k & k & & & 0 \\ 1 & -\sqrt{-1}(k-2) & k-1 & & \\ & 2 & \ddots & \ddots & \\ & & \ddots & \sqrt{-1}(k-2) & 1 \\ 0 & & & k & \sqrt{-1}k \end{pmatrix}$$

Then it follows by induction that the coefficient of v_{k-j} in z^pv $(0 \leq j \leq p \leq k)$ equals $\{p!(k-j)!(-\sqrt{-1})^{p-j}\}/\{(p-j)!(k-p)!\}$. In particular, the coefficient of $v=v_k$ in z^kv equals $k!(-\sqrt{-1})^k$. q.e.d.

Now let us prove Lemma 1. Since $-h=\tau(h)=-(h)_f^*$ by Remark 1, we have $(h)_f^*=h$.

Hence for h-weight vectors u_i, u_j of weights i, j respectively, we have

(a) $$f(u_i, u_j)=0 \quad \text{if} \quad i\neq j\,.$$

Since $yV_a\subset V_b$ and $yV_b\subset V_a$, we can take a basis $\{v^{(i)}\}_{i=1}^r$ of Ker y such that each $v^{(i)}$ is an h-weight vector contained in $V_a\cup V_b$. Suppose that there exist i, j with $i\neq j$ such that $f(v^{(i)}, v^{(j)})\neq 0$. Then it follows from $f(V_a, V_b)=0$ and the above (a) that $v^{(i)}$ and $v^{(j)}$ have the same h-weight and are contained in V_a or V_b simultaneously. If we put

$$\tilde{v}^{(j)}=v^{(j)}-\frac{f(v^{(i)}, v^{(j)})}{f(v^{(i)}, v^{(i)})}v^{(i)}\,,$$

we have $f(v^{(i)}, \tilde{v}^{(j)})=0$. By taking $\tilde{v}^{(j)}$ instead of $v^{(j)}$, we may assume that $\{v^{(i)}\}_{i=1}^r$ is f-orthogonal. Let V_i be the irreducible S-submodule generated by $v^{(i)}$. Then $V=\bigoplus_{i=1}^r V_i$ and it follows from $y=\tau(x)=-(x)_f^*$ that the decomposition $V=\bigoplus_{i=1}^r V_i$ is f-orthogonal. Since each V_i is an irreducible module over $S=Ch_{\boldsymbol{R}}+Cx_{\boldsymbol{R}}+Cy_{\boldsymbol{R}}$, V_i does not have an $x_{\boldsymbol{R}}$-stable non-trivial decomposition. Hence (1) follows.

Put $k+1=\dim V_i$ $(k\geq 0)$, $v=v^{(i)}$ and apply Lemma 3 to the irreducible S-module V_i. Then by the remark (a) above, we have

$$f(v, (x_{\boldsymbol{R}})^k v)=f\left(v, \frac{k!(-\sqrt{-1})^k}{2^k}v\right)=\frac{k!}{2^k}(-\sqrt{-1})^k f(v, v)\neq 0\,.$$

Let us express v as a sum of $h_{\boldsymbol{R}}$-weight vectors;

$$v=u_0+u_1+\cdots+u_k \quad \text{with} \quad h_{\boldsymbol{R}}u_j=-(k-2j)u_j\,.$$

Then $(x_{\boldsymbol{R}})^k v=(x_{\boldsymbol{R}})^k u_0$ and it follows from $h_{\boldsymbol{R}}=\tau(h_{\boldsymbol{R}})=-(h_{\boldsymbol{R}})_f^*$ that $f((x_{\boldsymbol{R}})^p u_0, (x_{\boldsymbol{R}})^q u_0)=0$ if $p+q\neq k$ $(p, q\geq 0)$. Therefore we have

$$f(v, (x_{\boldsymbol{R}})^k v)=f(u_0+u_1+\cdots+u_k, (x_{\boldsymbol{R}})^k u_0)=f(u_0, (x_{\boldsymbol{R}})^k u_0)=0$$

and hence

$$(-\sqrt{-1})^k f(u_0, (x_{\boldsymbol{R}})^k u_0)=\frac{k!(-1)^k}{2^k}f(v, v)=:c\,.$$

Here we note that $f(w, w)>0$ (resp. $f(w, w)<0$) if $w\in V_a\setminus\{0\}$ (resp. $w\in V_b\setminus\{0\}$). Then we have the following:

$$v\in V_a \quad \text{and} \quad k \quad \text{is even} \implies c>0 \quad \text{and} \quad \eta_i=\overbrace{ab\cdot\cdots\cdot ba}^{k+1},$$

$$v\in V_a \quad \text{and} \quad k \quad \text{is odd} \implies c<0 \quad \text{and} \quad \eta_i=\overbrace{ab\cdot\cdots\cdot ab}^{k+1},$$

$$v \in V_b \quad \text{and} \quad k \quad \text{is even} \quad \Longrightarrow c<0 \quad \text{and} \quad \eta_i = \overbrace{ba \cdot \cdots \cdot ab}^{k+1},$$

$$v \in V_b \quad \text{and} \quad k \quad \text{is odd} \quad \Longrightarrow c>0 \quad \text{and} \quad \eta_i = \overbrace{ba \cdot \cdots \cdot ba}^{k+1}.$$

Hence (2) follows. Thus the proof of Lemma 1 is completed.

(1.8) Proof of Lemma 2. Let us give the proof of Lemma 2. In the setting of Lemma 2, take an h-weight vector v such that $v \in (\mathrm{Ker}\, y) \cap (V_a \cup V_b)$. Let U be the irreducible S-submodule generated by v and put $\dim U = k+1$ $(k \geq 0)$: $U = \boldsymbol{C}v \oplus \boldsymbol{C}xv \oplus \cdots \oplus \boldsymbol{C}x^k v$, $x^{k+1}v = 0$. Since $v \in V_a \cup V_b$, U is s-stable. Here we note the following facts:

(F1) Since $\tau(h) = -h$, $x^j v$ and $\tau(x^j v)$ are h-weight vectors with the opposite weights: $h(x^j v) = -(k-2j)x^j v$, $h\tau(x^j v) = (k-2j)x^j v$.

(F2) For two h-weight vectors $v_i, v_j \in V$ with weights i and j respectively, if $f(v_i, v_j) \neq 0$, we have $i = -j$.

We first consider the following three cases:

(a) $\omega = 1$ (i.e., $f(V_a, V_b) = 0$) and $k+1$ is even.

(b) $\omega = -1$ (i.e., $f(V_a, V_a) = f(V_b, V_b) = 0$) and $k+1$ is odd.

(c) $\varepsilon(-1)^{k+1} = 1$.

Then it is easy to see that $f(v, x^k v) = 0$ (cf. [O2, Proof of Proposition 2]). It follows from (F2) that $f(U, U) = 0$. Moreover, since $\tau(h) = -h$, $\tau(x) = y$ (cf. Remark 1) and $\tau \circ s = s \circ \tau$, $\tau(U)$ is also an s-stable irreducible S-module. If $U = \tau(U)$, we must have $f(U, \tau(U)) = f(U, U) = 0$; in particular $f(v, \tau(v)) = 0$ which contradicts Remark 3. Hence $U \cap \tau(U) = 0$. Now we put $V_1 := U \oplus \tau(U)$. We have $f(x^p v, \tau(x^q v)) = (-1)^q f(y^q x^p v, \tau(v))$. Then it follows from (F1), (F2) and $f(v, \tau(v)) \neq 0$ that $f(x^p v, \tau(x^p v)) \neq 0$ and $f(x^p v, \tau(x^q v)) = 0$ $(p \neq q)$. Hence the restriction $f|_{V_1}$ is non-degenerate.

Let us show that V_1 is indecomposable in the sense of Lemma 2, (1). Suppose that V_1 has an $x_{\boldsymbol{R}}$-stable, τ-stable and f-orthogonal direct sum decomposition $V_1 = U_1 \oplus U_2$. Since $(x_{\boldsymbol{R}})^k V_1 \neq 0$, we may assume that there exists $u \in U_1$ such that $(x_{\boldsymbol{R}})^k u \neq 0$.

First suppose that $\tau^2 = -\mathrm{id}_V$ (i.e., $\varepsilon\omega = -1$). If there exists $c \in \boldsymbol{C}$ such that $\tau((x_{\boldsymbol{R}})^k u) = c(x_{\boldsymbol{R}})^k u$, we have

$$c\bar{c}(x_{\boldsymbol{R}})^k u = \bar{c}\tau((x_{\boldsymbol{R}})^k u) = \tau(c(x_{\boldsymbol{R}})^k u) = \tau^2((x_{\boldsymbol{R}})^k u) = -(x_{\boldsymbol{R}})^k u$$

which is a contradiction. Hence $\tau((x_{\boldsymbol{R}})^k u)$ and $(x_{\boldsymbol{R}})^k u$ are linearly independent. Then it follows that the $2(k+1)$ elements

$$u,\ x_{\boldsymbol{R}}u,\ \cdots,\ (x_{\boldsymbol{R}})^k u,\ \tau(u),\ \tau(x_{\boldsymbol{R}}u),\ \cdots,\ \tau((x_{\boldsymbol{R}})^k u)$$

of U_1 are linearly independent and hence $U_2 = 0$.

Secondly suppose that $\tau^2 = \mathrm{id}_V$ (i.e., $\varepsilon\omega = 1$). It follows from the assumptions (a), (b) or (c) that $\varepsilon(-1)^{k+1} = \omega(-1)^{k+1} = 1$. Then we have

$$f(u, (x_{\boldsymbol{R}})^k u) = (-1)^k f((x_{\boldsymbol{R}})^k u, u) = \varepsilon(-1)^k f(u, (x_{\boldsymbol{R}})^k u) = -f(u, (x_{\boldsymbol{R}})^k u)$$

and hence $f(u, (x_{\boldsymbol{R}})^k u) = 0$. Since $f|_{U_1}$ is non-degenerate, there exists $w \in U_1$ such that $f(w, (x_{\boldsymbol{R}})^k u) = (-1)^k f((x_{\boldsymbol{R}})^k w, u) \neq 0$. Thus $(x_{\boldsymbol{R}})^k u$ and $(x_{\boldsymbol{R}})^k w$ are linearly independent, which implies that $u, x_{\boldsymbol{R}} u, \cdots, (x_{\boldsymbol{R}})^k u, w, x_{\boldsymbol{R}} w, \cdots, (x_{\boldsymbol{R}})^k w$ are linearly independent. Hence we have $U_2 = 0$ as before. Therefore V_1 does not have a non-trivial $x_{\boldsymbol{R}}$-stable, τ-stable and f-orthogonal decomposition.

Next we suppose that none of (a), (b) and (c) is satisfied. This can happen only when $(\varepsilon, \omega) = (1, 1)$ and $k+1$ is odd or when $(\varepsilon, \omega) = (-1, -1)$ and $k+1$ is even. Then $\tau(x^k v) = y^k \tau(v)$ and v are h-weight vectors with the same weight and are contained in $\operatorname{Ker} y \cap V_a$ or $\operatorname{Ker} y \cap V_b$ simultaneously. Suppose that $\boldsymbol{C}v \neq \boldsymbol{C}\tau(x^k v)$. Define a positive real number c by $y^k x^k v = cv$ and put $v' := \sqrt{c}\, v + \tau(x^k v)$. Then v' has the same property as v. Moreover it is easily verified that $x^k v' = \sqrt{c}\, \tau(v')$. Therefore we may assume that $\boldsymbol{C}v = \boldsymbol{C}\tau(x^k v)$ by taking v' instead of v. Put $V_1 = U$. Then V_1 is an s-stable and τ-stable irreducible S-submodule of V. Moreover since $f(v, \tau(v)) \neq 0$ by Remark 3, $f|_{V_1}$ is non-degenerate. Since V_1 is an irreducible S-submodule, V_1 does not have an $x_{\boldsymbol{R}}$-stable, τ-stable and f-orthogonal decomposition.

If we take V_1 as above, the orthogonal complement of V_1 is also a τ-stable and s-stable S-submodule. By induction Lemma 2, (1) follows from this fact.

Let us show the statement (2) of Lemma 2 for the above V_1, where $V_1 = U \oplus \tau(U)$ or $V_1 = U$.

First suppose that $(\varepsilon, \omega) = (1, 1)$. Also suppose that $k = \dim U - 1$ is odd. Then $V_1 = U \oplus \tau(U)$. $v \in U$, $\tau(x^k v) \in \tau(U)$ are lowest weight vectors of V_1 such that $v \in V_a$ and $\tau(x^k v) \in V_b$, or that $v \in V_b$ and $\tau(x^k v) \in V_a$. Hence

$$\eta_1 = \begin{matrix} \overbrace{ba \cdot \cdots \cdot ba}^{k+1} \\ ab \cdot \cdots \cdot ab \end{matrix} .$$

On the other hand, since k is odd, we have $\Delta_1 = \Delta_k(0, 0)$. Suppose that k is even. Then $V_1 = U$ and

$$\eta_1 = \begin{cases} \overbrace{ab \cdot \cdots \cdot ba}^{k+1} & (v \in V_a) \\ ba \cdot \cdots \cdot ab & (v \in V_b) . \end{cases}$$

Since $\tau \circ s = s \circ \tau$, $(V_1)^\tau$ is decomposed as $(V_1)^\tau = (V_1)^\tau \cap V_a \oplus (V_1)^\tau \cap V_b$ with $\dim_{\boldsymbol{R}} (V_1)^\tau \cap V_a = n_a(\eta_1)$ and $\dim_{\boldsymbol{R}} (V_1)^\tau \cap V_b = n_b(\eta_1)$. Since the restriction of f to $(V_1)^\tau \cap V_a$ (resp. $(V_1)^\tau \cap V_b$) is positive definite (resp. negative definite), the signature of $f|_{(V_1)^\tau}$ is $(k/2+1, k/2)$ if $v \in V_a$ and $(k/2, k/2+1)$ if $v \in V_b$. Therefore we obtain the correspondence

$$\overbrace{ab\cdot\cdots\cdot ba}^{k+1} \longleftrightarrow \varDelta_k^+(0)\,, \qquad \overbrace{ba\cdot\cdots\cdot ab}^{k+1} \longleftrightarrow \varDelta_k^-(0)\,.$$

Secondly, suppose that $(\varepsilon, \omega)=(1, -1)$. Then we always have $V_1=U\oplus\tau(U)$ and $f(U, U)=f(\tau(U), \tau(U))=0$. Suppose that k is odd. Then v and $\tau(x^k v)$ are lowest weight vectors of V_1 contained in V_a or V_b simultaneously. Hence we have

$$\eta_1=\begin{cases} \overbrace{\begin{matrix} ab\cdot\cdots\cdot ab \\ ab\cdot\cdots\cdot ab\end{matrix}}^{k+1} & (v\in V_a) \\ \begin{matrix} ba\cdot\cdots\cdot ba \\ ba\cdot\cdots\cdot ba\end{matrix} & (v\in V_b)\,. \end{cases}$$

By (F1), (F2) and Lemma 3, we have the following:

$$f(v, (x_{\boldsymbol{R}})^k\tau(v))=f\left(v, \tau\left(\frac{k!}{2^k}(-\sqrt{-1})^k v\right)\right)=\frac{k!}{2^k}(\sqrt{-1})^k f(v, \tau(v))\,,$$

$$(\sqrt{-1})^{k-1}f_-(v, (x_{\boldsymbol{R}})^k v)=(\sqrt{-1})^{k-1}\{-\overline{f(v, \tau(x_{\boldsymbol{R}}^k v))}-\overline{f(v, x_{\boldsymbol{R}}^k v)j}\}$$

$$=-(\sqrt{-1})^{k-1}\overline{f(v, x_{\boldsymbol{R}}^k\tau(v))}=\frac{k!}{2^k}\sqrt{-1}\,\overline{f(v, \tau(v))}\,.$$

It follows from the definition of $f=(\ ,\ )$ in (1.3) that

$$f(v, \tau(v))=\begin{cases} -\sqrt{-1}\,|v|^2 & (v\in V_a) \\ \sqrt{-1}\,|v|^2 & (v\in V_b)\,, \end{cases}$$

where $|v|$ is the ordinary norm of $V=\boldsymbol{C}^{m+n}$. Hence we have

$$(\sqrt{-1})^{k-1}f_-(v, x_{\boldsymbol{R}}^k v)=\begin{cases} -\dfrac{k!}{2^k}|v|^2 & (v\in V_a) \\ \dfrac{k!}{2^k}|v|^2 & (v\in V_b) \end{cases}$$

and obtain the correspondence

$$\overbrace{\begin{matrix} ab\cdot\cdots\cdot ab \\ ab\cdot\cdots\cdot ab\end{matrix}}^{k+1} \longleftrightarrow \varDelta_k^-(0, 0)\,, \qquad \overbrace{\begin{matrix} ba\cdot\cdots\cdot ba \\ ba\cdot\cdots\cdot ba\end{matrix}}^{k+1} \longleftrightarrow \varDelta_k^+(0, 0)\,.$$

If k is even, we can easily verify that

$$\eta_1 = \begin{matrix} \overbrace{ab\cdots\cdots ba}^{k+1} \\ ba\cdots\cdots ab \end{matrix} \quad \text{and} \quad \Delta_1 = \Delta_k(0, 0)\,.$$

Thirdly, we suppose that $(\varepsilon, \omega) = (-1, 1)$. In this case, we always have $V_1 = U \oplus \tau(U)$ and $f(U, U) = f(\tau(U), \tau(U)) = 0$. Suppose that k is even. Then we have

$$\eta_1 = \begin{cases} \begin{matrix} \overbrace{ab\cdots\cdots ba}^{k+1} \\ ab\cdots\cdots ba \end{matrix} & (v \in V_a) \\ \begin{matrix} ba\cdots\cdots ab \\ ba\cdots\cdots ab \end{matrix} & (v \in V_b)\,. \end{cases}$$

As before we have

$$(\sqrt{-1})^k f_-(v, x_{\mathbf{R}}^k v) = -\frac{k!}{2^k}\overline{f(v, \tau(v))}\,, \qquad f(v, \tau(v)) = \begin{cases} -|v|^2 & (v \in V_a) \\ |v|^2 & (v \in V_b) \end{cases}$$

and hence

$$(\sqrt{-1})^k f_-(v, x_{\mathbf{R}}^k v) = \begin{cases} \dfrac{k!}{2^k}|v|^2 & (v \in V_a) \\ -\dfrac{k!}{2^k}|v|^2 & (v \in V_b)\,. \end{cases}$$

Therefore we obtain the correspondence

$$\begin{matrix} \overbrace{ab\cdots\cdots ba}^{k+1} \\ ab\cdots\cdots ba \end{matrix} \longleftrightarrow \Delta_k^+(0, 0)\,, \qquad \begin{matrix} \overbrace{ba\cdots\cdots ab}^{k+1} \\ ba\cdots\cdots ab \end{matrix} \longleftrightarrow \Delta_k^-(0, 0)\,.$$

On the other hand if k is odd, then it is easily verified that

$$\eta_1 = \begin{matrix} \overbrace{ba\cdots\cdots ba}^{k+1} \\ ab\cdots\cdots ab \end{matrix} \quad \text{and} \quad \Delta_1 = \Delta_k(0, 0)\,.$$

Fourthly, suppose that $(\varepsilon, \omega) = (-1, -1)$. Also suppose that k is odd. Then we have $V_1 = U$ and

$$\eta_1 = \begin{cases} \overbrace{ab\cdots\cdots ab}^{k+1} & (v \in V_a) \\ ba\cdots\cdots ba & (v \in V_b)\,. \end{cases}$$

Since $x_{\boldsymbol{R}}^k v \neq 0$ by Lemma 3, $v, x_{\boldsymbol{R}} v, \cdots, x_{\boldsymbol{R}}^k v$ form a basis of V_1. Choose $u \in (V_1)^\tau$ such that $x_{\boldsymbol{R}}^k u \neq 0$ and put $u = \sum_{i=0}^k c_i x_{\boldsymbol{R}}^i v$ ($c_i \in \boldsymbol{C}$, $c_0 \neq 0$). Then we have the following:

$$f(u, x_{\boldsymbol{R}}^k u) = f(u, x_{\boldsymbol{R}}^k \tau(u)) = f\left(u, \tau\left(\sum_{i=0}^k c_i x_{\boldsymbol{R}}^{i+k} v\right)\right) = \bar{c}_0 f(u, x_{\boldsymbol{R}}^k \tau(v))$$

$$= (-1)^k \bar{c}_0 f(x_{\boldsymbol{R}}^k u, \tau(v)) = (-1)^k \bar{c}_0 f(c_0 x_{\boldsymbol{R}}^k v, \tau(v)) = |c_0|^2 f(v, x_{\boldsymbol{R}}^k \tau(v)) .$$

Moreover, we have

$$(\sqrt{-1})^{k-1} f(u, x_{\boldsymbol{R}}^k u) = \frac{k!}{2^k} |c_0|^2 \sqrt{-1} f(v, \tau(v))$$

by (F1), (F2) and Lemma 3. By the definition of $f = (,)$ in (1.3), we have

$$f(v, \tau(v)) = \begin{cases} \sqrt{-1}\, |v|^2 & (v \in V_a) \\ -\sqrt{-1}\, |v|^2 & (v \in V_b) \end{cases}$$

and hence

$$(\sqrt{-1})^k f(u, x_{\boldsymbol{R}}^k u) = \begin{cases} -|c_0|^2 |v|^2 \dfrac{k!}{2^k} & (v \in V_a) \\ \\ |c_0|^2 |v|^2 \dfrac{k!}{2^k} & (v \in V_b) . \end{cases}$$

Therefore we obtain the correspondence

$$\overbrace{ab \cdot \cdots \cdot ab}^{k+1} \longleftrightarrow \Delta_k^-(0) , \qquad \overbrace{ba \cdot \cdots \cdot ba}^{k+1} \longleftrightarrow \Delta_k^+(0) .$$

If k is even, then $V_1 = U \oplus \tau(U)$ and it is easily verified that

$$\eta_1 = \begin{matrix} \overbrace{ab \cdot \cdots \cdot ba}^{k+1} \\ ba \cdot \cdots \cdot ab \end{matrix} \quad \text{and} \quad \Delta_1 = \Delta_k(0, 0) .$$

Thus the proof of Lemma 2 is completed.

