Tôhoku Math. J.
43 (1991), 361–374

CURVATURE PINCHING THEOREM FOR MINIMAL SURFACES WITH CONSTANT KAEHLER ANGLE IN COMPLEX PROJECTIVE SPACES

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(Received May 29, 1990, revised January 10, 1991)

Introduction. Let X be a complex space form with the complex structure J and the standard Kaehler metric $\langle\, ,\rangle$, M be an oriented 2-dimensional Riemannian manifold and $x: M \to X$ be an isometric minimal immersion of M into X. Then the Kaehler angle α of x, which is an invariant of the immersion x related to J, is defined by $\cos(\alpha) = \langle Je_1, e_2\rangle$, where $\{e_1, e_2\}$ is an orthonormal basis of M. The Kaehler angle gives a measure of the failure of x to be a holomorphic map. Indeed x is holomorphic if and only if $\alpha = 0$ on M, while x is anti-holomorphic if and only if $\alpha = \pi$ on M. In [4], Chern and Wolfson pointed out that the Kaehler angle of x plays an important role in the study of minimal surfaces in X. From this point of view, we would like to know all isometric minimal immersions of constant Kaehler angle in X.

In this paper, we shall mainly discuss this problem when X is a complex space form of positive constant holomorphic sectional curvature. So, let $P^n(\boldsymbol{C})$ be the complex projective space with the Fubini-Study metric of constnat holomorphic sectional curvature 4ρ. Let $S^2(K)$ be a 2-dimensional sphere of constant Gaussian curvature K. Examples of minimal surfaces of constant Kaehler angle in $P^n(\boldsymbol{C})$, are given in [1] and [2]: For each integer p with $0 \le p \le n$, there exist full isometric minimal immersions $\varphi_{n,p}: S^2(K_{n,p}) \to P^n(\boldsymbol{C})$, where $K_{n,p} = 4\rho/(n + 2p(n-p))$. Each $\varphi_{n,p}$ possesses holomorphic rigidity, that is to say, such two immersions differ by a holomorphic isometry of $P^n(\boldsymbol{C})$. The Kaehler angle $\alpha_{n,p}$ of $\varphi_{n,p}$ is given by $\cos(\alpha_{n,p}) = (n-2p)/(n+2p(n-p))$. Note that $K_{n,p} = 2\rho(1-(2p+1)\cos(\alpha_{n,p}))/p(p+1)$.

Characterizing minimal surfaces of constant Kaehler angle in $P^n(\boldsymbol{C})$, Ohnita [10] recently gave the following theorem: Let $\varphi: M \to P^n(\boldsymbol{C})$ be a full isometric minimal immersion of a 2-dimensional Riemannian manifold M into $P^n(\boldsymbol{C})$. Assume that the Gaussian curvature K of M and the Kaehler angle α of φ are both constant on M. Then the following hold.

(1) If $K > 0$, then there exists some p with $0 \le p \le n$ such that $K = 4\rho/(n + 2p(n-p))$, $\cos(\alpha) = (n-2p)/(n+2p(n-p))$ and $\varphi(M)$ is an open submanifold of $\varphi_{n,p}(S^2(K))$.

(2) If $K = 0$, then $\cos(\alpha) = 0$, that is to say, φ is totally real. Such φ's were already classified by Kenmotsu [6].

(3) The case of $K < 0$ is impossible.

In [10], Ohnita conjectured that the theorem will hold without the assumption that the Kaehler angle is constant. On the other hand, Bolton et al. [2] conjectured that, if the Kaehler angle of an isometric minimal immersion $x\colon M\to P^n(C)$ is constant, then the Gaussian curvature of x is also constant, when the immersion is neither holomorphic, anti-holomorphic nor totally real. They gave an affirmative answer to this conjecture for $n\leq 4$. We would like to discuss this conjecture under some additional conditions. We prove the following:

THEOREM. *Let X be a Kaehler manifold of complex dimension n of positive constant holomorphic sectional curvature 4ρ and M be a complete connected Riemannian 2-manifold. Let $x\colon M\to X$ be a full isometric minimal immersion with constant Kaehler angle α, which is neither holomorphic, anti-holomorphic nor totally real. If the J-invariant first osculating space of x is of constant dimension on M and the Gaussian curvature K of M satisfies $K\geq(1-7\cos(\alpha))\rho/6>0$ on M, then K is constant on M. Moreover, x is locally congruent to either $\varphi_{n,1}$, $\varphi_{n,2}$, or $\varphi_{n,3}$.*

COROLLARY. *Let $x\colon M\to X$ be a full isometric minimal immersion with constant Kaehler angle α, which is neither holomorphic, anti-holomorphic nor totally real. If the Gaussian curvature K of M satisfies $(1-5\cos(\alpha))\rho/3>K\geq(1-7\cos(\alpha))\rho/6$, then x is locally congruent to $\varphi_{n,3}$.*

The author would like to express particular thanks to Professor K. Kenmotsu for his advice and encouragement during the development of this work.

1. Preliminaries. Let X be a Kaehler manifold of complex dimension n of constant holomorphic sectional curvature 4ρ, and $\{\omega_\alpha\}$ be a local field of unitary coframes on X so that the metric is represented by $ds^2=\sum\omega_\alpha\bar{\omega}_\alpha$, where $\alpha, \beta, \gamma, \cdots$ run from 1 through n. We denote by $\{\omega_{\alpha\beta}\}$ the unitary connection forms with respect to $\{\omega_\alpha\}$. Then we have,

$$d\omega_\alpha=\sum\omega_{\alpha\beta}\wedge\omega_\beta\,,\qquad \omega_{\alpha\beta}+\bar{\omega}_{\beta\alpha}=0\,, \tag{1.1}$$

$$d\omega_{\alpha\beta}=\sum\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}+\Omega_{\alpha\beta}\,, \tag{1.2}$$

$$\Omega_{\alpha\beta}=-\rho(\omega_\alpha\wedge\bar{\omega}_\beta+\delta_{\alpha\beta}\sum\omega_\gamma\wedge\bar{\omega}_\gamma)\,. \tag{1.3}$$

We set $\omega_\alpha=\theta_{2\alpha-1}+i\theta_{2\alpha}$, $\omega_{\alpha\beta}=\theta_{2\alpha-1,2\beta-1}+i\theta_{2\alpha,2\beta-1}$. Then $\{\theta_{2\alpha-1}, \theta_{2\alpha}\}$ is a canonical 1-form of the underlying Riemannian structure of X and $\{\theta_{2\alpha-1,2\beta-1}, \theta_{2\alpha,2\beta-1}\}$ is the Riemannian connection form with respect to $\{\theta_{2\alpha-1}, \theta_{2\alpha}\}$. Let $\{e_{2\alpha-1}, e_{2\alpha}\}$ be the dual frame of $\{\theta_{2\alpha-1}, \theta_{2\alpha}\}$. Then it is an orthonormal frame with $Je_{2\alpha-1}=e_{2\alpha}$. Such a frame is called a *J-canonical frame.*

