Tôhoku Math. J.
46 (1994), 125–133

CLASSIFICATION OF EINSTEIN-KÄHLER TORIC FANO FOURFOLDS

YASUHIRO NAKAGAWA

(Received September 10, 1992, revised January 26, 1993)

Abstract. Earlier the author gave the classification of Einstein-Kähler toric Fano fourfolds except in one case. In the present paper, we prove the existence of Einstein-Kähler metrics on some family of Fano manifolds including the remaining toric Fano fourfold. In particular, we completely classify the Einstein-Kähler toric Fano fourfolds.

Introduction. The Futaki invariant (see Futaki [2]) is known as an obstruction to the existence of Einstein-Kähler metrics on a Fano r-fold. If a Fano r-fold is toric, then the Futaki invariant is explicitly calculated by Mabuchi's formula [4], where a toric Fano r-fold means an r-dimensional compact connected complex manifold, with $c_1 > 0$, admitting an effective almost homogeneous algebraic group action of an r-dimensional algebraic torus $(\boldsymbol{C}^*)^r$. In [6], the author studied Einstein-Kähler metrics on toric Fano fourfolds. In particular, he considered the following question:

QUESTION 0.1. *Does a toric Fano r-fold with vanishing Futaki invariant always admit an Einstein-Kähler metric?*

For $r \leqq 3$, this question was settled (cf. Mabuchi [4], Sakane [8], Siu [10], Tian and Yau [11]). In [6], the author gave an affirmative answer to Question 0.1 for $r = 4$ except in one case $X_{1;1}$ (see Section 3) basically by using Batyrev's classification of toric Fano fourfolds [1]. The main purpose of the present paper is to prove the existence of an Einstein-Kähler metric for the remaining case $X_{1;1}$ and then give an affirmative answer to Question 0.1 in the case $r \leqq 4$. More generally, we shall prove in Section 2 the existence of Einstein-Kähler metrics on some $\boldsymbol{P}^2(\boldsymbol{C})\sharp 3\overline{\boldsymbol{P}^2(\boldsymbol{C})}$-bundles over $M \times M$, for Kähler C-spaces M, employing Nadel's existence theorem [5].

Finally, the author wishes to thank Professor Toshiki Mabuchi for his valuable suggestions and constant encouragements.

1. Notation and preliminaries. Let M be a compact connected n-dimensional complex manifold and L a holomorphic line bundle over M. We denote by $\mathrm{Aut}(M)$ the group of holomorphic automorphisms of M. We put

$$\hat{L} := p_1^* L \otimes p_2^* L^{-1} ,$$

1991 *Mathematics Subject Classification*. Primary 53C25; Secondary 14M25, 32C10, 53C55.

where $p_i\colon M\times M\to M$ denotes the projection to the i-th factor, $i=1, 2$, and L^{-1} denotes the dual line bundle of L over M. We consider the holomorphic vector bundle

$$E(M, L):=\mathbf{1}_{M\times M}\oplus\mathbf{1}_{M\times M}\oplus\hat{L}$$

over $M\times M$, of rank 3, where $\mathbf{1}_{M\times M}:=M\times M\times \boldsymbol{C}$ means the holomorphic trivial line bundle over $M\times M$. Put $E^{\circ}(M, L):=E(M, L)\setminus$(zero-section). Then $E^{\circ}(M, L)$ has a natural scalar multiplication

$$E^{\circ}(M, L)\ni a\oplus b\oplus c\mapsto ta\oplus tb\oplus tc\in E^{\circ}(M, L)\,,$$

for $t\in\boldsymbol{C}^*$. Let $\boldsymbol{P}(E(M, L)):=E^{\circ}(M, L)/\boldsymbol{C}^*$ be the $\boldsymbol{P}^2(\boldsymbol{C})$-bundle over $M\times M$ associated to $E(M, L)$, and $\pi\colon E^{\circ}(M, L)\to\boldsymbol{P}(E(M, L))$ the natural projection. $\boldsymbol{P}(E(M, L))$ over $M\times M$ has three natural cross-sections Σ_1, Σ_2, Σ_3, corresponding to the direct summands

$$\{0\}\oplus\{0\}\oplus\hat{L}\,,\quad \{0\}\oplus\mathbf{1}_{M\times M}\oplus\{0\}\,,\quad \mathbf{1}_{M\times M}\oplus\{0\}\oplus\{0\}\,,$$

respectively. Let $\mathfrak{X}(M, L)$ be a complex manifolds obtained from $\boldsymbol{P}(E(M, L))$ by simultaneously blowing up three subvarieties Σ_1, Σ_2 and Σ_3 above. Note that $\mathfrak{X}(M, L)$ over $M\times M$ is a fiber bundle with each fiber isomorphic to

$$\boldsymbol{P}^2(\boldsymbol{C})\#3\overline{\boldsymbol{P}^2(\boldsymbol{C})}\,,$$

by which we mean the complex surface obtained from $\boldsymbol{P}^2(\boldsymbol{C})$ by blowing up three points $[1:0:0]$, $[0:1:0]$, $[0:0:1]$.

