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ESTIMATES FOR OPERATORS IN WEIGHTED $L^{p,q}$-SPACES

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Abstract. We give conditions on pairs of weight functions for which a certain operator defined on $\Omega\subseteq \boldsymbol{R}_+^2$ is bounded between weighted Lorentz spaces. The result is applied to obtain weighted estimates for the Laplace transform.

Introduction. Let f and w be nonnegative functions on $\boldsymbol{R}^+$. Then the distribution function of f relative to the measure $w(x)dx$ is defined by

$$f_w(s)=\int_{\{x:|f(x)|>s\}} w(x)dx=w(\{x:|f(x)|>s\})\,,$$

where $s>0$ and the decreasing rearrangement of $|f|$ relative to $w(x)dx$ is obtained by $f^w(t)=\inf\{s:f_w(s)\le t\}$ (see e.g. [4; 6, Chapter V]). Further if $0<p, q\le\infty$, then the weighted Lorentz spaces $L^{p,q}(w)$ are defined by

$$L^{p,q}(w)=\{f:\|f\|_{L^{p,q}(w)}<\infty\}\,,$$

where

$$(1)\qquad \|f\|_{L^{p,q}(w)}=\begin{cases}\left\{(q/p)\displaystyle\int_0^\infty [t^{1/p}f^w(t)]^q t^{-1}dt\right\}^{1/q}, & 0<p,\quad q<\infty\\ \displaystyle\sup_{t>0} t^{1/p}f^w(t)\,, \quad 0<p\le\infty\,,\quad q=\infty\,. & \end{cases}$$

In case either $1<p<\infty$ and $1<q<\infty$ or $p=q=\infty$, $L^{p,q}(w)$ is a Banach space with norm equivalent to the quasi-norm $\|f\|_{L^{p,q}(w)}$.

Clearly if $w\equiv 1$ then $L^{p,q}(w)=L^{p,q}$, where $L^{p,q}$ are the usual Lorentz spaces. We denote by $L_w^p(\Omega)$, $0<p\le\infty$, the space of weighted measurable functions f for which $\|f\|_{L_w^p(\Omega)}=\|w^{1/p}f\|_{L^p(\Omega)}$ is finite, where $\|\cdot\|_{L^p(\Omega)}$ denotes the usual Lebesgue norm. Note that $L^{q,q}(w)=L_w^q(\boldsymbol{R}^+)$.

If $1<p<\infty$, $1\le q\le\infty$ and $1/p+1/p'=1=1/q+1/q'$, then

$$(2)\qquad C^{-1}\|f\|_{L^{p,q}(w)}\le \sup_{\|g\|_{L^{p',q'}(w)}<1}\left|\int fgwdx\right|\le C\|f\|_{L^{p,q}(w)}$$

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(see e.g. [2, inequality (2.3)]).

Throughout, p' denotes the conjugate index of p and is related to p by $p+p'=pp'$ with $p'=+\infty$ if $p=1$. Similarly for other letters. Further, constants are denoted by C and may be different at different appearances but are always independent of the function in question. $\boldsymbol{Z}$ denotes the set of integers.

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LEMMA 1. *Suppose* $0<p, q<\infty$. *Then*

(a) $\|f\|^q_{L^{p,q}(w)}=q\int_0^\infty f_w(s)^{q/p}s^{q-1}ds.$

(b) *If* f *is non-increasing on* $\boldsymbol{R}^+$ *and if* $g(x)=\int_0^x w(s)ds<\infty$, *whenever* $w(x)>0$, *then*

$$\|f\|^q_{L^{p,q}(w)}=(q/p)\int_0^\infty f(x)^q g(x)^{q/p-1}w(x)dx\,.$$

Part (a) is due to Sawyer [5, Lemma 1] and Part (b) follows essentially along the same lines as in [5, Lemma 1].