Let U be a neighbourhood of a point of X. We choose and fix a local orthonormal system $\{\tilde{e}_1, \tilde{e}_2\}$ of vector fields on U which may not be a J-canonical frame. Generalizing the notion of the Kaehler angle of an immersion x, we use the same notation α defined by $\cos(\alpha)=\langle J\tilde{e}_1, \tilde{e}_2\rangle$. We denote by O_p^1 the subspace of the tangent space T_pX spanned by $\tilde{e}_1$, $\tilde{e}_2$, $J\tilde{e}_1$ and $J\tilde{e}_2$. If $\cos^2(\alpha)\neq 1$ on U, then the dimension of O_p^1 is equal to 4 for each $p\in U$. Let N_p^1 be the orthogonal complement of O_p^1 in T_pX so that $T_pX=O_p^1+N_p^1$. Since O_p^1 and N_p^1 are J-invariant subspaces of T_pX, we can define vectors $\tilde{e}_3$, $\tilde{e}_4$, e_1, e_2, e_3 and e_4 as follows:

$$
\begin{aligned}
&\tilde{e}_3=-\cot(\alpha)\tilde{e}_1-\operatorname{cosec}(\alpha)J\tilde{e}_2\,, \quad \tilde{e}_4=\operatorname{cosec}(\alpha)J\tilde{e}_1-\cot(\alpha)\tilde{e}_2\,,\\
&e_1=\cos\left(\frac{\alpha}{2}\right)\tilde{e}_1+\sin\left(\frac{\alpha}{2}\right)\tilde{e}_3\,, \quad e_2=\cos\left(\frac{\alpha}{2}\right)\tilde{e}_2+\sin\left(\frac{\alpha}{2}\right)\tilde{e}_4\,,\\
&e_3=\sin\left(\frac{\alpha}{2}\right)\tilde{e}_1-\cos\left(\frac{\alpha}{2}\right)\tilde{e}_3\,, \quad e_4=-\sin\left(\frac{\alpha}{2}\right)\tilde{e}_2+\cos\left(\frac{\alpha}{2}\right)\tilde{e}_4\,.
\end{aligned}
\tag{1.4}
$$

$\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ is an orthonormal basis of O_p^1 and $\{e_1, e_2, e_3, e_4\}$ is a J-canonical basis of O_p^1 for $p\in U$. This shows that starting from any orthonormal system $\{\tilde{e}_1, \tilde{e}_2\}$ of vectors satisfying $\langle J\tilde{e}_1, \tilde{e}_2\rangle\neq\pm 1$ on U, we can construct a 4-dimensional subspace O_p^1 of T_pX generated by $\{\tilde{e}_1, \tilde{e}_2, J\tilde{e}_1, J\tilde{e}_2\}$ which has a J-canonical basis $\{e_1, e_2, e_3, e_4\}$. Let $\{\tilde{e}_A\}$ be a local orthonormal frame on X which extends $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$, where A runs from 1 through $2n$. Let $\{\tilde{\theta}_A\}$ denote its dual frame. Then $\{e_1, e_2, e_3, e_4; \tilde{e}_\lambda, \lambda\geq 5\}$ is a local orthonormal frame such that $\{e_1, e_2, e_3, e_4\}$ is J-canonical. Putting $\omega_\alpha=\theta_{2\alpha-1}+i\theta_{2\alpha}$, we have, by (1.4),

$$
\begin{aligned}
&\tilde{\theta}_1+i\tilde{\theta}_2=\cos\left(\frac{\alpha}{2}\right)\omega_1+\sin\left(\frac{\alpha}{2}\right)\bar{\omega}_2\,,\\
&\tilde{\theta}_3+i\tilde{\theta}_4=\sin\left(\frac{\alpha}{2}\right)\omega_1-\cos\left(\frac{\alpha}{2}\right)\bar{\omega}_2\,,\\
&\tilde{\theta}_{2\lambda-1}+i\tilde{\theta}_{2\lambda}=\omega_\lambda \qquad (\lambda\geq 3)\,.
\end{aligned}
\tag{1.5}
$$

We set $\cos(\beta)=\langle J\tilde{e}_5, \tilde{e}_6\rangle$. If $\cos^2(\beta)\neq 1$ on an open subset U' of U, then in the same way as above the subspace N_p^1 has a splitting with respect to the $\{\tilde{e}_5, \tilde{e}_6\}$ such that $N_p^1=O_p^2+N_p^2$, $p\in U'$, O_p^2 is a J-invariant 4-dimensional subspace of N_p^1 spanned by $\{\tilde{e}_5, \tilde{e}_6, J\tilde{e}_5, J\tilde{e}_6\}$ and N_p^2 is its orthogonal complement in N_p^1. Then we have an orthonormal basis $\{\tilde{e}_5, \tilde{e}_6, \tilde{e}_7, \tilde{e}_8\}$ and a J-canonical basis $\{e_5, e_6, e_7, e_8\}$ of O_p^2 over U'. Let $\{e_{2\lambda-1}, e_{2\lambda}\}$ $(\lambda\geq 5)$ be a J-canonical basis of N^2 over U and put $\tilde{e}_{2\lambda-1}=e_{2\lambda-1}$ and $\tilde{e}_{2\lambda}=e_{2\lambda}$ for $\lambda\geq 5$. Let $\{\tilde{\theta}_{2\alpha-1}, \tilde{\theta}_{2\alpha}\}$ and $\{\theta_{2\alpha-1}, \theta_{2\alpha}\}$ be dual coframes of $\{\tilde{e}_{2\alpha-1}, \tilde{e}_{2\alpha}\}$ and $\{e_{2\alpha-1}, e_{2\alpha}\}$, respectively, over U. Putting $\omega_\alpha=\theta_{2\alpha-1}+i\theta_{2\alpha}$, we have the following relations, by (1.4):

$$
\begin{aligned}
&\tilde{\theta}_1+i\tilde{\theta}_2=\cos\left(\frac{\alpha}{2}\right)\omega_1+\sin\left(\frac{\alpha}{2}\right)\bar{\omega}_2\,,\\
&\tilde{\theta}_3+i\tilde{\theta}_4=\sin\left(\frac{\alpha}{2}\right)\omega_1-\cos\left(\frac{\alpha}{2}\right)\bar{\omega}_2\,,
\end{aligned}
$$

$$\tag{1.6}\begin{aligned}\tilde{\theta}_5+i\tilde{\theta}_6&=\cos\left(\frac{\beta}{2}\right)\omega_3+\sin\left(\frac{\beta}{2}\right)\bar{\omega}_4\,,\\ \tilde{\theta}_7+i\tilde{\theta}_8&=\sin\left(\frac{\beta}{2}\right)\omega_3-\cos\left(\frac{\beta}{2}\right)\bar{\omega}_4\,,\\ \tilde{\theta}_{2\lambda-1}+i\tilde{\theta}_{2\lambda}&=\omega_\lambda\,,\qquad (\lambda\geq 5)\,.\end{aligned}$$