(1.9) Closure relation of nilpotent orbits in $\mathfrak{g}_{\boldsymbol{R}}$. We describe the closure relation of nilpotent $G_{\boldsymbol{R}}$-orbits in $\mathfrak{g}_{\boldsymbol{R}}$ due to Djoković [D], who introduced the notion of chromosomes which correspond to the nilpotent $G_{\boldsymbol{R}}$-orbits in $\mathfrak{g}_{\boldsymbol{R}}$. He defined an ordering of chromosomes and described the closure relation of $[N(\mathfrak{g}_{\boldsymbol{R}})]_{G_{\boldsymbol{R}}}$ by means of this ordering. Let us define an ordering of *ab*-diagrams which is compatible with that of chromosomes as follows:

DEFINITION. (i) For an *ab*-diagram η, we denote by η' the *ab*-diagram which we obtain by erasing the first column from η. For an integer $k \geq 1$, we define the *ab*-diagram

$\eta^{(k)}$ by $\eta^{(k)} := (\eta^{(k-1)})'$ inductively.

(ii) For two ab-diagrams $\eta, \sigma \in D(m, n)$, we write $\eta \geq \sigma$ if $n_a(\eta^{(k)}) \geq n_a(\sigma^{(k)})$ and $n_b(\eta^{(k)}) \geq n_b(\sigma^{(k)})$ for all integer $k \geq 1$. We call such $\eta \geq \sigma$ a degeneration of ab-diagrams. If $\eta, \sigma \in D^{(\varepsilon, \omega)}(m, n)$, we call $\eta \geq \sigma$ an (ε, ω)-degeneration.

If we translate the main result of Djoković [D] in terms of ab-diagrams, we obtain the following:

THEOREM 2 (Djoković [D, Theorem 6]). *Let $G_{\mathbf{R}}$ be one of the real classical Lie groups which are constructed in* (1.3) *and are isomorphic to* $U(m, n)$, $O(m, n)$, $O^*(2n)$, $Sp(m, n)$ *and* $Sp(2n, \mathbf{R})$. *For two nilpotent $G_{\mathbf{R}}$-orbits $(\mathcal{O}_1)_{\mathbf{R}}, (\mathcal{O}_2)_{\mathbf{R}} \in [N(\mathfrak{g}_{\mathbf{R}})]_{G_{\mathbf{R}}}$, we denote by η_i $(i=1, 2)$ the ab-diagram of the nilpotent K_θ-orbit $(\mathcal{O}_i)_\theta$ in $\mathfrak{p}$ which corresponds to $(\mathcal{O}_i)_{\mathbf{R}}$ by Sekiguchi's bijection. Then we have $(\mathcal{O}_1)_{\mathbf{R}} \subset \overline{(\mathcal{O}_2)_{\mathbf{R}}}$ if and only if $\eta_1 \leq \eta_2$.*

By this result, to prove that Sekiguchi's bijection preserves the closure relation, it suffices to show

$$(\mathcal{O}_1)_\theta \subset \overline{(\mathcal{O}_2)_\theta} \quad \text{holds if and only if} \quad \eta_1 \leq \eta_2 ,$$

which we prove in the next section.

2. Closure relation of nilpotent orbits of the classical symmetric pairs. In this section, we determine the closure relation of nilpotent orbits in the classical symmetric pairs in terms of ab-diagrams. As a result, we see that Sekiguchi's bijection preserves the closure relation in our cases. In this section, we always consider the Zariski topology unless we specify otherwise.

(2.1) The main theorem of this section is the following:

THEOREM 3. *Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair of type* (AIII), (BDI), (DIII), (CII) *or* (CI). *For two nilpotent K_θ-orbits $\mathcal{O}_i$ $(i=1, 2)$ in the associated vector space $\mathfrak{p}$, we denote by η_i the ab-diagrams corresponding to $\mathcal{O}_i$. Then the Zariski closure $\overline{\mathcal{O}}_2$ contains $\mathcal{O}_1$ if and only if $\eta_1 \leq \eta_2$.*

By Theorem 2 and Theorem 3, we obtain the following:

COROLLARY. *For a symmetric pair $(\mathfrak{g}, \mathfrak{k})$ in Theorem* 3 *and the corresponding real reductive group $G_{\mathbf{R}}$, Sekiguchi's bijection preserves the closure relation.*

We will prove the "only if" part of Theorem 3 in (2, 2) and the "if" part in (2.3)–(2.8).

For a vector space V with an involution s and an ab-diagram $\eta \in D(\dim V_a, \dim V_b)$, we denote by C_η the nilpotent $\tilde{K}(V)$-orbit in $\tilde{\mathfrak{p}}(V)$ corresponding to η. On the other hand, for an (ε, ω)-space V and an (ε, ω)-diagram $\eta \in D^{(\varepsilon, \omega)}(\dim V_a, \dim V_b)$, we denote by $C_\eta^{(\varepsilon, \omega)}$ the nilpotent $K(V)$-orbit in $\mathfrak{p}(V)$ corresponding to η. Then we have $C_\eta^{(\varepsilon, \omega)} = C_\eta \cap \mathfrak{p}(V)$ by Proposition 1.

(2.2) Proof of the "only if" part of Theorem 3. We need the following lemma whose proof easily follows from the correspondence of nilpotent orbits and *ab*-diagrams.

LEMMA 4. *For a nilpotent element $X \in \tilde{\mathfrak{p}}(V)$ with an ab-diagram η, we have the following*:

$$\operatorname{rk}(X^{2i-1}|_{V_a}: V_a \to V_b) = n_b(\eta^{(2i-1)}), \qquad \operatorname{rk}(X^{2i-1}|_{V_b}: V_b \to V_a) = n_a(\eta^{(2i-1)}),$$
$$\operatorname{rk}(X^{2i}|_{V_a}: V_a \to V_a) = n_a(\eta^{(2i)}), \qquad \operatorname{rk}(X^{2i}|_{V_b}: V_b \to V_b) = n_b(\eta^{(2i)}),$$

where i is a positive integer.

Now let us prove the "only if" part of Theorem 3. First we consider the symmetric pair $(\mathfrak{gl}(V), \tilde{\mathfrak{k}}(V))$ of type (AIII) defined by a vector space V with an involution s. For two *ab*-diagrams $\eta, \sigma \in D(\dim V_a, \dim V_b)$, suppose that $C_\sigma \subset \bar{C}_\eta$. To prove $\sigma \leq \eta$, we consider the following $\tilde{K}(V)$-equivariant morphisms:

$$\varphi_b^{2i-1}: \tilde{\mathfrak{p}}(V) \longrightarrow \operatorname{Hom}_{\boldsymbol{C}}(V_a, V_b), \qquad X \longmapsto X^{2i-1}|_{V_a},$$
$$\varphi_a^{2i-1}: \tilde{\mathfrak{p}}(V) \longrightarrow \operatorname{Hom}_{\boldsymbol{C}}(V_b, V_a), \qquad X \longmapsto X^{2i-1}|_{V_b},$$
$$\varphi_a^{2i} \ : \tilde{\mathfrak{p}}(V) \longrightarrow \operatorname{Hom}_{\boldsymbol{C}}(V_a, V_a), \qquad X \longmapsto X^{2i}|_{V_a},$$
$$\varphi_b^{2i} \ : \tilde{\mathfrak{p}}(V) \longrightarrow \operatorname{Hom}_{\boldsymbol{C}}(V_b, V_b), \qquad X \longmapsto X^{2i}|_{V_b}.$$

We take $X \in C_\eta$, $Y \in C_\sigma$ and denote by φ one of the above morphisms. Since φ is $\tilde{K}(V)$-equivariant and $Y \in \bar{C}_\eta = \overline{\{\operatorname{Ad}(\tilde{K}(V))X\}}$, we have

$$\varphi(Y) \in \varphi(\overline{\{\operatorname{Ad}(\tilde{K}(V))X\}}) \subset \overline{\varphi(\{\operatorname{Ad}(\tilde{K}(V))X\})} = \overline{\tilde{K}(V)\varphi(X)}.$$

For example, if $\varphi = \varphi_b^{2i-1}$, we have

$$\varphi_b^{2i-1}(Y) = (Y^{2i-1}|_{V_a}: V_a \to V_b) \in \overline{\{\tilde{K}(V)(X^{2i-1}|_{V_a})\}}.$$

Therefore by Lemma 4, we have

$$n_b(\sigma^{(2i-1)}) = \operatorname{rk}(Y^{2i-1}|_{V_a}) \leq \operatorname{rk}(X^{2i-1}|_{V_a}) = n_b(\eta^{(2i-1)}).$$

By taking $\varphi = \varphi_a^{2i-1}$, φ_a^{2i}, φ_b^{2i} instead of φ_b^{2i-1}, we obtain $\sigma \leq \eta$.

The proof for the symmetric pair $(\mathfrak{g}(V), \mathfrak{k}(V))$ is similar.

(2.3) Proof of the "if" part of Theorem 3 for the symmetric pair of type (AIII). Let $\sigma < \eta$ be a degeneration of *ab*-diagrams. We have to show that $C_\sigma \subset \bar{C}_\eta$. Here we may assume that σ and η are adjacent, i.e., there exists no *ab*-diagram μ such that $\sigma < \mu < \eta$. To show $C_\sigma \subset \bar{C}_\eta$, it is sufficient to construct a morphism $z: \boldsymbol{C} \to \tilde{\mathfrak{p}}(V)$, $t \mapsto z(t)$ such that $z(0) \in C_\sigma$ and $z(t) \in C_\eta$ $(t \neq 0)$. First we construct a nilpotent element x_σ with the *ab*-diagram σ as follows:

For the i-th row σ_i of σ, let V_i be the complex vector space spanned by a basis $\{a_j^i; 1 \leq j \leq n_a(\sigma_i)\} \cup \{b_j^i; 1 \leq j \leq n_b(\sigma_i)\}$ and put $V = \bigoplus_{i=1}^r V_i$, where r is the number of rows in σ. Let V_a (resp. V_b) be the subspace of V spanned by $\mathscr{A} := \{a_j^i; 1 \leq i \leq r, 1 \leq j \leq$

$n_a(\sigma_i)\}$ (resp. $\mathscr{B} := \{b^i_j;\ 1 \le i \le r,\ 1 \le j \le n_b(\sigma_i)\}$) and define the linear involution s of V by $s|_{V_a} = \mathrm{id}_{V_a}$, $s|_{V_b} = -\mathrm{id}_{V_b}$. Thus we obtain a vector space V with an involution s. For two elements u, v of the basis $\mathscr{A} \cup \mathscr{B}$, we define $X(v \leftarrow u) \in \mathfrak{gl}(V)$ by

$$X(v \leftarrow u)u' = \begin{cases} v & (u' = u) \\ 0 & (u' \in \mathscr{A} \cup \mathscr{B} \setminus \{u\}) . \end{cases}$$

Then the associated vector space $\tilde{\mathfrak{p}}(V)$ is spanned by $\{X(b \leftarrow a),\ X(a \leftarrow b);\ a \in \mathscr{A},\ b \in \mathscr{B}\}$. For each σ_i, we define the nilpotent element x_i of $\tilde{\mathfrak{p}}(V)$ by

$$x_i = \begin{cases} X(a^i_p \leftarrow b^i_p) + X(b^i_p \leftarrow a^i_{p-1}) + \cdots + X(b^i_2 \leftarrow a^i_1) + X(a^i_1 \leftarrow b^i_1) & (\sigma_i = \overbrace{ba \cdots ba}^{2p}) \\ X(b^i_p \leftarrow a^i_p) + X(a^i_p \leftarrow b^i_{p-1}) + \cdots + X(a^i_2 \leftarrow b^i_1) + X(b^i_1 \leftarrow a^i_1) & (\sigma_i = \overbrace{ab \cdots ab}^{2p}) \\ X(a^i_{p+1} \leftarrow b^i_p) + X(b^i_p \leftarrow a^i_p) + \cdots + X(a^i_2 \leftarrow b^i_1) + X(b^i_1 \leftarrow a^i_1) & (\sigma_i = \overbrace{ab \cdots ba}^{2p+1}) \\ X(b^i_{p+1} \leftarrow a^i_p) + X(a^i_p \leftarrow b^i_p) + \cdots + X(b^i_2 \leftarrow a^i_1) + X(a^i_1 \leftarrow b^i_1) & (\sigma_i = \overbrace{ba \cdots ab}^{2p+1}) , \end{cases}$$

where we put $x_i = 0$ if $\sigma_i = a$ or $\sigma_i = b$. Define a nilpotent element x_σ of $\tilde{\mathfrak{p}}(V)$ by $x_\sigma = \sum_{i=1}^{r} x_i$. Then clearly the ab-diagram of x_σ is σ. Here we note the following lemma whose proof easily follows from [D, (11.3), (11.4), (11.5)] in view of the correspondence between mutations of chromsomes and degenerations of ab-diagrams.

LEMMA 5. *For an adjacent degeneration $\sigma < \eta$ of ab-diagrams, we denote by $\bar{\sigma} < \bar{\eta}$ the degeneration of ab-diagrams which we obtain from $\sigma < \eta$ by erasing all common rows. Then up to the change of a and b, $\bar{\sigma}$ and $\bar{\eta}$ are given as follows*:

(i)
$$\bar{\sigma} = \begin{matrix} \overbrace{\cdots\cdots\cdots ab}^{p} \\ \underbrace{\cdots ba}_{q} \end{matrix}, \quad \bar{\eta} = \begin{matrix} \overbrace{\cdots\cdots\cdots ba}^{p+1} \\ \underbrace{\cdots ab}_{q-1} \end{matrix} \quad (p \ge q \ge 1) .$$

(ii)
$$\bar{\sigma} = \begin{matrix} \overbrace{ba \cdots\cdots\cdots}^{p} \\ \underbrace{ab \cdots}_{q} \end{matrix}, \quad \bar{\eta} = \begin{matrix} \overbrace{ab \cdots\cdots\cdots}^{p+1} \\ \underbrace{ba \cdots}_{q-1} \end{matrix} \quad (p \ge q \ge 1) .$$

(iii)
$$\bar{\sigma}=\begin{array}{l}\overbrace{\cdots\cdots\cdots ba}^{p}\\ \underbrace{\cdots\cdots ba}_{q}\end{array},\quad \bar{\eta}=\begin{array}{l}\overbrace{\cdots\cdots\cdots ba}^{p+2}\\ \underbrace{\cdots ba}_{q-2}\end{array}\quad (p\geq q\geq 2,\, p-q:\text{even}).$$

Suppose that $\bar{\sigma}=\sigma_i+\sigma_j$ and put $V_{\bar{\sigma}}:=V_i\oplus V_j$, $W:=\bigoplus_{1\leq k\leq r, k\neq i,j}V_k$. Then $V_{\bar{\sigma}}$ is a vector space with an involution $s|_{V_{\bar{\sigma}}}$. *Let* $C_{\bar{\sigma}}$ and $C_{\bar{\eta}}$ be the nilpotent $\tilde{K}(V_{\bar{\sigma}})$-orbits in $\tilde{\mathfrak{p}}(V_{\bar{\sigma}})$ with *ab*-diagrams $\bar{\sigma}$ and $\bar{\eta}$, respectively. If we can prove $C_{\bar{\sigma}}\subset\bar{C}_{\bar{\eta}}$, it is easy to verify that $C_\sigma\subset\bar{C}_\eta$. Therefore we may assume that $\sigma=\bar{\sigma}$ and $\eta=\bar{\eta}$.

First we cosider the case (i):

$$\sigma=\begin{array}{l}\overbrace{\cdots\cdots\cdots ab}^{p}\\ \underbrace{\cdots\cdots ba}_{q}\end{array},\quad \eta=\begin{array}{l}\overbrace{\cdots\cdots\cdots ba}^{p+1}\\ \underbrace{\cdots\cdots ab}_{q-1}\end{array}\quad (p\geq q\geq 1).$$

We define a map $z\colon \boldsymbol{C}\rightarrow\tilde{\mathfrak{p}}(V)$ by $z(t)=x_\sigma+tX(a^2_{n_a(\sigma_2)}\leftarrow b^1_{n_b(\sigma_1)})$. Then we have $z(0)\in C_\sigma$ and $z(t)\in C_\eta$ $(t\in\boldsymbol{C}^\times)$. For example, suppose that $p=2p'$ is even and $q=2q'+1$ is odd. Then

$$\{z(t)^k a^1_1\,;\,0\leq k\leq 2p'\}\cup\{z(t)^k(ta^2_1-a^1_{p'-q'+1})\,;\,0\leq k\leq 2q'-1\}$$

is a basis of V and $z(t)^{2p'+1}a^1_1=z(t)^{2q'}(ta^2_1-a^1_{p'-q'+1})=0$ for $t\in\boldsymbol{C}^\times$. This means that $z(t)\in C_\eta$ $(t\in\boldsymbol{C}^\times)$. In such a way, we can show that $C_\sigma\subset\bar{C}_\eta$.

As for the case (ii) (resp. (iii)), we define a map $z\colon\boldsymbol{C}\rightarrow\tilde{\mathfrak{p}}(V)$ by

$$z(t)=x_\sigma+tX(b^1_1\leftarrow a^2_1)\quad(\text{resp. } x_\sigma+tX(b^2_{n_b(\sigma_2)}\leftarrow a^1_{n_a(\sigma_1)}))\,.$$

By using this, we can show that $C_\sigma\subset\bar{C}_\eta$ as before.

Therefore Theorem 3 is proved for the symmetric pairs of type (AIII).

(2.4) Reduction lemmas. Let $\sigma<\eta$ be an (ε,ω)-degeneration. We have to show that $C^{(\varepsilon,\omega)}_\sigma\subset\overline{C^{(\varepsilon,\omega)}_\eta}$. As before, we may assume that σ and η are adjacent, i.e., there exists no (ε,ω)-diagram μ such that $\sigma<\mu<\eta$. As the first reduction, we note the following lemma whose proof easily follows from [D, Section 12] in view of the correspondence between mutations of chromosomes and (ε,ω)-degenerations.

LEMMA 6. *For an adjacent (ε,ω)-degeneration $\sigma<\eta$, we denote by $\bar{\sigma}<\bar{\eta}$ the (ε,ω)-degeneration which we obtain from $\sigma<\eta$ by erasing all common rows. Then up to the change of a and b, $\bar{\sigma}$ and $\bar{\eta}$ are as in Table* V.

The second reduction lemma is the following:

LEMMA 7. *Let $\sigma \leq \eta$ be an (ε, ω)-degeneration. Suppose that the first columns of σ and η coincide. By erasing this common column from $\sigma \leq \eta$, we obtain a $(-\varepsilon, -\omega)$-degeneration $\sigma' \leq \eta'$ (cf.* (1.4), *Table* II). *Then if $C_\sigma^{(\varepsilon,\omega)} \subset \overline{C_\eta^{(\varepsilon,\omega)}}$, we have $C_{\sigma'}^{(-\varepsilon,-\omega)} \subset \overline{C_{\eta'}^{(-\varepsilon,-\omega)}}$.*

We will prove Lemma 7 in (3.6) by using the classical invariant theory.

Now by Lemma 6, to prove $C_\sigma^{(\varepsilon,\omega)} \subset \overline{C_\eta^{(\varepsilon,\omega)}}$, we may assume that $\sigma < \eta$ is an (ε, ω)-degeneration in Table V as in the case of the symmetric pair of type (AIII). Let $\sigma < \eta$ be an (ε, ω)-degeneration of type (i) in Table V ($1 \leq i \leq 5$ if $(\varepsilon, \omega) = \pm(1, -1)$ and $1 \leq i \leq 10$ if $(\varepsilon, \omega) = \pm(1, 1)$). Then the $(-\varepsilon, -\omega)$-degeneration $\sigma' < \eta'$ (which we obtain from $\sigma < \eta$ by erasing the common first column) has the same form just as the $(-\varepsilon, -\omega)$-degeneration of type (i). Therefore it is sufficient to prove $C_\sigma^{(\varepsilon,\omega)} \subset \overline{C_\eta^{(\varepsilon,\omega)}}$ in the cases $(\varepsilon, \omega) = (1, -1)$ and $(\varepsilon, \omega) = (-1, -1)$.

