Tôhoku Math. J.
46 (1994), 83–104

HEAT-DIFFUSION AND POISSON INTEGRALS FOR LAGUERRE EXPANSIONS

Dedicated to Jarosław Gela—may high school mathematics teacher.

KRZYSZTOF STEMPAK[1]

(Received August 31, 1992, revised May 10, 1993)

Abstract. We investigate the heat-diffusion and Poisson integrals for expansions with respect to three different systems of Laguerre functions. The main achievements of the paper are the weak type (1, 1) estimates for the associated maximal functions.

1. Introduction. Muckenhoupt [Mu 1] studied Poisson integrals for Hermite and Laguerre polynomial expansions. The aim of this paper is to go further and discuss Poisson integrals for expansions with respect to three different systems of Laguerre functions. More specifically, we successively consider the systems of functions $l_n^a(x)$, $\mathscr{L}_n^a(x)$ and $\varphi_n^a(x)$, $n=0, 1, \ldots$, (cf. the beginnings of the sections that follow for the definitions and the range of the parameter a) forming an orthonormal basis in $L^2(\mathscr{R}_+, x^a dx)$ in the first, and in $L^2(\mathscr{R}_+, dx)$ in the second and third cases.

Muckenhoupt [Mu 1] proved that if the Poisson integral $g(r, x)$ of a function $f(x)$ in $L^1(\mathscr{R}_+, e^{-x}x^a dx)$ is defined by

$$g(r, x)=\int_0^\infty K(r, x, y)f(y)e^{-y}y^a dy\,, \qquad 0<r<1\,,$$

with the Poisson kernel

$$K(r, x, y)=\sum_0^\infty r^n \frac{n!}{\Gamma(n+a+1)} L_n^a(x)L_n^a(y)\,,$$

then $g(r, x)$ converges to $f(x)$ almost everywhere and in L^p-norm as $r\to 1^-$ whenever $f\in L^p(\mathscr{R}_+, e^{-x}x^a dx)$, $1\le p<\infty$.

A similar question concerning a.e. and L^p-convergence may be raised when discussing Poisson integrals for expansions with respect to a system of Laguerre functions. One could expect an interplay between these and the Laguerre polynomial expansion and, in fact, this is the case at least when the a.e. convergence is concerned. To show how it goes consider, for instance, the expansion with respect to the system $l_n^a(x)$, $a>-1, n=0, 1, \ldots$, and for any $f_1\in L^p(x^a dx)$, $1\le p\le\infty$, define its Poisson integral

[1] The paper was written while the author was visiting the Department of Mathematics, University of Georgia, during the 1990–91 academic year.

1991 *Mathematics Subject Classification*. Primary 42C05; Secondary 42B25.

$$g_1(r, x) = \int_0^\infty K_1(r, x, y) f_1(y) e^{-y} y^a dy , \qquad 0 < r < 1 ,$$

with the Poisson kernel

$$K_1(r, x, y) = \sum_0^\infty r^n l_n^a(x) l_n^a(y) = K(r, x, y) e^{-(x+y)/2} .$$

Now, setting $f(x) = f_1(x) e^{x/2}$, by Hölder's inequality f is in $L^1(e^{-x} x^a dx)$ and the corresponding Poisson integrals are related by

$$g(r, x) = e^{x/2} g_1(r, x) .$$

Therefore, Muckenhoupt's result immediately gives $e^{x/2} g_1(r, x) \to e^{x/2} f_1(x)$ almost everywhere and thus also $g_1(r, x)$ converges to $f_1(x)$ a.e. as $r \to 1$. It is however difficult to imagine the existence of a similar argument when considering L^p-convergence (it would be probably possible when having a suitable weighted result for the L^p-convergence of Poisson integrals in the polynomial case).

The main achievements of this paper are the weak type (1, 1) estimates for maximal functions associated with the heat-diffusion and the Poisson integrals for all Laguerre function expansions we consider. Such estimates give a.e. convergence as an immediate by-product but at the same time they are important for its own sake. Also, as one can note following our arguments, proving L^p-convergence of Poisson integrals requires equivalent amount of efforts by using identical estimates as for the weak type (1, 1) result.

In this paper we consequently look at considered expansions through the fact that they are spectral decompositions of a suitable second order differential operators. This explains an apparent ambiguity between the semigroup terminology we use (the heat-diffusion and Poisson integrals) and that used by Muckenhoupt [Mu 1] (the Poisson and alternate Poisson integrals correspondingly). Also, we would like to emphasise that in this paper we closely follow Muckenhoupt's ideas from [Mu 1]. However, the polynomial case had some advantages (also some disadvantages to be fair) and therefore considering Laguerre function cases requires additional efforts.

The weak type (1, 1) estimate for the maximal Poisson function in the polynomial case was obtained by majorizing Poisson integrals by suitable Hardy-Littlewood maximal function. To do this Muckenhoupt used the following simple but insightful device.

LEMMA 1.1 ([Mu 1, p. 233]). *Let μ be a positive absolutely continuous measure on $(0, \infty)$ and for any $f \in L^p(d\mu)$, $1 \le p \le \infty$, let $h(x) = \int_0^\infty K(x, y) f(y) d\mu(y)$ with the kernel $K(x, y)$ making the integral absolutely convergent and satisfying the properties*:

(a) $|K(x, y)| \le L(x, y)$;

(b) $\int_0^\infty L(x, y) d\mu(y) \le C$ *with C independent of x*;

(c) *$L(x, y)$ is monotone increasing in y for $y \le x$ and monotone decreasing for $y \ge x$.*

Then $|h(x)| \leq Cf^*(x)$ *where*

$$f^*(x) = \sup_{x \neq z > 0} \left(\int_x^z |f(y)| d\mu(y) \Big/ \int_x^z d\mu(y) \right). \tag{1.1}$$

Originally it was assumed that μ is a finite measure but it may be checked that this assumption is irrelevant.

To prove the weak type (1, 1) result in §2 we will use the Hardy-Littlewood maximal function (1.1) with $d\mu(y) = y^a dy$, where $a > -1$ is a fixed parameter. If we denote $B_\varepsilon(c) = \{y \in (0, \infty) : |x-y| < \varepsilon\}$ then the measure μ satisfies the doubling condition

$$\mu(B_{2\varepsilon}(x)) \leq C\mu(B_\varepsilon(x))$$

with $C > 0$ independent of $x \in (0, \infty)$ and $\varepsilon > 0$. This implies that the maximal function

$$Mf(x) = \sup_{\varepsilon > 0} \mu(B_\varepsilon(x))^{-1} \int_{B_\varepsilon(x)} |f| d\mu$$

is of weak type (1, 1). It may be also easily proved that Mf dominates f^*. Therefore we obtain:

LEMMA 1.2. *Let* $a > -1$ *and* f^* *be given by* (1.1) *with* $d\mu(y) = y^a dy$. *Then*

$$\mu(\{x : f^*(x) > \lambda\}) \leq \frac{C}{\lambda} \int_0^\infty |f| d\mu,$$

that is, $f \to f^*$ *is of weak type* (1, 1).