Let $\{\tilde{\theta}_{2\alpha-1,2\beta-1}, \tilde{\theta}_{2\alpha-1,2\alpha}, \tilde{\theta}_{2\alpha,2\beta}\}$ be the Riemannian connection form with respect to the orthonormal coframe $\{\tilde{\theta}_{2\alpha-1}, \tilde{\theta}_{2\alpha}\}$. By taking the exterior derivative of $(1.6)_1$ and using (1.1) and (1.6), we get

$$\tag{1.7}\begin{aligned}\tilde{\theta}_{12}&=i\left\{\cos^2\left(\frac{\alpha}{2}\right)\omega_{11}-\sin^2\left(\frac{\alpha}{2}\right)\omega_{22}\right\},\\ \tilde{\theta}_{13}+i\tilde{\theta}_{23}&=-\left(\omega_{12}+\frac{1}{2}(d\alpha-\sin(\alpha)(\omega_{11}+\omega_{22}))\right\},\\ \tilde{\theta}_{14}+i\tilde{\theta}_{24}&=i\left\{\omega_{12}-\frac{1}{2}(d\alpha-\sin(\alpha)(\omega_{11}+\omega_{22}))\right\}\\ \tilde{\theta}_{15}+i\tilde{\theta}_{25}&=\left\{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\omega_{13}+\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\omega_{14}\right.\\ &\qquad\left.+\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\bar{\omega}_{23}+\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\bar{\omega}_{24}\right\},\\ \tilde{\theta}_{16}+i\tilde{\theta}_{26}&=i\left\{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\omega_{13}-\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\omega_{14}\right.\\ &\qquad\left.-\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\bar{\omega}_{23}+\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\bar{\omega}_{24}\right\},\\ \tilde{\theta}_{17}+i\tilde{\theta}_{27}&=\left\{\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\omega_{13}-\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\omega_{14}\right.\\ &\qquad\left.+\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\bar{\omega}_{23}-\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\bar{\omega}_{24}\right\},\\ \tilde{\theta}_{18}+i\tilde{\theta}_{28}&=i\left\{\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\omega_{13}+\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\omega_{14}\right.\\ &\qquad\left.-\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\bar{\omega}_{23}-\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\bar{\omega}_{24}\right\},\end{aligned}$$

$$\tilde\theta_{1,2\lambda-1}+i\tilde\theta_{2,2\lambda-1}=\cos\left(\frac{\alpha}{2}\right)\omega_{1\lambda}+\sin\left(\frac{\alpha}{2}\right)\bar\omega_{2\lambda}\,,$$

$$\tilde\theta_{1,2\lambda}+i\tilde\theta_{2,2\lambda}=i\left\{\cos\left(\frac{\alpha}{2}\right)\omega_{1\lambda}-\sin\left(\frac{\alpha}{2}\right)\bar\omega_{2\lambda}\right\}, \qquad (\lambda\geq 5)\,.$$

By taking the exterior derivatives of $(1.6)_2$–$(1.6)_5$, we get other identities related to $\tilde\theta_{\lambda\nu}$ and $\omega_{\lambda\nu}$, which we omit to show.

2. Minimal surfaces of Kaehler manifold. Let M be an oriented 2-dimensional Riemannian manifold and $x\colon M\to X$ be an isometric immersion of M into a Kaehler manifold X of constant holomorphic sectional curvature 4ρ. Let $\{\tilde e_1,\tilde e_2\}$ be a local orthonormal frame on M. By definition, $\cos(\alpha)=\langle J\tilde e_1,\tilde e_2\rangle$ is the *Kaehler function* (α is a *Kaehler angle*) of x (cf. [4]). The immersion is said to be *totally real* if $\cos(\alpha)=0$ on M. It is said to be *complex* if $\cos^2(\alpha)=1$ on M. We assume that x is not a complex immersion at a point $p\in M$. In the open subset $\cos^2(\alpha)\neq 1$, we extend $\{\tilde e_1,\tilde e_2\}$ to a neighbourhood of X and using results of Section 1, we get canonical 1-forms $\{\tilde\theta_1,\tilde\theta_2,\tilde\theta_3,\tilde\theta_4\}$ defined on the neighbourhood of X. Let $\{\tilde\theta_A\}$, $A=1,\cdots,2n$, be a local orthonormal frame on X which contain the $\{\tilde\theta_1,\tilde\theta_2,\tilde\theta_3,\tilde\theta_4\}$. We denote the restriction of $\{\tilde\theta_A\}$ to M by the same letters. Then we have $\tilde\theta_t=0$ $(3\leq t\leq 2n)$ on M. Putting $\phi=\tilde\theta_1+i\tilde\theta_2$, the induced metric of M is written as $ds^2=\phi\bar\phi$. By taking the exterior derivative of (1.5) restricted to M, we get

$$(2.1)\qquad \begin{aligned}
&\frac{1}{2}\{d\alpha+\sin(\alpha)(\omega_{11}+\omega_{22})\}=a\phi+b\bar\phi\,,\\
&\omega_{12}=b\phi+c\bar\phi\,,\\
&\cos\left(\frac{\alpha}{2}\right)\omega_{\lambda 1}=a_\lambda\phi+b_\lambda\bar\phi\,,\\
&\sin\left(\frac{\alpha}{2}\right)\omega_{\lambda 2}=b_\lambda\phi+c_\lambda\bar\phi\,,\qquad 3\leq\lambda\leq n\,,
\end{aligned}$$

where $a, b, c, a_\lambda, b_\lambda$ and c_λ are complex-valued smooth functions defined locally on M and depend only on the choice of $\{\tilde e_1,\tilde e_2\}$. Let $\{h_{tij}\}$ be the components of the second fundamental form so that $\tilde\theta_{it}=\sum_j h_{tij}\tilde\theta_j$. By using (1.7) and (2.1), all h_{tij}'s can be expressed in terms of by $a, b, c, a_\lambda, b_\lambda$ and c_λ. Indeed, we have

$$h_{311}=-\frac{1}{2}\{a+\bar a+2(b+\bar b)+c+\bar c\}\,,$$

$$h_{312}=\frac{i}{2}(-a+\bar a+c-\bar c)\,,$$

$$
\begin{aligned}
h_{322} &= -\frac{1}{2}\{-a-\bar{a}+2(b+\bar{b})-c-\bar{c}\}\,, \\
h_{411} &= \frac{i}{2}\{a-\bar{a}+2(b-\bar{b})+c-\bar{c}\}\,, \\
h_{412} &= \frac{1}{2}(-a-\bar{a}+c+\bar{c})\,, \\
h_{422} &= \frac{i}{2}\{-a+\bar{a}+2(b-\bar{b})-c+\bar{c}\}\,, \\
h_{2\lambda-1,11} &= -\frac{1}{2}\{a_\lambda+\bar{a}_\lambda+2(b_\lambda+\bar{b}_\lambda)+c_\lambda+\bar{c}_\lambda\}\,, \\
h_{2\lambda-1,12} &= \frac{i}{2}\{-a_\lambda+\bar{a}_\lambda+c_\lambda-\bar{c}_\lambda\}\,, \\
h_{2\lambda-1,22} &= -\frac{1}{2}\{-a_\lambda-\bar{a}_\lambda+2(b_\lambda+\bar{b}_\lambda)-c_\lambda-\bar{c}_\lambda\}\,, \\
h_{2\lambda,11} &= \frac{i}{2}\{a_\lambda-\bar{a}_\lambda+2(b_\lambda-\bar{b}_\lambda)+c_\lambda-\bar{c}_\lambda\}\,, \\
h_{2\lambda,12} &= \frac{1}{2}(-a_\lambda-\bar{a}_\lambda+c_\lambda+\bar{c}_\lambda)\,, \\
h_{2\lambda,22} &= \frac{i}{2}\{-a_\lambda+\bar{a}_\lambda+2(b_\lambda-\bar{b}_\lambda)-c_\lambda+\bar{c}_\lambda\}\,.
\end{aligned}
\tag{2.2}
$$