REMARK 4. To prove $C_\sigma^{(\varepsilon,\omega)} \subset \overline{C_\eta^{(\varepsilon,\omega)}}$ for the (ε, ω)-degenerations $\sigma < \eta$ in Table

TABLE V

(DIII) $(\varepsilon, \omega) = (1, -1)$				(CII) $(\varepsilon, \omega) = (-1, 1)$			
	$\bar{\sigma}$	$\bar{\eta}$			$\bar{\sigma}$	$\bar{\eta}$	
(1)	$2p-1$ ab·····ba ba·····ab ab···ba ba···ab $2q-1$	$2p$ ab·····ab ab·····ab ba···ba ba···ba $2q-2$	$(p \geq q \geq 1)$	(1)	$2p$ ba·····ba ab·····ab ba···ba ab···ab $2q$	$2p+1$ ba·····ab ba·····ab ab···ba ab···ba $2q-1$	$(p \geq q \geq 1)$
(2)	$2p$ ba·····ba ba·····ba ab···ba ba···ab $2q-1$	$2p+1$ ab·····ba ba·····ab ba···ba ba···ba $2q-2$	$(p \geq q \geq 1)$	(2)	$2p+1$ ab·····ba ab·····ba ba···ba ab···ab $2q$	$2p+2$ ba·····ba ab·····ab ab···ba ab···ba $2q-1$	$(p \geq q \geq 1)$
(3)	$2p-1$ ab·····ba ba·····ab ba···ba ba···ba $2q$	$2p$ ba·····ba ba·····ba ab···ba ba···ab $2q-1$	$(p > q \geq 1)$	(3)	$2p$ ba·····ba ab·····ab ab···ba ab···ba $2q+1$	$2p+1$ ab·····ba ab·····ba ba···ba ab···ab $2q$	$(p > q \geq 0)$
(4)	$2p-1$ ab·····ab ab·····ab ba···ba ba···ba $2q$	$2p$ ab·····ba ba·····ab ab···ba ba···ab $2q-1$	$(p \geq q \geq 1)$	(4)	$2p$ ba·····ab ba·····ab ab···ba ab···ba $2q+1$	$2p+1$ ba·····ba ab·····ab ba···ba ab···ab $2q$	$(p \geq q \geq 0)$
(5)	$2p$ ba·····ba ba·····ba ba···ba ba···ba $2q$	$2p+2$ ba·····ba ba·····ba ba···ba ba···ba $2q-2$	$(p \geq q \geq 1)$	(5)	$2p+1$ ab·····ba ab·····ba ab···ba ab···ba $2q+1$	$2p+3$ ab·····ba ab·····ba ab···ba ab···ba $2q-1$	$(p \geq q \geq 1)$

(CI) $(\varepsilon, \omega)=(-1, -1)$				(BDI) $(\varepsilon, \omega)=(1, 1)$			
	$\bar{\sigma}$	$\bar{\eta}$			$\bar{\sigma}$	$\bar{\eta}$	
(1)	$\overbrace{ab\cdots ba}^{2p-1}$ $\underbrace{ba\cdots ab}_{2p-1}$	$\overbrace{ab\cdots ab}^{2p}$ $\underbrace{ba\cdots ba}_{2p-2}$	$(p=q\geq 1)$	(1)	$\overbrace{ba\cdots ba}^{2p}$ $\underbrace{ab\cdots ab}_{2p}$	$\overbrace{ba\cdots ab}^{2p+1}$ $\underbrace{ab\cdots ba}_{2p-1}$	$(p=q\geq 1)$
(2)	$\overbrace{ba\cdots ba}^{2p}$ $\underbrace{ba\cdots ba}_{2q}$	$\overbrace{ba\cdots ba}^{2p+2}$ $\underbrace{ba\cdots ba}_{2q-2}$	$(p\geq q\geq 1)$	(2)	$\overbrace{ab\cdots ba}^{2p+1}$ $\underbrace{ab\cdots ba}_{2q+1}$	$\overbrace{ab\cdots ba}^{2p+3}$ $\underbrace{ab\cdots ba}_{2q-1}$	$(p\geq q\geq 1)$
(3)	$\overbrace{ba\cdots ba}^{2p}$ $\underbrace{ab\cdots ab}_{2q}$	$\overbrace{ba\cdots ba}^{2p+2}$ $\underbrace{ab\cdots ab}_{2q-2}$	$(p\geq q\geq 1)$	(3)	$\overbrace{ab\cdots ba}^{2p+1}$ $\underbrace{ba\cdots ba}_{2q+1}$	$\overbrace{ab\cdots ba}^{2p+3}$ $\underbrace{ab\cdots ab}_{2q-1}$	$(p\geq q\geq 1)$
(4)	$\overbrace{ba\cdots ba}^{2p}$ $ab\cdots ab$ $\underbrace{ba\cdots ba}_{2q}$	$\overbrace{ab\cdots ba}^{2p+1}$ $ba\cdots ab$ $\underbrace{ba\cdots ba}_{2q-2}$	$(p\geq q\geq 1)$	(4)	$\overbrace{ab\cdots ba}^{2p+1}$ $ba\cdots ab$ $\underbrace{ab\cdots ba}_{2q+1}$	$\overbrace{ba\cdots ba}^{2p+2}$ $ab\cdots ab$ $\underbrace{ab\cdots ba}_{2q-1}$	$(p\geq q\geq 1)$
(5)	$\overbrace{ba\cdots ba}^{2p}$ $ba\cdots ba$ $\underbrace{ab\cdots ab}_{2q}$	$\overbrace{ba\cdots ba}^{2p+2}$ $ba\cdots ab$ $\underbrace{ab\cdots ba}_{2q-1}$	$(p\geq q\geq 1)$	(5)	$\overbrace{ab\cdots ba}^{2p+1}$ $ab\cdots ba$ $\underbrace{ba\cdots ab}_{2q+1}$	$\overbrace{ab\cdots ba}^{2p+3}$ $ab\cdots ab$ $\underbrace{ba\cdots ba}_{2q}$	$(p\geq q\geq 0)$
(6)	$\overbrace{ba\cdots ba}^{2p}$ $ab\cdots ab$ $ba\cdots ba$ $\underbrace{ab\cdots ab}_{2q}$	$\overbrace{ba\cdots ab}^{2p+1}$ $ab\cdots ba$ $ba\cdots ab$ $\underbrace{ab\cdots ba}_{2q-1}$	$(p\geq q\geq 1)$	(6)	$\overbrace{ab\cdots ba}^{2p+1}$ $ba\cdots ab$ $ab\cdots ba$ $\underbrace{ba\cdots ab}_{2q+1}$	$\overbrace{ab\cdots ab}^{2p+2}$ $ba\cdots ba$ $ab\cdots ab$ $\underbrace{ba\cdots ba}_{2q}$	$(p\geq q\geq 0)$
(7)	$\overbrace{ab\cdots ab}^{2p}$ $ab\cdots ab$ $ba\cdots ba$ $\underbrace{ba\cdots ba}_{2q}$	$\overbrace{ab\cdots ba}^{2p+1}$ $ba\cdots ab$ $ab\cdots ba$ $\underbrace{ba\cdots ab}_{2q-1}$	$(p\geq q\geq 1)$	(7)	$\overbrace{ba\cdots ab}^{2p+1}$ $ba\cdots ab$ $ab\cdots ba$ $\underbrace{ab\cdots ba}_{2q+1}$	$\overbrace{ba\cdots ba}^{2p+2}$ $ab\cdots ab$ $ba\cdots ba$ $\underbrace{ab\cdots ab}_{2q}$	$(p\geq q\geq 0)$
(8)	$\overbrace{ab\cdots ab}^{2p}$ $\underbrace{ba\cdots ba}_{2q}$	$\overbrace{ba\cdots ba}^{2p+2}$ $\underbrace{ab\cdots ab}_{2q-2}$	$(p\geq q\geq 1)$	(8)	$\overbrace{ba\cdots ab}^{2p+1}$ $\underbrace{ab\cdots ba}_{2q+1}$	$\overbrace{ab\cdots ba}^{2p+3}$ $\underbrace{ba\cdots ab}_{2q-1}$	$(p\geq q\geq 1)$
(9)	$\overbrace{ab\cdots ab}^{2p}$ $ba\cdots ba$ $\underbrace{ba\cdots ba}_{2q}$	$\overbrace{ba\cdots ba}^{2p+2}$ $ab\cdots ba$ $\underbrace{ba\cdots ab}_{2q-1}$	$(p\geq q\geq 1)$	(9)	$\overbrace{ba\cdots ab}^{2p+1}$ $ab\cdots ba$ $\underbrace{ab\cdots ba}_{2q+1}$	$\overbrace{ab\cdots ba}^{2p+3}$ $ba\cdots ba$ $\underbrace{ab\cdots ab}_{2q}$	$(p\geq q\geq 0)$
(10)	$\overbrace{ab\cdots ab}^{2p}$ $ab\cdots ab$ $\underbrace{ba\cdots ba}_{2q}$	$\overbrace{ab\cdots ba}^{2p+1}$ $ba\cdots ab$ $\underbrace{ab\cdots ab}_{2q-2}$	$(p\geq q\geq 1)$	(10)	$\overbrace{ba\cdots ab}^{2p+1}$ $ba\cdots ab$ $\underbrace{ab\cdots ba}_{2q+1}$	$\overbrace{ba\cdots ba}^{2p+2}$ $ab\cdots ab$ $\underbrace{ba\cdots ab}_{2q-1}$	$(p\geq q\geq 1)$

V, we may assume that q is sufficiently large by Lemma 7.

(2.5) Construction of x_σ. In view of (2.4), we only consider the cases $(\varepsilon, \omega)=(1, -1)$ or $(-1, -1)$ in (2.5)–(2.8).

In this subsection, starting from an (ε, ω)-diagram σ, we will construct an (ε, ω)-space V and a nilpotent element $x_\sigma \in \mathfrak{p}(V)$ with the (ε, ω)-diagram σ.

Let $\sigma=\sum_{i=1}^{r}\sigma_i$ be an (ε, ω)-diagram which is a sum of primitive (ε, ω)-diagrams σ_i. According as $(\varepsilon, \omega)=(1, -1)$ or $(-1, -1)$, we define an (ε, ω)-space V as follows:

First suppose that $(\varepsilon, \omega)=(1, -1)$. Then σ_i has one of the following forms:

$$\text{(i)}\quad \sigma_i=\begin{matrix}\overbrace{ba\cdots ba}^{2p_i}\\ ba\cdots ba\end{matrix}\qquad \text{(ii)}\quad \sigma_i=\begin{matrix}\overbrace{ab\cdots ab}^{2p_i}\\ ab\cdots ab\end{matrix}\qquad \text{(iii)}\quad \sigma_i=\begin{matrix}\overbrace{ab\cdots ba}^{2p_i-1}\\ ba\cdots ab\end{matrix}.$$

According to the types (i)–(iii) of σ_i, let V_i be a vector space spanned by the following basis:

(i) $\{b_1^{i+}, a_1^{i+}, \cdots, b_{p_i}^{i+}, a_{p_i}^{i+}\}\cup\{b_{p_i}^{i-}, a_{p_i}^{i-}, \cdots, b_1^{i-}, a_1^{i-}\}$

(ii) $\{a_1^{i+}, b_1^{i+}, \cdots, a_{p_i}^{i+}, b_{p_i}^{i+}\}\cup\{a_{p_i}^{i-}, b_{p_i}^{i-}, \cdots, a_1^{i-}, b_1^{i-}\}$

(iii) $\{a_1^{i+}, b_1^{i+}, \cdots, b_{p_i-1}^{i+}, a_{p_i}^{i+}\}\cup\{b_{p_i}^{i-}, a_{p_i-1}^{i-}, \cdots, a_1^{i-}, b_1^{i-}\}$.

Put $V:=\bigoplus_{i=1}^{r}V_i$, $\mathscr{A}:=\{a_j^{i+}, a_j^{i-}\}$ and $\mathscr{B}:=\{b_j^{i+}, b_j^{i-}\}$. Let V_a (resp. V_b) be the subspace of V spanned by $\mathscr{A}$ (resp. $\mathscr{B}$) and s the linear involution of V such that $s|_{V_a}=\mathrm{id}_{V_a}$ and $s|_{V_b}=-\mathrm{id}_{V_b}$. We define an involution $v\mapsto\bar{v}$ of the set $\mathscr{A}\cup\mathscr{B}$ by $a_j^{i+}\mapsto b_j^{i-}$, $b_j^{i-}\mapsto a_j^{i+}$, $b_j^{i+}\mapsto a_j^{i-}$ $a_j^{i-}\mapsto b_j^{i+}$. We define a non-degenerate symmetric bilinear form $(\ ,\ )$ on V by

$$(u, v)=\begin{cases}1 & (v=\bar{u})\\ 0 & (v\in\mathscr{A}\cup\mathscr{B}\setminus\{\bar{u}\}).\end{cases}$$

Then V is a $(1, -1)$-space with respect to s and $(\ ,\ )$.

Secondly, suppose that $(\varepsilon, \omega)=(-1, -1)$. Then σ_i has one of the following forms:

$$\text{(i)}\quad \sigma_i=\overbrace{ba\cdots ba}^{2p_i}\qquad \text{(ii)}\quad \sigma_i=\overbrace{ab\cdots ab}^{2p_i}\qquad \text{(iii)}\quad \sigma_i=\begin{matrix}\overbrace{ab\cdots ba}^{2p_i-1}\\ ba\cdots ab\end{matrix}.$$

According to the types of σ_i, let V_i be a complex vector space spanned by the following basis:

(i) $\{b_1^i, a_1^i, \cdots, b_{p_i}^i, a_{p_i}^i\}$ (ii) $\{a_1^i, b_1^i, \cdots, a_{p_i}^i, b_{p_i}^i\}$

(iii) $\{a_1^{i+}, b_1^{i+}, \cdots, b_{p_i-1}^{i+}, a_{p_i}^{i+}\}\cup\{b_{p_i}^{i-}, a_{p_i-1}^{i-}, \cdots, a_1^{i-}, b_1^{i-}\}$.

Put $V:=\bigoplus_{i=1}^{r}V_i$, $\mathscr{A}:=\{a_j^i, a_j^{i+}, a_j^{i-}\}$ and $\mathscr{B}:=\{b_j^i, b_j^{i+}, b_j^{i-}\}$. Define V_a, V_b and $s\colon V\to V$ as before. We define an involution $v\mapsto\bar{v}$ of $\mathscr{A}\cup\mathscr{B}$ by $a_j^i\mapsto b_{p_i-j+1}^i$, $b_j^i\mapsto a_{p_i-j+1}^i$, $a_j^{i+}\mapsto b_j^{i-}$, $b_j^{i-}\mapsto a_j^{i+}$, $b_j^{i+}\mapsto a_j^{i-}$, $a_j^{i-}\mapsto b_j^{i+}$ and define a non-degenerate skew-symmetric bilinear form $(\ ,\ )$ on V by

$$(v,\bar{v})=\begin{cases}1 & (v\in\mathscr{A})\\ -1 & (v\in\mathscr{B})\end{cases},\quad (u,v)=0\quad (v\in\mathscr{A}\cup\mathscr{B}\setminus\{\bar{u}\}).$$

Then the adjoint $X(v\leftarrow u)^*$ of $X(v\leftarrow u)$ (cf. (2.3)) is given as follows (see [O2, Lemma 10]):

$$X(v\leftarrow u)^*=X(\bar{u}\leftarrow\bar{v})\qquad \text{if}\quad (\varepsilon,\omega)=(1,-1),$$

$$X(v\leftarrow u)^*=\begin{cases}X(\bar{u}\leftarrow\bar{v}) & (u,v\in\mathscr{A}\ \text{ or }\ u,v\in\mathscr{B})\\ -X(\bar{u}\leftarrow\bar{v}) & (u\in\mathscr{A},v\in\mathscr{B}\ \text{ or }\ u\in\mathscr{B},v\in\mathscr{A})\end{cases}\qquad \text{if}\quad (\varepsilon,\omega)=(-1,-1).$$

We note that $\mathfrak{p}(V)$ is spanned by

$$\{X(v\leftarrow u)-X(v\leftarrow u)^*\,;\quad (v,u)\in\mathscr{A}\times\mathscr{B}\quad\text{or}\quad (v,u)\in\mathscr{B}\times\mathscr{A}\}.$$

For each primitive (ε,ω)-diagram σ_i, we define a nilpotent element x_i of $\mathfrak{p}(V)$ as in Table VI.

TABLE VI

(ε,ω)	σ_i	x_i
$(1,-1)$	(i)	$X(a^{i+}_{p_i}\leftarrow b^{i+}_{p_i})+X(b^{i+}_{p_i}\leftarrow a^{i+}_{p_i-1})+\cdots+X(b^{i+}_2\leftarrow a^{i+}_1)+X(a^{i+}_1\leftarrow b^{i+}_1)$ $-\{X(a^{i-}_1\leftarrow b^{i-}_1)+X(b^{i-}_1\leftarrow a^{i-}_2)+\cdots+X(b^{i-}_{p_i-1}\leftarrow a^{i-}_{p_i})+X(a^{i-}_{p_i}\leftarrow b^{i-}_{p_i})\}$
	(ii)	$X(b^{i+}_{p_i}\leftarrow a^{i+}_{p_i})+X(a^{i+}_{p_i}\leftarrow b^{i+}_{p_i-1})+\cdots+X(a^{i+}_2\leftarrow b^{i+}_1)+X(b^{i+}_1\leftarrow a^{i+}_1)$ $-\{X(b^{i-}_1\leftarrow a^{i-}_1)+X(a^{i-}_1\leftarrow b^{i-}_2)+\cdots+X(a^{i-}_{p_i-1}\leftarrow b^{i-}_{p_i})+X(a^{i-}_{p_i}\leftarrow b^{i-}_{p_i})\}$
	(iii)	$X(a^{i+}_{p_i}\leftarrow b^{i+}_{p_i-1})+X(b^{i+}_{p_i-1}\leftarrow a^{i+}_{p_i-1})+\cdots+X(a^{i+}_2\leftarrow b^{i+}_1)+X(b^{i+}_1\leftarrow a^{i+}_1)$ $-\{X(b^{i-}_1\leftarrow a^{i-}_1)+X(a^{i-}_1\leftarrow b^{i-}_2)+\cdots+X(b^{i-}_{p_i-1}\leftarrow a^{i-}_{p_i-1})+X(a^{i-}_{p_i-1}\leftarrow b^{i-}_{p_i})\}$
$(-1,-1)$	(i)	$X(a^{i}_{p_i}\leftarrow b^{i}_{p_i})+X(b^{i}_{p_i}\leftarrow a^{i}_{p_i-1})+\cdots+X(b^{i}_2\leftarrow a^{i}_1)+X(a^{i}_1\leftarrow b^{i}_1)$
	(ii)	$X(b^{i}_{p_i}\leftarrow a^{i}_{p_i})+X(a^{i}_{p_i}\leftarrow b^{i}_{p_i-1})+\cdots+X(a^{i}_2\leftarrow b^{i}_1)+X(b^{i}_1\leftarrow a^{i}_1)$
	(iii)	$X(a^{i+}_{p_i}\leftarrow b^{i+}_{p_i-1})+X(b^{i+}_{p_i-1}\leftarrow a^{i+}_{p_i-1})+X(a^{i+}_2\leftarrow b^{i+}_1)+X(b^{i+}_1\leftarrow a^{i+}_1)$ $+X(b^{i-}_1\leftarrow a^{i-}_1)+X(a^{i-}_1\leftarrow b^{i-}_2)+\cdots+X(b^{i-}_{p_i-1}\leftarrow a^{i-}_{p_i-1})+X(a^{i-}_{p_i-1}\leftarrow b^{i-}_{p_i})$

Put $x_\sigma=\sum_{i=1}^{r}x_i$. Then clearly x_σ is a nilpotent element of $\mathfrak{p}(V)$ and the (ε,ω)-diagram of x_σ is σ.

(2.6) Here we give the proof of $C^{(\varepsilon,\omega)}_\sigma\subset\overline{C^{(\varepsilon,\omega)}_\eta}$ for the (ε,ω)-degeneration (1–5, $(\varepsilon,\omega)=(1,-1)$) and (1, 7, 8, 9, 10, $(\varepsilon,\omega)=(-1,-1)$).

As we have seen in (2.4), to complete the proof of Theorem 3, it is sufficient to prove $C^{(\varepsilon,\omega)}_\sigma\subset\overline{C^{(\varepsilon,\omega)}_\eta}$ for the (ε,ω)-degenerations $\sigma<\eta$ in Table V ($(\varepsilon,\omega)=(1,-1)$, $(-1,-1)$). What we would like to construct is a morphism $z\colon \boldsymbol{C}\to\mathfrak{p}(V)$ such that $z(0)\in C^{(\varepsilon,\omega)}_\sigma$ and $z(t)\in C^{(\varepsilon,\omega)}_\eta$ ($t\in\boldsymbol{C}^\times$). Let $\sigma<\eta$ be an (ε,ω)-degeneration in Table V

($(\varepsilon, \omega)=(1,-1)$ or $(-1,-1)$) and $x_\sigma \in \mathfrak{p}(V)$ the nilpotent element constructed in (2.5). We first consider the (ε, ω)-degenerations (1)–(5), $(\varepsilon, \omega)=(1,-1)$ and (1), (7), (8), (9), (10), $(\varepsilon, \omega)=(-1,-1)$ in Table V. In these cases, the construction of $z: \boldsymbol{C} \to \mathfrak{p}(V)$ is rather easily seen as follows:

We define $z(t)$ $(t \in \boldsymbol{C})$ as in Table VII, where the number (i) corresponds to that in Table V.

TABLE VII

	$(\varepsilon, \omega)=(1,-1)$
(1)	$z(t)=x_\sigma+t\{X(b_1^{2-} \leftarrow a_p^{1+})-X(b_p^{1-} \leftarrow a_1^{2+})\}$
(2)	$z(t)=x_\sigma+t\{X(b_1^{2-} \leftarrow a_p^{1+})-X(b_p^{1-} \leftarrow a_1^{2+})\}$
(3)	$z(t)=x_\sigma+t\{X(a_1^{1+} \leftarrow b_1^{2+})-X(a_1^{2-} \leftarrow b_1^{1-})\}$
(4)	$z(t)=x_\sigma+t\{X(a_1^{2-} \leftarrow b_p^{1+})-X(a_p^{1-} \leftarrow b_1^{2+})\}$
(5)	$z(t)=x_\sigma+t\{X(b_1^{2-} \leftarrow a_p^{1+})-X(b_1^{1-} \leftarrow a_1^{2+})\}$
	$(\varepsilon, \omega)=(-1,-1)$
(1)	$z(t)=x_\sigma+t\{X(b_1^{1-} \leftarrow a_p^{1+})+X(b_p^{1-} \leftarrow a_1^{1+})\}$
(7)	$z(t)=x_\sigma+t\{X(a_q^3 \leftarrow b_p^1)+X(a_1^1 \leftarrow b_1^3)\}+t\{X(a_q^4 \leftarrow b_p^1)+(a_1^1 \leftarrow b_1^4)\}$ $+\sqrt{-1}\,t\{X(a_q^3 \leftarrow b_p^2)+X(a_1^2 \leftarrow b_1^3)\}+\sqrt{-1}\,t\{X(a_q^4 \leftarrow b_p^2)+X(a_1^2 \leftarrow b_1^4)\}$
(8)	$z(t)=x_\sigma+t\{X(a_q^2 \leftarrow b_p^1)+X(a_1^1 \leftarrow b_1^2)\}$
(9)	$z(t)=x_\sigma+t\{X(a_q^2 \leftarrow b_p^1)+X(a_1^1 \leftarrow b_1^2)\}+t\{X(a_q^3 \leftarrow b_p^1)+X(a_1^1 \leftarrow b_1^3)\}$
(10)	$z(t)=x_\sigma+t\{X(a_q^3 \leftarrow b_p^1)+X(a_1^1 \leftarrow b_1^3)\}+\sqrt{-1}\,t\{X(a_q^3 \leftarrow b_p^2)+X(a_1^2 \leftarrow b_1^3)\}$

Then it is easy to see that $z(t) \in \mathfrak{p}(V)$ by (2.5) and $z(0) \in C_\sigma^{(\varepsilon,\omega)}$. To prove $z(t) \in C_\eta^{(\varepsilon,\omega)}$ $(t \in \boldsymbol{C}^\times)$, we may assume that q is sufficiently large by Remark 4. Then we can verify $z(t) \in C_\eta^{(\varepsilon,\omega)}$ $(t \in \boldsymbol{C}^\times)$ as follows:

For example let $\sigma < \eta$ be the $(-1,-1)$-degeneration (7) in Table V. Then $z(t)$ $(t \neq 0)$ acts on V in the following manner:

$$a_1^1 \to b_1^1 \to \cdots \to a_p^1 \to b_p^1 \to t(a_q^3+a_q^4) \to 0\,,$$
$$b_1^3 \to a_1^3+t(a_1^1+\sqrt{-1}\,a_1^2) \to \cdots \to a_q^3+t(a_q^1+\sqrt{-1}\,a_q^2) \to t(b_q^1+\sqrt{-1}\,b_q^2) \to \cdots$$
$$\to t(b_p^1+\sqrt{-1}\,b_p^2) \to t\{t(a_q^3+a_q^4)+(\sqrt{-1})^2 t(a_q^3+a_q^4)\}=0\,,$$
$$a_1^3 \to b_2^3 \to \cdots \to a_q^3 \to 0\,,$$
$$b_{p-q+1}^1-t(b_1^3+b_1^4) \to a_{p-q+2}^1-t(a_1^3+a_1^4) \to \cdots \to b_p^1-t(b_q^3+b_q^4)$$
$$\to t(a_q^3+a_q^4)-t(a_q^3+a_q^4)=0\,.$$

Here the non-zero elements in the above sequences form a basis of V. Hence the ab-diagram of $z(t)$ $(t \neq 0)$ is η. The other cases can be shown similarly.