FACT 1.1 (see, e.g., Oda [7]). *$\boldsymbol{P}^2(\boldsymbol{C})\#3\overline{\boldsymbol{P}^2(\boldsymbol{C})}$ admits two automorphisms ρ_1, ρ_2 induced respectively by the automorphisms*

$$\rho_1'\colon [x_0:x_1:x_2]\mapsto[x_1:x_0:x_2]\,,$$
$$\rho_2'\colon [x_0:x_1:x_2]\mapsto[x_0^{-1}:x_1^{-1}:x_2^{-1}]\,,$$

on $\boldsymbol{C}^*\times\boldsymbol{C}^*=\{[x_0:x_1:x_2]\in\boldsymbol{P}^2(\boldsymbol{C});\ x_i\neq 0$ for all $i\}\subset\boldsymbol{P}^2(\boldsymbol{C})$.

We consider the same situation in a more general context (see Lemma 1.3). Let E_1, E_2 and E_3 be the irreducible reduced exceptional divisors on $\mathfrak{X}(M, L)$ sitting over Σ_1, Σ_2 and Σ_3, respectively. We also define irreducible reduced divisors D_1', D_2', D_3' on $\boldsymbol{P}(E(M, L))$ by

$$D_1':=\pi(\{\mathbf{1}_{M\times M}\oplus\mathbf{1}_{M\times M}\oplus\{0\}\}\cap E^{\circ}(M, L))\,,$$
$$D_2':=\pi(\{\mathbf{1}_{M\times M}\oplus\{0\}\oplus\hat{L}\}\cap E^{\circ}(M, L))\,,$$
$$D_3':=\pi(\{\{0\}\oplus\mathbf{1}_{M\times M}\oplus\hat{L}\}\cap E^{\circ}(M, L))\,.$$

Let D_1, D_2 and D_3 denote, respectively, the strict transforms of D_1', D_2' and D_3' in $\mathfrak{X}(M, L)$. Moreover, for $i, j\in\{1, 2, 3\}$ with $i\neq j$, we put

$$F_{i,j}:=D_i\cap E_j\,.$$

We define a fiberwise automorphism τ_1' of $\boldsymbol{P}(E(M, L))$ over $M \times M$ obtained by interchanging the two factors of $\mathbf{1}_{M\times M} \oplus \mathbf{1}_{M\times M}$ as follows:

$$\tau_1'(\pi(a \oplus b \oplus c)) := \pi(b \oplus a \oplus c),$$

with $\pi(a \oplus b \oplus c) \in \boldsymbol{P}(E(M, L))$. We put

$$\begin{aligned} \mathfrak{X}_0(M, L) &:= \boldsymbol{P}(E(M, L)) \setminus (D_1' \cup D_2' \cup D_3') \\ &\cong \mathfrak{X}(M, L) \setminus (D_1 \cup D_2 \cup D_3 \cup E_1 \cup E_2 \cup E_3). \end{aligned} \tag{1.2}$$

Then τ_1' can be regarded as an automorphism of $\mathfrak{X}_0(M, L)$. Furthermore, we set $\hat{L}^\circ := \hat{L} \setminus$(zero-section). Let $\hat{L}_{(\xi,\eta)}$ be the fiber of $\hat{L}$ over $(\xi, \eta) \in M \times M$. Take an arbitrary $a \in \hat{L}_{(\xi,\eta)} \cap \hat{L}^\circ$ for each $(\xi, \eta) \in M \times M$. We then define $\zeta_1(a) \in (\hat{L}^{-1})_{(\xi,\eta)}$ by

$$\zeta_1(a)(b) = b/a \in \boldsymbol{C}, \qquad b \in \hat{L}_{(\xi,\eta)}.$$

Therefore, it allows us to define an isomorphism

$$\zeta_1 : \hat{L}^\circ \ni a \mapsto \zeta_1(a) \in \hat{L}^{-1} \setminus \text{(zero-section)}$$

of $\boldsymbol{C}^*$-bundles over $M \times M$. Moreover, via the identification $\mathbf{1}_{m\times m} \setminus$(zero-section)$=$ $M \times M \times \boldsymbol{C}^*$, we define an automorphism ζ_2 of $\mathbf{1}_{m\times m} \setminus$(zero-section) by

$$\zeta_2(\xi, \eta, a) := (\xi, \eta, a^{-1}), \quad (\xi, \eta, a) \in M \times M \times \boldsymbol{C}^*.$$