PROOF. Part (a) follows, on evaluating the two integral of $qs^{q-1}(q/p)t^{q/p-1}$ over the set $\{(t,s):0<s<f^w(t), 0<t\}$. By permuting the s integration first we obtain the right hand side of (1) to the q-th power, i.e.,

$$\int_{\{(t,s):0<s<f^w(t),0<t\}} qs^{q-1}(q/p)t^{q/p-1}dsdt=q\int_0^\infty s^{q-1}\left(\int_0^{f_w(s)}(q/p)t^{q/p-1}dt\right)ds$$
$$=q\int_0^\infty s^{q-1}f_w(s)^{q/p}ds\,.$$

Hence, Part (a) follows.

(b) is established by evaluating the two iterated integrals of

$$qx^{q-1}(q/p)g(y)^{q/p-1}w(y)\,, \tag{3}$$

over the set $M=\{(x,y):0<x<f(y);0<y\}$. Performing the x integration over M on (3) first yields the right hand side of (b)

$$\int_0^\infty\left[q\int_0^{f(y)}x^{q-1}dx\right](q/p)g(y)^{q/p-1}w(y)dy=(q/p)\int_0^\infty f(y)^q g(y)^{q/p-1}w(y)dy$$

and performing the y integration first yields the right hand side of (a)

$$\int_0^\infty \left[q \int_0^{f(y)} x^{q-1} dx \right] (q/p) g(y)^{q/p-1} w(y) dy \tag{4}$$

$$= (q^2/p) \int_0^\infty x^{q-1} dx \int_0^\infty \chi_{(0,f(y))}(x) g(y)^{q/p-1} w(y) dy ,$$

where

$$\chi_{(0,f(y))}(x) = \begin{cases} 1 , & \text{if } x \in (0, f(y)) \\ 0 , & \text{if } x \notin (0, f(y)) . \end{cases}$$

But $M = \{(x, y) : x > 0, 0 < y < S(x)\}$ where $S(x) = \inf\{y : f(y) \le x\}$. Therefore, the right hand side of (4) is equal to

$$(q^2/p) \int_0^\infty x^{q-1} dx \int_{(0,S(x))} g(y)^{q/p-1} w(y) dy \tag{5}$$

$$= (q^2/p) \int_0^\infty x^{q-1} dx \int_0^{S(x)} \left(\int_0^y w(t) dt \right)^{q/p-1} d\left(\int_0^y w(t) dt \right) = q \int_0^\infty x^{q-1} g(S(x))^{q/p} dx .$$

But

$$f_w(x) = w(\{y : f(y) > x\}) = \int_{\{y : f(y) > x\}} w(y) dy = \int_0^{S(x)} w(y) dy = g(S(x)) . \tag{6}$$

So that, from (5) and (6) we obtain

$$\|f\|^q_{L^{p,q}(w)} = q \int_0^\infty x^{q-1} f_w(x)^{q/p} dx = (q/p) \int_0^\infty f(x)^q g(x)^{q/p-1} w(x) dx .$$

This completes the proof of the lemma.

Now, we state and prove the weighted Lorentz inequality for the K-operator for $1 < s < r \le q < \infty$, $0 < p < \infty$.

THEOREM 2. *Suppose* $1 < s < r \le q < \infty$, $0 < p < \infty$. *Define*

$$(Kf)(x) = \int_0^\infty k(x, y) f(y) dy ,$$

$0 \le k(x, y) \le Ch(y)$, *for* x, y on $\mathbf{R}^+$, *so that there exists a sequence* $\{x_j\}_{j \in \mathbf{Z}}$ *satisfying*

$$\int_0^\infty k(x_j, t) f(t) dt = 2^{-j} ,$$

for all $f(x) \ge 0$ *with* $\|f\|_{L^{r,s}(v)} < \infty$. *If* w, v *are non-negative weight functions on* $\mathbf{R}^+$, *then*

$$\|Kf\|_{L^{p,q}(w)} \le C \|f\|_{L^{r,s}(v)} , \tag{7}$$

implies

$$\sup_{x>0}\left(\int_0^x w(t)dt\right)^{1/p}\|k(x,\cdot)/v\|_{L^{r',s'}(v)}=C<\infty\,. \tag{8}$$

Conversely, the condition

$$\sup_{x>0}\left(\int_0^x w(t)dt\right)^{1/p}\|h/v\|_{L^{r',s'}(k(x,\cdot)v/h)}=C<\infty \tag{9}$$

implies (7).