(2.7) In (2.7) and (2.8), we prove $C_\sigma^{(\varepsilon,\omega)} \subset \overline{C_\eta^{(\varepsilon,\omega)}}$ for the remaining $(-1,-1)$-degenerations. Let us begin with the $(-1,-1)$-degeneration (6):

$$\sigma = \underbrace{\overbrace{\begin{matrix} ba\cdots\cdots ba \\ ab\cdots\cdots ab \\ ba\cdots ba \\ ab\cdots ab \end{matrix}}^{2n}}_{2m} < \eta = \underbrace{\overbrace{\begin{matrix} ba\cdots\cdots ab \\ ab\cdots\cdots ba \\ ba\cdots ab \\ ab\cdots ba \end{matrix}}^{2n+1}}_{2m-1} .$$

We consider the following element Z of $\mathfrak{p}(V)$:

$$\begin{aligned} Z := {} & x_\sigma + a_1\{X(b^1_1 \leftarrow a^1_1) + X(b^1_n \leftarrow a^1_n)\} + a_2\{X(b^1_1 \leftarrow a^1_2) + X(b^1_{n-1} \leftarrow a^1_n)\} + \cdots \\ & + a_{n-1}\{X(b^1_1 \leftarrow a^1_{n-1}) + X(b^1_2 \leftarrow a^1_n)\} + a_n X(b^1_1 \leftarrow a^1_n) + b_1\{X(a^2_1 \leftarrow b^2_1) \\ & + X(a^2_n \leftarrow b^2_n)\} + b_2\{X(a^2_1 \leftarrow b^2_2) + X(a^2_{n-1} \leftarrow b^2_n)\} + \cdots + b_{n-1}\{X(a^2_1 \leftarrow b^2_{n-1}) \\ & + X(a^2_2 \leftarrow b^2_n)\} + b_n X(a^2_1 \leftarrow b^2_n) + c_1\{X(b^3_1 \leftarrow a^3_1) + X(b^3_m \leftarrow a^3_m)\} \\ & + c_2\{X(b^3_1 \leftarrow a^3_2) + X(b^3_{m-1} \leftarrow a^3_m)\} + \cdots + c_{m-1}\{X(b^3_1 \leftarrow a^3_{m-1}) + X(b^3_2 \leftarrow a^3_m)\} \\ & + c_m X(b^3_1 \leftarrow a^3_m) + d_1\{X(a^4_1 \leftarrow b^4_1) + X(a^4_m \leftarrow b^4_m)\} + d_2\{X(a^4_1 \leftarrow b^4_2) \\ & + X(a^4_{m-1} \leftarrow b^4_m)\} + \cdots + d_{m-1}\{X(a^4_1 \leftarrow b^4_{m-1}) + X(a^4_2 \leftarrow b^4_m)\} + d_m X(a^4_1 \leftarrow b^4_m) \\ & + p_1\{X(b^1_1 \leftarrow a^2_1) + X(b^2_n \leftarrow a^1_n)\} + p_2\{X(b^1_1 \leftarrow a^2_2) + X(b^2_{n-1} \leftarrow a^1_n)\} + \cdots \\ & + p_n\{X(b^1_1 \leftarrow a^2_n) + X(b^2_1 \leftarrow a^1_n)\} + q_1\{X(b^1_1 \leftarrow a^3_1) + X(b^3_m \leftarrow a^1_n)\} \\ & + q_2\{X(b^1_1 \leftarrow a^3_2) + X(b^3_{m-1} \leftarrow a^1_n)\} + \cdots + q_m\{X(b^1_1 \leftarrow a^3_m) + X(b^3_1 \leftarrow a^1_n)\} \\ & + r_1\{X(b^1_1 \leftarrow a^4_1) + X(b^4_m \leftarrow a^1_n)\} + r_2\{X(b^1_1 \leftarrow a^4_2) + X(b^4_{m-1} \leftarrow a^1_n)\} + \cdots \\ & + r_m\{X(b^1_1 \leftarrow a^4_m) + X(b^4_1 \leftarrow a^1_n)\} + s_1\{X(a^2_1 \leftarrow b^3_1) + X(a^3_m \leftarrow b^2_n)\} \\ & + s_2\{X(a^2_1 \leftarrow b^3_2) + X(a^3_{m-1} \leftarrow b^2_n)\} + \cdots + s_m\{X(a^2_1 \leftarrow b^3_m) + X(a^3_1 \leftarrow b^2_n)\} \\ & + t_1\{X(a^2_1 \leftarrow b^4_1) + X(a^4_m \leftarrow b^2_n)\} + t_2\{X(a^2_1 \leftarrow b^4_2) + X(a^4_{m-1} \leftarrow b^2_n)\} + \cdots \\ & + t_m\{X(a^2_1 \leftarrow b^4_m) + X(a^4_1 \leftarrow b^2_n)\} + u_1\{X(b^3_1 \leftarrow a^4_1) + X(b^4_m \leftarrow a^3_m)\} \\ & + u_2\{X(b^3_1 \leftarrow a^4_2) + X(b^4_{m-1} \leftarrow a^3_m)\} + \cdots + u_m\{X(b^3_1 \leftarrow a^4_m) + X(b^4_1 \leftarrow a^3_m)\} . \end{aligned}$$

If we express Z in terms of a matrix with respect to the basis $\{b^1_1, a^1_1, b^1_2, a^1_2, \cdots, b^1_n, a^1_n, a^2_1, b^2_1, \cdots, a^2_n, b^2_n, b^3_1, a^3_1, \cdots, b^3_m, a^3_m, a^4_1, b^4_1, \cdots, a^4_m, b^4_m\}$ of V, we have

$$Z = \left(\begin{array}{c|c|c|c} A & \begin{matrix} \boldsymbol{p} \\ 0 \end{matrix} & \begin{matrix} \boldsymbol{q} \\ 0 \end{matrix} & \begin{matrix} \boldsymbol{r} \\ 0 \end{matrix} \\ \hline 0 \quad {}^t\boldsymbol{p}' & B & \begin{matrix} \boldsymbol{s} \\ 0 \end{matrix} & \begin{matrix} \boldsymbol{t} \\ 0 \end{matrix} \\ \hline 0 \quad {}^t\boldsymbol{q}' & 0 \quad {}^t\boldsymbol{s}' & C & \begin{matrix} \boldsymbol{u} \\ 0 \end{matrix} \\ \hline 0 \quad {}^t\boldsymbol{r}' & 0 \quad {}^t\boldsymbol{t}' & 0 \quad {}^t\boldsymbol{u}' & D \end{array}\right),$$

where we put

$$A=\begin{pmatrix} 0 & a_1 & 0 & a_2 \cdots 0 & a_{n-1} & 0 & a_n \\ 1 & & & & & & 0 \\ & 1 & & & & & a_{n-1} \\ & & \ddots & & 0 & & \vdots \\ & & & \ddots & & & a_2 \\ & 0 & & & \ddots & & 0 \\ & & & & & 1 & a_1 \\ & & & & & 1 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & b_1 & 0 & b_2 \cdots 0 & b_{n-1} & 0 & b_n \\ 1 & & & & & & 0 \\ & 1 & & & & & b_{n-1} \\ & & \ddots & & 0 & & \vdots \\ & & & \ddots & & & b_2 \\ & 0 & & & \ddots & & 0 \\ & & & & & 1 & b_1 \\ & & & & & 1 & 0 \end{pmatrix}$$

$$C=\begin{pmatrix} 0 & c_1 & 0 & c_2 \cdots 0 & c_{m-1} & 0 & c_m \\ 1 & & & & & & 0 \\ & 1 & & & & & c_{m-1} \\ & & \ddots & & 0 & & \vdots \\ & & & \ddots & & & c_2 \\ & 0 & & & \ddots & & 0 \\ & & & & & 1 & c_1 \\ & & & & & 1 & 0 \end{pmatrix}, \quad D=\begin{pmatrix} 0 & d_1 & 0 & d_2 \cdots 0 & d_{m-1} & 0 & d_m \\ 1 & & & & & & 0 \\ & 1 & & & & & d_{m-1} \\ & & \ddots & & 0 & & \vdots \\ & & & \ddots & & & d_2 \\ & 0 & & & \ddots & & 0 \\ & & & & & 1 & d_1 \\ & & & & & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{p}=(p_1, 0, p_2, 0, \cdots, p_n, 0), \qquad \boldsymbol{q}=(0, q_1, 0, q_2, \cdots, q_{m-1}, 0, q_m),$$

$$\boldsymbol{r}=(r_1, 0, r_2, 0, \cdots, r_m, 0), \qquad \boldsymbol{s}=(s_1, 0, s_2, 0, \cdots, s_m, 0),$$

$$\boldsymbol{t}=(0, t_1, 0, t_2, \cdots, t_{m-1}, 0, t_m), \qquad \boldsymbol{u}=(u_1, 0, u_2, 0, \cdots, u_m, 0),$$

and denote $\boldsymbol{v}'=(v_l, v_{l-1}, \cdots, v_2, v_1)$ for a vector $\boldsymbol{v}=(v_1, v_2, \cdots, v_{l-1}, v_l)\in \boldsymbol{C}^l$. Let us consider the condition that Z is nilpotent and the Young diagram of Z is $(2n+1, 2n+1, 2m-1, 2m-1)$.

Let T be a variable and $M_l(\boldsymbol{C}[T])$ the ring of $l\times l$-matrices with coefficients in $\boldsymbol{C}[T]$. For two matrices $X(T)$, $Y(T)\in M_l(\boldsymbol{C}[T])$, we write $X(T)\sim Y(T)$ if there are two invertible matrices $M_1(T)$, $M_2(T)\in M_l(\boldsymbol{C}[T])$ such that $X(T)=M_1(T)Y(T)M_2(T)$. We denote by I_l the identity matrix of degree l. Then we have by computation

$$Z-TI_{4(m+n)}\sim\begin{pmatrix} I_{4(m+n-1)} & 0 \\ 0 & M(T) \end{pmatrix},$$

where

$$M(T)=\begin{pmatrix} A(T) & f_{21}(T) & f_{31}(T) & f_{41}(T) \\ f_{21}(T) & B(T) & f_{32}(T) & f_{42}(T) \\ f_{31}(T) & f_{32}(T) & C(T) & f_{43}(T) \\ f_{41}(T) & f_{42}(T) & f_{43}(T) & D(T) \end{pmatrix},$$

$$A(T)=-T^{2n}+\sum_{i=m}^{n-1}\left(2a_{n-i}-\sum_{l=1}^{n-i-1}a_l a_{n-i-l}-\sum_{l=1}^{n-1}p_l p_{n-i-l+1}\right)T^{2i}$$

$$+\left(2a_{n-m+1}-\sum_{l=1}^{n-m}a_l a_{n-m-l+1}-\sum_{l=1}^{n-m+1}p_l p_{n-m-l+2}-r_1^2\right)T^{2(m-1)}$$

$$+\sum_{i=0}^{m-2}\left(2a_{n-i}-\sum_{l=1}^{n-i-1}a_l a_{n-i-l}-\sum_{l=1}^{n-i}p_l p_{n-i-l+1}-\sum_{l=1}^{m-i}r_l r_{m-i-l+1}\right.$$

$$\left.-\sum_{l=1}^{m-i-1}q_l q_{m-i-l}\right)T^{2i}-a_n\,,$$

$$B(T)=-T^{2n}+\sum_{i=m}^{n-1}\left(2b_{n-i}-\sum_{l=1}^{n-i-1}b_l b_{n-i-l}\right)T^{2i}$$

$$+\left(2b_{n-m+1}-\sum_{l=1}^{n-m}b_l b_{n-m-l+1}-s_1^2\right)T^{2(m-1)}$$

$$+\sum_{i=0}^{m-2}\left(2b_{n-i}-\sum_{l=1}^{n-i-1}b_l b_{n-i-l}-\sum_{l=1}^{m-i}s_l s_{m-i-l+1}-\sum_{l=1}^{m-i-1}t_l t_{m-i-l}\right)T^{2i}-b_n\,,$$

$$C(T)=-T^{2m}+\sum_{i=0}^{m-1}\left(2c_{m-i}-\sum_{l=1}^{m-i-1}c_l c_{m-i-l}-\sum_{l=1}^{m-i}u_l u_{m-i-l+1}\right)T^{2i}-c_m\,,$$

$$D(T)=-T^{2m}+\sum_{i=0}^{m-1}\left(2d_{m-i}-\sum_{l=1}^{m-i-1}d_l d_{m-i-l}\right)T^{2i}-d_m\,,$$

$$f_{21}(T)=\sum_{i=m-1}^{n-1}\left(p_{n-i}-\sum_{l=1}^{n-i-1}p_l b_{n-i-l}\right)T^{2i+1}$$

$$+\sum_{i=0}^{m-2}\left(p_{n-i}-\sum_{l=1}^{n-i-1}p_l b_{n-i-l}-\sum_{l=1}^{m-i-1}q_l s_{m-i-l}-\sum_{l=1}^{m-i-1}r_l t_{m-i-l}\right)T^{2i+1}$$

$$f_{31}(T)=\sum_{i=0}^{m-1}\left(q_{m-i}-\sum_{l=1}^{m-i-1}q_l c_{m-i-l}-\sum_{l=1}^{m-i}r_l u_{m-i-l+1}\right)T^{2i}\,,$$

$$f_{41}(T)=\sum_{i=0}^{m-1}\left(r_{m-i}-\sum_{l=1}^{m-i-1}r_l d_{m-i-l}\right)T^{2i+1}\,,$$

$$f_{32}(T)=s_1T^{2m-1}+\sum_{i=0}^{m-2}\left(s_{m-i}-\sum_{l=1}^{m-i-1}s_l c_{m-i-l}-\sum_{l=1}^{m-i-1}t_l u_{m-i-l}\right)T^{2i+1}\,,$$

$$f_{42}(T)=\sum_{i=0}^{m-1}\left(t_{m-i}-\sum_{l=1}^{m-i-1}t_l d_{m-i-l}\right)T^{2i}\,,$$

$$f_{43}(T)=\sum_{i=0}^{m-1}\left(u_{m-i}-\sum_{l=1}^{m-i-1}u_l d_{m-i-l}\right)T^{2i+1}.$$

In order that the relation $M(T)\sim\mathrm{diag}(T^{2m-1}, T^{2m-1}, T^{2n+1}, T^{2n+1})$ holds, we must have

$$(1)\quad \begin{cases} 2a_{n-m+1}-\sum_{l=1}^{n-m}a_l a_{n-m+1-l}-\sum_{l=1}^{n-m+1}p_l p_{n-m+2-l}-r_1^2=0\,, \\ (2-\delta_{k,n})a_k-\sum_{l=1}^{k-1}a_l a_{k-l}-\sum_{l=1}^{k}p_l p_{k-l+1}-\sum_{l=1}^{m-n+k}r_l r_{m-n+k-l+1} \\ \qquad -\sum_{l=1}^{m-n+k-1}q_l q_{m-n+k-l}=0 \qquad (n-m+2\le k\le n) \end{cases}$$

$$(2)\quad \begin{cases} 2b_{n-m+1}-\sum_{l=1}^{n-m}b_l b_{n-m+1-l}-s_1^2=0 \\ (2-\delta_{k,n})b_k-\sum_{l=1}^{k-1}b_l b_{k-l}-\sum_{l=1}^{m-n+k}s_l s_{m-n+k-l+1}-\sum_{l=1}^{m-n+k-1}t_l t_{m-n+k-l}=0 \\ \qquad (n-m+2\le k\le n) \end{cases}$$

$$(3)\quad (2-\delta_{k,m})c_k-\sum_{l=1}^{k-1}c_l c_{k-l}-\sum_{l=1}^{k}u_l u_{k-l+1}=0 \qquad (1\le k\le m)$$

$$(4)\quad (2-\delta_{k,m})d_k-\sum_{l=1}^{k-1}d_l d_{k-l}=0 \qquad (1\le k\le m)$$

$$(5)\quad p_k-\sum_{l=1}^{k-1}p_l p_{k-l}-\sum_{l=1}^{m-n+k-1}q_l s_{m-n+k-l}-\sum_{l=1}^{m-n+k-1}r_l t_{m-n+k-l}=0 \qquad (n-m+2\le k\le n)$$

$$(6)\quad q_k-\sum_{l=1}^{k-1}q_l c_{k-l}-\sum_{l=1}^{k}r_l u_{k-l+1}=0 \qquad (1\le k\le m)$$

$$(7)\quad r_k-\sum_{l=1}^{k-1}r_l d_{k-l}=0 \qquad (2\le k\le m)$$

$$(8)\quad s_k-\sum_{l=1}^{k-1}s_l c_{k-l}-\sum_{l=1}^{k-1}t_l u_{k-l}=0 \qquad (2\le k\le m)$$

$$(9)\quad t_k-\sum_{l=1}^{k-1}t_l d_{k-l}=0 \qquad (1\le k\le m)$$

$$(10)\quad u_k-\sum_{l=1}^{k-1}u_l d_{k-l}=0 \qquad (2\le k\le m)\,.$$

Suppose that the above equalities (1)–(10) hold and that $u_1 \neq 0 \neq s_1$. Then we can compute the following:

$$M(T)=\begin{pmatrix} A(T) & f_{21}(T) & 0 & r_1T^{2m-1} \\ f_{21}(T) & B(T) & s_1T^{2m-1} & 0 \\ 0 & s_1T^{2m-1} & -T^{2m} & u_1T^{2m-1} \\ r_1T^{2m-1} & 0 & u_1T^{2m-1} & -T^{2m} \end{pmatrix} \sim \begin{pmatrix} u_1T^{2m-1} & -T^{2m} & r_1T^{2m-1} & 0 \\ -T^{2m} & u_1T^{2m-1} & 0 & s_1T^{2m-1} \\ 0 & r_1T^{2m-1} & A(T) & f_{21}(T) \\ s_1T^{2m-1} & 0 & f_{21}(T) & B(T) \end{pmatrix}$$

$$\sim \begin{pmatrix} u_1T^{2m-1} & 0 & 0 & 0 \\ 0 & u_1T^{2m-1}-T^{2m+1}/u_1 & r_1T^{2m}/u_1 & s_1T^{2m-1} \\ 0 & r_1T^{2m-1} & A(T) & f_{21}(T) \\ 0 & s_1T^{2m}/u_1 & f_{21}(T)-s_1r_1T^{2m-1}/u_1 & B(T) \end{pmatrix}$$

$$\sim \begin{pmatrix} u_1T^{2m-1} & 0 & 0 & 0 \\ 0 & s_1T^{2m-1} & 0 & 0 \\ 0 & f_{21}(T) & (T^2-u_1^2)f_{21}(T)/s_1+u_1r_1T^{2m-1} & u_1A(T)-r_1f_{21}(T)T/s_1 \\ 0 & B(T) & (T^2-u_1^2)B(T)/s_1+s_1T^{2m} & u_1f_{21}(T)-s_1r_1T^{2m-1}-r_1B(T)T/s_1 \end{pmatrix}$$

$$\sim \begin{pmatrix} u_1T^{2m-1} & 0 & 0 & 0 \\ 0 & s_1T^{2m-1} & 0 & 0 \\ 0 & 0 & g_1(T) & g_3(T) \\ 0 & 0 & g_2(T) & g_4(T) \end{pmatrix},$$

where

$$g_1(T)=(T^2-u_1^2)f_{21}(T)+r_1s_1u_1T^{2m-1}, \quad g_3(T)=s_1u_1A(T)-r_1f_{21}(T)T,$$
$$g_2(T)=(T^2-u_1^2)B(T)+s_1^2T^{2m}, \quad g_4(T)=s_1u_1f_{21}(T)-r_1B(T)-s_1^2r_1T^{2m-1}.$$

If we write

$$P_k:=p_k-\sum_{l=1}^{k-1} b_lp_{k-l} \quad (1\leq k\leq n-m+1), \qquad B_k:=2b_k-\sum_{l=1}^{k-1} b_lb_{k-l} \quad (1\leq k\leq n-m),$$

$$A_k:=2a_k-\sum_{l=1}^{k-1} a_la_{k-l}-\sum_{l=1}^{k} p_lp_{k-l+1} \quad (1\leq k\leq n-m),$$

then we have

$$g_1(T)=P_1T^{2n+1}+(P_2-u_1^2P_1)T^{2n-1}+\cdots+(P_{n-m+1}-u_1^2P_{n-m})T^{2m+1}$$
$$-u_1(u_1P_{n-m+1}-r_1s_1)T^{2m-1},$$

$$g_2(T)=-T^{2n+2}+(B_1+u_1^2)T^{2n}+(B_2-u_1^2B_1)T^{2n-2}+\cdots+(B_{n-m}-u_1^2B_{n-m+1})T^{2m+2}$$
$$+(s_1^2-u_1^2)T^{2m},$$

$$g_3(T)=-(s_1u_1+r_1P_1)T^{2n}+(s_1u_1A_1-r_1P_2)T^{2n-2}+\cdots+(s_1u_1A_{n-m}-r_1P_{n-m+1})T^{2m},$$

$$g_4(T)=r_1T^{2n+1}+(s_1u_1P_1-r_1B_1)T^{2n-1}+\cdots+(s_1u_1P_{n-m}-r_1B_{n-m})T^{2m+1}$$
$$+s_1(u_1P_{n-m+1}-s_1r_1)T^{2m-1}\,.$$