Let $\iota : M \times M \ni (\xi, \eta) \mapsto (\eta, \xi) \in M \times M$ be the involutive automorphism of $M \times M$. Then this ι naturally induces two biholomorphic maps

$$\begin{aligned} &\iota' : \hat{L}^{-1} (= p_1^* L^{-1} \otimes p_2^* L) = \iota^* \hat{L} \to \hat{L}, \\ &\iota'' : \mathbf{1}_{M\times M} = \iota^* \mathbf{1}_{M\times M} \to \mathbf{1}_{M\times M}, \end{aligned}$$

such that the following diagrams commute:

$$\begin{array}{ccc} \hat{L}^{-1} & \xrightarrow{\iota'} & \hat{L} \\ \downarrow & & \downarrow \\ M \times M & \xrightarrow{\iota} & M \times M, \end{array} \qquad \begin{array}{ccc} \mathbf{1}_{M\times M} & \xrightarrow{\iota''} & \mathbf{1}_{M\times M} \\ \downarrow & & \downarrow \\ M \times M & \xrightarrow{\iota} & M \times M, \end{array}$$

where all vertical arrows mean natural projections. We now define the automorphism τ_2' of $\mathfrak{X}_0(M, L)$ covering the involution ι on $M \times M$ by

$$\tau_2'(\pi(a \oplus b \oplus c)) := \pi((\iota'' \circ \zeta_2)(a) \oplus (\iota'' \circ \zeta_2)(b) \oplus (\iota' \circ \zeta_1)(c)),$$

for each $\pi(a \oplus b \oplus c) \in \mathfrak{X}_0(M, L)$. In view of (1.2), we obtain:

LEMMA 1.3. *The above τ_1' and τ_2' on $\mathfrak{X}_0(M, L)$ extend naturally to automorphisms (denoted by τ_1 and τ_2, respectively) on $\mathfrak{X}(M, L)$.*

PROOF. For an arbitrary $(\xi_0, \eta_0) \in M \times M$, we take in M sufficiently small open neighborhoods V_1 and V_2 of ξ_0 and η_0, respectively. By taking local trivializations of L over V_1, V_2, we can regard $\mathfrak{X}_0(M, L)$ locally as $V\times \boldsymbol{C}^*\times \boldsymbol{C}^*(\subset V\times \boldsymbol{P}^2(\boldsymbol{C}))$ over V for $V=V_1\times V_2$ or $V_2\times V_1$. Then τ_1' and τ_2' can be locally defined in the form

$$\tau_1' : (\xi, \eta, [x_0 : x_1 : x_2]) \mapsto (\xi, \eta, [x_1 : x_0 : x_2])\,,$$
$$\tau_2' : (\xi, \eta, [x_0 : x_1 : x_2]) \mapsto (\eta, \xi, [x_0^{-1} : x_1^{-1} : x_2^{-1}])\,,$$

where $(\xi, \eta)\in V_1\times V_2$ and $[x_0 : x_1 : x_2]\in \boldsymbol{C}^*\times \boldsymbol{C}^*(\subset \boldsymbol{P}^2(\boldsymbol{C}))$. Therefore, we can extend τ_1' and τ_2' on $\mathfrak{X}_0(M, L)$ to automorphisms on $\mathfrak{X}(M, L)$ in a manner similar to Fact 1.1. ■

Note that $\tau_1^2=\tau_2^2=\mathrm{id}_{\mathfrak{X}(M,L)}$, $\tau_1\circ\tau_2=\tau_2\circ\tau_1$, and that

$$(1.4)\qquad \begin{aligned}
&\tau_1(D_i)=D_{\nu(i)}\,, && \tau_1(E_j)=E_{\nu(j)}\,, && i, j\in\{1, 2, 3\}\,, \\
&\tau_1(F_{k,l})=F_{\nu(k),\nu(l)}\,, && && k, l\in\{1, 2, 3\}\,, \quad \text{with } k\neq l\,, \\
&\tau_2(D_i)=E_i\,, && \tau_2(E_j)=D_j\,, && i, j\in\{1, 2, 3\}\,, \\
&\tau_2(F_{k,l})=F_{l,k}\,, && && k, l\in\{1, 2, 3\}\,, \quad \text{with } k\neq l\,,
\end{aligned}$$

where ν denotes the permutation of $\{1, 2, 3\}$ fixing 1 and interchanging 2 and 3.