By (2.2), the mean curvature vector of this immersion is written as $-(\bar{b}(\tilde{e}_3+i\tilde{e}_4)+\sum\bar{b}_\lambda(\tilde{e}_{2\lambda-1}+i\tilde{e}_{2\lambda})+[\text{conjugate}])$. The immersion x is said to be *minimal* if $h_{t11}+h_{t22}=0$ on M for any t, or equivalently, if $b=b_\lambda=0$ on M for any λ. x is said to be *superminimal* if it is minimal and $c=0$ on M (cf. [4], [6]). Note that a complex immersion is always minimal and $|c|^2$ is a scalar invariant of x.

From now on, we assume that x is minimal. Let K be the Gaussian curvature of M, defined by $d\tilde{\theta}_{12}=-(i/2)K\phi\wedge\bar{\phi}$. By virtue of $(1.6)_1$ and $(2.1)_1$, the Gauss equation of x becomes (cf. [6, Prop. 1])

$$K=(1+3\cos^2(\alpha))\rho-2(|a|^2+|c|^2+\textstyle\sum_\lambda|a_\lambda|^2+\sum_\lambda|c_\lambda|^2)\,. \tag{2.3}$$

By taking the exterior derivative of (2.1) and using the structure equation, we get, for some locally defined functions a_i, c_i, $a_{\lambda,i}$ and $c_{\lambda,i}$ $(i=1, 2)$,

$$da - ia\tilde{\theta}_{12} = a_1\phi + a_2\bar{\phi} ,$$

$$\text{with} \quad a_2 = |a|^2\cot(\alpha) - \sum_\lambda |a_\lambda|^2 \tan\left(\frac{\alpha}{2}\right) + \sum_\lambda |c_\lambda|^2 \cot\left(\frac{\alpha}{2}\right) + \frac{3}{4}\rho\sin(2\alpha) ,$$

$$(2.4) \quad dc + 3ic\tilde{\theta}_{12} = c_1\phi + c_2\bar{\phi} , \qquad \text{with} \quad c_1 = -ac\cot(\alpha) ,$$

$$da_\lambda - 2ia_\lambda\tilde{\theta}_{12} - \sum_\mu a_\mu\omega_{\lambda\mu} = a_{\lambda,1}\phi + a_{\lambda,2}\bar{\phi} , \qquad \text{with} \quad a_{\lambda,2} = -\bar{c}a_\lambda\cot\left(\frac{\alpha}{2}\right) ,$$

$$dc_\lambda + 2ic_\lambda\tilde{\theta}_{12} - \sum_\mu c_\mu\omega_{\lambda\mu} = c_{\lambda,1}\phi + c_{\lambda,2}\bar{\phi} , \qquad \text{with} \quad c_{\lambda,1} = ca_\lambda\tan\left(\frac{\alpha}{2}\right) .$$

We put $\tilde{\phi} = e^{ik}\phi$ and $\tilde{\omega}_\lambda = \sum_\lambda a_{\lambda\mu}\omega_\mu$, where k is a locally defined real-valued function and $(a_{\lambda\mu})$ is a unitary matrix ($\lambda, \mu \geq 3$). Then we have $\tilde{\omega}_1 = e^{ik}\omega_1$, $\tilde{\omega}_2 = e^{-ik}\omega_2$ and hence, by (1.1), we get $\tilde{\omega}_{11} = idk + \omega_{11}$, $\tilde{\omega}_{22} = -idk + \omega_{22}$, $\tilde{\omega}_{12} = e^{2ik}\omega_{12}$, $\sum_\mu \tilde{\omega}_{1\mu}a_{\mu\nu} = e^{ik}\omega_{1\nu}$ and $\sum_\mu \tilde{\omega}_{2\mu}a_{\mu\nu} = e^{-ik}\omega_{2\nu}$. By (2.1), we have $\tilde{a} = e^{-ik}a$, $\tilde{c} = e^{3ik}c$, $\tilde{a}_\lambda = e^{-2ik}a_{\lambda\mu}a_\mu$ and $\tilde{c}_\lambda = e^{2ik}a_{\lambda\mu}c_\mu$. Thus $|a|^2$, $|c|^2$, $\sum|a_\lambda|^2$ and $\sum|c_\lambda|^2$ are scalar invariants of x. We wish to compute the Laplacians of these functions. Let Δ be the Laplacian for the metric of M.

LEMMA 2.1. *Let $x: M \to X$ be an isometric minimal immersion of M into a Kaehler manifold X of constant holomorphic sectional curvature* 4ρ *with the Kaehler angle α. Then we have*

$$\Delta\alpha = 4|a|^2\cot(\alpha) - 4\sum|a_\lambda|^2\tan\left(\frac{\alpha}{2}\right) + 4\sum|c_\lambda|^2\cot\left(\frac{\alpha}{2}\right) + 3\rho\sin(2\alpha) ,$$

$$\Delta\log|c|^2 = 6K + 8|a|^2 + 4\sum|a_\lambda|^2\cos(\alpha)\sec^2\left(\frac{\alpha}{2}\right)$$

$$- 4\sum|c_\lambda|^2\cos(\alpha)\operatorname{cosec}^2\left(\frac{\alpha}{2}\right) - 12\rho\cos^2(\alpha) .$$

PROOF. By adding $(2.1)_1$ to its conjugate, we get $d\alpha = a\phi + \bar{a}\bar{\phi}$. Hence $d^c\alpha = i(\bar{a}\bar{\phi} - a\phi)$. Because of $dd^c\alpha = (i/2)(\Delta\alpha)\phi \wedge \bar{\phi}$, we get the formula for $\Delta\alpha$ by $(2.4)_1$. By $(2.4)_2$, we get the formula for $\Delta\log|c|^2$.

REMARK. The first formula in Lemma 2.1 was also proved by Chern and Wolfson [4, p. 72]. Using this, we get formulas for $\Delta\log(\sin(\alpha/2))$ and $\Delta\log(\cos(\alpha/2))$, which coincide with the formulas (2.1) and (2.2) in [5], if $n = 2$.

Using Lemma 2.1, we have $\Delta\log(|c|^2\sin^2\alpha) = 6K$, which coincides with (2.2) in [6]. Hence, in the same way as Theorem 3 in [6], we get the following.

PROPOSITION 2.2. *Let X be a complex n-dimensional Kaehler manifold of positive* *constant holomorphic sectional curvature* 4ρ *and* M *a complete connected* 2-*dimensional Riemannian manifold. Let* $x: M \to X$ *be an isometric minimal immersion which is not complex. If* $K \geq 0$, *then either* $c=0$ *or* $K=0$ *on* M.

Note that Proposition 2.2 is an extension of Theorem 3 in [6] and Theorem 6.1 in [5].