In order that the relation

$$\begin{pmatrix} g_1(T) & g_3(T) \\ g_2(T) & g_4(T) \end{pmatrix} \sim \begin{pmatrix} T^{2n+1} & 0 \\ 0 & T^{2n+1} \end{pmatrix}$$

holds, we must have the following:

(11) $P_{k+1}=u_1^2P_k \Longleftrightarrow P_{k+1}-\sum_{l=1}^{k} b_lp_{k+1-l}=u_1^2\left(p_k-\sum_{l=1}^{k-1} b_lp_{k-l}\right) \quad (1\le k\le n-m)$

(11′) $u_1P_{n-m+1}-r_1s_1=0 \Longleftrightarrow r_1s_1=u_1(p_{n-m+1}-\sum_{l=1}^{n-m} b_lp_{n-m+1-l})$

(12) $B_1=2b_1=-u_1^2$

(13) $B_{k+1}=u_1^2B_k \Longleftrightarrow 2b_{k+1}-\sum_{l=1}^{k} b_lb_{k+1-l}=u_1^2\left(2b_k-\sum_{l=1}^{k-1} b_lb_{k-l}\right) \quad (1\le k\le n-m-1)$

(14) $s_1^2=u_1^2B_{n-m}=u_1^2\left(2b_{n-m}-\sum_{l=1}^{n-m-1} b_lb_{n-m-l}\right)$

(15) $s_1u_1=-r_1P_1=-rp_1$

(16) $r_1P_{k+1}=s_1u_1A_k \Longleftrightarrow r_1\left(p_{k+1}-\sum_{l=1}^{k} b_lp_{k+1-l}\right)$

$$=s_1u_1\left(2a_k-\sum_{l=1}^{k-1} a_la_{k-l}-\sum_{l=1}^{k} p_lp_{k-l+1}\right) \quad (1\le k\le n-m)$$

(17) $s_1u_1P_k=r_1B_k \Longleftrightarrow s_1u_1\left(p_k-\sum_{l=1}^{k-1} b_lp_{k-l}\right)=r_1\left(2b_k-\sum_{l=1}^{k-1} b_lb_{k-l}\right) \quad (1\le k\le n-m)$

(18) $u_1P_{n-m+1}=s_1r_1 \Longleftrightarrow u_1\left(p_{n-m+1}-\sum_{l=1}^{n-m} b_lp_{n-m+1-l}\right)=s_1r_1\,.$

Then from (4), (7), (9) and (10), we have $d_1=d_2=\cdots=d_m=r_2=\cdots=r_m=t_1=\cdots=t_m=u_2=\cdots=u_m=0$. Now we put $u_1=t, p_1=-t, s_1=r_1=\sqrt{-1}\,t^{n-m+1}$ for $t\in \boldsymbol{C}^\times$ and define b_1 by (12); $b_1=-t^2/2$. Then (15) and (17, $k=1$) hold. Define $c_1, c_2, \cdots, c_m$ by (3) and q_1 by (6, $k=1$); $c_1=t^2/2$, $q_1=r_1u_1=\sqrt{-1}\,t^{n-m+2}$. Define $b_2, \cdots, b_{n-m}$ by (13); $B_k=-t^{2k}$ $(1\le k\le n-m)$. Then (14) holds. Define $p_2, p_3, \cdots, p_{n-m+1}$ by (11); $P_k=-t^{2k-1}$ $(1\le k\le n-m+1)$. Then (11′)$\Leftrightarrow$(18) and (17) hold, since $u_1P_{n-m+1}-r_1s_1=t(-t^{2n-2m+1})-(-t^{2(n-m+1)})=0$ and $s_1u_1P_k-r_1B_k=s_1(u_1P_k-B_k)=s_1\{t(-t^{2k-1})-(-t^{2k})\}=0$ $(1\le k\le n-m)$. Define $a_1, \cdots, a_{n-m}$ by (16). We define

$s_2, \cdots, s_m, q_2, \cdots, q_m$ and $b_{n-m+1}, \cdots, b_n$ by (8), (6, $2 \leq k \leq m$) and (2), respectively. Finally, we define $p_{n-m+2}, \cdots, p_n$ and $a_{n-m+1}, \cdots, a_n$ by (5) and (1), respectively. Then $a_i, b_i, c_i, d_i, p_i, r_i, s_i, t_i, u_i$ are all polynomials in t. We denote by $z(t)$ the element Z which is parametrized by t as above. Then since

$$z(t) - TI_{4(m+n)} \sim \mathrm{diag}(T^{2m-1}, T^{2m-1}, T^{2n+1}, T^{2n+1}, \overbrace{1, \cdots, 1}^{4(m+n-1)}) \quad (t \in \boldsymbol{C}^\times),$$

$z(t)$ is nilpotent and the Young diagram of $z(t)$ is $(2n+1, 2n+1, 2m-1, 2m-1)$. But since η is the unique $(-1, -1)$-diagram whose Young diagram is $(2n+1, 2n+1, 2m-1, 2m-1)$ (cf. Proposition 1), we must have $z(t) \in C_\eta^{(-1,-1)}$ if $t \in \boldsymbol{C}^\times$. Moreover, since $z(0) = x_\sigma \in C_\sigma^{(-1,-1)}$, we have $C_\sigma^{(-1,-1)} \subset \overline{C_\eta^{(-1,-1)}}$ which is what we had to show for the $(-1, -1)$-degeneration (6).

(2.8) We can also prove $C_\sigma^{(\varepsilon,\omega)} \subset \overline{C_\eta^{(\varepsilon,\omega)}}$ for the $(-1, -1)$-degenerations (2), (3), (4) and (5) just like for the $(-1, -1)$-degeneration (6).

For each $(-1, -1)$-degeneration (2), (3), (4) or (5), we consider the element $Z \in \mathfrak{p}(V)$ which has the following matrix expression with respect to the following basis of V:

(2)
$$\sigma = \begin{matrix} \overbrace{ba \cdots\cdots ba}^{2n} \\ \underbrace{ba \cdots ba}_{2m} \end{matrix} < \eta = \begin{matrix} \overbrace{ba \cdots\cdots ba}^{2n+2} \\ \underbrace{ba \cdots ba}_{2m-2} \end{matrix}$$

basis: $\{b_1^1, a_1^1, \cdots, b_n^1, a_n^1, b_1^2, a_1^2, \cdots, b_m^2, a_m^2\}$

$$Z = \left(\begin{array}{c|c} A & \begin{matrix} \boldsymbol{q} \\ 0 \end{matrix} \\ \hline 0 \quad {}^t\boldsymbol{q}' & C \end{array}\right)$$

(3)
$$\sigma = \begin{matrix} \overbrace{ba \cdots\cdots ba}^{2n} \\ \underbrace{ab \cdots ab}_{2m} \end{matrix} < \eta = \begin{matrix} \overbrace{ba \cdots\cdots ba}^{2n+2} \\ \underbrace{ab \cdots ab}_{2m-2} \end{matrix}$$

basis: $\{b_1^1, a_1^1, \cdots, b_n^1, a_n^1, a_1^2, b_1^2, \cdots, a_m^2, b_m^2\}$

$$Z=\left(\begin{array}{c|c} A & \begin{matrix}\mathbf{r}\\ 0\end{matrix} \\ \hline 0\quad {}^t\mathbf{r}' & D \end{array}\right)$$

(4)
$$\sigma=\overbrace{\begin{matrix} ba\cdot\cdots\cdot ba \\ ab\cdot\cdots\cdot ab \\ \underbrace{ba\cdot\cdot\cdot ba}_{2m} \end{matrix}}^{2n} \quad < \eta = \overbrace{\begin{matrix} ba\cdot\cdots\cdot ab \\ ab\cdot\cdots\cdot ba \\ \underbrace{ba\cdot\cdot\cdot ba}_{2m-2} \end{matrix}}^{2n+1}$$

basis: $\{b_1^1, a_1^1, \cdots, b_n^1, a_n^1, a_1^2, b_1^2, \cdots, a_n^2, b_n^2, b_1^3, a_1^3, \cdots, b_m^3, a_m^3\}$

$$Z=\left(\begin{array}{c|c|c} A & \begin{matrix}\mathbf{p}\\ 0\end{matrix} & \begin{matrix}\mathbf{q}\\ 0\end{matrix} \\ \hline 0\quad {}^t\mathbf{p}' & B & \begin{matrix}\mathbf{s}\\ 0\end{matrix} \\ \hline 0\quad {}^t\mathbf{q}' & 0\quad {}^t\mathbf{s}' & C \end{array}\right)$$

(5)
$$\sigma=\overbrace{\begin{matrix} ba\cdot\cdots\cdot ba \\ ba\cdot\cdot\cdot ba \\ \underbrace{ab\cdot\cdot\cdot ab}_{2m} \end{matrix}}^{2n} \quad < \eta = \overbrace{\begin{matrix} ba\cdot\cdots\cdot ba \\ ba\cdot\cdot\cdot ab \\ \underbrace{ab\cdot\cdot\cdot ba}_{2m-1} \end{matrix}}^{2n+2}$$

basis: $\{b_1^1, a_1^1, \cdots, b_n^1, a_n^1, b_1^2, a_1^2, \cdots, b_m^2, a_m^2, a_1^3, b_1^3, \cdots, a_m^3, b_m^3\}$

$$Z=\left(\begin{array}{c|c|c} A & \begin{matrix}\mathbf{q}\\ 0\end{matrix} & \begin{matrix}\mathbf{r}\\ 0\end{matrix} \\ \hline 0\quad {}^t\mathbf{q}' & C & \begin{matrix}\mathbf{u}\\ 0\end{matrix} \\ \hline 0\quad {}^t\mathbf{r}' & 0\quad {}^t\mathbf{u}' & D \end{array}\right).$$

Here the matrices A, B, C, D and the vectors $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{u}$ are those in (2.7).

In the case (2), we can construct a morphism $z: \boldsymbol{C}\to\mathfrak{p}(V)$ such that $z(t)$ is nilpotent, that $z(0)=x_\sigma\in C_\sigma^{(-1,-1)}$, and that the Young diagram of $z(t)$ ($t\in\boldsymbol{C}^\times$) is $(2n+2, 2m-2)$ by considering the condition for

$$Z-TI_{2(m+2)}\sim\operatorname{diag}(\overbrace{1,\cdots\cdots,1}^{2(m+n-1)}, T^{2m-2}, T^{2n-2})$$

as in the case (6). Since the $(-1,-1)$-diagrams with the Young diagram $(2n+2, 2m-2)$ are

$$\eta, \eta_1=\begin{matrix}\overbrace{ba\cdots\cdots ba}^{2n+2},\\ \underbrace{ab\cdots ab}_{2m-2}\end{matrix}\quad \eta_2=\begin{matrix}\overbrace{ab\cdots\cdots ab}^{2n+2},\\ \underbrace{ba\cdots ba}_{2m-2}\end{matrix}\quad \eta_3=\begin{matrix}\overbrace{ab\cdots\cdots ab}^{2n+2},\\ \underbrace{ab\cdots ab}_{2m-2}\end{matrix}$$

we have $\{z(t); t\in\boldsymbol{C}^\times\}\subset C_\eta^{(-1,-1)}\cup C_{\eta_1}^{(-1,-1)}\cup C_{\eta_2}^{(-1,-1)}\cup C_{\eta_3}^{(-1,-1)}$. But since $\{z(t); t\in\boldsymbol{C}^\times\}$ is connected and these $K(V)$-orbits have the same dimension (cf. (3.7), Remark 7), $\{z(t); t\in\boldsymbol{C}^\times\}$ must be contained in one of the above $K(V)$-orbits. If $\{z(t); t\in\boldsymbol{C}^\times\}\subset C_{\eta_i}^{(-1,-1)}$ ($i=1, 2$ or 3), we must have $C_\sigma^{(-1,-1)}\subset\overline{C_{\eta_i}^{(-1,-1)}}$ and hence $\sigma\leq\eta_i$ by the "only if" part of Theorem 3. This contradicts the definition of the ordering $\leq$ of (ε,ω)-diagrams. Therefore $\{z(t); t\in\boldsymbol{C}^\times\}\subset C_\eta^{(-1,-1)}$ and hence we have $C_\sigma^{(-1,-1)}\subset\overline{C_\eta^{(-1,-1)}}$.

As for the $(-1,-1)$-degenerations (4) and (5), we can prove $C_\sigma^{(-1,-1)}\subset\overline{C_\eta^{(-1,-1)}}$ similarly.

Now we consider the remaining $(-1,-1)$-degeneration (3). By letting $b_i=c_i=p_i=q_i=s_i=t_i=u_i=0$ in the case (6), we get

$$Z-TI_{2(m+n)}\sim\left(\begin{array}{c|cc} I_{2(m+n-1)} & \multicolumn{2}{c}{0}\\ \hline 0 & \begin{matrix}A(T)\\ f_{41}(T)\end{matrix} & \begin{matrix}f_{41}(T)\\ D(T)\end{matrix}\end{array}\right),$$

where

$$A(T)=-T^{2n}+\sum_{i=m}^{n-1}\left(2a_{n-i}-\sum_{l=1}^{n-i-1}a_la_{n-i-l}\right)T^{2i}+\sum_{i=0}^{m-1}\left\{(2-\delta_{i,0})a_{n-i}\right.$$
$$\left.-\sum_{l=1}^{n-i-1}a_la_{n-i-l}-\sum_{l=1}^{m-i}r_lr_{m-i-l+1}\right\}T^{2i},$$

$$D(T)=-T^{2m}+\sum_{i=0}^{m-1}\left\{(2-\delta_{i,0})d_{m-i}-\sum_{l=1}^{m-i-1}d_ld_{m-i-l}\right\}T^{2i},$$

$$f_{41}(T)=\sum_{i=0}^{m-1}\left(r_{m-i}-\sum_{l=1}^{m-i-1} r_l d_{m-i-l}\right)T^{2i+1}.$$

In order that the relation

$$\begin{pmatrix} A(T) & f_{41}(T) \\ f_{41}(T) & D(T) \end{pmatrix} \sim \mathrm{diag}(T^{2m-2}, T^{2n+2})$$

holds, we must have

$$(20) \qquad (2-\delta_{k,n})a_k-\sum_{l=1}^{k-1} a_l a_{k-l}-\sum_{l=1}^{m-n+k} r_l r_{m-n+k-l+1} \qquad (n-m+2\le k\le n)$$

$$(21) \qquad (2-\delta_{k,m})d_k-\sum_{l=1}^{k-1} d_l d_{k-l}=0 \qquad (2\le k\le m)$$

$$(22) \qquad r_k-\sum_{l=1}^{k-1} r_l d_{k-l}=0 \qquad (2\le k\le m).$$

Suppose that the above equalities (20)–(22) hold and that $d_1\neq 0\neq r_1$. Then we get

$$\begin{pmatrix} A(T) & f_{41}(T) \\ f_{41}(T) & D(T) \end{pmatrix} \sim \mathrm{diag}(d_1T^{2m-2},\ d_1A(T)-T(r_1^2T^{2m-1}+TA(T))).$$

If we write $A_k:=2a_k-\sum_{l=1}^{k-1} a_l a_{k-l}$ $(1\le k\le n-m+1)$, we have

$$d_1A(T)-T(r_1^2T^{2m-1}+TA(T))=T^{2n+2}-(A_1+d_1)T^{2n}+(d_1A_1-A_2)T^{2n-2}+\cdots$$
$$+(d_1A_{n-m}-A_{n-m+1})T^{2m}+(A_{n-m+1}-r_1^2)T^{2m-2}.$$

Therefore in order that the relation

$$\begin{pmatrix} A(T) & f_{41}(T) \\ f_{41}(T) & D(T) \end{pmatrix} \sim \mathrm{diag}(T^{2m-2}, T^{2n+2})$$

holds, it is sufficient to hold the following equalities:

$$(23) \quad A_1+d_1=0 \Longleftrightarrow 2a_1+d_1=0$$

$$(24) \quad A_{k+1}-d_1A_k=0 \Longleftrightarrow 2a_{k+1}-\sum_{l=1}^{k} a_l a_{k+1-l}=d_1\left(2a_k-\sum_{l=1}^{k-1} a_l a_{k-l}\right) \quad (1\le k\le n-m)$$

$$(25) \quad A_{n-m+1}-r_1^2=0 \Longleftrightarrow 2a_{n-m+1}-\sum_{l=1}^{n-m} a_l a_{n-m+1-l}=r_1^2.$$

Now we put $d_1=t^2$, $a_1=-t^2/2$, $r_1=\sqrt{-1}\,t^{n-m+1}$ for $t\in \boldsymbol{C}$ and define $a_2,\cdots,a_{n-m+1}$ by (24); $A_k=-t^{2k}$ $(1\le k\le n-m+1)$. Then the equalities (23) and (25) hold. Define $d_2,\cdots,d_m$, $r_2,\cdots,r_m$ and $a_{n-m+2},\cdots,a_n$ by (21), (22) and (20), respectively. We denote by $z(t)$ the element Z which is parametrized by t as above. Then $z(t)$ is nilpotent and the Young diagram of $z(t)$ is $(2n+2, 2m-2)$ if $t\in \boldsymbol{C}^{\times}$. As before, we have $\{z(t); t\in \boldsymbol{C}^{\times}\}\subset C_{\eta}^{(-1,-1)}$ or $\{z(t); t\in \boldsymbol{C}^{\times}\}\subset C_{\eta_1}^{(-1,-1)}$, where

$$\eta_1 = \begin{matrix} \overbrace{ab\cdots\cdots ab}^{2n+2} . \\ \underbrace{ba\cdots ba}_{2m-2} \end{matrix}$$

To prove $\{z(t); t\in \boldsymbol{C}^{\times}\}\subset C_{\eta}^{(-1,-1)}$, it is sufficient to show that $z(t)^{2n+1}b_1^1\neq 0$. We consider the action of $z(t)$ on the basis:

$$z(t)b_i^1 = a_i^1\ (1\leq i\leq n),\ z(t)a_i^1 = a_i b_1^1 + b_{i+1}^1\ (1\leq i\leq n-1)\,,$$

$$z(t)a_n^1 = a_n b_1^1 + a_{n-1}b_2^1 + \cdots + a_1 b_n^1 + r_m b_1^2 + r_{m-1}b_2^2 + \cdots + r_1 b_m^2\,,$$

$$z(t)b_i^2 = d_i a_1^2 + a_{i+1}^2\ (1\leq i\leq m-1)\,,$$

$$z(t)b_m^2 = d_m a_1^2 + d_{m-1}a_2^2 + \cdots + d_1 a_m^2\,.$$

Therefore we have

$$z(t)^{2n-2}b_1^1 \in b_n^1 + \boldsymbol{C}\{b_1^1, b_2^1, \cdots, b_{n-1}^1\}\,,$$

$$z(t)^{2n-1}b_1^1 \in a_n^1 + \boldsymbol{C}\{a_1^1, a_2^1, \cdots, a_{n-1}^1\}\,,$$

$$z(t)^{2n}b_1^1 \in \sum_{k=1}^{m} r_{m-k+1}b_k^2 + \boldsymbol{C}\{b_1^1, b_2^1, \cdots, b_n^1\}\,,$$

$$\begin{aligned} z(t)^{2n+1}b_1^1 \in\ & r_m(d_1 a_1^2 + a_2^2) + r_{m-1}(d_2 a_1^2 + a_3^2) + \cdots + r_2(d_{m-1}a_1^2 + a_m^2) \\ & + r_1(d_m a_1^2 + d_{m-1}a_2^2 + \cdots + d_1 a_m^2) + \boldsymbol{C}\{a_1^1, a_2^1, \cdots, a_n^1\}\,, \end{aligned}$$

where $\boldsymbol{C}\{v_1, \cdots, v_l\}$ is the $\boldsymbol{C}$-span of vectors $v_1, \cdots, v_l\in V$. Since the coefficient of a_m^2 in $z(t)^{2n+1}b_1^1$ is

$$r_2 + r_1 d_1 = r_1 d_1 + r_1 d_1 = 2\sqrt{-1}\, t^{n-m+3}\neq 0\,,$$

we have $z(t)^{2n+1}b_1^1\neq 0$. Hence we conclude $C_{\sigma}^{(-1,-1)}\subset \overline{C_{\eta}^{(-1,-1)}}$.

Thus the proof of Theorem 3 is completed.

(2.9) Connection with Sekiguchi's Problem. Let $\mathfrak{g}$ a complex simple Lie algebra and G the adjoint group of $\mathfrak{g}$. Let θ be an involution of the algebraic group G. We consider the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ defined by (G, θ). Let $N(\mathfrak{p})_{\mathrm{reg}}$ (resp. $N(\mathfrak{p})_{\mathrm{pr}}$, resp. $N(\mathfrak{p})_{\mathrm{sing}}$) be the smooth part (resp. the principal K_θ-orbit, resp. the singular locus) of $N(\mathfrak{p})$. Let $N(\mathfrak{p})'_{\mathrm{sing}}$ be the union of open K_θ-orbits in $N(\mathfrak{p})_{\mathrm{sing}}$. Let $\chi: \mathfrak{p}\to\mathfrak{a}/W\simeq \boldsymbol{C}^l$ be the invariant morphism, where $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{p}$, $W=N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{k})$ and $l=\dim \mathfrak{a}$ (cf. [Sel]). We also consider the open subvariety

$$N(\mathfrak{p})_1 = \{X\in N(\mathfrak{p}); \mathrm{rk}(d\chi)_X\geq l-1\}\,.$$

Then the following problems and conjecture were posed by Sekiguchi ([Sel]).

PROBLEM I. Determine the K_θ-orbits in $N(\mathfrak{p})$.

PROBLEM II. Determine the closure relation of K_θ-orbits in $N(\mathfrak{p})$.

PROBLEM III. Determine the union $N(\mathfrak{p})'_{\text{sing}}$ of open K_θ-orbits in $N(\mathfrak{p})_{\text{sing}}$.