Furthermore, consider a 2-dimensional compact real torus $G_1 := U(1)\times U(1)$, where $U(1) := \{t\in \boldsymbol{C};\, |t|=1\}$. Then its complexification $G_1^{\boldsymbol{C}}=\boldsymbol{C}^*\times \boldsymbol{C}^*$ acts biholomorphically on $\boldsymbol{P}(E(M, L))$ by

$$\boldsymbol{P}(E(M, L))\ni \pi(a\oplus b\oplus c)\mapsto \pi(a\oplus t_1 b\oplus t_2 c)\in \boldsymbol{P}(E(M, L))\,,$$

for all $(t_1, t_2)\in G_1^{\boldsymbol{C}}$. This $G_1^{\boldsymbol{C}}$-action on $\boldsymbol{P}(E(M, L))$ extends naturally to the $G_1^{\boldsymbol{C}}$-action on $\mathfrak{X}(M, L)$. Note that the subvarieties D_i, E_j, $F_{k,l}$, with $i, j, k, l\in\{1, 2, 3\}$ and $k\neq l$, are all $G_1^{\boldsymbol{C}}$-invariant.

In this paper, we only consider the case where M is a Kähler C-space, that is, a simply connected compact complex homogeneous space with a Kähler metric. By a result of Wang [12], M can be written as $M=G/U$, where G is a simply connected complex semisimple Lie group and U is a parabolic subgroup of G. Recall that every holomorphic line bundle L over a Kähler C-space G/U is homogeneous (cf. Ise [3]). Namely, L can be written in the form $L=G\times_\rho \boldsymbol{C}$ for some 1-dimensional holomorphic representation $\rho : U\to GL(1, \boldsymbol{C})=\boldsymbol{C}^*$ of U on $\boldsymbol{C}$. Therefore, G acts naturally on L inducing a $(G\times G)$-action on $\mathfrak{X}(G/U, L)$. Then, for a maximal compact subgroup G_c of G, the product $G_c\times G_c$ acts naturally on $\mathfrak{X}(G/U, L)$. By $K(G/U, L)$, we denote the compact subgroup in $\mathrm{Aut}(\mathfrak{X}(G/U, L))$ generated by G_1, $G_c\times G_c$ and $\{\tau_1, \tau_2\}$. Now, the following lemma is straightforward from (1.4):

LEMMA 1.5. *Any reduced $K(G/U, L)^{\boldsymbol{C}}$-invariant closed analytic subspace Y of $\mathfrak{X}(G/U, L)$, with $\varnothing\neq Y\subsetneq \mathfrak{X}(G/U, L)$, is one of the following seven subspaces*:

$$\Gamma_1 := F_{2,1} \cup F_{1,3} \cup F_{1,2} \cup F_{3,1} ,$$
$$\Gamma_2 := F_{2,3} \cup F_{3,2} ,$$
$$\Gamma_3 := \Gamma_1 \cup \Gamma_2 = F_{2,1} \cup F_{2,3} \cup F_{1,3} \cup F_{1,2} \cup F_{3,2} \cup F_{3,1} ,$$
$$\Psi_1 := D_2 \cup D_3 \cup E_2 \cup E_3 ,$$
$$\Psi_2 := D_1 \cup E_1 ,$$
$$\Psi_3 := \Psi_2 \cup \Gamma_2 = D_1 \cup E_1 \cup F_{2,3} \cup F_{3,2} ,$$
$$\Psi_4 := \Psi_1 \cup \Psi_2 = D_1 \cup D_2 \cup D_3 \cup E_1 \cup E_2 \cup E_3 .$$

2. The existence of Einstein-Kähler metrics on $\mathfrak{X}(G/U, L)$. In this section, we shall show the existence theorem for Einstein-Kähler metrics on $\mathfrak{X}(G/U, L)$. Namely, we shall prove:

THEOREM 2.1. *Let G/U be a Kähler C-space. For every holomorphic line bundle L over G/U, the complex manifold $\mathfrak{X}(G/U, L)$ defined in Section* 1 *admits an Einstein-Kähler metric, provided the first Chern class $c_1(\mathfrak{X}(G/U, L))$ of $\mathfrak{X}(G/U, L)$ is positive.*

To prove this theorem, we quote the following fact on the existence of Einstein-Kähler metrics:

FACT 2.2. (Nadel [5]). *Let X be a Fano r-fold and K a compact subgroup of* $\mathrm{Aut}(X)$. *Assume that X admits no Einstein-Kähler metrics. Then there exists a $K^{\boldsymbol{C}}$-invariant closed analytic subspace $\varnothing \neq Z \subsetneqq X$, called the multiplier ideal subscheme of X, satisfying the following conditions*:

(1) $\dim_{\boldsymbol{C}}(H^i(Z, \mathcal{O}_Z)) = 0$, *for all* $i > 0$, *and* $\dim_{\boldsymbol{C}}(H^0(Z, \mathcal{O}_Z)) = 1$;

(2) *The logarithmic-geometric genus of $X \setminus Z$ vanishes.*

PROOF OF THEOREM 2.1. Suppose, for contradiction, that $\mathfrak{X}(G/U, L)$ admits no Einstein-Kähler metrics. Then there exists a $K(G/U, L)^{\boldsymbol{C}}$-invariant multiplier ideal subscheme Z of $\mathfrak{X}(G/U, L)$ by Fact 2.2, where $K(G/U, L)$ is the compact subgroup of $\mathrm{Aut}(\mathfrak{X}(G/U, L))$ defined in Section 1. Since Z is $K(G/U, L)^{\boldsymbol{C}}$-invariant, Z_{red} is one of the seven analytic subspaces Γ_1, Γ_2, Γ_3, Ψ_1, Ψ_2, Ψ_3, Ψ_4 by Lemma 1.5, where Z_{red} is the reduced analytic subspace of $\mathfrak{X}(G/U, L)$ associated to Z. By definition, Γ_1, Γ_2, Γ_3, Ψ_1, Ψ_2 and Ψ_3 are not connected. Therefore by Fact 2.2, (1), Z_{red} can be none of these six. Hence, $Z_{\mathrm{red}} = \Psi_4$. By K_1, we denote the finite subgroup of $K(G/U, L)^{\boldsymbol{C}}$ generated by τ_1 and τ_2, so that $K_1 := \{\mathrm{id}_{\mathfrak{X}(G/U, L)}, \tau_1, \tau_2, \tau_1 \circ \tau_2\} (\cong \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z})$. Let $\mathcal{I}_Z$ be the defining ideal sheaf of Z in $\mathfrak{X}(G/U, L)$, and let

$$\mathcal{I}_Z = \mathcal{P}_1 \cap \mathcal{P}_2 \cap \cdots \cap \mathcal{P}_r$$

be a primary decomposition (see, e.g., Siu [9]) of $\mathcal{I}_Z$ with primary ideal subsheaves

$\mathscr{P}_k$'s of $\mathcal{O}_{\mathfrak{X}(G/U,L)}$. We define $\mathscr{I}'_{D_i}$, $\mathscr{I}'_{E_j}$, as the intersections of all $\mathscr{P}_k$'s such that the support $\mathrm{Supp}(\mathcal{O}_{\mathfrak{X}(G/U,L)}/\mathscr{P}_k)$ of $\mathcal{O}_{\mathfrak{X}(G/U,L)}/\mathscr{P}_k$ is contained in D_i, E_j, respectively. We put

$$\mathscr{I}_{\tilde{D}_i} := \mathscr{I}'_{D_i} \cap \tau_1^* \mathscr{I}'_{D_{\nu(i)}} \cap \tau_2^* \mathscr{I}'_{E_i} \cap (\tau_1 \circ \tau_2)^* \mathscr{I}'_{E_{\nu(i)}}, \qquad i=1,2,3,$$

$$\mathscr{I}_{\tilde{E}_j} := \mathscr{I}'_{E_j} \cap \tau_1^* \mathscr{I}'_{E_{\nu(j)}} \cap \tau_2^* \mathscr{I}'_{D_j} \cap (\tau_1 \circ \tau_2)^* \mathscr{I}'_{D_{\nu(j)}}, \qquad j=1,2,3,$$

where ν is the permutation in (1.4). Then by the K_1-invariance of $\mathscr{I}_Z$, $\mathscr{I}_{\tilde{D}_i}$ and $\mathscr{I}_{\tilde{E}_j}$ are coherent ideal subsheaves of $\mathcal{O}_{\mathfrak{X}(G/U,L)}$ such that

$$\mathscr{I}_Z = \mathscr{I}_{\tilde{D}_1} \cap \mathscr{I}_{\tilde{D}_2} \cap \mathscr{I}_{\tilde{D}_3} \cap \mathscr{I}_{\tilde{E}_1} \cap \mathscr{I}_{\tilde{E}_2} \cap \mathscr{I}_{\tilde{E}_3}.$$