PROOF. Let us prove (7) implies (8). Let f be a non-negative function. Then

$$(Kf)_w(\xi)=\int_{\{t\in(0,\infty):(kf)(t)>\xi\}} w(t)dt\geq\int_{\{t\in(0,x):(Kf)(t)>\xi\}} w(t)dt$$

which implies

$$(Kf)_w(\xi)\geq\int_0^x w(t)dt\,,\qquad \text{for}\quad 0<\xi<(Kf)(x)\,. \tag{10}$$

Inequalities (7) and (10) together with Lemma 1 yield

$$\begin{aligned}\|f\|_{L^{r,s}(v)}&\geq C^{-1}\|Kf\|_{L^{p,q}(w)}=C^{-1}\left[\int_0^\infty qt^{q-1}(Kf)_w^{q/p}(t)dt\right]^{1/q}\\&\geq C^{-1}\left[\int_0^{(Kf)(x)}\left(\int_0^x w(t)dt\right)^{q/p}qt^{q-1}dt\right]^{1/q}\\&=C^{-1}\left(\int_0^x w(t)dt\right)^{1/p}\left(\int_0^\infty (k(x,t)/v(t))f(t)v(t)dt\right).\end{aligned}$$

Thus

$$\sup_{x>0}\left(\int_0^x w(t)dt\right)^{1/p}\|k(x,\cdot)/v\|_{L^{r',s'}(v)}=C<\infty\,,$$

which proves that (7) implies (8).

Conversely, fix $f\geq 0$ in $L^{r,s}(v)$, $s\leq r$ and choose $\{x_j\}$ such that

$$(Kf)(x_j)=\int_0^\infty k(x_j,t)f(t)dt=2^{-j}$$

for all $j\in\boldsymbol{Z}$. Then, $\{x_j\}_{j\in\boldsymbol{Z}}$ is an increasing sequence of positive numbers (note that $(Kf)(x)$ is decreasing)

$$2^{-(j+1)}=2^{-j}-2^{-(j+1)}=(Kf)(x_j)-(Kf)(x_{j+1})=\int_0^\infty (k(x_j,t)-k(x_{j+1},t))f(t)dt\,.$$

By Hölder's inequality and Lemma 1 we obtain

$$\begin{aligned}(11)\qquad \|Kf\|^q_{L^{p,q}(w)} &= (q/p)\sum_j \int_{x_{j-1}}^{x_j} (Kf)^q(x) g(x)^{q/p-1} w(x)dx \\ &\le (q/p)\sum_j (Kf)^q(x_{j-1}) \int_0^{x_j} g(x)^{q/p-1} w(x)dx \\ &= 2^{2q}\sum_j \left(\int_0^{x_j} w(x)dx\right)^{q/p} \left\{\int_0^\infty [k(x_j,t)-k(x_{j+1},t)](h(t)/v(t))(v(t)/h(t))f(t)dt\right\}^q \\ &\le 2^{2q}\sum_j \left(\int_0^{x_j} w(x)dx\right)^{q/p} \|h/v\|^q_{L^{r',s'}((k(x_j,\cdot)-k(x_{j+1},\cdot))v/h)} \|f\|^q_{L^{r,s}((k(x_j,\cdot)-k(x_{j+1},\cdot))v/h)}\,.\end{aligned}$$

Since $k(x_j,x)-k(x_{j+1},x)\le k(x_j,x)$ we obtain

$$\begin{aligned}&\|h/v\|_{L^{r',s'}((k(x_j,\cdot)-k(x_{j+1},\cdot))v/h)} \\ &\le \left[\int_0^\infty s' t^{s'-1}\left(\int_{\{x\in(0,\infty):(h(x)/v(x))>t\}} (k(x_j,x)/h(x))v(x)dx\right)^{s'/r'} dt\right]^{1/s'} = \|h/v\|_{L^{r',s'}(k(x_j,\cdot)v/h)}\,.\end{aligned}$$