PROBLEM IV. Determine the smooth equivalence classes $\text{Sing}(N(\mathfrak{p}), K_\theta X)$ (cf. (3.1)) for $X \in N(\mathfrak{p})'_{\text{sing}}$.

CONJECTURE I. $N(\mathfrak{p})_1$ contains $N(\mathfrak{p})'_{\text{sing}}$.

For the symmetric pairs $(\mathfrak{sl}(n, C), \mathfrak{o}(n, C))$, $(\mathfrak{sl}(n, C), \mathfrak{sp}(n, C))$, $(\mathfrak{sl}(m+n, C), \mathfrak{sl}(m, C)+\mathfrak{sl}(n, C))$, these problems are already solved in [Se1]. So let us consider the problems and the conjecture for the remaining classical symmetric pairs $(\mathfrak{o}(m+n, C), \mathfrak{o}(m, C)+\mathfrak{o}(n, C))$, $(\mathfrak{o}(2n, C), \mathfrak{gl}(n, C))$, $(\mathfrak{sp}(m+n, C), \mathfrak{sp}(m, C)+\mathfrak{sp}(n, C))$, $(\mathfrak{sp}(2n, C), \mathfrak{gl}(n, C))$.

Problems I and II are almost solved by Proposition 1 and Theorem 3. Only the group $\text{Ad}(K(V))$ in (1.2) and the above K_θ (which act on $\mathfrak{p}$ and have the same identity component) are a little bit different.

Let us consider Problems III and IV. Let V be an (ε, ω)-space such that $\dim V_a = m$ and $\dim V_b = n$. Note that $m=n$ if $\omega=-1$. Recall that the symmetric pair $(\mathfrak{g}(V), \mathfrak{k}(V))$ defined by the (ε, ω)-space V is given as follows:

$$(\mathfrak{g}(V), \mathfrak{k}(V)) = \begin{cases} (\mathfrak{o}(m+n, C), \mathfrak{o}(m, C)+\mathfrak{o}(n, C)) & ((\varepsilon, \omega)=(1, 1)) \\ (\mathfrak{o}(2n, C), \mathfrak{gl}(n, C)) & ((\varepsilon, \omega)=(1, -1)) \\ (\mathfrak{sp}(m+n, C), \mathfrak{sp}(m, C)+\mathfrak{sp}(n, C)) & ((\varepsilon, \omega)=(-1, 1)) \\ (\mathfrak{sp}(2n, C), \mathfrak{gl}(n, C)) & ((\varepsilon, \omega)=(-1, -1)) . \end{cases}$$

To consider problems III and IV, we can take a sufficiently large group which acts on $\mathfrak{p}=\mathfrak{p}(V)$ and contains K_θ. If $m=n$, it is easily verified that there exists an element $g_c \in G(V)$ such that $g_c V_a = V_b$, $g_c V_b = V_a$ and $\text{Ad}(g_c) \in K_\theta$. Moreover such an element g_c is unique up to the conjugation by $K(V)$. If we put $K(V)' := \langle K(V) \cup \{g_c\} \rangle$, then it turns out that $K_\theta \subset \text{Ad}(K(V)')$. $\text{Ad}(g_c)$ acts on $[N(\mathfrak{p})]_{K(V)} \simeq D^{(\varepsilon, \omega)}(n, n)$ by the change of a and b. On the other hand if $m \neq n$, then $K_\theta \subset \text{Ad}(K(V))$. Now we put

$$\tilde{K}_\theta := \begin{cases} \text{Ad}(K(V)) & (m \neq n) \\ \text{Ad}(K(V)') & (m=n) . \end{cases}$$

Then $\tilde{K}_\theta$ acts on $\mathfrak{p}$ and contains K_θ. From now on, we consider $\tilde{K}_\theta$-orbits instead of K_θ-orbits.

In Table VIII, we summarize the $\tilde{K}_\theta$-orbits contained in $N(\mathfrak{p})_{\text{pr}}$, $N(\mathfrak{p})_{\text{reg}} \setminus N(\mathfrak{p})_{\text{pr}}$, $N(\mathfrak{p})'_{\text{sing}}$ and $\text{Sing}(N(\mathfrak{p}), \mathcal{O}_i)$ for the $\tilde{K}_\theta$-orbits $\mathcal{O}_i \subset N(\mathfrak{p})'_{\text{sing}}$ ($i=1$ or $i=2$). The $\tilde{K}_\theta$-orbits contained in $N(\mathfrak{p})_{\text{pr}}$ (resp. $N(\mathfrak{p})_{\text{reg}} \setminus N(\mathfrak{p})_{\text{pr}}$, resp. $N(\mathfrak{p})'_{\text{sing}}$) are given in the first (resp.

TABLE VIII

$(\mathfrak{o}(2m, \boldsymbol{C}), \mathfrak{o}(m, \boldsymbol{C})+\mathfrak{o}(m, \boldsymbol{C}))$ $(m\geq 4)$			
$\begin{matrix}\overbrace{ab\cdots ba}^{2m-1}\\ a\end{matrix}\Big/\sim$	$\varnothing$	$\begin{matrix}\overbrace{ab\cdots ba}^{2m-3}\\ bab\end{matrix}\Big/\sim$	$x^{m-1}+xy^2=0$
$(\mathfrak{o}(2m+k, \boldsymbol{C}), \mathfrak{o}(m+k, \boldsymbol{C})+\mathfrak{o}(m, \boldsymbol{C}))$ $(k\geq 2)$			
$k\left\{\begin{matrix}\overbrace{ab\cdots ba}^{2m+1}\\ a\\ \vdots\\ a\end{matrix}\right.$	$\varnothing$	(1) $k\left\{\begin{matrix}\overbrace{ab\cdots ba}^{2m-1}\\ aba\\ \vdots\\ a\end{matrix}\right.$	$x^m+y^2=0$
		(2) $k+1\left\{\begin{matrix}\overbrace{ba\cdots ab}^{2m-1}\\ a\\ \vdots\\ a\end{matrix}\right.$	$x_1^2+x_2^2+\cdots+x_{k+1}^2=0$
$(\mathfrak{o}(2m+1, \boldsymbol{C}), \mathfrak{o}(m+1, \boldsymbol{C})+\mathfrak{o}(m, \boldsymbol{C}))$			
$\overbrace{ab\cdots ba}^{2m+1}$	$\varnothing$	(1) $\begin{matrix}\overbrace{ab\cdots ba}^{2m-1}\\ a\\ b\end{matrix}$	$x^{2m}+y^2=0$
		(2) $\begin{matrix}\overbrace{ba\cdots ab}^{2m-1}\\ a\\ a\end{matrix}$	$xy=0$
$(\mathfrak{o}(2n, \boldsymbol{C}), \mathfrak{gl}(n, \boldsymbol{C}))$ $(n=2m)$			
$\begin{matrix}\overbrace{ab\cdots ab}^{2m}\\ ab\cdots ab\end{matrix}\Big/\sim$	$\varnothing$	(1) $\begin{matrix}\overbrace{ab\cdots ba}^{2m-1}\\ ba\cdots ab\\ a\\ b\end{matrix}$	$xy=0$
		(2) $\begin{matrix}\overbrace{ab\cdots ab}^{2m-2}\\ ab\cdots ab\\ ab\\ ab\end{matrix}$	$x^m+u_1v_1+u_2v_2=0$
$(\mathfrak{o}(2n, \boldsymbol{C}), \mathfrak{gl}(n, \boldsymbol{C}))$ $(n=2m+1)$			
$\begin{matrix}\overbrace{ab\cdots ba}^{2m+1}\\ ba\cdots ab\end{matrix}\Big/\sim$	$\begin{matrix}\overbrace{ab\cdots ab}^{2m}\\ ab\cdots ab\end{matrix}\Big/\sim$ a b	$\begin{matrix}\overbrace{ab\cdots ba}^{2m-1}\\ ba\cdots ab\end{matrix}\Big/\sim$ ab ab	$x^m+u_1v_1+u_1v_2=0$

	(𝔰𝔭(4m, C), 𝔰𝔭(2m, C)+𝔰𝔭(2m, C))		
$\overbrace{ab\cdots ab}^{2m}$ $ba\cdots ba$ $/\sim$	$\varnothing$	$\overbrace{ab\cdots ba}^{2m-1}$ $ab\cdots ba$ $/\sim$ b b	$xy+zw=0$
	(𝔰𝔭(4m+2, C), 𝔰𝔭(2m+2, C)+𝔰𝔭(2m, C))		
$\overbrace{ab\cdots ba}^{2m-1}$ $ab\cdots ba$	$\overbrace{ab\cdots ab}^{2m}$ $ba\cdots ba$ a a	(1) $\overbrace{ab\cdots ba}^{2m-1}$ $ab\cdots ba$ ab ba (2) $\overbrace{ba\cdots ab}^{2m-1}$ $ba\cdots ab$ a a a a	$x^m+u_1v_1+u_2v_2=0$ $\sum_{i=1}^{4} u_iv_i=0$
	(𝔰𝔭(4m+2k, C), 𝔰𝔭(2m+2k, C)+𝔰𝔭(2m, C))		
$2k\left\{\begin{array}{l}\overbrace{ab\cdots ba}^{2m+1}\\ ab\cdots ba\\ a\\ \vdots\\ a\end{array}\right.$	$2k+2\left\{\begin{array}{l}\overbrace{ba\cdots ba}^{2m}\\ ab\cdots ab\\ a\\ \vdots\\ a\end{array}\right.$	(1) $2k\left\{\begin{array}{l}\overbrace{ab\cdots ba}^{2m-1}\\ ab\cdots ba\\ aba\\ aba\\ \vdots\\ a\end{array}\right.$ (2) $2k+4\left\{\begin{array}{l}\overbrace{ba\cdots ab}^{2m-1}\\ ba\cdots ab\\ a\\ \vdots\\ a\end{array}\right.$	$x^m+yz+uv=0$ $\sum_{i=1}^{2k+2} u_iv_i=0$
	(𝔰𝔭(2n, C), gl(n, C))		
$\overbrace{ab\cdots ab}^{2n}$ $/\sim$	$\varnothing$	(1) $\overbrace{ab\cdots ab}^{2n-2}$ ab $/\sim$ (2) $\overbrace{ab\cdots ab}^{2n-2}$ ba $/\sim$	$x^n+y^2=0$ $x^n+xy^2=0$

second, resp. third) column and Sing(N(𝔭), $\mathcal{O}_i$) are given in the fourth column. For an ab-diagram η such that $n_a(\eta)=n_b(\eta)$, $\eta/\sim$ corresponds to the $\tilde{K}_\theta$-orbit which contains the $K(V)$-orbit with the ab-diagram η. Table VIII is obtained by Proposition 1, Theorem

3 and [Se1, Theorem 4 and Table IV]. We mention that Conjecture I is true in our cases. We should note that the singularity in $N(\mathfrak{p})$ at $X \in N(\mathfrak{p})'_{\text{sing}}$ is smoothly equivalent to the simple singularity in the sense of Arnol'd [A] in every case.

3. Singularities in the closure of nilpotent orbits.

(3.1) Smooth equivalence classes.

DEFINITION ([KP3]). Consider two varieties X, Y and let $x \in X$, $y \in Y$. The singularity of X at x is said to be smoothly equivalent to the singularity of Y at y if there exists a variety Z, a point $z \in Z$ and two morphism $Y \xleftarrow{\psi} Z \xrightarrow{\varphi} X$ such that $\varphi(z)=x$, $\psi(z)=y$ and φ, ψ are smooth at z. This clearly defines an equivalence relation among pointed varieties (X, x). We denote by $\mathrm{Sing}(X, x)$ the equivalence class to which (X, x) belongs.

Suppose that an algebraic group G acts on a variety X. Then $\mathrm{Sing}(X, x)=\mathrm{Sing}(X, x')$ if x and x' belong to the same orbit $\mathcal{O}$. In this case, we denote the equivalence class also by $\mathrm{Sing}(X, \mathcal{O})$.

REMARK 5. Let (X, x) and (Y, y) be pointed varieties over $\boldsymbol{C}$. Suppose that $\dim_x X = \dim_y Y + r$ for some integer $r \geq 0$. Then $\mathrm{Sing}(X, x)=\mathrm{Sing}(Y, y)$ if and only if some neighbourhoods (in the classical topology) of $x \in X$ and $(y, 0) \in Y \times \boldsymbol{C}^r$ are analytically isomorphic. Therefore various geometric properties of X at x depend only on the equivalence class $\mathrm{Sing}(X, x)$ (cf. [KP3, 12.2]).

The following theorem is the main result of this section.

THEOREM 4. *Let $\sigma \leq \eta$ be a degeneration of ab-diagrams. Suppose that the first k rows and the first l columns of η and σ coincide. Denote by $\bar{\eta}$ and $\bar{\sigma}$ the ab-diagrams which we obtain by erasing these coincident rows and columns of η and σ, respectively. Then we have the following*:

(1) $\bar{\sigma} \leq \bar{\eta}$ *and* $\mathrm{Sing}(\bar{C}_\eta, C_\sigma)=\mathrm{Sing}(\bar{C}_{\bar{\eta}}, C_{\bar{\sigma}})$.

(2) *Furthermore, suppose that σ and η are (ε, ω)-diagrams and that the sum of the coincident k rows forms an (ε, ω)-diagram. Then $\bar{\sigma} \leq \bar{\eta}$ is an $(\varepsilon', \omega') := (-1)^l(\varepsilon, \omega)$-degeneration and*

$$\mathrm{Sing}(\overline{C_\eta^{(\varepsilon,\omega)}}, C_\sigma^{(\varepsilon,\omega)})=\mathrm{Sing}(\overline{C_{\bar{\eta}}^{(\varepsilon',\omega')}}, C_{\bar{\sigma}}^{(\varepsilon',\omega')}) .$$

This is an analogue of the results of Kraft and Procesi [KP2, Proposition 3.1] and [KP3, Proposition 12.3]. We will treat separately the two steps "cancelling columns" and "cancelling rows".

(3.2) Construction of morphisms $\tilde{\rho}$ and $\tilde{\pi}$. Let V and U be vector spaces with involutions s_V and s_U, respectively. Put

$$n_a := \dim V_a\,, \quad n_b := \dim V_b\,, \quad m_a := \dim U_a\,, \quad m_b := \dim U_b\,,$$
$$L^+(U, V) := \{A \in \mathrm{Hom}_{\boldsymbol{C}}(U, V)\,; s_V A s_U = A\}\,,$$
$$L^-(V, U) := \{B \in \mathrm{Hom}_{\boldsymbol{C}}(V, U)\,; s_U B s_V = -B\}\,,$$
$$\tilde{L}(V, U) := L^+(U, V) \times L^-(V, U)\,.$$

Then $\tilde{K}(V) \times \tilde{K}(U)$ acts on $\tilde{L}(V, U)$ by

$$(g, h)(A, B) = (gAh^{-1}, hBg^{-1}) \qquad ((g, h) \in \tilde{K}(V) \times \tilde{K}(U), (A, B) \in \tilde{L}(V, U))\,.$$

We define two morphisms

$$\tilde{\mathfrak{p}}(V) \xleftarrow{\tilde{\rho}} \tilde{L}(V, U) \xrightarrow{\tilde{\pi}} \tilde{\mathfrak{p}}(U)\,, \qquad \tilde{\rho}(A, B) = AB\,, \quad \tilde{\pi}(A, B) = BA\,.$$

Then $\tilde{\rho}$ (resp. $\tilde{\pi}$) is clearly $\tilde{K}(V)$-equivariant (resp. $\tilde{K}(U)$-equivariant).

DEFINITION ([KP1]). Let X be an affine variety with an action of a reductive algebraic group G and Y an affine variety. A morphism $\varphi: X \to Y$ is called a quotient map under G if, via φ, the coordinate ring of Y is identified with the ring of G-invariant functions on X.

REMARK 6. If $\varphi: X \to Y$ is a quotient map under G and X_1 is a G-invariant closed subset of X, then $\varphi(X_1)$ is closed in Y (cf. [MF, Chap. 1, §2]).

PROPOSITION 3. *In the above setting, suppose that* $\min\{n_a, n_b\} \geq \max\{m_a, m_b\}$. *Then*

(1) $\tilde{\pi}$ is surjective and

$$\mathrm{Im}\,\tilde{\rho} = \{X \in \tilde{\mathfrak{p}}(V)\,; \mathrm{rk}(X|_{V_a}: V_a \to V_b) \leq m_b,\ \mathrm{rk}(X|_{V_b}: V_b \to V_a) \leq m_a\}\,.$$

(2) $\tilde{\pi}: \tilde{L}(V, U) \to \tilde{\mathfrak{p}}(U)$ and $\tilde{\rho}: \tilde{L}(V, U) \to \mathrm{Im}\,\tilde{\rho}$ are quotient maps under $\tilde{K}(V)$ and $\tilde{K}(U)$, respectively.

Proposition 3, (1) easily follows from elementary computation of matrices. (2) follows from Theorem 5, (1) below.

THEOREM 5 (Weyl, [W]). *Let* $\mathrm{Mat}(m, n)$ *(resp.* $\mathrm{Sym}(n)$*, resp.* $\mathrm{Skew}(n)$*) be the set of all* $m \times n$*-matrices (resp.* $n \times n$*-symmetric matrices, resp.* $n \times n$*-skew-symmetric matrices) over* $\boldsymbol{C}$*. Let* J_m *be a non-degenerate* $m \times m$*-skew-symmetric matrix and* $Sp(m, \boldsymbol{C})$ *the symplectic group defined by* J_m*.*

(1) *$GL(m, \boldsymbol{C})$ acts on* $\mathrm{Mat}(l, m) \times \mathrm{Mat}(m, n)$ *by* $g(A, B) = (Ag^{-1}, gB)$*. Then the image of the comorphism of the morphism*

$$\mathrm{Mat}(l, m) \times \mathrm{Mat}(m, n) \longrightarrow \mathrm{Mat}(l, n)\,, \quad (A, B) \longmapsto AB$$

coincides with the ring of $GL(m, \boldsymbol{C})$-invariant polynomials on $\mathrm{Mat}(l, m) \times \mathrm{Mat}(m, n)$.

(2) *$O(m, \boldsymbol{C})$ and $Sp(m, \boldsymbol{C})$ act on* $\mathrm{Mat}(m, n)$ *by left multiplication. Then the image of the comorphism of the morphism*

$$\mathrm{Mat}(m, n) \longrightarrow \mathrm{Sym}(n), \qquad A \longmapsto {}^tAA$$

$$(\textit{resp.}\ \mathrm{Mat}(m, n) \longrightarrow \mathrm{Skew}(n), \qquad A \longmapsto {}^tAJ_mA)$$

coincides with the ring of $O(m, \mathbf{C})$ *(resp.* $Sp(m, \mathbf{C})$*)-invariant polynomials on* $\mathrm{Mat}(m, n)$.

(3.4) Proof of "cancelling columns" of Theorem 4, (1). Let V be a vector space with an involution s_V and $D \in \tilde{\mathfrak{p}}(V)$ a nilpotent element with an *ab*-diagram η: $C_\eta = \mathrm{Ad}(K(V))D$. Put $U := \mathrm{Im}\, D \subset V$. Since $s_V D = -D s_V$, s_V stabilizes U. Hence $s_U := s_V|_U$ defines an involution of U and U is a vector space with an involution. In this situation, we consider the morphisms in (3.3): $\tilde{\mathfrak{p}}(V) \xleftarrow{\tilde{\rho}} \tilde{L}(V, U) \xrightarrow{\tilde{\pi}} \tilde{\mathfrak{p}}(U)$. Then we easily see the following:

LEMMA 8. *Let* $I: U \hookrightarrow V \in \mathrm{Hom}_{\mathbf{C}}(U, V)$ *be the inclusion and* $D_0 := [D: V \to U] \in \mathrm{Hom}_{\mathbf{C}}(V, U)$. *Then we have*:

(1) $(I, D_0) \in \tilde{L}(V, U)$, $\tilde{\rho}(I, D_0) = D$, $\tilde{\pi}(I, D_0) = [D|_U: U \to U]$.

(2) *The ab-diagram of* $D|_U \in \tilde{\mathfrak{p}}(U)$ *is* η' *(cf.* (1.9)).

REMARK 7. For an *ab*-diagram η, we have

$$\min\{n_a(\eta), n_b(\eta)\} \geq \max\{n_a(\eta'), n_b(\eta')\}.$$

This is easily verified by considering the case that η has only one row.

As before, we put $\dim V_a = n_a$, $\dim V_b = n_b$, $\dim U_a = m_a$, $\dim U_b = m_b$. Then by Lemma 8, (2) and Remark 7, V and U satisfy the assumption in Proposition 3. Now we put

$$L^+(U, V)' := \{A \in L^+(U, V); \mathrm{rk}\, A = m_a + m_b \text{ (i.e., } A: U \to V \text{ is injective)}\},$$

$$L^-(V, U)' := \{B \in L^-(V, U); \mathrm{rk}\, B = m_a + m_b \text{ (i.e., } B: V \to U \text{ is surjective)}\},$$

$$\tilde{L}' := L^+(U, V)' \times L^-(V, U)' \subset \tilde{L}(V, U),$$

$$\tilde{\mathfrak{p}}(V)' := \{X \in \tilde{\mathfrak{p}}(V); \mathrm{rk}(X|_{V_a}) = m_b, \mathrm{rk}(X|_{V_b}) = m_a\}.$$

Then we have the following:

LEMMA 9. (1) $\tilde{\pi}|_{\tilde{L}'}: \tilde{L}' \to \tilde{\mathfrak{p}}(U)$ *is smooth.*

(2) $\tilde{\rho}(\tilde{L}') = \tilde{\mathfrak{p}}(V)'$ *and the map* $\tilde{\rho}|_{\tilde{L}'}: \tilde{L}' \to \tilde{\mathfrak{p}}(V)'$ *is locally trivial in the classical topology with typical fibre* $\tilde{K}(U)$.

Since the proof of Lemma 9 is similar to that of [KP2, Lemma 5.2], we omit it.

LEMMA 10. *Let* $C_\sigma \subset \tilde{\mathfrak{p}}(V)$ *be a nilpotent orbit with an ab-diagram* σ *such that* $\sigma \leq \eta$ *and that the first columns of* η *and* σ *coincide. Then we have*

(1) $\tilde{\rho}^{-1}(C_\sigma)$ *is a single orbit under* $\tilde{K}(V) \times \tilde{K}(U)$ *contained in* $\tilde{L}'$.