Let $\tilde{D}_i$, $\tilde{E}_j$, $i, j \in \{1, 2, 3\}$, denote the closed analytic subspaces of $\mathfrak{X}(G/U, L)$ defined by $\mathscr{I}_{\tilde{D}_i}$, $\mathscr{I}_{\tilde{E}_j}$, respectively. Then Z is expressible in the form

$$Z = \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3 \cup \tilde{E}_1 \cup \tilde{E}_2 \cup \tilde{E}_3,$$

and $\tilde{D}_i$, $\tilde{E}_j$, $i, j \in \{1, 2, 3\}$, satisfy $(\tilde{D}_i)_{\mathrm{red}} = D_i$, $(\tilde{E}_j)_{\mathrm{red}} = E_j$, respectively. Let

$$Z' := \tilde{D}_1 \amalg \tilde{D}_2 \amalg \tilde{D}_3 \amalg \tilde{E}_1 \amalg \tilde{E}_2 \amalg \tilde{E}_3$$

be the disjoint union of $\tilde{D}_i$, $i=1,2,3$, and $\tilde{E}_j$, $j=1,2,3$, and let $\varpi\colon Z' \to Z$ be the natural projection. Then we have a short exact sequence

$$0 \to \mathcal{O}_Z \to \varpi_* \mathcal{O}_{Z'} \to \mathscr{F} := (\varpi_* \mathcal{O}_{Z'}/\mathcal{O}_Z) \to 0, \tag{2.3}$$

where the support $\mathrm{Supp}(\mathscr{F})$ of $\mathscr{F}$ is just $\Gamma_3 = F_{2,1} \cup F_{2,3} \cup F_{1,3} \cup F_{1,2} \cup F_{3,2} \cup F_{3,1}$, and $\mathscr{F}$ is K_1-invariant. Note that $F_{2,3}$ and $F_{3,2}$ are K_1-congruent, and that $F_{2,1}$, $F_{1,3}$, $F_{1,2}$ and $F_{3,1}$ are also K_1-congruent. Moreover, all $F_{i,j}$'s, with $i, j \in \{1, 2, 3\}$ and $i \neq j$, are mutually disjoint. Now from (2.3), we obtain a long exact sequence

$$\{0\} \to H^0(Z, \mathcal{O}_Z) \to H^0(Z', \mathcal{O}_{Z'}) \to H^0(Z, \mathscr{F}) \to H^1(Z, \mathcal{O}_Z) \to \cdots. \tag{2.4}$$

Since $\tilde{D}_1$ and $\tilde{E}_1$ are K_1-congruent, and since $\tilde{D}_2$, $\tilde{D}_3$, $\tilde{E}_2$ and $\tilde{E}_3$ are also K_1-congruent, there exist non-negative integers p, q, r and s such that

$$\dim_{\mathbf{C}}(H^0(Z', \mathcal{O}_{Z'})) = 2p + 4q \quad \text{and} \quad \dim_{\mathbf{C}}(H^0(Z, \mathscr{F})) = 2r + 4s. \tag{2.5}$$

By (2.4) and (2.5) together with Fact 2.2, (1), we obtain $2p + 4q = 2r + 4s + 1$, in contradiction. Thus we can conclude that $\mathfrak{X}(G/U, L)$ admits an Einstein-Kähler metric. ■

3. The classification of Einstein-Kähler toric Fano fourfolds. First, we introduce some notation. For a positive integer n, let $\{e_i;\ i = 1, 2, \ldots, n\}$ denote the standard basis for $\mathbf{R}^n$, and put $e_0 := -(e_1 + e_2 + \cdots + e_n)$, i.e.,

$$e_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0), \qquad i = 1, 2, \ldots, n,$$

$$e_0 = (-1, -1, \ldots, -1).$$

By viewing $\boldsymbol{R}^{2n+2}$ as $\boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R} \times \boldsymbol{R}$, we consider the following vectors in $\boldsymbol{R}^{2n+2}$:

$$
\begin{array}{llll}
x_0 := (e_0, \mathbf{0}, 0, -m), & y_0 := (\mathbf{0}, e_0, 0, m), \\
x_1 := (e_1, \mathbf{0}, 0, 0), & y_1 := (\mathbf{0}, e_1, 0, 0), \\
x_2 := (e_2, \mathbf{0}, 0, 0), & y_2 := (\mathbf{0}, e_2, 0, 0), \\
\vdots & \vdots \\
x_n := (e_n, \mathbf{0}, 0, 0), & y_n := (\mathbf{0}, e_n, 0, 0), \\
z_1 := (\mathbf{0}, \mathbf{0}, 1, 0), & z_2 := (\mathbf{0}, \mathbf{0}, 0, 1), \\
z_3 := (\mathbf{0}, \mathbf{0}, -1, 1), & z_4 := (\mathbf{0}, \mathbf{0}, -1, 0), \\
z_5 := (\mathbf{0}, \mathbf{0}, 0, -1), & z_6 := (\mathbf{0}, \mathbf{0}, 1, -1),
\end{array}
$$

where m is a non-negative integer. For vectors $\mu_1, \mu_2, \ldots, \mu_l \in \boldsymbol{Z}^{2n+2} (\subset \boldsymbol{R}^{2n+2})$, let

$$\langle \mu_1, \mu_2, \ldots, \mu_l \rangle := \{a_1\mu_1 + a_2\mu_2 + \cdots + a_l\mu_l;\ a_i \in \boldsymbol{R},\ a_i \geqq 0 \text{ for all } i\}$$

be the *strongly convex rational polyhedral cone* in $\boldsymbol{R}^{2n+2}$ (see [7; p. 1]) generated by $\mu_1, \mu_2, \ldots, \mu_l$. We introduce the following strongly convex rational polyhedral cones in $\boldsymbol{R}^{2n+2}$ by using the notation in [7; p. 2]:

$$
\begin{aligned}
&\sigma_{1,i} := \langle x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle, \qquad i = 0, 1, \ldots, n, \\
&\sigma_{2,j} := \langle y_0, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n \rangle, \qquad j = 0, 1, \ldots, n, \\
&\sigma_{3,k} := \langle z_k, z_{k+1} \rangle, \qquad k = 1, 2, \ldots, 6, \\
&\Delta_1 := \{\text{the faces of } \sigma_{1,i};\ i = 0, 1, \ldots, n\}, \\
&\Delta_2 := \{\text{the faces of } \sigma_{2,j};\ j = 0, 1, \ldots, n\}, \\
&\Delta_3 := \{\text{the faces of } \sigma_{3,k};\ k = 1, 2, \ldots, 6\},
\end{aligned}
$$

where we set $z_7 := z_1$. Furthermore, define a *fan* $\Delta_{n;m}$ of $\boldsymbol{Z}^{2n+2}$ by

$$\Delta_{n;m} := \{\sigma' + \sigma'' + \sigma''';\ \sigma' \in \Delta_1,\ \sigma'' \in \Delta_2,\ \sigma''' \in \Delta_3\}.$$

Then, a fundamental result on toric varieties [7; Theorems 1.4, 1.10, 1.11] allows us to obtain a compact connected non-singular toric $(2n+2)$-fold $X_{n;m}$ corresponding to the fan $\Delta_{n;m}$ of $\boldsymbol{Z}^{2n+2}$. The following lemma is relevant to our purpose:

LEMMA 3.1. (a) *Let H be the hyperplane line bundle over $\boldsymbol{P}^n(\boldsymbol{C})$. Then the toric $(2n+2)$-fold $X_{n;m}$ is expressible as $\mathfrak{X}(\boldsymbol{P}^n(\boldsymbol{C}), H^m)$, for all n and m.*

(b) *If $m \leqq n$, then $c_1(X_{n;m}) > 0$, i.e., $X_{n;m}$ is a toric Fano $(2n+2)$-fold.*

PROOF. The statement (a) is straightforward from [7; Propositions 1.26, 1.33], and (b) also follows from [7; Lemma 2.20, (e)]. ■

REMARK 3.2. For each $\alpha \in \{1, 2\}$, the automorphism τ_α of $X_{n;m} = \mathfrak{X}(\boldsymbol{P}^n(\boldsymbol{C}), H^m)$ defined in Lemma 1.3 can be interpreted as the equivariant automorphism of $X_{n;m}$ associated to the automorphism of the fan $\Delta_{n;m}$ (see [7; p. 19]) given by the next matrix $A_\alpha \in GL(2n+2, \boldsymbol{Z})$ (I_n being the identity matrix of degree n):

$$A_1 := \left(\begin{array}{c|c|cc} \mathrm{I}_n & 0 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0 & \mathrm{I}_n & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0 \cdots 0 & 0 \cdots 0 & -1 & 0 \\ 0 \cdots 0 & 0 \cdots 0 & 1 & 1 \end{array}\right),$$

$$A_2 := \left(\begin{array}{c|c|cc} 0 & \mathrm{I}_n & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline \mathrm{I}_n & 0 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0 \cdots 0 & 0 \cdots 0 & -1 & 0 \\ 0 \cdots 0 & 0 \cdots 0 & 0 & -1 \end{array}\right).$$