We will also need the following weak version of a density result proved by Muckenhoupt.

LEMMA 1.3 ([Mu 2, Lemma 1]). *If* $1 \leq p < \infty$ *and* d *is a fixed real number, then the functions* $x^d \exp(-x/2)P(x)$, $P(x)$ *being polynomial, are dense in* $L^p(\mathcal{R}_+, dx)$.

The expansions with respect to the Laguerre polynomials (as well as those of Hermite) in some way behave very badly. For instance, not only do partial sums fail to converge in L^p-norm unless $p = 2$, which was proved by Pollard [Po], but even Cesàro means of any order $\delta > 0$ fail to converge unless $p = 2$, as proved by Askey and Hirschman [A-H]. A deep reason for such bad behavior of polynomial expansions is due to the fact that for $p > 2$, L^p-norms of L^2-normalized Laguerre (or Hermite) polynomials grow exponentially. More precisely, if $(n!/\Gamma(n+a+1))^{1/2} L_n^a(x)$, $a > -1$, is the n-th normalized Laguerre polynomial of order a it may be read off from [Po], that for $p > 2$ and every $1 < A < p-1$

$$\limsup_{n \to \infty} \frac{1}{A^n} \left(\int_0^\infty \left| \left(\frac{n!}{\Gamma(n+a+1)} \right)^{1/2} L_n^a(x) \right|^p e^{-x} x^a dx \right)^{1/p} = \infty.$$

Here are details. Assuming that for all n

$$\|(n!/\Gamma(n+a+1))^{1/2}L_n^a(x)\|_{L^p(e^{-x}x^a dx)} \leq CA^n$$

we have for any $f \in L^q(e^{-x}x^a dx)$, $1/p+1/q=1$,

$$(1.2) \qquad |\hat{f}(n)| = \left|(n!/\Gamma(n+a+1))^{1/2}\int_0^\infty L_n^a(x)f(x)e^{-x}x^a dx\right|$$
$$\leq \|(n!/\Gamma(n+a+1))^{1/2}L_n^a\|_p \|f\|_q \leq CA^n\|f\|_q .$$

Meanwhile, the function $f(x)=e^{cx}$, $1/2<c<1/q$, belongs to $L^q(e^{-x}x^a dx)$ and, as one can check,

$$\hat{f}(n)=(-1)^n\frac{1}{(1-c)^{a+1}}\left(\frac{\Gamma(n+a+1)}{\Gamma(n+1)}\right)^{1/2}\left(\frac{c}{1-c}\right)^n .$$

Choosing c, $1/2<c<1/q$, such that $c/(1-c)>A$ we get a contradiction with (1.2).

In this paper, for any of the three systems of functions we consider, $l_n^a(x)$, $\mathscr{L}_n^a(x)$ and $\varphi_n^a(x)$, $n=0, 1, 2, \ldots$, the L^p-norms of corresponding Laguerre functions have polynomial growth, as opposed to the polynomial expansion case. This, for instance, allows us to define heat-diffusion and Poisson integrals by using series and speak about the Abel-Poisson summability instead of convergence of Poisson integrals. In the polynomial expansion case, for the reason described above, merely defining Poisson integral by a suitable series is not satisfactory and an integral definition must be used. As pointed out by Muckenhoupt, the function considered by Pollard $f(x)=e^{cx}$, $1/2<c<1/q$, is, for every $1\leq q<2$, an example of function in $L^q(e^{-x}x^a dx)$ such that for all $r<1$ sufficiently close to 1 the series $\sum r^n a_n L_n^a(x)$ diverges for every x. Here

$$a_n=\frac{n!}{\Gamma(n+a+1)}\int_0^\infty f(x)L_n^a(x)e^{-x}x^a dx .$$

Finally, we would like to note that for ranges of parameter a smaller than what we consider, some of the results contained in this paper were proved by different authors using different, sometimes sophisticated and involved methods (cf. remarks in the following sections). In this paper, in each case discussed, we consider the largest possible range of parameter a. Furthermore, purely real-variable method we use following Muckenhoupt's ideas, is in our opinion the most elementary and straightforward.

The author would like to thank Ryszard Szwarc for a valuable remark.

2. The Laguerre functions $l_n^a(x)$. Throughout this section it will be assumed that a is a fixed number greater than -1. The symbol $\|f\|_p$ will be used to denote p-th norm of a function $f(x)$ on $(0, \infty)$ with respect to the measure $x^a dx$. The functions

$$l_n^a(x)=(n!/\Gamma(n+a+1))^{1/2}L_n^a(x)e^{-x/2} ,$$

$n=0, 1, \ldots$, form an orthonormal basis in $L^2(\mathscr{R}_+, x^a dx)$. Expansions with respect to this system were considered by the author in [St 1], [St 2] and by Thangavelu, for instance, in [T2]. We will need the following estimate

$$\|l_n^a\|_p \le C n^{\varepsilon(p,a)}, \tag{2.1}$$

$1 \le p \le \infty$, which is a consequence, after a correction, of an estimate by Markett [Ma, Lemma 1]. More precisely, from [Ma] we have ($a_n \sim b_n$ will stand for $a_n = O(b_n)$ and $b_n = O(a_n)$ as $n \to \infty$):

(A) Case $1 \le p < 4$:

(a) If $a > 0$

$$\|l_n^a\|_p \sim \begin{cases} n^{(a+1)(1/p-1/2)}, & 1 \le p < 4(a+1)/(2a+1), \\ n^{-1/4}(\log n)^{1/p}, & p = 4(a+1)/(2a+1), \\ n^{(a+1)(1/2-1/p)-1/2}, & 4(a+1)/(2a+1) < p < 4. \end{cases}$$

(b) If $-1 < a \le 0$

$$\|l_n^a\|_p \sim n^{(a+1)(1/p-1/2)}.$$

(B) Case $p = 4$:

$$\|l_n^a\|_4 \sim \begin{cases} n^{-(a+1)/4}(\log n)^{1/4}, & a \le 0, \\ n^{(a+1)/4-1/2}, & a > 0. \end{cases}$$

(C) Case $4 < p \le \infty$:

(a) If $a \ge 0$

$$\|l_n^a\|_p \sim n^{(a+1)(1/2-1/p)-1/2}.$$

(b) If $-1/3 \le a < 0$

$$\|l_n^a\|_p \sim \begin{cases} n^{(a+1)(1/p-1/2)+(1-4/p)/6}, & 4 < p \le 2(3a+2)/(3a+1), \\ n^{(a+1)(1/2-1/p)-1/2}, & p > 2(3a+2)/(3a+1). \end{cases}$$

(c) If $-1 < a < -1/3$

$$\|l_n^a\|_p \sim n^{(a+1)(1/p-1/2)+(1-4/p)/6}.$$

We take $\alpha = 2a/p$ and $\beta = (1-2/p)a$, $1 \le p < \infty$, in Lemma 1 of [Ma] to obtain Case (A), Case (B) part $a > 0$ and Case (C) except $p = \infty$. When $p = \infty$ then the case $a \ge 0$ is covered by Lemma 1 of [Ma] and remaining part $a < 0$ is proved by using exactly the same arguments. Part (B), Case $a \le 0$, comes from our correction of Lemma 1, [Ma] (there the second line in (2.9) with $p = 4$, i.e. the case $\beta < 0$, should read $n^{\alpha/2-1/4}(\log n)^{1/4}$ rather than $n^{\alpha/2-1/4}$ only).

The function l_n^a is an eigenfunction of the differential operator $L = xd^2/dx^2 + (a+1)d/dx - x/4$ with eigenvalue $-(n+(a+1)/2)$. The operator $-L$ is positive and symmetric in $L^2(x^a dx)$. For any function $f(x)$ in $L^p(x^a dx)$, $1 \le p \le \infty$, with the expansion

$\sum b_n l_n^a(x)$, where

$$b_n = \int_0^\infty f(y) l_n^a(y) y^a dy ,$$

we define its heat-diffusion integral $g(t, x)$, $t>0$, by

$$g(t, x) = \sum_0^\infty \exp\left(-t\left(n + \frac{a+1}{2}\right)\right) b_n l_n^a(x). \tag{2.2}$$

For any fixed $t>0$ the series in (2.2) converges uniformly. This is because $\|l_n^a\|_\infty \le Cn^{\varepsilon(a)}$ and by Hölder's inequality and (2.1)

$$|b_n| \le \|f\|_p \|l_n^a\|_q \le Cn^{\varepsilon(q)} .$$

Equivalently,

$$g(t, x) = \int_0^\infty P(t, x, y) f(y) y^a dy , \tag{2.3}$$

where

$$P(t, x, y) = \sum_0^\infty \exp\left(-t\left(n + \frac{a+1}{2}\right)\right) l_n^a(x) l_n^a(y) .$$

Interchanging the order of integration and summation in (2.3) is justified by the dominated convergence theorem, since, by (2.1)

$$\sum_0^\infty \exp\left(-t\left(n + \frac{a+1}{2}\right)\right) \|l_n^a\|_\infty \|l_n^a\|_q \|f\|_p < \infty .$$

In particular, for f equal to 1 identically, by use of

$$\int_0^\infty L_n^a(y) e^{-y/2} y^a dy = (-1)^n 2^{a+1} \frac{\Gamma(a+n+1)}{n!}$$

(cf. [GR, p. 845]) and the fact that

$$\sum_0^\infty z^n L_n^a(x) = (1-z)^{-a-1} \exp\left(\frac{xz}{z-1}\right),$$

we obtain

$$\int_0^\infty P(t, x, y) y^a dy = \frac{1}{(\cosh t/2)^{a+1}} \cdot \exp\left(-\frac{x}{2} \tanh t/2\right) . \tag{2.4}$$

LEMMA 2.1. *For every $f(x)$ in $L^p(x^a dx)$, $1 \le p \le \infty$, the heat-diffusion integral $g(t, x)$ is a C^∞-function satisfying the differential equation*

$$\left(L_x - \frac{\partial}{\partial t}\right) g(t, x) = 0 . \tag{2.5}$$

PROOF. Simple calculus shows that for $k = 1, 2, \ldots$

$$\|(d/dx)^k l_n^a\|_\infty \leq C n^\varepsilon$$

with an ε depending on k and a only. This is because

$$(d/dx) l_n^a = -n^{1/2} l_{n-1}^{a+1} - \frac{1}{2} l_n^a$$

and similarly for higher derivatives. Therefore, we can differentiate the series in (2.2) term by term with respect to the x-variable. Differentiation with respect to the t-variable is even easier so $g(t, x)$ is a C^∞-function satisfying (2.5).

It is now convenient to switch to the kernel

$$R(r, x, y) = \sum r^n l_n^a(x) l_n^a(y) ,$$

$0 < r < 1$, rather than work with $P(t, x, y)$. A formula from [Sz, p. 102], then gives

$$R(r, x, y) = \frac{1}{1-r} \frac{1}{(rxy)^{a/2}} \exp\left(-\frac{1}{2}\frac{1+r}{1-r}(x+y)\right) I_a\left(\frac{2(rxy)^{1/2}}{1-r}\right),$$

where $I_a(x) = i^{-a} J_a(ix)$ is the usual Bessel function of an imaginary argument. Note that this gives positivity of $P(t, x, y)$. Applying Muckenhoupt's argument, [Mu 1, p. 238], then produces

$$cH(r, x, y) \leq R(r, x, y) \leq CH(r, x, y)$$

with positive constants c and C depending only on a where

$$H(r, x, y) = \begin{cases} \dfrac{1}{(1-r)^{a+1}} \exp\left(-\dfrac{1}{2}\dfrac{1+r}{1-r}(x+y)\right) \\ \dfrac{(4rxy)^{-(a+1/2)/2}}{e(1-r)^{1/2}} \exp\left(-\dfrac{1}{2}\dfrac{1+r}{1-r}(x+y) + \dfrac{2(rxy)^{1/2}}{1-r}\right) \end{cases}$$

for $y \leq (1-r)^2/4rx$ and $y \geq (1-r)^2/4rx$ correspondingly.

The following lemma is crucial in what follows.

LEMMA 2.2. *There exists a function $L(r, x, y)$ satisfying the following properties:*

(i) *$H(r, x, y) \leq L(r, x, y)$;*

(ii) *for every $0 < r < 1$ and $x > 0$, $L(r, x, y)$ as a function of y is monotone increasing on $[0, x]$ and monotone decreasing on $[x, \infty)$;*

(iii) *$\int_0^\infty L(r, x, y) y^a dy \leq C$ independently of $x > 0$ and $0 < r < 1$.*

PROOF. We simply define $L(r, x, y)$ to be the least majorant of $H(r, x, y)$ satisfying (ii). Therefore $L(r, x, y)$ is the largest of $H(r, x, y)$ and any maxima of $H(r, x, t)$ occurring for t on the opposite side of y from x. So we have to check (iii) only.