From this estimate, (9) and Minkowski's inequality the previous inequality (11) shows that

$$\begin{aligned}&\|Kf\|^q_{L^{p,q}(w)} \le C\sum_j \|f\|^q_{L^{r,s}((k(x_j,\cdot)-k(x_{j+1},\cdot))v/h)} \\ &= C\sum_j\left[\int_0^\infty st^{s-1}\left(\int_{\{x\in(0,\infty):f(x)>t\}} (k(x_j,x)-k(x_{j+1},x))(v(x)/h(x))dx\right)^{s/r} dt\right]^{q/s} \\ &\le C\left[\sum_j\left[\int_0^\infty st^{s-1}\left(\int_{\{x\in(0,\infty):f(x)>t\}} (k(x_j,x)-k(x_{j+1},x))(v(x)/h(x))dx\right)^{s/r} dt\right]^{r/s}\right]^{q/r} \\ &\le C\left[\int_0^\infty st^{s-1}\left(\int_{\{x\in\mathbf{R}^+:f(x)>t\}} \sum_j (k(x_j,x)-k(x_{j+1},x))(v(x)/h(x))dx\right)^{s/r} dt\right]^{q/s} \\ &\le C\left[\int_0^\infty st^{s-1}\left(\int_{\{x\in\mathbf{R}^+;f(x)>t\}} v(x)dx\right)^{s/r} dt\right]^{q/s} = C\|f\|^q_{L^{r,s}(v)}\,,\end{aligned}$$

where the second, the third and the last inequalities follow from $r\le q$, Minkowski's inequality and

$$\sum_j (k(x_j,x)-k(x_{j+1},x)) \le Ch(x)\,,$$

respectively. This completes the proof of the theorem.

We now state and prove the 2-dimensional weighted Lebesgue inequality for the K-operator for $1<p\le q<\infty$.

THEOREM 3. *Suppose* $1<p\le q<\infty$ *and that* $k(x,y)=\prod_{i=1}^{2}k_i(x_i,y_i)$, $h(x)=\prod_{i=1}^{2}h_i(x_i)$, $0\le k_i(x_i,y_i)\le Ch_i(y_i)$, $i=1,2$. *Define*

$$(Kf)(x)=\int_{\boldsymbol{R}_+^2}k(x,y)f(y)dy\,,$$

so that there exists a sequence $\{x_j\}_{j\in\boldsymbol{Z}}$ *satisfying*

$$\int_0^\infty k_1(x_j,y_1)(K_2f)(y_1,x_2)dy_1=2^{-j}\,,\qquad \int_0^\infty k_2(x_j,y_2)f(y_1,y_2)dy_2=2^{-j}$$

where

$$(K_2f)(y_1,x_2)=\int_0^\infty k_2(x_2,y_2)f(y_1,y_2)dy_2$$

for all $f(y_1,y_2)\ge 0$ *with* $(\int_{\boldsymbol{R}_+^2}w(x)f(x)^pdx)^{1/p}<\infty$. *Suppose further that* $w(x)=w(x_1,x_2)$ *is a non-negative function defined on* $\boldsymbol{R}_+^2$ *and satisfies*

$$\sup_{s>0}\left(\int_0^s w(x_1,x_2)^{q/p}dx_i\right)^{1/q}\left(\int_0^\infty k_i(s,x_i)h_i^{p'/p}(x_i)w(x_1,x_2)^{1-p'}dx_i\right)=C<\infty \tag{12}$$

for $i=1,2$. *Then*

$$\left(\int_{\boldsymbol{R}_+^2}w(x)^{q/p}(Kf)^q(x)dx\right)^{1/q}\le C\left(\int_{\boldsymbol{R}_+^2}w(x)f(x)^pdx\right)^{1/p}. \tag{13}$$