We assume that $K>0$ on M, hence $c=0$ by Proposition 2.2. Let $H(t)=h_{t11}+ih_{t12}$ with $t=3, 4, \cdots, 2n$, and we put $H=\sum_t (H(t))^2$. Then we get $H=4\sum_\lambda \bar{a}_\lambda c_\lambda$ by (2.2). Hence, $|H|^2$ is a globally defined smooth function on M. Using (2.4), we get $dH+4iH\tilde{\theta}_{12}=\bar{H}_2\bar{\phi}$, where we put $H_2=4\sum(\bar{a}_\lambda c_{\lambda,2}+\bar{a}_{\lambda,1}c_\lambda)$. Hence $\Delta|H|^2=2(4K|H|^2+2|H_2|^2)$. On the other hand, we have $|H|^2 \leq 4(\sum|a_\lambda|^2+\sum|c_\lambda|^2)^2$ by Schwarz's inequality. From these and the Gauss equation (2.3), if $K>0$, $|H|^2$ is a subharmonic function on M bounded above, hence is constnt $(=0)$. We put $V_{11}=\sum_t h_{t11}\tilde{e}_t$ and $V_{12}=\sum_t h_{t12}\tilde{e}_t$. Then, by (2.2), we have

$$V_{11}=-\frac{1}{2}\sum(a_\lambda+\bar{a}_\lambda+c_\lambda+\bar{c}_\lambda)\tilde{e}_{2\lambda-1}+\frac{i}{2}\sum(a_\lambda-\bar{a}_\lambda+c_\lambda-\bar{c}_\lambda)\tilde{e}_{2\lambda}, \tag{2.5}$$

$$V_{12}=-\frac{i}{2}\sum(a_\lambda-\bar{a}_\lambda-c_\lambda+\bar{c}_\lambda)\tilde{e}_{2\lambda-1}-\frac{1}{2}\sum(a_\lambda+\bar{a}_\lambda-c_\lambda-\bar{c}_\lambda)\tilde{e}_{2\lambda}.$$

V_{11} and V_{12} are independent of the choice of the normal frame field $\{\tilde{e}_t\}$ $(t \geq 3)$. The subspace O^2 spanned by $\{V_{11}, V_{12}, JV_{11}, JV_{12}\}$ is called that *J-invariant first osculating space* of x. The geometric meaning of $|H|^2$ follows from the identity $|H|^2=(\|V_{11}\|^2-\|V_{12}\|^2)^2+4\langle V_{11}, V_{12}\rangle^2$. We define a subset of M by $\Omega_{(2)}=\{p\in M, V_{11}(p)=0$ or $V_{12}(p)=0\}$. For the set $T_p^1(M)$ of unit tangent vectors of $T_p(M)$, we define a subset of $N_p(M)$ by $A(T_p^1(M))=\{\sum h_{tij}X_iX_j\tilde{e}_t, \sum X_i\tilde{e}_i \in T_p^1(M)\}$, which is called the *ellipse of curvature in the first osculating space* ([5]). Summarizing these computations, we have the following:

PROPOSITION 2.3. *Under the same assumption as in Proposition* 2.2, *if* $K>0$ *on* M *and* $\Omega_{(2)}=0$, *then the ellipse of curvature in the first osculating space is a circle.*

3. Minimal surfaces with constant Kaehler angle. We wish to study a minimal immersion $x: M \to X$ with constant Kaehler angle α, which implies $a=0$. Suppose that x is not complex and $K>0$ on M. Then, by Lemma 2.1 and Proposition 2.2, we have $-4\tan(\alpha/2)\sum|a_\lambda|^2+4\cot(\alpha/2)\sum|c_\lambda|^2+3\rho\sin(2\alpha)=0$ and $c=0$. Hence, the Gauss equation (2.3) is expressed as $\sum|a_\lambda|^2+\sum|c_\lambda|^2=(1/2)(1+3\cos^2(\alpha))\rho-(1/2)K$. These equations give

$$\sum|a_\lambda|^2=\frac{1}{2}\cos^2\left(\frac{\alpha}{2}\right)(\rho+3\rho\cos(\alpha)-K), \tag{3.1}$$

$$\sum |c_\lambda|^2 = \frac{1}{2}\sin^2\left(\frac{\alpha}{2}\right)(\rho - 3\rho\cos(\alpha) - K).$$

If $K \geq (1-3\cos(\alpha))\rho > 0$, we then have $K = (1-3\cos(\alpha))\rho$, which means that K is constant. Hence, by Ohnita's theorem [10], we conclude that x is locally congruent to $\varphi_{n,1}$. Summarizing these facts, we get:

THEOREM 3.1. *Let M be a complete connected oriented 2-dimensional Riemannian manifold and X a Kaehler manifold of complex dimension n of positive constant holomorphic sectional curvature 4ρ. Let $x: M \to X$ be a full isometric minimal immersion with constant Kaehler angle α which is not complex. If $K \geq (1-3\cos(\alpha))\rho > 0$, then K is constant and x is locally congruent to $\varphi_{n,1}$. If $K \geq (1+3\cos(\alpha))\rho > 0$, then K is constant and x is locally congruent to $\varphi_{n,n-1}$.*

By (3.1), we have $\sum |a_\lambda|^2 - \sum |c_\lambda|^2 = (1/2)(4\rho - K)\cos(\alpha)$. By this and (2.5), we have $\Omega_{(2)} = \phi$ if $\cos(\alpha) \neq 0$ on M. From now on, we assume that x is not totally real, i.e., $\cos(\alpha) \neq 0$.

LEMMA 3.2. *Under the same assumptions as in Theorem 3.1 we have*

$$\Delta(\sum |a_\lambda|^2) = 2(3K - \rho - 5\rho\cos(\alpha))(\sum |a_\lambda|^2) + 4\sum |a_{\lambda,1}|^2,$$

$$\Delta(\sum |c_\lambda|^2) = 2(3K - \rho + 5\rho\cos(\alpha))(\sum |c_\lambda|^2) + 4\sum |c_{\lambda,2}|^2.$$

PROOF. We only give the proof for the formula for $\Delta(\sum |c_\lambda|^2)$, because the other can be shown in a similar way. By $(2.4)_4$, we have $d(\sum |c_\lambda|^2) = \sum_\lambda \{(c_\lambda \bar{c}_{\lambda,2} + \bar{c}_\lambda c_{\lambda,1})\phi + (c_\lambda \bar{c}_{\lambda,1} + \bar{c}_\lambda c_{\lambda,2})\bar{\phi}\}$ and $dc_{\lambda,1} + ic_{\lambda,1}\tilde{\theta}_{12} - \sum_\mu c_{\mu,1}\omega_{\lambda\mu} = (\tan(\alpha/2)a_\lambda c_1 + \tan(\alpha/2)a_{\lambda,1}c + (1/2)\sec^2(\alpha/2)ac a_\lambda)\phi + (\tan(\alpha/2)a_\lambda c_2 + \tan(\alpha/2)a_{\lambda,2}c + (1/2)\sec^2(\alpha/2)\bar{a}ca_\lambda)\bar{\phi}$. Hence, we get

$$dd^c(\sum |c_\lambda|^2) = 2i\left\{\sum |c_\lambda|^2 K + \sum(|c_{\lambda,1}|^2 + |c_{\lambda,2}|^2) + (L + \bar{L}) + \sec^2\left(\frac{\alpha}{2}\right)|\sum a_\lambda \bar{c}_\lambda|^2 \right.$$
$$\left. - \operatorname{cosec}^2\left(\frac{\alpha}{2}\right)(\sum |c_\lambda|^2)^2 + \rho\cos(\alpha)\sum |c_\lambda|^2\right\}\phi \wedge \bar{\phi},$$

where we put $L = \sum\{\tan(\alpha/2)\bar{a}_\lambda \bar{c}_2 + \tan(\alpha/2)\bar{a}_{\lambda,2}\bar{c} + (1/2)\sec^2(\alpha/2)a\bar{a}_\lambda \bar{c}\}c_\lambda$. By Theorem 2.1, Proposition 2.3 and $(3.1)_2$, we have $c_{\lambda,1} = 0$, $\sum a_\lambda \bar{c}_\lambda = 0$ and $L = 0$.