For fixed r, x the function $H(r, x, y)$ is decreasing on $0<y\leq(1-r)^2/4rx$ and continuous at $y=(1-r)^2/4rx$. Therefore, if m is a point where a local maximum of $H(r, x, y)$ is taken then $m>(1-r)^2/4rx$. By differentiating the second function in the definition of $H(r, x, y)$ one can check that possible local extrema occur at

$$\left(\frac{(rx)^{1/2}\pm(rx-(a+1/2)(1-r^2))^{1/2}}{1+r}\right)^2.$$

By analysing the sign of the derivative we then conclude that, if exists, the single local maximum of $H(r, x, y)$ is taken at

$$m=\left(\frac{(rx)^{1/2}+(rx-(a+1/2)(1-r^2))^{1/2}}{1+r}\right)^2. \tag{2.6}$$

It is clear now that

$$\int_0^\infty L(r, x, y)y^a dy\leq\int_0^\infty H(r, x, y)y^a dy+I_0+I_m$$

where $I_0=H(r, x, 0)\int_0^x y^a dy$, and $I_m=H(r, x, m)\left|\int_x^m y^a dy\right|$.

Using (2.4) we are now reduced to showing that I_0, I_m are bounded independently of r and x. Estimating I_0 gives

$$I_0=\frac{1}{(1-r)^{a+1}}\exp\left(-\frac{1}{2}\frac{1+r}{1-r}x\right)x^{a+1}/(a+1)\leq C\left(\frac{x}{1-r}\right)^{a+1}\exp\left(-\frac{1}{2}\frac{x}{1-r}\right),$$

which is less than a constant depending only on a. The estimate of I_m is more elaborate. We start with recalling that

$$m>(1-r)^2/4rx. \tag{2.7}$$

By elementary inspection one can find that this implies

$$x>\frac{1}{6}\frac{1-r^2}{r} \tag{2.8}$$

(if we assume the opposite inequality then $m\leq6(1-r)/4(1+r)$ and together these would give a contradiction). Consequently, (2.8) produces

$$\frac{rx}{(1+r)^2}\leq m\leq9\frac{rx}{(1+r)^2}$$

or, rather,

$$\frac{1}{4}rx\leq m\leq9rx. \tag{2.9}$$

A bit of calculus also shows that $\exp(-(1-r)x/2(1+r))$ is the maximum value of

$$f(y)=\exp\left(-\frac{1}{2}\frac{1+r}{1-r}(x+y)+\frac{2(rxy)^{1/2}}{1-r}\right)$$

attained at $y=4rx/(1+r)^2$. This and (2.9) now shows that I_m is less than

$$\frac{C}{(1-r)^{1/2}(rx)^{a+1/2}}\cdot\exp\left(-\frac{1}{2}\frac{1-r}{1+r}x\right)|m-x|\cdot\max\{m^a, x^a\}. \tag{2.10}$$

Now we need also a good estimate for $|m-x|$ which is

$$|m-x|\leq C(1-r)[1+x(1-r)]. \tag{2.11}$$

To get this we write

$$\left|m-\frac{4r}{(1+r)^2}x\right|=\left|\sqrt{m}+\frac{2\sqrt{rx}}{1+r}\right|\left|\sqrt{m}-\frac{2\sqrt{rx}}{1+r}\right|$$

and by (2.9) the first term is bounded by $C(rx)^{1/2}$. Using (2.6) the second part is estimated by

$$\left|\frac{(rx)^{1/2}-(rx-(a+1/2)(1-r^2))^{1/2}}{1+r}\right|=\frac{|a+1/2|(1-r^2)}{(1+r)[(rx)^{1/2}+(rx-(a+1/2)(1-r^2))^{1/2}]}$$
$$\leq C\frac{1-r}{(rx)^{1/2}}.$$

Together this gives

$$\left|m-\frac{4r}{(1+r)^2}x\right|\leq C(1-r).$$

Next,

$$\left|\frac{4r}{(1+r)^2}x-x\right|=x\left(\frac{1-r}{1+r}\right)^2\leq x(1-r)^2,$$

so by triangle inequality we get (2.11). Coming back to (2.10) the estimate of I_m becomes

$$I_m\leq C(1-r)^{1/2}(1+x(1-r))\exp\left(-\frac{1}{4}(1-r)x\right)\max\{m^a, x^a\}/(rx)^{a+1/2}.$$

To show that the above is bounded independently on x and r we consider two cases.
Case 1: $x<m$. Then by (2.9) $r\geq 1/9$ and hence

$$\max\{m^a, x^a\}\leq Cx^a.$$

Therefore using (2.8) gives $(1-r)/x\leq C$, so

$$I_m \le C\left(\frac{1-r}{x}\right)^{1/2}(1+x(1-r))\exp\left(-\frac{1}{4}(1-r)x\right) \le C$$

since the function $(1+t)\exp(-t/4)$ is bounded.
Case 2: $x \ge m$. If $a<0$ then $\max\{m^a, x^a\} = m^a \le C(rx)^a$ and therefore

$$I_m \le C\left(\frac{1-r}{rx}\right)^{1/2}(1+x(1-r))\exp\left(-\frac{1}{4}(1-r)x\right) \le C .$$

If $a \ge 0$ then $\max\{m^a, x^a\} = x^a$, so

(i) for $r \ge 1/2$, $I_m \le C((1-r)/rx)^{1/2}(1+x(1-r))\exp(-x(1-r)/4) \le C$;

(ii) for $0<r<1/2$, $rx>(1-r^2)/6>C$ and

$$I_m \le C(1+x(1-r))x^a \exp\left(-\frac{1}{4}(1-r)x\right) \le C(1+x)x^a \exp\left(-\frac{x}{8}\right) \le C .$$

This completes the proof of Lemma 2.2.

We are now ready to prove the main theorem concerning heat-diffusion integrals for expansions we consider.

THEOREM 2.3. *Let $a>-1$, $1 \le p \le \infty$ and $f \in L^p(x^a dx)$. Let $g(t,x)$ denote its heat-diffusion integral* (2.1) *and $f^*(x)$ the maximal function* (1.1) *with $d\mu(x) = x^a dx$. Then*

$$|g(t,x)| \le C\exp(-t(a+1)/2)f^*(x) \tag{2.12}$$

and, consequently, the associated maximal function is of weak-type (1,1), *i.e.*

$$\mu\left(\left\{x : \sup_{t>0} |g(t,x)| > \lambda\right\}\right) \le \frac{C}{\lambda}\int_0^\infty |f| d\mu ,$$

which implies that $\lim_{t\to 0} g(t,x) = f(x)$ almost everywhere. Moreover

$$\|g(t,x)\|_p \le (\cosh t/2)^{-(a+1)}\|f\|_p \tag{2.13}$$

which gives $\|g(t,x)-f(x)\|_p \to 0$ as $t \to 0$, for $1 \le p < \infty$.