PROOF. Suppose first that (12) holds. Then

$$\begin{aligned}
\int_{\boldsymbol{R}_+^2}w(x)^{q/p}(Kf)^q(x)dx&=\int_{\boldsymbol{R}_+^2}w(x)^{q/p}\left[\int_0^\infty\int_0^\infty k_1(x_1,y_1)k_2(x_2,y_2)f(y_1,y_2)dy_2dy_1\right]^q dx \qquad (14)\\
&=\int_{\boldsymbol{R}_+^2}w(x)^{q/p}\left[\int_0^\infty k_1(x_1,y_1)\left(\int_0^\infty k_2(x_2,y_2)f(y_1,y_2)dy_2\right)dy_1\right]^q dx\\
&=\int_{\boldsymbol{R}_+^2}w(x)^{q/p}\left[\int_0^\infty k_1(x_1,y_1)(K_2f)(y_1,x_2)dy_1\right]^q dx\,,
\end{aligned}$$

where

$$(K_2f)(y_1,x_2)=\int_0^\infty k_2(x_2,y_2)f(y_1,y_2)dy_2\,.$$

Now, fix $f\ge 0$ in $L_w^p(\boldsymbol{R}_+^2)$ and a positive increasing sequence $\{x_j\}_{j\in\boldsymbol{Z}}$ and define

$$\int_0^\infty k_1(x_j,y_1)(K_2f)(y_1,x_2)dy_1=2^{-j}\,.$$

Then

$$(15)\qquad 2^{-(j+1)}=2^{-j}-2^{-(j+1)}=\int_0^\infty (k_1(x_j, y_1)-k_1(x_{j+1}, y_1))(K_2f)(y_1, x_2)dy_1 .$$

By combining (14) and (15) one obtains

$$\begin{aligned}
&\int_{\boldsymbol{R}_+^2} w(x)^{q/p}(Kf)^q(x)dx \\
&=\int_0^\infty \sum_j\left(\int_{x_{j-1}}^{x_j} w^{q/p}(x_1, x_2)\left[\int_0^\infty k_1(x_1, y_1)(K_2f)(y_1, x_2)dy_1\right]^q dx_1\right)dx_2 \\
&\le\int_0^\infty\left(\sum_j\left[\int_0^\infty k_1(x_{j-1}, y_1)(K_2f)(y_1, x_2)dy_1\right]^q\int_{x_{j-1}}^{x_j} w^{q/p}(x_1, x_2)dx_1\right)dx_2 \\
&\le 2^{2q}\int_0^\infty\sum_j\left(\int_0^{x_j} w^{q/p}(x_1, x_2)dx_1\right)\left(\int_0^\infty[k_1(x_j, y_1)-k_1(x_{j+1}, y_1)]\right. \\
&\qquad\left.\times(h_1^{1/p}(y_1)/w(y_1, x_2))((K_2f)(y_1, x_2)/h_1^{1/p}(y_1))w(y_1, x_2)dy_1\right)^q dx_2 .
\end{aligned}$$

Hölder's inequality is applied to the second integral of the previous integral and one gets

$$\begin{aligned}
&\int_{\boldsymbol{R}_+^2} w(x)^{q/p}(Kf)^q(x)dx\le 2^{2q}\int_0^\infty\sum_j\left(\int_0^{x_j} w^{q/p}(x_1, x_2)dx_1\right) \\
&\times\left[\int_0^\infty[k_1(x_j, y_1)-k_1(x_{j+1}, y_1)](h_1^{1/p}(y_1)/w(y_1, x_2))^{p'}w(y_1, x_2)dy_1\right]^{q/p'} \\
&\times\left[\int_0^\infty[k_1(x_j, y_1)-k_1(x_{j+1}, y_1)]((K_2f)(y_1, x_2)/h^{1/p}(y_1))^p w(y_1, x_2)dy_1\right]^{q/p} dx_2 .
\end{aligned}$$