PROPOSITION 3.3. *Let $x: M \to X$ be a full isometric minimal immersion with constant Kaehler angle α, which is neither complex nor totally real. If there exists an open subset U of M such that $K|_U < (1-3\cos(\alpha))\rho$, then we have $n \geq 4$.*

PROOF. By (3.1), we have $V_{11} \neq 0$ and $V_{12} \neq 0$ on U, and $\tilde{e}_1, \tilde{e}_2, J\tilde{e}_1, J\tilde{e}_2, V_{11}, V_{12}, JV_{11}, JV_{12}$ are linearly independent on U. This means that $n \geq 4$.

Using the second formula in Lemma 3.2 and $(3.1)_2$, we have:

THEOREM 3.4. *Let X be a Kaehler manifold of complex dimension n of positive constant holomorphic sectional curvature 4ρ and M a complete connected Riemannian 2-manifold. Let $x: M\to X$ be a full isometric minimal immersion with constant Kaehler angle α, which is neither complex nor totally real. If $K\geq(1-5\cos(\alpha))\rho/3$ (>0) on M, then K is constant, and we have $K=(1-5\cos(\alpha))\rho/3$ or $\sum|c_\lambda|^2=0$. In case $K=(1-5\cos(\alpha))\rho/3$, x is locally congruent to $\varphi_{n,2}$, and in case $\sum|c_\lambda|^2=0$, x is locally congruent to $\varphi_{n,1}$.*

COROLLARY 3.5. *Under the same assumption as in Theorem 3.4, if $(1-3\cos(\alpha))\rho>K\geq(1-5\cos(\alpha))\rho/3$, then x is locally congruent to $\varphi_{n,2}$.*

REMARK. Using the first formula in Lemma 3.2, we get a result analogous to Theorem 3.4: If $K\geq(1+5\cos(\alpha))\rho/3$ (>0) on M, then K is constant, so that x is locally congruent to $\varphi_{n,n-1}$ or $\varphi_{n,n-2}$. Hence, we can estimate $(\sum|c_\lambda|^2)$ when $\cos(\alpha)>0$, or $(\sum|a_\lambda|^2)$ when $\cos(\alpha)<0$. Hence, we may assume $\cos(\alpha)>0$.

Because of Proposition 2.3 and the assumption that x is not totally real, V_{11} and V_{12} are perpendicular to each other and of the same lengths. Normalizing these vectors, we adopt them as a basis of O^2, so that $\tilde{e}'_5=V_{11}/\|V_{11}\|$ and $\tilde{e}'_6=V_{12}/\|V_{12}\|$. We put $\cos(\beta)=\langle J\tilde{e}'_5, \tilde{e}'_6\rangle$. Then we have $\cos(\beta)=(\sum|a_\lambda|^2-\sum|c_\lambda|^2)/(\sum|a_\lambda|^2+\sum|c_\lambda|^2)$. If $\cos(\beta)=\pm 1$ on M, then we have $\sum|a_\lambda|^2=0$ or $\sum|c_\lambda|^2=0$, and this case is reduced to Theorem 3.1. Now we assume $\cos(\beta)\neq\pm 1$ at a point of M. Then $\dim(O^2)=4$ in a neighbourhood U of this point. So, as in Section 1, we get the equations (1.4) and (1.5) on U. With respect to this new frame, we have $V_{11}=h'_{511}\tilde{e}'_5$, $V_{12}=h'_{612}\tilde{e}'_6$ and $h'_{611}=h'_{t11}=h'_{512}=h'_{t12}=0$ $(t\geq 7)$. From these equations, (1.6) and (2.1), we have

$$c_3=\cot\left(\frac{\beta}{2}\right)\bar{a}_4\,,\qquad c_4=\tan\left(\frac{\beta}{2}\right)\bar{a}_3\quad\text{and}\quad a_\lambda=c_\lambda=0\,,\qquad(\lambda\geq 5)\,. \tag{3.2}$$

Moreover, because of $\|V_{11}\|=\|V_{12}\|$, c_3 and c_4 are both real-valued and $c_3c_4=0$. We may assume $c_3\neq 0$. Hence $h'_{511}=-\sec(\beta/2)c_3$ and $h'_{612}=\sec(\beta/2)c_3$. Using (2.1), (2.4) and the facts mentioned above, we get

$$\begin{aligned}
&\sin\left(\frac{\alpha}{2}\right)\omega_{32}=c_3\bar{\phi}\,,\qquad \cos\left(\frac{\alpha}{2}\right)\omega_{41}=a_4\phi\,,\\
&\omega_{31}=\omega_{42}=\omega_{\lambda 1}=\omega_{\lambda 2}=0\,,\qquad(\lambda\geq 5)\,,\\
&dc_3+2ic_3\tilde{\theta}_{12}-c_3\omega_{33}=c_{3,2}\bar{\phi}\,,\\
&c_3\omega_{43}=-c_{4,2}\bar{\phi}\,,\qquad c_3\omega_{\lambda 3}=-c_{\lambda,2}\bar{\phi}\,,\qquad(\lambda\geq 5)\,,\\
&da_4-2ia_4\tilde{\theta}_{12}-a_4\omega_{44}=a_{4,1}\phi\,,\\
&a_4\omega_{34}=-a_{3,1}\phi\,,\qquad a_4\omega_{\lambda 4}=-a_{\lambda,1}\phi\,,\qquad(\lambda\geq 5)\,.
\end{aligned}\tag{3.3}$$

From now on $\lambda, \mu\cdots$ run from 5 to through n. By taking the exterior derivative of (3.3) and using the structure equations, we have

$$dc_{4,2}+3ic_{4,2}\tilde{\theta}_{12}-c_{4,2}\omega_{44}=c_{4,22}\bar{\phi}\,,$$

$$dc_{\lambda,2}+3ic_{\lambda,2}\tilde{\theta}_{12}-\sum_{\mu}c_{\mu,2}\omega_{\lambda\mu}=c_{\lambda,21}\phi+c_{\lambda,22}\bar{\phi}\,,\qquad \text{with}\quad c_{\lambda,21}=-c_{4,2}a_{\lambda,1}/a_4\,,$$

$$da_{3,1}-3ia_{3,1}\tilde{\theta}_{12}-a_{3,1}\omega_{33}=a_{3,11}\phi\,, \tag{3.4}$$

$$da_{\lambda,1}-3ia_{\lambda,1}\tilde{\theta}_{12}-\sum_{\mu}a_{\mu,1}\omega_{\lambda\mu}=a_{\lambda,11}\phi+a_{\lambda,12}\bar{\phi}\,,\qquad \text{with}\quad a_{\lambda,12}=-a_{3,1}c_{\lambda,2}/c_3\,.$$