PROOF. (2.12) is an immediate consequence of Lemma 2.2, Lemma 1.1 and the identity

$$P(t,x,y) = \exp\left(-t\frac{a+1}{2}\right)R(e^{-t},x,y) .$$

Then the weak type (1,1) estimate follows by Lemma 1.2. To prove (2.13) we note that by Hölder's inequality, using (2.4), we get

$$|g(t,x)|^p \le ((\cosh t/2)^{-(a+1)})^{p/q}\int_0^\infty |f(y)|^p P(t,x,y)y^a dy .$$

Integrating with respect to x and interchanging the order of integration gives (2.13), $1 \leq p < \infty$. The case $p = \infty$ is obvious. To prove a.e. and L^p-convergence we consider the linear space of polynomials multiplied by $e^{-x/2}$ which is dense in $L^p(x^a dx)$, $1 \leq p < \infty$. This easily follows by applying Lemma 1.3 with $d = a/p$. For any function of this type the series in (2.1) is finite and converges to this function both, in L^p-norm, $1 \leq p < \infty$, and almost everywhere.

We now pass to the Poisson integrals. For a function $f(x)$ in $L^p(x^a dx)$, $1 \leq p \leq \infty$, with the expansion $\sum b_n l_n^a(x)$ we define its Poisson integral $f(t, x)$, $t > 0$, by

$$f(t, x) = \sum_0^\infty \exp\left(-t\left(n + \frac{a+1}{2}\right)^{1/2}\right) b_n l_n^a(x) . \tag{2.14}$$

Equivalently,

$$f(t, x) = \int_0^\infty Q(t, x, y) f(y) y^a dy , \tag{2.15}$$

where

$$\begin{aligned} Q(t, x, y) &= \sum_0^\infty \exp\left(-t\left(n + \frac{a+1}{2}\right)^{1/2}\right) l_n^a(y) l_n^a(x) \\ &= \frac{t}{\sqrt{4\pi}} \int_0^\infty P(s, x, y) s^{-3/2} e^{-t^2/4s} ds . \end{aligned} \tag{2.16}$$

The last identity is obtained by using the well-known formula

$$e^{-\beta} = \frac{\beta}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-s} e^{-\beta^2/4s} ds .$$

Interchanging the order of integration and summation in (2.16) as well as in (2.15) is justified by the dominated convergence theorem.

COROLLARY 2.4. *Let $f \in L^p(x^a dx)$, $1 \leq p \leq \infty$, $a > -1$ and $f(t, x)$ denote its Poisson integral. Then $f(t, x)$ is a C^∞-function satisfying the differential equation*

$$\left(L_x + \frac{\partial^2}{\partial t^2}\right) f(t, x) = 0 .$$

Moreover the conclusions of Theorem 2.3 *are valid with $g(t, x)$ replaced by $f(t, x)$ and the factors* $\exp(-t(a+1)/2)$ *and* $(\cosh t/2)^{-(a+1)}$ *replaced by* 1.

PROOF. Using (2.16) and interchanging the order of integration produces

$$f(t, x) = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-t^2/4s} g(s, x) ds .$$

This, the fact that

$$\frac{t}{\sqrt{4\pi}}\int_0^\infty s^{-3/2}e^{-t^2/4s}ds=1$$

and Minkowski's integral inequality together imply (2.12) and (2.13) in Theorem 2.3 with $g(t,x)$ replaced by $f(t,x)$ and the factors $\exp(-t(a+1)/2)$ and $(\cosh t/2)^{-(a+1)}$ dropped.

REMARK 2.5. In case $a\geq 0$ the results from Theorem 2.3 where proved by the author in [St 1], [St 2] (cf. also [T2]) by using a generalized convolution structure defined in $L^1(x^a dx)$. That approach did not allow to cover the case $-1<a<0$.

3. The Laguerre functions $\mathscr{L}_n^a(x)$. In this section we work with the Laguerre functions

$$\mathscr{L}_n^a(x)=(n!/\Gamma(n+a+1))^{1/2}L_n^a(x)e^{-x/2}x^{a/2},$$

forming an orthonormal basis in $L^2(0,\infty)$. Since these functions belong to all L^p-spaces, $1\leq p\leq\infty$, only if $a\geq 0$, it will be assumed now that a is a fixed nonnegative number. Then, by [Ma, Lemma 1],

$$\|\mathscr{L}_n^a\|_p\sim\begin{cases} n^{1/p-1/2}, & 1\leq p<4, \\ n^{-1/4}(\log n)^{1/4}, & p=4, \\ n^{-1/p}, & 4<p\leq\infty. \end{cases} \tag{3.1}$$

The symbol $\|f\|_p$ is now used to denote the p-th norm of a function $f(x)$ on $(0,\infty)$ with respect to the Lebesgue measure dx. The function $\mathscr{L}_n^a$ is an eigenfunction of the differential operator

$$L=x\frac{d^2}{dx^2}+\frac{d}{dx}-\left(\frac{x}{4}+\frac{a^2}{4x}\right)$$

with eigenvalue $-(n+(a+1)/2)$. The operator $-L$ is positive and symmetric in $L^2(0,\infty)$. With any $f(x)$ in $L^p(0,\infty)$, $1\leq p\leq\infty$, we associate its expansion given by the formal series $\sum c_n\mathscr{L}_n^a(x)$ where

$$c_n=\int_0^\infty f(y)\mathscr{L}_n^a(y)dy.$$

Then we define its heat-diffusion integral $g(t,x)$, $t>0$,

$$g(t,x)=\sum_0^\infty \exp\left(-t\left(n+\frac{a+1}{2}\right)\right)c_n\mathscr{L}_n^a(x) \tag{3.2}$$

or, equivalently,

$$g(t, x)=\int_0^\infty P(t, x, y)f(y)dy\,, \tag{3.3}$$

where

$$P(t, x, y)=\sum_0^\infty \exp\left(-t\left(n+\frac{a+1}{2}\right)\right)\mathscr{L}_n^a(x)\mathscr{L}_n^a(y)\,.$$

Due to the estimate (3.1), for any fixed $t>0$ the series in (3.2) converges uniformly and interchanging of order of integration and summation in (3.2) is justified by the dominated convergence theorem.