Again since $k_1(x_j, y_1)-k(x_{j+1}, y_1)\le k_1(x_j, y_1)$ and on applying (12)

$$\sup_{s>0}\left(\int_0^s w^{q/p}w(x_1, x_2)dx_1\right)^{1/q}\left(\int_0^\infty k_1(s, y_1)h_1(y_1)^{p'/p}w(y_1, x_2)^{1-p'}dy_1\right)^{1/p'}\le C ,$$

for fixed x_2 one gets

$$\begin{aligned}
(16)\qquad \int_{\boldsymbol{R}_+^2} w(x)^{q/p}(Kf)^q(x)dx &\le C\int_0^\infty\left[\int_0^\infty\left[\sum_j(k_1(x_j, y_1)-k_1(x_{j+1}, y_1))\right]\right. \\
&\qquad\left.\times((K_2f)(y_1, x_2)/h_1^{1/p}(y_1))^p w(y_1, x_2)dy_1\right]^{q/p} dx_2 \\
&\le C\int_0^\infty\left[\int_0^\infty(K_2f)^p w(y_1, x_2)dy_1\right]^{q/p} dx_2 ,
\end{aligned}$$

where the last inequality follows from

$$\sum_j (k_1(x_j, y_1) - k_1(x_{j+1}, y_1)) \le Ch_1(y_1) .$$

Again choose a positive increasing sequence $\{x_j\}_{j\in \boldsymbol{Z}}$ such that

$$\int_0^\infty k_2(x_j, y_2) f(y_1, y_2) dy_2 = 2^{-j} .$$

Then

$$2^{-(j+1)} = \int_0^\infty [k_2(x_j, y_2) - k_2(x_{j+1}, y_2)] f(y_1, y_2) dy_2 .$$

By Minkowski's inequality for $p \le q$ applied to the last integral of (16) and on proceeding as in the beginning of the proof we obtain

$$\begin{aligned}\int_{\boldsymbol{R}_+^2} w(x)^{q/p}(Kf)^q(x)dx &\le C\left(\int_0^\infty\left(\int_0^\infty (K_2 f)^q(y_1, x_2) w^{q/p}(y_1, x_2) dx_2\right)^{p/q} dy_1\right)^{q/p} \\ &= C\left(\int_0^\infty\left[\sum_j \int_{x_{j-1}}^{x_j}\left[\int_0^\infty k_2(x_2, y_2) f(y_1, y_2) dy_2\right]^q w^{q/p}(y_1, x_2) dx_2\right]^{p/q} dy_1\right)^{q/p} \\ &\le C2^{2q}\left(\int_0^\infty\left[\sum_j\left[\int_0^\infty [k_2(x_j, y_2) - k_2(x_{j+1}, y_2)](h_2^{1/p}(y_2)/w(y_1, y_2))\right.\right.\right. \\ &\quad \left.\left.\left.\times (f(y_1, y_2)/h_2^{1/p}(y_2)) w(y_1, y_2) dy_2\right]^q\left(\int_0^{x_j} w^{q/p}(y_1, x_2) dx_2\right)\right]^{p/q} dy_1\right)^{q/p} .\end{aligned}$$

Next, from Hölder's inequality, it follows that

$$\begin{aligned}(17)\quad \int_{\boldsymbol{R}_+^2} w(x)^{q/p}(Kf)^q(x)dx &\le C\left[\int_0^\infty\left[\sum_j\left(\int_0^{x_j} w^{q/p}(y_1, x_2) dx_2\right)\right.\right. \\ &\quad \times\left[\int_0^\infty [k_2(x_j, y_2) - k_2(x_{j+1}, y_2)](h_2^{1/p}(y_2)/w(y_1, y_2))^{p'} w(y_1, y_2) dy_2\right]^{q/p'} \\ &\quad \left.\left.\times\left[\int_0^\infty [k_2(x_j, y_2) - k_2(x_{j+1}, y_2)](f(y_1, y_2)/h_2^{1/p}(y_2))^p w(y_1, y_2) dy_2\right]^{q/p}\right]^{p/q} dy_1\right]^{q/p} \\ &\le C\left[\int_0^\infty\left[\sum_j\left(\int_0^{x_j} w^{q/p}(y_1, x_2) dx_2\right)\left[\int_0^\infty k_2(x_j, y_2) h_2^{p'/p}(y_2) w^{1-p'}(y_1, y_2) dy_2\right]^{q/p'}\right.\right. \\ &\quad \left.\left.\times\left[\int_0^\infty [k_2(x_j, y_2) - k_2(x_{j+1}, y_2)](f(y_1, y_2)/h_2^{1/p}(y_2))^p w(y_1, y_2) dy_2\right]^{q/p}\right]^{p/q} dy_1\right]^{q/p} .\end{aligned}$$