By the definition of $\tilde{e}'_5$ and $\tilde{e}'_6$, we have $\tilde{\theta}_{i,2\lambda-1}=\tilde{\theta}_{i,2\lambda}=0$ ($\lambda\geq 5$). By taking the exterior derivative of these forms and using the structure equations, we can introduce the quantities defined by the following equations:

$$\begin{aligned}
&h'_{511}\tilde{\theta}_{5,2\lambda-1}=h_{2\lambda-1,111}\tilde{\theta}_1+h_{2\lambda-1,112}\tilde{\theta}_2\,,\\
&h'_{612}\tilde{\theta}_{6,2\lambda-1}=h_{2\lambda-1,112}\tilde{\theta}_1-h_{2\lambda-1,111}\tilde{\theta}_2\,,\\
&h'_{511}\tilde{\theta}_{5,2\lambda}=h_{2\lambda,111}\tilde{\theta}_1+h_{2\lambda,112}\tilde{\theta}_2\,,\\
&h'_{612}\tilde{\theta}_{6,2\lambda}=h_{2\lambda,112}\tilde{\theta}_1-h_{2\lambda,111}\tilde{\theta}_2\,,\qquad \lambda\geq 5\,.
\end{aligned} \tag{3.5}$$

By taking the exterior derivative of $(1.6)_3$, we get

$$\tilde{\theta}_{5,2\lambda-1}+i\tilde{\theta}_{6,2\lambda-1}=\cos\left(\frac{\beta}{2}\right)\omega_{3\lambda}+\sin\left(\frac{\beta}{2}\right)\bar{\omega}_{4\lambda}\,,$$

$$\tilde{\theta}_{5,2\lambda}+i\tilde{\theta}_{6,2\lambda}=i\left(\cos\left(\frac{\beta}{2}\right)\omega_{3\lambda}-\sin\left(\frac{\beta}{2}\right)\bar{\omega}_{4\lambda}\right).$$

Hence, by (3.3), $h_{2\lambda-1,111}$, $h_{2\lambda-1,112}$, $h_{2\lambda,111}$ and $h_{2\lambda,112}$ are expressed in terms of $a_{\lambda,1}$ and $c_{\lambda,2}$ because of $h'_{511}=-h'_{612}=-\sec(\beta/2)c_3$. Indeed, we have

$$\begin{aligned}
&h_{2\lambda-1,111}=-\frac{1}{2}(a_{\lambda,1}+\bar{a}_{\lambda,1}+c_{\lambda,2}+\bar{c}_{\lambda,2})\,,\\
&h_{2\lambda-1,112}=-\frac{i}{2}(a_{\lambda,1}-\bar{a}_{\lambda,1}-c_{\lambda,2}+\bar{c}_{\lambda,2})\,,\\
&h_{2\lambda,111}=\frac{i}{2}(a_{\lambda,1}-\bar{a}_{\lambda,1}+c_{\lambda,2}-\bar{c}_{\lambda,2})\,,\\
&h_{2\lambda,112}=-\frac{1}{2}(a_{\lambda,1}+\bar{a}_{\lambda,1}-c_{\lambda,2}-\bar{c}_{\lambda,2})\,.
\end{aligned} \tag{3.6}$$

Using these quantities, we define normal vectors V_{111} and V_{112} in the following way: $V_{111}=\sum(h_{2\lambda-1,111}\tilde{e}_{2\lambda-1}+h_{2\lambda,111}\tilde{e}_{2\lambda})$ and $V_{112}=\sum(h_{2\lambda-1,112}\tilde{e}_{2\lambda-1}+h_{2\lambda,112}\tilde{e}_{2\lambda})$. By (3.6), V_{111} and V_{112} are of the following forms:

$$(3.7)\qquad V_{111}=-\frac{1}{2}\sum(a_{\lambda,1}+\bar{a}_{\lambda,1}+c_{\lambda,2}+\bar{c}_{\lambda,2})\tilde{e}_{2\lambda-1}+\frac{i}{2}\sum(a_{\lambda,1}-\bar{a}_{\lambda,1}+c_{\lambda,2}-\bar{c}_{\lambda,2})\tilde{e}_{2\lambda}\,,$$

$$V_{112}=-\frac{i}{2}\sum(a_{\lambda,1}-\bar{a}_{\lambda,1}-c_{\lambda,2}+\bar{c}_{\lambda,2})\tilde{e}_{2\lambda-1}-\frac{1}{2}\sum(a_{\lambda,1}+\bar{a}_{\lambda,1}-c_{\lambda,2}-\bar{c}_{\lambda,2})\tilde{e}_{2\lambda}\,.$$

THEOREM 3.6. *Let X be a Kaehler manifold of complex dimension n of positive constant holomorphic sectional curvature 4ρ and M a complete connected 2-dimensional Riemannian manifold. Let $x: M\to X$ be a full isometric minimal immersion of constant Kaehler angle α, which is neither complex nor totally real. If there exists an open subset U of M such that $0<K|_U<(1-5\cos(\alpha))\rho/3$, then we have $n\geq 5$.*

PROOF. By assumption, we get $K<(1-3\cos(\alpha))\rho$ on U. Hence, by Proposition 3.3, we get $n\geq 4$ and $\sum|c_\lambda|^2\neq 0$. Assume that $\sum|c_{\lambda,2}|^2=0$ on U. Then we have $d(\sum|c_\lambda|^2)=0$. On the other hand, by Lemma 3.2, we have $\Delta\sum|c_\lambda|^2\neq 0$, which contradicts the constancy of $\sum|c_\lambda|^2$. Hence, we have $\sum|c_{\lambda,2}|^2\neq 0$. Using (3.6), we have $V_{111}\neq 0$ or $V_{112}\neq 0$ at a point of U. This shows that $n\geq 5$.

REMARK. Combining Theorem 3.4 and Theorem 3.6, we can give another proof of the fact that the conjecture by Bolton et al. [2] is affirmative if $n\leq 4$.

Let $\{\tilde{e}'_1, \tilde{e}'_2\}$ be another local orthonormal frame on M such that $\tilde{e}'_1=\cos(k)\tilde{e}_1-\sin(k)\tilde{e}_2$ and $\tilde{e}'_2=\sin(k)\tilde{e}_1+\cos(k)\tilde{e}_2$. Then we have $V'_{11}=\cos(2k)V_{11}-\sin(2k)V_{12}$ and $V'_{12}=\sin(2k)V_{11}+\cos(2k)V_{12}$. On the other hand, by the definition of c_3, we have $V_{11}=-\sec(\beta/2)c_3e_5$ and $V_{12}=\sec(\beta/2)c_3e_6$. So, under such a change, we have, by (3.3), $c'_3=c_3$, $a'_4=a_4$, $c'_{4,2}=e^{5ik}c_{4,2}$ and $c'_{\lambda,2}=e^{3ik}(\sum a_{\lambda\mu}c_{\mu,2})$, where we put $\omega'_\lambda=\sum a_{\lambda\mu}\omega_\mu$ for a unitary matrix $(a_{\lambda\mu})$ $(5\leq\lambda,\ \mu\leq n)$. Hence $|c_{4,2}|^2$ and $\sum|c_{\lambda,2}|^2$ are scalar invariants of x.