LEMMA 3.1. *Let $f\in L^p(0, \infty)$, $1\le p\le\infty$. Then its heat-diffusion integral* (3.2) *is a C^∞-function satisfying the differential equation*

$$\left(L_x-\frac{\partial}{\partial t}\right)g(t, x)=0\,. \tag{3.4}$$

PROOF. Clearly $g(t, x)$ is differentiable with respect to the t-variable and

$$\frac{\partial^k}{\partial t^k}g(t, x)=\sum_0^\infty\left(-\left(n+\frac{a+1}{2}\right)\right)^k\exp\left(-t\left(n+\frac{a+1}{2}\right)\right)c_n\mathscr{L}_n^a(x)\,. \tag{3.5}$$

Then we show that $(\partial^k/\partial t^k)g(t, x)$ is differentiable with respect to the x-variable on $x>\varepsilon$. This is, since

$$\frac{d}{dx}\mathscr{L}_n^a(x)=-n^{1/2}\mathscr{L}_{n-1}^{a+1}(x)x^{-1/2}-\frac{1}{2}\mathscr{L}_n^a(x)+\frac{a}{2}\mathscr{L}_n^a(x)x^{-1}\,,$$

and using (3.1) we get $|(d/dx)\mathscr{L}_n^a(x)|\le Cn^{1/2}$ on (ε, ∞). Therefore for fixed $t>0$ we can differentiate the series in (3.5) term by term. Similarly for higher derivatives.

In the same way as in §2 we will work with the kernel

$$R(r, x, y)=\sum r^n\mathscr{L}_n^a(x)\mathscr{L}_n^a(y)\,,\qquad 0<r<1\,,$$

rather than with $P(t, x, y)$. This time

$$R(r, x, y)=\frac{r^{-a/2}}{1-r}\exp\left(-\frac{1}{2}\frac{1+r}{1-r}(x+y)\right)I_a\left(\frac{2(rxy)^{1/2}}{1-r}\right)$$

and

$$cH(r, x, y)\le R(r, x, y)\le CH(r, x, y)$$

with

$$(3.6)\quad H(r,x,y)=\begin{cases}(1-r)^{-a-1}(xy)^{a/2}\exp\left(-\frac{1}{2}\frac{1+r}{1-r}(x+y)\right)\\ (e(4r)^{a/2+1/4}(1-r)^{1/2}(xy)^{1/4})^{-1}\exp\left(-\frac{1}{2}\frac{1+r}{1-r}(x+y)+\frac{2(rxy)^{1/2}}{1-r}\right)\end{cases}$$

for $y\leq(1-r)^2/4rx$ and $y\geq(1-r)^2/4rx$ correspondingly.

We now show that

$$(3.7)\qquad \int_0^\infty P(t,x,y)dy\leq \exp\left(-t\,\frac{a+1}{2}\right)$$

with C independent of $x\geq 0$ which will follow from the estimate

$$(3.8)\qquad \int_0^\infty R(r,x,y)dy\leq C$$

with a constant C independent of $0<r<1$ and $x>0$. If $0<r<1/2$ then using (3.1)

$$\int_0^\infty R(r,x,y)dy\leq C\sum r^n\|\mathscr{L}_n^a\|_1\leq C\sum r^n n^{1/2}\leq C\,.$$

If $1/2<r<1$ we check, equivalently, that

$$(3.9)\qquad \int_0^\infty H(r,x,y)dy\leq C\,.$$

Using the transition point $(1-r)^2/4rx$ and splitting up the interval of integration in (3.9) into two intervals one can note that the first resulting integral is uniformly bounded. For the uniform boundedness of the second integral we observe that on $y\geq(1-r)^2/4rx$ the function $H(r,x,y)$ is equal, up to the factor $(4r)^{-a/2}$ which is bounded on $1/2<r<1$, to the function $H(r,x,y)$ treated in §2 with the parameter a equal 0. Therefore, using the results from §2 we get (3.9) and finish the proof of (3.8).

LEMMA 3.2. *Let $H(r,x,y)$ be given by* (3.6). *There exists a function $L(r,x,y)$ satisfying the following properties*

(i) $H(r,x,y)\leq L(r,x,y)$;

(ii) *for every $0<r<1$ and $x>0$, $L(r,x,y)$ as a function of y is monotone increasing on $[0,x]$ and monotone decreasing on $[x,\infty)$*;

(iii) $\int_0^\infty L(r,x,y)dy\leq Cr^{-a/2}$ *with C independent of $x>0$.*

PROOF. We define $L(r,x,y)$ in the same way as in the proof of Lemma 2.2. There are at most three points m_1, m_2, m_3 suspected to be points of local maximum for $H(r,x,y)$ as a function of y-variable:

m_1 is a point where a relative maximum of

$$h_1(y)=y^{-1/4}\exp\left(-\frac{1}{2}\frac{1+r}{1-r}y+\frac{2(rx)^{1/2}}{1-r}y^{1/2}\right)$$

occurs; $m_2=(1-r)a/(1+r)$ is the point where a relative maximum of

$$h_2(y)=y^{a/2}\cdot\exp\left(-\frac{1}{2}\frac{1+r}{1-r}y\right)$$

is taken; $m_3=(1-r)^2/4rx$ is the transition point.

It is clear now that

$$\int_0^\infty L(r, x, y)dy\leq\int_0^\infty H(r, x, y)dy+\sum_1^3 I_j\,,$$

where $I_j=H(r, x, m_j)\cdot|m_j-x|$. To prove (iii) we use (3.8) and show that $I_1\leq Cr^{-a/2}$, $I_2\leq C$, $I_3\leq C$ with C independent of $0<r<1$ and $x>0$.

Estimating I_1 observe that, up to the factor $(4r)^{-a/2}$, the job was done in §2 in the special case $a=0$. To prove that $I_2\leq C$ note that

$$H(r, x, m_2)\leq C\frac{1}{1-r}\left(\frac{x}{1-r}\right)^{a/2}\exp\left(-\frac{1}{2}\frac{x}{1-r}\right).$$

Now, if $x\leq m_2$ then $|m_2-x|\leq m_2\leq C(1-r)$ so $H(r, x, m_2)|m_2-x|$ is bounded, and, if $x>m_2$ then $|m_2-x|<x$ so we write

$$H(r, x, m_2)|m_2-x|\leq C\left(\frac{x}{1-r}\right)^{1+a/2}\exp\left(-\frac{1}{2}\frac{x}{1-r}\right)$$

which is once more bounded.