Here, we used $k_2(x_j, y_2) - k_2(x_{j+1}, y_2) \le k_2(x_j, y_2)$ in the second integral of the last inequality.

On applying (12),

$$\sup_{s>0}\left(\int_0^s w^{q/p}(y_1, x_2)dx_2\right)^{1/q}\left(\int_0^\infty k_2(s, y_2)h_2^{p'/p}(y_2)w^{1-p'}(y_1, y_2)dy_2\right)^{1/p'} \leq C ,$$

for fixed y_1 and

$$\sum_j [k_2(x_j, y_2)-k_2(x_{j+1}, y_2)] \leq h_2(y_2)$$

to the above inequality (17) one obtains

$$\int_{\boldsymbol{R}_+^2} w(x)^{q/p}(Kf)^q(x)dx \leq C\left(\int_0^\infty\int_0^\infty w(y_1, y_2)f^p(y_1, y_2)dy_2dy_1\right)^{q/p} = C\left(\int_{\boldsymbol{R}_+^2} w(x)f(x)^p dx\right)^{q/p} ,$$

which proves the theorem.

Here, we apply Theorems 2 and 3 to the Laplace transforms

(18) $$(Lf)(x) = \int_0^\infty e^{-xt}f(t)dt , \qquad x>0$$

and

(19) $$(Lf)(x) = \int_{\boldsymbol{R}_+^2} e^{-\langle x,y\rangle}f(y)dy , \qquad x\in \boldsymbol{R}_+^2 ,$$

respectively, where $\langle x, y\rangle = x_1y_1 + x_2y_2$, $x_i, y_i \in \boldsymbol{R}_+$, $i=1, 2$, complementing those results obtained in [1], [3].

THEOREM 4. *Suppose* $1<s<r\leq q<\infty$, $0<p<\infty$. *If* w, v *are non-negative weight functions on* $\boldsymbol{R}^+$, *then*

(20) $$\|Lf\|_{L^{p,q}(w)} \leq C\|f\|_{L^{r,s}(v)} , \qquad \textit{for all} \quad f\geq 0$$

implies

$$\sup_{x>0}\left(\int_0^x w(t)dt\right)^{1/p}\|e^{-x\cdot}/v\|_{L^{r',s'}(v)} = C < \infty .$$

Conversely, the condition

$$\sup_{x>0}\left(\int_0^x w(t)dt\right)^{1/p}\|1/v\|_{L^{r',s'}(e^{-x\cdot}v)} = C < \infty$$

implies (20).

The proof follows from Theorem 2 and (18) with $K=L$.

THEOREM 5. *Suppose $1<p\le q<\infty$. If w and v are non-negative weights and satisfy*

$$\sup_{s>0}\left(\int_0^s w(x_1, x_2)^{q/p}dx_i\right)^{1/q}\left(\int_0^\infty e^{-sx_i}w(x_1, x_2)^{1-p'}dx_i\right)^{1/p'}=C<\infty ,$$

$i=1, 2$. *Then,*

$$\left(\int_{\boldsymbol{R}_+^2} w(x)^{q/p}(Lf)^q(x)dx\right)^{1/q}\le C\left(\int_{\boldsymbol{R}_+^2} w(x)f(x)^p dx\right)^{1/p} .$$

Again, the proof follows from (19) and Theorem 3 with $K=L$.

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