LEMMA 3.7. *Let $x: M\to X$ be an isometric minimal immersion with constant Kaehler angle α, which is neither complex nor totally real. On an open subset U of M such that $\cos(\beta)\neq\pm 1$, we have*

$$\Delta|c_{4,2}|^2=6K|c_{4,2}|^2+4|c_{4,22}|^2$$
$$+4|c_{4,2}|^2\left\{\sec^2\left(\frac{\alpha}{2}\right)a_4^2-|c_{4,2}|^2/c_3^2-\sum|a_{\lambda,1}|^2/a_4^2+\rho\cos(\alpha)\right\},$$

$$\Delta\sum|c_{\lambda,2}|^2=6K\sum|c_{\lambda,2}|^2+4(\sum|c_{\lambda,21}|^2+\sum|c_{\lambda,22}|^2)+4\rho\cos(\alpha)\sum|c_{\lambda,2}|^2$$
$$-4(\sum|c_{\lambda,2}|^2)^2/c_3^2+4|\sum\bar{c}_{\lambda,2}a_{\lambda,1}|^2/a_4^2-8|c_{4,2}|^2\sum|c_{\lambda,2}|^2/c_3^2$$
$$-4\bar{c}_{4,22}\sum c_{\lambda,2}\bar{a}_{\lambda,1}/a_4-4c_{4,22}\sum\bar{c}_{\lambda,2}a_{\lambda,1}/a_4\,,$$

where λ runs from 5 through n.

PROOF. We only prove the formula for $\Delta(\sum|c_{\lambda,2}|^2)$ here, because the other can be shown in a similar way. By (3.3) and $(3.4)_2$, we have

$$d(\sum|c_{\lambda,2}|^2)=\sum(c_{\lambda,2}\bar{c}_{\lambda,22}+\bar{c}_{\lambda,2}c_{\lambda,21})\phi+\sum(c_{\lambda,2}\bar{c}_{\lambda,21}+\bar{c}_{\lambda,2}c_{\lambda,22})\bar{\phi}\,,$$

$$dc_{\lambda,21}+2ic_{\lambda,21}\tilde{\theta}_{12}-\sum c_{\mu,21}\omega_{\lambda\mu}$$

$$=\left(-c_{4,2}\frac{a_{\lambda,11}}{a_4}+c_{4,2}\frac{a_{4,1}a_{\lambda,1}}{a_4^2}\right)\phi+\left(-c_{4,22}\frac{a_{\lambda,1}}{a_4}-c_{4,2}\frac{a_{\lambda,12}}{a_4}\right)\bar{\phi}\,.$$

Hence, we can directly calculate $dd^c(\sum|c_{\lambda,2}|^2)$.

PROPOSITION 3.8. *Let M be a complete 2-dimensional Riemannian manifold and $x: M\to X$ be an isometric minimal immersion of constant Kaehler angle α, which is neither complex nor totally real. If $\cos(\beta)\neq\pm 1$ on M and K is strictly positive on M (hence M is compact), then we have $|c_{4,2}|^2=0$ on M.*

PROOF. By (3.2), (3.3), Lemma 3.2 and Lemma 3.7, we have $\Delta(a_4^2|c_{4,2}|^2)=10Ka_4^2|c_{4,2}|^2+4|a_4c_{4,22}+\bar{a}_{4,1}c_{4,2}|^2$, which shows that $a_4^2|c_{4,2}|^2$ is constant. Hence, we get $|c_{4,2}|^2=0$.

Let $H^{(2)}(t)=h_{t111}+ih_{t112}$ with $t=9, 10, \cdots, 2n$, and we put $H^{(2)}=\sum_t(H^{(2)}(t))^2$. Then we get $H^{(2)}=4\sum\bar{a}_{\lambda,1}c_{\lambda,2}$ by (3.7), where λ runs from 5 through n. $|H^{(2)}|^2$ is a globally defined smooth function on M. By (3.3), (3.4) and Proposition 3.8, we have $dH^{(2)}+6iH^{(2)}\tilde{\theta}_{12}=4\sum(\bar{a}_{\lambda,1}c_{\lambda,22}+\bar{a}_{\lambda,11}c_{\lambda,2})\bar{\phi}$ because of $\sum(\bar{a}_{\lambda,1}c_{\lambda,21}+\bar{a}_{\lambda,12}c_{\lambda,2})=0$. By the same calculation as in the proof of Proposition 2.3, we have the following:

PROPOSITION 3.9. *Under the same assumptions as in Proposition 3.8, we have $H^{(2)}=0$ on M.*

$c_3^2\sum|c_{\lambda,2}|^2$ $(5\leq\lambda\leq n)$ is independent of the choice of normal vectors $\tilde{e}_t$, $5\leq t\leq 2n$. By Lemmas 3.2 and 3.7 as well as Propositions 3.8 and 3.9, we have

$$\Delta\{c_3^2\sum|c_{\lambda,2}|^2\}=2c_3^2\sum|c_{\lambda,2}|^2\{6K-\rho+7\rho\cos(\alpha)\}+4\sum|c_3c_{\lambda,22}+c_{3,2}c_{\lambda,2}|^2\,, \tag{3.8}$$

from which we obtain:

THOREM 3.10. *Let X be a Kaehler manifold of complex dimension n of positive constant holomorphic sectional curvature 4ρ and M be a complete connected Riemannian 2-manifold. Let $x: M\to X$ be a full isometric minimal immersion of constant Kaehler angle α, which is neither complex nor totally real. If the J-invariant first osculating space of x is of constant dimension on M and $K\geq(1-7\cos(\alpha))\rho/6>0$ on M, then K is constant so that x is locally congruent to either $\varphi_{n,1}$, $\varphi_{n,2}$ or $\varphi_{n,3}$.*

PROOF. By Theorem 3.4, we may assume that there exists an open subset U such that $K<(1-5\cos(\alpha))\rho/3$ on U. Hence, by Theorem 3.6, we get $\sum|c_\lambda|^2\neq 0$ and $\sum|c_{\lambda,2}|^2\neq 0$ at a point of U. Hence by assumption we have $\cos(\beta)\neq\pm 1$ on M. By (3.8) we have $6K-\rho+7\rho\cos(\alpha)=0$, which shows that x is locally congruent to $\varphi_{n,3}$.

COROLLARY 3.11. *Let $x: M \to X$ be a full isometric minimal immersion with constant Kaehler angle α, which is neither complex nor totally real. If* $(1-5\cos(\alpha))\rho/3 > K \geq (1-7\cos(\alpha))\rho/6$, then x is locally congruent to $\varphi_{n,3}$.

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