To finish the proof we check that $I_3\leq C$. Clearly we can assume that $(1-r)a/(1+r)=m_2>m_3$ which implies $m_3<C(1-r)$. Since

$$H(r, x, m_3)\leq C\frac{1}{1-r}\left(\frac{x}{1-r}\right)^{a/2}\exp\left(-\frac{1}{2}\frac{x}{1-r}\right),$$

if $x\leq m_3$ we then get

$$I_3\leq H(r, x, m_3)m_3\leq C\,.$$

If $x\geq m_3$ then

$$I_3\leq H(r, x, m_3)x\leq C\left(\frac{x}{1-r}\right)^{1+a/2}\exp\left(-\frac{1}{2}\frac{x}{1-r}\right)\leq C\,.$$

This concludes the proof of Lemma 3.2.

We now come to the following theorem concerning the heat-diffusion integrals for expansions with respect to the functions $\mathscr{L}_n^a(x)$.

THEOREM 3.3. *Let* $a\geq 0$, $1\leq p\leq\infty$ *and* $f\in L^p(0,\infty)$. *Let* $g(t,x)$ *denote its heat-diffusion integral* (3.2) *and* $f^*(x)$ *the maximal function* (1.1) *with* μ *being the Lebesgue measure on* $(0,\infty)$. *Then*

$$|g(t,x)|\leq C\exp(-t/2)f^*(x) \tag{3.10}$$

and, consequently, the associated maximal function is of weak-type (1, 1), *i.e.*

$$\mu\left(\left\{x:\sup_{t>0}|g(t,x)|>\lambda\right\}\right)\leq\frac{C}{\lambda}\int_0^\infty|f|d\mu,$$

which implies that $\lim_{t\to 0}g(t,x)=f(x)$ *almost everywhere. Moreover,*

$$\|g(t,x)\|_p\leq C\exp\left(-t\frac{a+1}{2}\right)\|f\|_p \tag{3.11}$$

which gives $\|g(t,x)-f(x)\|_p\to 0$ *as* $t\to 0$, *if* $1\leq p<\infty$.

PROOF. We repeat the argument from the proof of Theorem 2.3. To prove (3.10) we use the identity

$$P(t,x,y)=\exp\left(-t\frac{a+1}{2}\right)R(e^{-t},x,y).$$

In the proof of (3.11) the estimate (3.7) comes in handy. As a dense linear subspace in $L^p(0,\infty)$ we now choose the space of polynomials multiplied by $e^{-x/2}x^{a/2}$, cf. Lemma 1.3.

In the same way as in §2, for any function $f(x)$ in $L^p(dx)$, $1\leq p\leq\infty$, with the expansion $\sum c_n\mathscr{L}_n^a(x)$ we define its Poisson integral $f(t,x)$, $t>0$, by

$$f(t,x)=\sum_0^\infty\exp\left(-t\left(n+\frac{a+1}{2}\right)^{1/2}\right)c_n\mathscr{L}_n^a(x). \tag{3.12}$$

The corresponding result for the Poisson integrals is the following.

COROLLARY 3.4. *Let* $f\in L^p(0,\infty)$, $1\leq p\leq\infty$, *and* $f(t,x)$ *denote its Poisson integral given by* (3.12). *Then* $f(t,x)$ *is a* C^∞*-function satisfying the differential equation*

$$\left(L_x+\frac{\partial^2}{\partial t^2}\right)f(t,x)=0.$$

Furthermore the conclusions of Theorem 3.3 *are valid with* $g(t,x)$ *replaced by* $f(t,x)$ *and the exponentials in* (3.10) *and* (3.11) *dropped.*

REMARK 3.5. In case $a=0,1,2,\ldots$ the a.e. and L^p-convergence of heat-diffusion integrals in Theorem 3.3 was proved by Długosz [Dł] by using a group-theoretic method.

4. The Laguerre functions $\varphi_n^a(x)$. In this section we consider still another Laguerre functions

$$\varphi_n^a(x)=\mathscr{L}_n^a(x^2)(2x)^{1/2}$$

that form an orthonormal basis in $L^2(0, \infty)$. Only if $a\geq -1/2$ do these functions belong to all L^p-spaces on $(0, \infty)$, $1\leq p\leq\infty$, therefore we assume throughout this section that a is a fixed number greater than or equal to $-1/2$. Expansions with respect to this system were considered by Markett [Ma] and Thangavelu [T1]. In particular, it was observed in [Ma] that expansions with respect to $\varphi_n^a(x)$ behave much better than those with respect to the system of Laguerre functions $\mathscr{L}_n^a(x)$. Using once more Lemma 1 of [Ma] together with our correction mentioned in §1 we obtain

$$\|\varphi_n^a\|_p\sim\begin{cases} n^{(1/p-1/2)/2}, & 1\leq p<4, \\ n^{-1/8}(\log n)^{1/4}, & p=4, \\ n^{-(1/p+1/2)/6}, & 4<p<\infty. \end{cases} \tag{4.1}$$

Here the meaning of the symbol $\|\ \ \|_p$ is the same as in §3. The function $\varphi_n^a(x)$ is an eigenfunction of the differential operator

$$L=\frac{d^2}{dx^2}-x^2-\frac{1}{x^2}\left(a^2-\frac{1}{4}\right)$$

with eigenvalue $-\lambda_n$, $\lambda_n=4n+2a+2$. The operator $-L$ is symmetric in $L^2(0, \infty)$ and positive if $a\geq 1/2$. With any $f(x)$ in $L^p(0, \infty)$, $1\leq p\leq\infty$, we now associate its expansion given by the formal series $\sum d_n\varphi_n^a(x)$ where

$$d_n=\int_0^\infty f(y)\varphi_n^a(y)dy\,.$$

Next, we define its heat-diffusion integral $g(t, x)$, $t>0$,

$$g(t, x)=\sum_0^\infty \exp(-t\lambda_n)d_n\varphi_n^a(x)\,. \tag{4.2}$$

In an equivalent form

$$g(t, x)=\int_0^\infty P(t, x, y)f(y)dy \tag{4.3}$$

where

$$P(t, x, y)=\sum_0^\infty \exp(-t\lambda_n)\varphi_n^a(x)\varphi_n^a(y)\,.$$

Uniform convergence of (4.2) and interchanging of order of integration and summation in (4.3) is a consequence of (4.1).