Since $\boldsymbol{P}^n(\boldsymbol{C}) = SU(n+1)/S(U(1) \times U(n))$ is a Kähler C-space, Lemma 3.1 allows us to apply Theorem 2.1 to $X_{n;m}$ with $m \leqq n$. We thus obtain:

THEOREM 3.3. *If $m \leqq n$, then the toric Fano $(2n+2)$-fold $X_{n;m} = \mathfrak{X}(\boldsymbol{P}^n(\boldsymbol{C}), H^m)$ always admits an Einstein-Kähler metric.*

In particular, for $n = m = 1$, the toric Fano fourfold $X_{1;1} = \mathfrak{X}(\boldsymbol{P}^1(\boldsymbol{C}), H)$ admits an Einstein-Kähler metric. Therefore by using the notation in [6], we infer from [6] the following classification of Einstein-Kähler toric Fano fourfolds:

THEOREM 3.4. *An Einstein-Kähler toric Fano fourfold is equivariantly isomorphic to one of the following eleven toric Fano fourfolds*:

$$
\begin{array}{ll}
\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(1,-1))\times\boldsymbol{P}^1(\boldsymbol{C}), & \boldsymbol{P}^2(\boldsymbol{C})\times\boldsymbol{P}^2(\boldsymbol{C}), \\
(\boldsymbol{P}^2(\boldsymbol{C})\sharp 3\overline{\boldsymbol{P}^2(\boldsymbol{C})})\times(\boldsymbol{P}^2(\boldsymbol{C})\sharp 3\overline{\boldsymbol{P}^2(\boldsymbol{C})}), & \boldsymbol{P}^3(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C}), \\
(\boldsymbol{P}^2(\boldsymbol{C})\sharp 3\overline{\boldsymbol{P}^2(\boldsymbol{C})})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C}), & X_{P_1}, \\
(\boldsymbol{P}^2(\boldsymbol{C})\sharp 3\overline{\boldsymbol{P}^2(\boldsymbol{C})})\times\boldsymbol{P}^2(\boldsymbol{C}), & X_{P_2}=X_{1;1}, \\
\boldsymbol{P}^2(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C}), & \boldsymbol{P}^4(\boldsymbol{C}), \\
\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C}). &
\end{array}
$$

As a corollary to this theorem, we can give an affirmative answer to Question 0.1 for $r\leqq 4$. Namely, we obtain:

COROLLARY 3.5. *For a toric Fano r-fold X with* $r\leqq 4$, *the following are equivalent*:

(1) *The Futaki invariant* F_X *of X vanishes*;

(2) *X admits an Einstein-Kähler metric.*

REFERENCES

[1] V. V. BATYREV, On the classification of toric Fano 4-folds, preprint.

[2] A. FUTAKI, An obstruction to the existence of Kähler Einstein metrics, Invent. Math. 73 (1983), 437–443.

[3] M. ISE, Some properties of complex analytic vector bundles over compact complex homogeneous spaces, Osaka Math. J. 12 (1962), 217–252.

[4] T. MABUCHI, Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties, Osaka J. Math. 24 (1987), 705–737.

[5] A. M. NADEL, Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. 132 (1990), 549–596.

[6] Y. NAKAGAWA, Einstein-Kähler toric Fano fourfolds, Tohoku Math. J. 45 (1993), 297–310.

[7] T. ODA, "Convex bodies and algebraic geometry: An introduction to the theory of toric varieties," Ergebnisse der Math. (3) 15, Springer-Verlag, Heidelberg, 1988.

[8] Y. SAKANE, Example of compact Einstein Kähler manifolds with positive Ricci tensor, Osaka J. Math. 23 (1986), 585–616.

[9] Y.-T. SIU, Noether-Lasker decomposition of coherent analytic subsheaves, Trans. Amer. Math. Soc. 135 (1969), 375–385.

[10] Y.-T. SIU, The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group, Ann. of Math. 127 (1988), 585–627.

[11] G. TIAN AND S.-T. YAU, Kähler-Einstein metrics on complex surfaces with $C_1>0$, Commun. Math. Phys. 112 (1987), 175–203.

[12] H.-C. WANG, Closed manifolds with homogeneous complex structure, Amer. J. Math. 76 (1954), 1–32.

MATHEMATICAL INSTITUTE
FACULTY OF SCIENCE
TOHOKU UNIVERSITY
SENDAI 980
JAPAN