(2) $\tilde{\pi}(\tilde{\rho}^{-1}(C_\sigma)) = C_{\sigma'}$.

(3) *Put* $\tilde{N}_\eta := \tilde{\pi}^{-1}(\bar{C}_{\eta'})$. *Then* $\tilde{\rho}(\tilde{N}_\eta) = \bar{C}_\eta$.

(4) $\tilde{\rho}(\tilde{L}'\cap\tilde{N}_\eta)=\tilde{\mathfrak{p}}(V)'\cap\bar{C}_\eta$.

PROOF. (1) Take $X\in C_\sigma$. Since the first columns of η and σ coincide, we have

$$\operatorname{rk}(X|_{V_a}: V_a\to V_b)=n_b(\sigma')=n_b(\eta')=\operatorname{rk}(D|_{V_a}: V_a\to V_b)=\dim U_b=m_b\,,$$
$$\operatorname{rk}(X|_{V_b}: V_b\to V_a)=n_a(\sigma')=n_a(\eta')=\operatorname{rk}(D|_{V_b}: V_b\to V_a)=\dim U_a=m_a$$

(cf. Lemma 4). Hence $X\in\tilde{\mathfrak{p}}(V)'=\tilde{\rho}(\tilde{L}')$. For any $(P, Q)\in\tilde{L}(V, U)$ such that $\tilde{\rho}(P, Q)=PQ=X$, since $\operatorname{rk}(PQ)=\operatorname{rk}(X)=m_a+m_b$, we have $(P, Q)\in\tilde{L}'$ and hence $\tilde{\rho}^{-1}(C_\sigma)\subset\tilde{L}'$. Therefore we have

$$\tilde{\rho}^{-1}(C_\sigma)=\tilde{\rho}^{-1}(\operatorname{Ad}(\tilde{K}(V))X)=\tilde{K}(V)\tilde{\rho}^{-1}(X)=\tilde{K}(V)(\tilde{\rho}|_{\tilde{L}'})^{-1}(X)\,.$$

Since $(\tilde{\rho}|_{\tilde{L}'})^{-1}(X)$ is a single $\tilde{K}(U)$-orbit by Lemma 9, (2), $\tilde{\rho}^{-1}(C_\sigma)$ is a single orbit under $\tilde{K}(V)\times\tilde{K}(U)$.

(2) Take $(P, Q)\in\tilde{\rho}^{-1}(C_\sigma)$. Since $\operatorname{rk}(PQ)=m_a+m_b$, we see that $P|_{U_a}: U_a\to V_a$, $P|_{U_b}: U_b\to V_b$ are injective and $Q|_{V_a}: V_a\to U_b$, $Q|_{V_b}: V_b\to U_a$ are surjective. Since $\tilde{\rho}(P, Q)=PQ$ is nilpotent, $\tilde{\pi}(P, Q)=QP$ is also nilpotent. Let us denote by ν the *ab*-diagram of $QP\in C_\nu\subset\tilde{\mathfrak{p}}(U)$. For an even integer $h>0$, let us compare the ranks of the following two maps:

$$[(PQ)^h|_{V_a}: V_a\to V_a]=[\overbrace{V_a\xrightarrow{Q}U_b\xrightarrow{P}V_b\xrightarrow{Q}U_a\xrightarrow{P}V_a\to\cdots\to V_b\xrightarrow{Q}U_a\xrightarrow{P}V_a}^{2h}]\,,$$
$$[(QP)^{h-1}|_{U_b}: U_b\to U_a]=[\overbrace{U_b\xrightarrow{P}V_b\xrightarrow{Q}U_a\xrightarrow{P}V_a\to\cdots\to V_b\xrightarrow{Q}U_a}^{2(h-1)}]\,.$$

Since $Q: V_a\to U_b$ is surjective and $P: U_a\to V_a$ is injective, we have $n_a((\sigma')^{(h-1)})=n_a(\sigma^{(h)})=\operatorname{rk}((PQ)^h|_{V_a}: V_a\to V_a)=\operatorname{rk}((QP)^{h-1}|_{U_b}: U_b\to U_a)=n_a(\nu^{(h-1)})$ (cf. Lemma 4). Similarly, we have $n_b((\sigma')^{(h-1)})=n_b(\nu^{(h-1)})$ and the same equalities hold for any odd integers $h>0$. Therefore we have $\nu=\sigma'$, i.e., $\tilde{\pi}(P, Q)\in C_{\sigma'}$ and hence $\tilde{\pi}(\tilde{\rho}^{-1}(C_\sigma))=C_{\sigma'}$.

(3) Since $\tilde{\rho}(I, D_0)\in C_\eta$ and $\tilde{\pi}(I, D_0)\in C_{\eta'}$ by Lemma 8, we have $C_\eta\subset\tilde{\rho}(\tilde{\pi}^{-1}(C_{\eta'}))\subset\tilde{\rho}(\tilde{N}_\eta)$. Since $\tilde{N}_\eta$ is a $\tilde{K}(U)$-stable closed subset of $\tilde{L}(V, U)$ and $\tilde{\rho}$ is a quotient map under $\tilde{K}(U)$, $\tilde{\rho}(\tilde{N}_\eta)$ is closed. Hence $\bar{C}_\eta\subset\tilde{\rho}(\tilde{N}_\eta)$.

Conversely, take $Y=(P, Q)\in\tilde{N}_\eta$. Since $\tilde{\pi}(Y)=QP\in\bar{C}_{\eta'}$, $\tilde{\rho}(Y)=PQ$ is also nilpotent. Let μ (resp. ν) be the *ab*-diagram of $\tilde{\rho}(Y)$ (resp. $\tilde{\pi}(Y)$). Then $C_\nu\subset\tilde{\pi}(\tilde{N}_\eta)=\bar{C}_{\eta'}$ and hence $\nu\leq\eta'$. For any even integer $h>0$, we have

$$n_a(\mu^{(h)})=\operatorname{rk}((PQ)^h|_{V_a}: V_a\to V_a)=\operatorname{rk}\left(V_a\xrightarrow{Q}U_b\xrightarrow{(QP)^{h-1}}U_a\xrightarrow{P}V_a\right)$$
$$\leq\operatorname{rk}((QP)^{h-1}|_{U_b}: U_b\to U_a)=n_a(\nu^{(h-1)})\leq n_a(\eta^{(h)})\,.$$

Similarly, we have $n_b(\mu^{(h)})\leq n_b(\eta^{(h)})$ and the same inequalities hold for any odd integer $h>0$. Therefore $\mu\leq\eta$ and $\tilde{\rho}(Y)\in C_\mu\subset\bar{C}_\eta$. Hence $\tilde{\rho}(\tilde{N}_\eta)\subset\bar{C}_\eta$.

(4) Since $\tilde{\mathfrak{p}}(V)'=\tilde{\rho}(\tilde{L}')$, we have $\tilde{\rho}(\tilde{L}'\cap\tilde{N}_\eta)\subset\tilde{\mathfrak{p}}(V)'\cap\bar{C}_\eta$.

Conversely, take a $\tilde{K}(V)$-orbit $C_\mu\subset\tilde{\mathfrak{p}}(V)'\cap\bar{C}_\eta$ and $X\in C_\mu$ $(\mu\leq\eta)$. Since $n_b(\mu')=\mathrm{rk}(X|_{V_a})=m_b=n_b(\eta')$ and $n_a(\mu')=\mathrm{rk}(X|_{V_b})=m_a=n_a(\eta')$, the first cloumns of η and μ coincide. Therefore $\tilde{\pi}(\tilde{\rho}^{-1}(C_\mu))=C_{\mu'}\subset\bar{C}_{\eta'}$ by (2) and $\tilde{\rho}^{-1}(C_\mu)\subset\tilde{N}_\eta\cap\tilde{L}'$ by (1). Thus $C_\mu\subset\tilde{\rho}(\tilde{N}_\eta\cap\tilde{L}')$ and hence $\tilde{\mathfrak{p}}(V)'\cap\bar{C}_\eta\subset\tilde{\rho}(\tilde{L}'\cap\tilde{N}_\eta)$. q.e.d.

Now let us give the proof of "cancelling columns" of Theorem 4, (1). For a degeneration $\sigma\leq\eta$ of ab-diagrams with a coincident first column, we have constructed the morphisms

$$\bar{C}_\eta\xleftarrow{\tilde{\rho}_r}\tilde{N}_\eta\xrightarrow{\tilde{\pi}_r}\bar{C}_{\eta'}\quad(\tilde{\pi}_r:=\tilde{\pi}|_{\tilde{N}_\eta},\ \tilde{\rho}_r:=\tilde{\rho}|_{\tilde{N}_\eta})$$

such that $\tilde{\pi}_r(\tilde{\rho}_r{}^{-1}(C_\sigma))=C_{\sigma'}$. Therefore it is sufficient to show that $\tilde{\pi}_r$ and $\tilde{\rho}_r$ are smooth at a point $Y\in\tilde{\rho}_r^{-1}(C_\sigma)$.

Since $\tilde{\pi}\colon\tilde{L}(V,U)\to\tilde{\mathfrak{p}}(U)$ is smooth at $Y\in\tilde{L}$ (cf. Lemma 9, (1)) and

$$\begin{array}{ccc}\tilde{N}_\eta=\tilde{\pi}^{-1}(\bar{C}_{\eta'}) & \xrightarrow{\tilde{\pi}_r} & \bar{C}_{\eta'}\\ \downarrow & & \downarrow\\ \tilde{L}(V,U) & \xrightarrow{\tilde{\pi}} & \tilde{\mathfrak{p}}(U)\end{array}$$

is a fibre product, $\tilde{\pi}_r\colon\tilde{N}_\eta\to\tilde{\mathfrak{p}}(U)$ is also smooth at Y.

On the other hand, since $\tilde{\rho}|_{\tilde{L}'}\colon\tilde{L}'\to\tilde{\mathfrak{p}}(V)'$ is locally trivial with typical fibre $\tilde{K}(U)$ (cf. Lemma 9, (2)) and $\tilde{L}'\cap\tilde{N}_\eta$ is a $\tilde{K}(U)$-invariant closed subset of $\tilde{L}'$,

$$\tilde{\rho}_r|_{\tilde{L}'\cap\tilde{N}_\eta}\colon\tilde{L}'\cap\tilde{N}_\eta\to\tilde{\rho}(\tilde{L}'\cap\tilde{N}_\eta)=\tilde{\mathfrak{p}}(V)'\cap\bar{C}_\eta\ \text{(cf. Lemma 10, (4))}$$

is also locally trivial and hence $\tilde{\rho}_r\colon\tilde{N}_\eta\to\bar{C}_\eta$ is smooth at Y. Therefore the "cancelling columns" of Theorem 4, (1) is proved.

(3.5) Construction of morphisms ρ and π. Let V (resp. U) be an (ε_V,ω_V)-space (resp. (ε_U,ω_U)-space) with an involution s_V (resp. s_U) and a bilinear form $(\ ,\ )_V$ (resp. $(\ ,\ )_U$). Put

$$L(V,U):=\mathrm{Hom}_{\boldsymbol{C}}(V,U)\,,\qquad L^-(V,U):=\{X\in L(V,U);s_UXs_V=-X\}$$

and define the adjoint $X^*\in L(U,V)$ of $X\in L(V,U)$ by

$$(Xv,u)_U=(v,X^*u)_V\quad(u\in U,v\in V)\,.$$

Then $K(U)\times K(V)$ acts on $L^-(V,U)$ by $(g,h)X=gXh^{-1}$ $(X\in L^-(V,U),\ (g,h)\in K(U)\times K(V))$. For an element Y of $\mathfrak{gl}(V)$, we also consider the adjoint $Y^*\in\mathfrak{gl}(V)$ defined in (1.2). Then we easily see the following:

LEMMA 11. (1) $s_V^*=\omega_Vs_V$.

(2) *For an element* $X\in L(V,U)$, *we have* $(X^*)^*=\varepsilon_U\varepsilon_VX$. *In particular*, $(XX^*)^*=\varepsilon_U\varepsilon_VXX^*$ *and* $(X^*X)^*=\varepsilon_U\varepsilon_VX^*X$.

(3) *For an element* $X\in L^-(V,U)$, *we have* $s_UXX^*s_U=\omega_U\omega_VXX^*$ *and* $s_VX^*Xs_V=$

$\omega_U \omega_V X^* X$.

From now on, we suppose that $(\varepsilon_V, \omega_V) = (\varepsilon, \omega)$, $(\varepsilon_U, \omega_U) = (-\varepsilon, -\omega)$ and put $n_a := \dim V_a$, $n_b := \dim V_b$, $m_a := \dim U_a$, $m_b := \dim U_b$. By Lemma 11, we can define two morphisms

$$\mathfrak{p}(V) \xleftarrow{\rho} L^-(V, U) \xrightarrow{\pi} \mathfrak{p}(U), \quad \pi(X) = XX^*, \quad \rho(X) = X^* X .$$

Moreover, ρ (resp. π) is $K(V)$-equivariant (resp. $K(U)$-equivariant).

PROPOSITION 4. *Suppose that* $\min\{n_a, n_b\} \geq \max\{m_a, m_b\}$. *Then* π *is a surjective quotient map under* $K(V)$. *On the other hand,*

$$\operatorname{Im} \rho = \{X \in \mathfrak{p}(V)\,;\, \operatorname{rk}(X|_{V_a}) \leq m_b,\ \operatorname{rk}(X|_{V_b}) \leq m_a\}$$

and $\rho: L^-(V, U) \to \operatorname{Im} \rho$ *is a quotient map under* $K(U)$.

PROOF. The statements with respect to the images of π and ρ follow from elementary computation of matrices. π and ρ are quotient maps in view of Theorem 5. q.e.d.

(3.6) Proof of "cancelling columns" of Theorem 4, (2). Let V be an (ε, ω)-space with an involution s_V and a bilinear form $(\ ,\ )_V$. Let $D \in \mathfrak{p}(V)$ be a nilpotent element with an (ε, ω)-diagram η: $D \in C_\eta^{(\varepsilon,\omega)} \subset \mathfrak{p}(V)$. Put $U := \operatorname{Im} D \subset V$. Then s_V stabilizes U as before and so we can define an involution s_U of U by $s_U := s_V|_U$. Let us consider a bilinear form $|u, v| := (u, Dv)_V$ $(u, v \in V)$ on V. Since

$$|u, v| = (D^* u, v)_V = (-Du, v)_V = -\varepsilon(v, Du)_V = -\varepsilon(v, Du)_V = -\varepsilon|v, u| \quad (u, v \in V)$$

and the radical of $|\ ,\ |$ is precisely $\operatorname{Ker} D$, $|\ ,\ |$ induces a non-degenerate $-\varepsilon$-form $(\ ,\ )_U$ on $U = \operatorname{Im} D = V/\operatorname{Ker} D$:

$$(Du, Dv)_U = (u, Dv)_V \qquad (u, v \in V) .$$

Then we can easily verify that

$$(s_U Du, Dv)_U = -\omega(Du, s_U Dv)_U \qquad (u, v \in V) .$$

Hence U is a $(-\varepsilon, -\omega)$-space with respect to s_U and $(\ ,\ )_U$.

In this situation, we consider the morphisms ρ and π in (3.5). Let I: $U \subsetneq V \in L(U, V)$ be the inclusion and $D_0 := [D: V \to U] \in L(V, U)$. Then we have the following:

LEMMA 12. (1) $(D_0)^* = I$.
(2) $\rho(D_0) = D$, $\pi(D_0) = [D|_U: U \to U] \in C_{\eta'}^{(-\varepsilon, -\omega)} \subset \mathfrak{p}(U)$.

Let us put

$$L' := \{Y \in L^-(V, U)\,;\, \operatorname{rk} Y = m_a + m_b\} ,$$

$$\mathfrak{p}(V)' := \{X \in \mathfrak{p}(V)\,;\, \operatorname{rk}(X|_{V_a}) = m_b,\ \operatorname{rk}(X|_{V_b}) = m_a\} .$$

Then we have the following:

LEMMA 13. (1) *$\pi|_{L'}: L' \to \mathfrak{p}(U)$ is smooth.*

(2) *$\rho(L') = \mathfrak{p}(V)'$ and $\rho^{-1}(\rho(Y))$ is a single orbit under $K(U)$. Moreover $\rho|_{L'}: L' \to \mathfrak{p}(V)'$ is locally trivial in the classical topology with typical fibre $K(U)$.*

PROOF. The smoothness of $\pi|_{L'}: L' \to \mathfrak{p}(U)'$ (i.e., the surjectivity of $(d\pi)_Y: L^-(V, U) \to \mathfrak{p}(U)$, $P \mapsto PY^* + YP^*$ for any $Y \in L'$) and the fact $\rho(L') = \mathfrak{p}(V)'$ follow from elementary computation of matrices.

Let us prove the local triviality of $\rho|_{L'}$. The group $\tilde{K}(V) = GL(V_a) \times GL(V_b)$ acts on $L^-(V, U)$ (resp. $\mathfrak{p}(V)$) on the right by $(Y, g) \mapsto Yg$ (resp. $(X, g) \mapsto g^*Xg$). Clearly, $\rho: L^-(V, U) \to \mathfrak{p}(V)$ is $\tilde{K}(V)$-equivariant with respect to these actions. Moreover, we can verify that L' and $\mathfrak{p}(V)'$ are single orbits under $\tilde{K}(V)$ and hence $\rho|_{L'}: L' \to \mathfrak{p}(V)'$ is locally trivial.

Take $Y \in L'$ and $Z \in L^-(V, U)$ so that $\rho(Y) = \rho(Z)$ (i.e., $Y^*Y = Z^*Z$). Since Y is surjective and Y^* is injective, $\mathrm{rk}(Z^*Z) = \mathrm{rk}(Y^*Y) = m_a + m_b$. Hence Z is surjective and Z^* is injective: $Z \in L'$. Therefore $\mathrm{Ker}\, Y = \mathrm{Ker}(Y^*Y) = \mathrm{Ker}(Z^*Z) = \mathrm{Ker}\, Z$ and hence we can take an element $h \in \tilde{K}(U) = GL(U_a) \times GL(U_b)$ in such a way that $Y = hZ$. Then $Z^*Z = Y^*Y = Z^*h^*hZ$. Since Z^* is injective and Z is surjective, we have $h^*h = 1$, i.e., $h \in K(U)$. q.e.d.

LEMMA 14. *Let $C_\sigma^{(\varepsilon,\omega)} \subset \mathfrak{p}(V)$ be a nilpotent orbit with an (ε, ω)-diagram σ such that $\sigma \leq \eta$. Suppose that the first columns of η and σ coincide. Then we have the following:*

(1) *$\rho^{-1}(C_\sigma^{(\varepsilon,\omega)})$ is a single $K(U) \times K(V)$-orbit contained in L'.*

(2) *$\pi(\rho^{-1}(C_\sigma^{(\varepsilon,\omega)})) = C_{\sigma'}^{(-\varepsilon,-\omega)}$.*

(3) *Put $N_\eta := \pi^{-1}(\overline{C_{\eta'}^{(-\varepsilon,-\omega)}})$. Then $\rho(N_\eta) = \overline{C_\eta^{(\varepsilon,\omega)}}$.*

(4) *$\rho(L' \cap N_\eta) = \mathfrak{p}(V)' \cap \overline{C_\eta^{(\varepsilon,\omega)}}$.*

By using Lemma 13 and Lemma 14, one can easily deduce the proof of the "cancelling columns" of Theorem 4, (2) from that of Theorem 4, (1) in (3.4). The proofs of Lemma 14, (1), (2) are similar to those of Lemma 10, (1) (2). We can also prove (3) and (4) similarly, if we assume Theorem 3. But since we have not proved Lemma 7 which we need to prove Theorem 3 yet, let us give the proof of Lemma 7 here.

Put $N := \rho^{-1}(C_\eta^{(\varepsilon,\omega)})$. Since ρ is continuous, we have $\rho(\bar{N}) \subset \overline{\rho(N)}$. On the other hand, since $\bar{N}$ is a $K(U)$-invariant closed subset of $L^-(V, U)$ and ρ is a quotient map under $K(U)$, $\rho(\bar{N})$ is closed (cf. Remark 6) and hence $\rho(\bar{N}) = \overline{\rho(N)} = \overline{C_\eta^{(\varepsilon,\omega)}}$. Similarly, we have $\pi(\bar{N}) = \overline{\pi(N)} = \overline{C_{\eta'}^{(-\varepsilon,-\omega)}}$ by using Lemma 14, (2). Let σ be an (ε, ω)-diagram in Lemma 7. Since $\rho(\bar{N}) = \overline{C_\eta^{(\varepsilon,\omega)}} \supset C_\sigma^{(\varepsilon,\omega)}$ by assumption, there exists $Y \in \bar{N}$ such that $\rho(Y) \in C_\sigma^{(\varepsilon,\omega)}$. Again by Lemma 14, (2), we have $\pi(Y) \in C_{\sigma'}^{(-\varepsilon,-\omega)} \cap \pi(\bar{N}) \subset \overline{C_{\eta'}^{(-\varepsilon,-\omega)}}$. Hence $C_{\sigma'}^{(-\varepsilon,-\omega)} \subset \overline{C_{\eta'}^{(-\varepsilon,-\omega)}}$. Thus Lemma 7 is proved.

(3.7) Proof of "cancelling rows" of Theorem 4, (1). To prove the remaining part of theorem 4, we need the following concept. Let V be a vector space with a linear action of an algebraic group G and X a closed G-invariant subvariety of V. Let N be a subspace of V complementary to the tangent space $T_x(Gx)\subset V$ for a point $x\in X$. Put $S:=(N+x)\cap X$. Then the map $G\times S\to X$, $(g, s)\mapsto gs$ is smooth at (e, x) and hence $\mathrm{Sing}(X, x)=\mathrm{Sing}(S, x)$. S is called a cross section of X at a point $x\in X$.

REMARK 8. In the above setting, if X is irreducible or equidimensional, then we have $\dim_x S=\mathrm{codim}(X, Gx)$ (cf. [KP3, 12.4]).

Let us give some remarks on the connection of ab-diagrams and Young diagrams.

REMARK 9. (1) For an ab-diagram ν, let us denote by $Y(\nu)$ the Young diagram which we obtain by replacing a and b by the block $\square$. Then for a nilpotet element $x_\nu\in\tilde{\mathfrak{p}}(V)$ with an ab-diagram ν, $\mathrm{Ad}(GL(V))x_\nu$ is just the nilpotent orbit in $\mathfrak{gl}(V)$ corresponding to the Young diagram $Y(\nu)$.

(2) If $\nu\leq\mu$ is a degeneration of ab-diagrams, then clearly we have $Y(\nu)\leq Y(\mu)$ (for the definition of the ordering of Young diagrams, see [KP1]).

(3) Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair defined by (G, θ) and x an element of the associated vector space $\mathfrak{p}$. Then we have

$$\dim\mathfrak{k}^x-\dim\mathfrak{p}^x=\dim\mathfrak{k}-\dim\mathfrak{p}$$

by [KR, Proposition 5], where $\mathfrak{k}^x$ and $\mathfrak{p}^x$ are the centralizers of x in $\mathfrak{k}$ and $\mathfrak{p}$, respectively. It follows from the above equality that $\dim\mathrm{Ad}(G)x=2\dim\mathrm{Ad}(K_\theta)x$. In particular, in the setting of (1), we have $\dim\mathrm{Ad}(GL(V))x_\nu=2\dim C_\nu$.

Now let us give the proof of the "cancelling rows" of Theorem 4, (1). Let V be a vector space with an involution s and C_η (resp. C_σ) be a nilpotent $\tilde{K}(V)$-orbit in $\tilde{\mathfrak{p}}(V)$ with an ab-diagram η (resp. σ) such that $\sigma\leq\eta$. Moreover, we suppose that the first k rows of η and σ coincide. Let ν be the ab-diagram which consists of the coincident k rows and $\bar{\eta}$ (resp. $\bar{\sigma}$) the ab-diagram which we obtain by erasing ν from η (resp. σ): $\eta=\nu+\bar{\eta}$, $\sigma=\nu+\bar{\sigma}$. Let us denote by σ_i the i-th row of σ: $\nu=\sum_{i=1}^k\sigma_i$, $\bar{\sigma}=\sum_{k<i}\sigma_i$. For $x_\sigma\in C_\sigma$, we can take an x_σ-stable and s-stable direct sum decomposition $V=\bigoplus_{i=1}^r V_{\sigma_i}$ such that the ab-diagram of $x_\sigma|_{V_{\sigma_i}}\in\tilde{\mathfrak{p}}(V_{\sigma_i})$ is σ_i, where r is the number of rows of σ. Put $V_\nu:=\bigoplus_{i=1}^k V_{\sigma_i}$ and $V_{\bar{\sigma}}:=\bigoplus_{i=k+1}^r V_{\sigma_i}$. Then V_ν and $V_{\bar{\sigma}}$ are also vector spaces with involutions $s|_{V_\nu}$ and $s|_{V_{\bar{\sigma}}}$, respectively. Moreover, x_σ is decomposed as $x_\sigma=(x_\nu, x_{\bar{\sigma}})\in\tilde{\mathfrak{p}}(V_\nu)\oplus\tilde{\mathfrak{p}}(V_{\bar{\sigma}})$ with $x_\nu\in C_\nu$ and $x_{\bar{\sigma}}\in C_{\bar{\sigma}}$. Take $x_{\bar{\eta}}\in C_{\bar{\eta}}\subset\tilde{\mathfrak{p}}(V_{\bar{\sigma}})$ and put $x_\eta:=(x_\nu, x_{\bar{\eta}})\in C_\eta$.

Let us construct four cross sections of the closures of the orbits of x_η at x_σ. First we put

$$Y:=\{A\in\tilde{\mathfrak{p}}(V); AV_\nu\subset V_{\bar{\sigma}}, AV_{\bar{\sigma}}\subset V_\nu\},\ X:=\{A\in\tilde{\mathfrak{k}}(V); AV_\nu\subset V_{\bar{\sigma}}, AV_{\bar{\sigma}}\subset V_\nu\}.$$

Then we have the following:

$$\tilde{\mathfrak{p}}(V)=\tilde{\mathfrak{p}}(V_\nu)\oplus\tilde{\mathfrak{p}}(V_{\bar{\sigma}})\oplus Y, \qquad \tilde{\mathfrak{k}}(V)=\tilde{\mathfrak{k}}(V_\nu)\oplus\tilde{\mathfrak{k}}(V_{\bar{\sigma}})\oplus X,$$

$$[x_\sigma, \tilde{\mathfrak{k}}(V_\nu)\oplus\tilde{\mathfrak{k}}(V_{\bar\sigma})]\subset\tilde{\mathfrak{p}}(V_\nu)\oplus\tilde{\mathfrak{p}}(V_{\bar\sigma}), \qquad [x_\sigma, X]\subset Y,$$
$$[x_\sigma, \tilde{\mathfrak{p}}(V_\nu)\oplus\tilde{\mathfrak{p}}(V_{\bar\sigma})]\subset\tilde{\mathfrak{k}}(V_\nu)\oplus\tilde{\mathfrak{k}}(V_{\bar\sigma}), \qquad [x_\sigma, Y]\subset X.$$

Take subspaces N_1, N_2, N_3, N_4 of $\mathfrak{gl}(V)$ such that

$$\tilde{\mathfrak{p}}(V_\nu)\oplus\tilde{\mathfrak{p}}(V_{\bar\sigma})=[x_\sigma, \tilde{\mathfrak{k}}(V_\nu)\oplus\tilde{\mathfrak{k}}(V_{\bar\sigma})]\oplus N_1, \qquad Y=[x_\sigma, X]\oplus N_3,$$
$$\tilde{\mathfrak{k}}(V_\nu)\oplus\tilde{\mathfrak{k}}(V_{\bar\sigma})=[x_\sigma, \tilde{\mathfrak{p}}(V_\nu)\oplus\tilde{\mathfrak{p}}(V_{\bar\sigma})]\oplus N_2, \qquad X=[x_\sigma, Y]\oplus N_4$$

and put $N:=N_1\oplus N_2\oplus N_3\oplus N_4$, $N':=N_1\oplus N_2$, $N_0:=N_1\oplus N_3$, $N'_0:=N_1$,

$$S:=(N+x_\sigma)\cap\overline{\{\mathrm{Ad}(GL(V)x_\eta\}}, \qquad S':=(N'+x_\sigma)\cap\overline{\{\mathrm{Ad}(GL(V_\nu)\times GL(V_{\bar\sigma}))x_\eta\}},$$
$$S_0:=(N_0+x_\sigma)\cap\overline{\{\mathrm{Ad}(\tilde K(V))x_\eta\}}, \qquad S'_0:=(N'_0+x_\sigma)\cap\overline{\{\mathrm{Ad}(\tilde K(V_\nu)\times\tilde K(V_{\bar\sigma}))x_\eta\}}.$$

Then S (resp. S', resp. S_0, resp. S'_0) is a cross section of the closure of the orbit under $GL(V)$ (resp. $GL(V_\nu)\times GL(V_{\bar\sigma})$, resp. $\tilde K(V)$, resp. $\tilde K(V_\nu)\times\tilde K(V_{\bar\sigma})$) of x_η at x_σ. By Remark 8, we have

$$\dim_{x_\sigma}S=\mathrm{codim}(\overline{\{\mathrm{Ad}(GL(V))x_\eta\}}, \ \mathrm{Ad}(GL(V))x_\sigma),$$
$$\dim_{x_\sigma}S'=\mathrm{codim}(\overline{\{\mathrm{Ad}(GL(V_\nu)\times GL(V_{\bar\sigma}))x_\eta\}}, \ \mathrm{Ad}(GL(V_\nu)\times GL(V_{\bar\sigma}))x_\sigma)$$
$$=\mathrm{codim}(\overline{\{\mathrm{Ad}(GL(V_{\bar\sigma}))x_{\bar\eta}\}}, \ \mathrm{Ad}(GL(V_{\bar\sigma}))x_{\bar\sigma}).$$

Then by [KP2, Proposition 3.1], we have $\dim_{x_\sigma}S=\dim_{x_\sigma}S'$. By the normality of the closures of $GL(V)$-orbits in $\mathfrak{gl}(V)$ ([KP1]), $\overline{\{\mathrm{Ad}(GL(V))x_\eta\}}$ is normal at x_σ. But since $\mathrm{Sing}(S, x_\sigma)=\mathrm{Sing}(\overline{\{\mathrm{Ad}(GL(V))x_\eta\}}, x_\sigma)$, S is normal at x_σ (cf. Remark 5). Since S' is a closed subset of S, S' and S coincide in a suitable neighbourhood of x_σ. By the closure relation of nilpotent $GL(V)$-orbits in $\mathfrak{gl}(V)$ (cf. [KP1]), we have

$$S'\cap\tilde{\mathfrak{p}}(V)=(N'_0+x_\sigma)\cap[(\tilde{\mathfrak{p}}(V_\nu)\cap\overline{\{\mathrm{Ad}(GL(V_\nu))x_\nu\}})\times(\tilde{\mathfrak{p}}(V_{\bar\sigma})\cap\overline{\{\mathrm{Ad}(GL(V_{\bar\sigma}))x_{\bar\eta}\}})]$$
$$=(N'_0+x_\sigma)\cap\left[\left(\bigcup_{\mu_1\in D(\le Y(\nu))}C_{\mu_1}\right)\times\left(\bigcup_{\mu_2\in D(\le Y(\bar\eta))}C_{\mu_2}\right)\right],$$

where we write $D(\le Y(\nu)):=\{\mu_1\in D(n_a(\nu), n_b(\nu)); Y(\mu_1)\le Y(\nu)\}$. To show that $S'_0=(N'_0+x_\sigma)\cap(\bar C_\nu\times\bar C_{\bar\eta})$ and S_0 coincide in a suitable neighbourhood of x_σ, we need the following lemma:

LEMMA 15. *In the above setting, $(N'_0+x_\sigma)\cap(C_\nu\times\bar C_{\bar\eta})$ is open in $S'\cap\tilde{\mathfrak{p}}(V)\cap\bar C_\eta=S'\cap\bar C_\eta$.*

PROOF. Put $W:=\{(\bigcup_{\mu_1\in D(\le Y(\nu))}C_{\mu_1})\times(\bigcup_{\mu_2\in D(\le Y(\bar\eta))}C_{\mu_2})\}\cap\bar C_\eta$. We consider the projection

$$p_1: W\to\left(\bigcup_{\mu_1\in D(\le Y(\nu))}C_{\mu_1}\right)=:W_1, \qquad p_1(y_1, y_2)=y_1.$$

Since $W_1 \subset \tilde{\mathfrak{p}}(V_\nu)$ is a finite union of the closures of $\tilde{K}(V_\nu)$-orbits and $\bar{C}_\nu$ is an irreducible component of W_1 of the maximum dimension (cf. Remark 9, (3)), C_ν is open in W_1.

q.e.d.

By Lemma 15, there exists a neighbourhood U of x_σ in S such that $S'_0 \cap U = S' \cap \bar{C}_\eta \cap U$. Then we have $S \cap \bar{C}_\eta \cap U \supset S_0 \cap U \supset S'_0 \cap U = S' \cap \bar{C}_\eta \cap U$. Since S and S' coincide in a suitable neighbourhood of x_σ, S_0 and S'_0 also coincide in a suitable neighbourhood of x_σ, S_0 and S'_0 also coincide in a suitable neighbourhood of x_σ. Therefore we have

$$\mathrm{Sing}(\bar{C}_\eta, C_\sigma) = \mathrm{Sing}(S_0, x_\sigma) = \mathrm{Sing}(S'_0, x_\sigma) = \mathrm{Sing}(\bar{C}_\nu \times \bar{C}_{\bar{\eta}}, (x_\nu, x_{\bar{\sigma}})) .$$

Since C_ν is smooth at x_ν, we have

$$\mathrm{Sing}(\bar{C}_\eta, C_\sigma) = \mathrm{Sing}(\bar{C}_{\bar{\eta}}, x_{\bar{\sigma}}) = \mathrm{Sing}(\bar{C}_{\bar{\eta}}, C_{\bar{\sigma}}) .$$

Thus the proof of Theorem 4, (1) is completed.

(3.8) Proof of the "cancelling rows" of Theorem 4, (2). For an (ε, ω)-space V with an involution s and a bilinear form $(\, ,)$, we put

$$\mathfrak{q}^+(V) := \{X \in \tilde{\mathfrak{k}}(V); X^* = X\} , \quad \mathfrak{q}^-(V) := \{X \in \tilde{\mathfrak{p}}(V); X^* = X\} .$$

Then we have

$$\tilde{\mathfrak{k}}(V) = \mathfrak{k}(V) \oplus \mathfrak{q}^+(V) , \quad \tilde{\mathfrak{p}}(V) = \mathfrak{p}(V) \oplus \mathfrak{q}^-(V) ,$$

$$\mathfrak{gl}(V) = \mathfrak{k}(V) \oplus \mathfrak{q}^+(V) \oplus \mathfrak{p}(V) \oplus \mathfrak{q}^-(V) , \quad [\mathfrak{p}(V), \mathfrak{q}^+(V)] \subset \mathfrak{q}^-(V) .$$

Now let us prove the "cancelling rows" of Theorem 4, (2). Let V be an (ε, ω)-space and $C_\eta^{(\varepsilon,\omega)}$ (resp. $C_\sigma^{(\varepsilon,\omega)}$) be a nilpotent $K(V)$-orbit in $\mathfrak{p}(V)$ with an (ε, ω)-diagram η (resp. σ) such that $\sigma \leq \eta$. Moreover, we suppose that the first k rows of η and σ coincide and that the sum ν of the coincident k rows is also an (ε, ω)-diagram. Let us denote by $\bar{\sigma} \leq \bar{\eta}$ the (ε, ω)-degeneration which we obtain by erasing ν from $\sigma \leq \eta$: $\eta = \nu + \bar{\eta}$, $\sigma = \nu + \bar{\sigma}$. Let us decompose σ as $\sigma = \sum_{i=1}^{r'} \sigma_i$ so that $\nu = \sum_{i=1}^{k'} \sigma_i$ and $\bar{\sigma} = \sum_{i=k'+1}^{r'} \sigma_i$, where σ_i $(1 \leq i \leq r')$ are primitive (ε, ω)-diagrams.

Take $x_\sigma \in C_\sigma^{(\varepsilon,\omega)}$. Then by the proof of [O2, Proposition 2], we can take an x_σ-stable, s-stable and $(\, ,)$-orthogonal direct sum decomposition $V = \bigoplus_{i=1}^{r'} V_{\sigma_i}$ (therefore each V_{σ_i} is also an (ε, ω)-space with respect to the restrictions of s and $(\, ,)$) such that the (ε, ω)-diagram of $x_\sigma|_{V_{\sigma_i}} \in \mathfrak{p}(V_{\sigma_i})$ is σ_i. Put $V_\nu := \bigoplus_{i=1}^{k'} V_{\sigma_i}$ and $V_{\bar{\sigma}} := \bigoplus_{i=k'+1}^{r'} V_{\sigma_i}$. Then V_ν and $V_{\bar{\sigma}}$ are also (ε, ω)-spaces and x_σ is decomposed as $x_\sigma = (x_\nu, x_{\bar{\sigma}}) \in \mathfrak{p}(V_\nu) \oplus \mathfrak{p}(V_{\bar{\sigma}})$ with $x_\nu \in C_\nu^{(\varepsilon,\omega)}$ and $x_{\bar{\sigma}} \in C_{\bar{\sigma}}^{(\varepsilon,\omega)}$. We denote by X_1 (resp. Y_1, resp. X_2, resp. Y_2) the subspace of $\mathfrak{k}(V)$ (resp. $\mathfrak{p}(V)$, resp. $\mathfrak{q}^+(V)$, resp. $\mathfrak{q}^-(V)$) consisting of endmorphisms A such that $AV_\nu \subset V_{\bar{\sigma}}$, $AV_{\bar{\sigma}} \subset V_\nu$. Then we have the following:

$$\mathfrak{k}(V) = \mathfrak{k}(V_\nu) \oplus \mathfrak{k}(V_{\bar{\sigma}}) \oplus X_1 , \qquad \mathfrak{p}(V) = \mathfrak{p}(V_\nu) \oplus \mathfrak{p}(V_{\bar{\sigma}}) \oplus Y_1 ,$$

$$\mathfrak{q}^+(V) = \mathfrak{q}^+(V_\nu) \oplus \mathfrak{q}^+(V_{\bar{\sigma}}) \oplus X_2 , \quad \mathfrak{q}^-(V) = \mathfrak{q}^-(V_\nu) \oplus \mathfrak{q}^-(V_{\bar{\sigma}}) \oplus Y_2 ,$$

$$\tilde{\mathfrak{k}}(V)=\tilde{\mathfrak{k}}(V_\nu)\oplus\tilde{\mathfrak{k}}(V_{\bar\sigma})\oplus(X_1\oplus X_2)\,,\quad \tilde{\mathfrak{p}}(V)=\mathfrak{p}(V_\nu)\oplus\mathfrak{p}(V_{\bar\sigma})\oplus(Y_1\oplus Y_2)\,,$$

$$[x_\sigma, X_1]\subset Y_1\,,\quad [x_\sigma, X_2]\subset Y_2\,.$$

Take subspaces $N_1^+, N_1^-, N_3^+, N_3^-$ such that

$$\mathfrak{p}(V_\nu)\oplus\mathfrak{p}(V_{\bar\sigma})=[\mathfrak{k}(V_\nu)\oplus\mathfrak{k}(V_{\bar\sigma}), x_\sigma]\oplus N_1^+\,,$$

$$\mathfrak{q}^-(V_\nu)\oplus\mathfrak{q}^-(V_{\bar\sigma})=[\mathfrak{q}^+(V_\nu)\oplus\mathfrak{q}^+(V_{\bar\sigma}), x_\sigma]\oplus N_1^-\,,$$

$$Y_1=[X_1, x_\sigma]\oplus N_3^+\,,\quad Y_2=[X_2, x_\sigma]\oplus N_3^-$$

and put $N_0:=N_1^+\oplus N_1^-\oplus N_3^+\oplus N_3^-$, $N'_0:=N_1^+\oplus N_1^-$, $N_0^+:=N_1^+\oplus N_3^-$, $N_0^{+\prime}:=N_1^+$. Then we have

$$\tilde{\mathfrak{p}}(V)=[\tilde{\mathfrak{k}}(V), x_\sigma]\oplus N_0\,,\qquad \tilde{\mathfrak{p}}(V_\nu)\oplus\tilde{\mathfrak{p}}(V_{\bar\sigma})=[\tilde{\mathfrak{k}}(V_\nu)\oplus\tilde{\mathfrak{k}}(V_{\bar\sigma}), x_\sigma]\oplus N'_0\,,$$

$$\mathfrak{p}(V)=[\mathfrak{k}(V), x_\sigma]\oplus N_0^+\,,\qquad \mathfrak{p}(V_\nu)\oplus\mathfrak{p}(V_{\bar\sigma})=[\mathfrak{k}(V_\nu)\oplus\mathfrak{k}(V_{\bar\sigma}), x_\sigma]\oplus N_0^{+\prime}$$

Take $x_{\bar\eta}\in C_{\bar\eta}^{(\varepsilon,\omega)}\subset\mathfrak{p}(V_{\bar\sigma})$ and put $x_\eta:=(x_\nu, x_{\bar\sigma})\in C_\eta^{(\varepsilon,\omega)}$,

$$S_0:=(N_0+x_\sigma)\cap\overline{\{\mathrm{Ad}(\tilde K(V))x_\eta\}}\,,\qquad S'_0:=(N'_0+x_\sigma)\cap\overline{\{\mathrm{Ad}(\tilde K(V_\nu)\times\tilde K(V_{\bar\sigma}))x_\eta\}}\,,$$

$$S_0^+:=(N_0^++x_\sigma)\cap\overline{\{\mathrm{Ad}(K(V))x_\eta\}}\,,\quad S_0^{+\prime}:=(N_0^{+\prime}+x_\sigma)\cap\overline{\{\mathrm{Ad}(K(V_\nu)\times K(V_{\bar\sigma}))x_\eta\}}\,.$$

Then S_0 (resp. S'_0, resp. S_0^+, resp. $S_0^{+\prime}$) is a cross section of $\bar C_\eta$ (resp. $\bar C_\nu\times\bar C_{\bar\sigma}$, resp. $\overline{C_\eta^{(\varepsilon,\omega)}}$, resp. $\overline{C_\nu^{(\varepsilon,\omega)}}\times\overline{C_{\bar\sigma}^{(\varepsilon,\omega)}}$) at x_σ.

Here we note that S_0 and S'_0 are constructed in the same manner as those in (3.7). Therefore S_0 and S'_0 coincide in a suitable neighbourhood of x_σ. By Theorem 3, we have

$$S'_0\cap\mathfrak{p}(V)=(N_0^{+\prime}+x_\sigma)\cap\{(\mathfrak{p}(V_\nu)\cap\bar C_\nu)\times(\mathfrak{p}(V_{\bar\sigma})\cap\bar C_{\bar\eta})\}=(N_0^{+\prime}+x_\sigma)\cap(\overline{C_\nu^{(\varepsilon,\omega)}}\times\overline{C_{\bar\eta}^{(\varepsilon,\omega)}})=S_0^{+\prime}$$

and hence $S_0\cap\mathfrak{p}(V)\supset S_0^+\supset S_0^{+\prime}=S'_0\cap\mathfrak{p}(V)$. Therefore S_0^+ and $S_0^{+\prime}$ also coincide in a suitable neighbourhood of x_σ. Hence we have

$$\mathrm{Sing}(\overline{C_\eta^{(\varepsilon,\omega)}}, C_\sigma^{(\varepsilon,\omega)})=\mathrm{Sing}(S_0^+, x_\sigma)=\mathrm{Sing}(S_\sigma^{+\prime}, x_\sigma)$$

$$=\mathrm{Sing}(\overline{C_\nu^{(\varepsilon,\omega)}}\times\overline{C_{\bar\sigma}^{(\varepsilon,\omega)}}, (x_\nu, x_{\bar\sigma}))=\mathrm{Sing}(\overline{C_{\bar\eta}^{(\varepsilon,\omega)}}, C_{\bar\sigma}^{(\varepsilon,\omega)})\,,$$

where the last equality follows from the smoothness of $\overline{C_\nu^{(\varepsilon,\omega)}}$ at x_ν. Therefore the proof of Theorem 4 is completed.

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