LEMMA 4.1. *Let $f \in L^p(0, \infty)$, $1 \le p \le \infty$. Then its heat-diffusion integral* (4.2) *is a C^∞-function that satisfies the differential equation*

$$\left(L_x - \frac{\partial}{\partial t}\right) g(t, x) = 0 . \tag{4.4}$$

PROOF. Clearly

$$\frac{\partial^k}{\partial t^k} g(t, x) = \sum_0^\infty (-\lambda_n)^k \exp(-t\lambda_n) d_n \varphi_n^a(x) . \tag{4.5}$$

Since

$$\frac{d}{dx} \varphi_n^a(x) = -2n^{1/2} \varphi_{n-1}^{a+1}(x) + \left(\frac{2a-1}{2x} - x\right) \varphi_n^a(x) ,$$

by using (4.1) we obtain $|(d/dx)\varphi_n^a(x)| \le Cn^{1/2}$ on $(\varepsilon, 1/\varepsilon)$. Therefore we can differentiate the series in (4.5) with respect to the x-variable term by term. Similarly for higher derivatives.

As in previous sections we now switch to the kernel

$$R(r, x, y) = \sum_0^\infty r^n \varphi_n^a(x) \varphi_n^a(y) ,$$

$0 < r < 1$. Then

$$R(r, x, y) = \frac{2(xy)^{1/2}}{(1-r)r^{a/2}} \exp\left(-\frac{1}{2} \frac{1+r}{1-r} (x^2 + y^2)\right) I_a\left(\frac{2r^{1/2}}{1-r} xy\right)$$

and

$$cH(r, x, y) \le R(r, x, y) \le CH(r, x, y)$$

with

$$H(r, x, y) = \begin{cases} 2(1-r)^{-a-1}(xy)^{a+1/2} \exp\left(-\frac{1}{2}\frac{1+r}{1-r}(x^2+y^2)\right) \\ 2(e(4r)^{a/2+1/4}(1-r)^{1/2})^{-1} \exp\left(-\frac{1}{2}\frac{1+r}{1-r}(x^2+y^2) + \frac{2r^{1/2}}{1-r} xy\right) \end{cases} \tag{4.6}$$

for $y \le (1-r)/2r^{1/2}x$ and $y > (1-r)/2r^{1/2}x$ correspondingly. We will use the estimate

$$\int_0^\infty P(t, x, y) dy \le C \exp(-t(2a+2)) \tag{4.7}$$

which follows from

$$\int_0^\infty R(t, x, y)dy \leq C. \tag{4.8}$$

To prove this note that for $0<r<1$

$$\int_0^{m_3} H(r, x, y)dy \leq C(x/(1-r)^{1/2})^{a+1/2} \cdot \exp(-(x/(1-r)^{1/2})/2) \tag{4.9}$$

where $m_3=(1-r)/2r^{1/2}x$, and, for $1/2<r<1$,

$$\int_{m_3}^\infty H(r, x, y)dy \leq C \exp\left(-\frac{1}{2}\frac{1-r}{1+r}x^2\right) \cdot \int_A^\infty \exp\left(-\frac{1}{2}\left(u-\frac{1}{A}\right)^2\right)du \tag{4.10}$$

with $A=(1-r^2)^{1/2}/2r^{1/2}x$. Since $u^{a+1/2}\exp(-u^2/2) \leq C$ and the integral in (4.10) is bounded independently of $A>0$, combining (4.9) and (4.10) gives (4.8) for $1/2<r<1$. On the other hand, if $0<r\leq 1/2$, by using (4.1) we estimate

$$\int_0^\infty R(r, x, y)dy \leq C\sum r^n n^{-1/12}\|\varphi_n^a\|_1 \leq C\sum r^n n^{1/6} \leq C.$$

This finishes the proof of (4.7).

To prove the main theorem of this section the following lemma is needed.

LEMMA 4.2. *Let $H(r, x, y)$ be given by* (4.6). *The conclusions of Lemma* 3.2 *are valid if a in* (iii) *is replaced by* $b=\max\{a+1/2, 1\}$.

PROOF. Repeating the argument used in the proof of Lemma 3.2 and using the estimate

$$\int_0^\infty H(r, x, y)dy \leq C$$

which follows from (4.8), we are reduced to proving $I_1+I_2+I_3 \leq Cr^{-b/2}$. Here $I_j=H(r, x, m_j)|m_j-x|$ and $m_1=2r^{1/2}x/(1+r)$ is the point where a relative maximum of

$$h_1(y)=-\frac{1}{2}\frac{1+r}{1-r}y^2+\frac{2r^{1/2}}{1-r}xy$$

is taken; $m_2=((a+1/2)(1-r)/(1+r))^{1/2}$ is the point where a relative maximum of

$$h_2(y)=y^{a+1/2}\exp\left(-\frac{1}{2}\frac{1+r}{1-r}y^2\right)$$

occurs; $m_3=(1-r)/2r^{1/2}x$ is the transition point.

First we show that $I_1 \leq Cr^{-(a+1/2)/2}$ clearly assuming that $m_1>(1-r)/2r^{1/2}x$. We note that $|m_1-x|=x(1-r)^2/(1+r)(1+r^{1/2})^2$ and $H(r, x, y)$ is bounded on $[(1-r)/2r^{1/2}x, \infty)$ by

$$C\frac{1}{r^{(a+1/2)/2}}\frac{1}{(1-r)^{1/2}}\exp\left(-\frac{1}{2}\frac{1-r}{1+r}x^2\right).$$

Therefore I_1 is less than

$$C\frac{1}{r^{(a+1/2)/2}}(1-r)(x(1-r)^{1/2})\exp\left(-\frac{1}{4}(x(1-r)^{1/2})^2\right)$$

which gives the desired estimate since $u\exp(-u^2/4)$, $u>0$ is a bounded function.

The next estimate is $I_2 \le Cr^{-1/2}$ and we assume that $m_2 \le (1-r)/2r^{1/2}x$. Then

$$H(r, x, m_2) \le C\frac{1}{(1-r)^{1/2}}\left(\frac{x^2}{1-r}\right)^{(a+1/2)/2}\exp\left(-\frac{1}{2}\frac{x^2}{1-r}\right)$$

and we are reduced to showing that $|m_2-x| \le C((1-r)/r)^{1/2}$. If $x \le m_2$ then $|m_2-x| \le m_2 \le C(1-r)^{1/2}$, if $x \ge m_2$ then $|m_2-x| \le x \le C((1-r)/r)^{1/2}$, since, from

$$m_2=\left(\frac{1-r}{1+r}(a+1/2)\right)^{1/2} \le \frac{1-r}{2r^{1/2}x}$$

we obtain $x \le C((1-r)/r)^{1/2}$.

The last estimate we prove is $I_3 \le C$. We assume that $m_1 \le (1-r)/2r^{1/2}x \le m_2$ and this produces

$$C_1((1-r)/r)^{1/2} \le x \le C_2((1-r)/r)^{1/2}.$$

Therefore

$$\begin{aligned} H(r, x, m_3) &\le \frac{1}{(1-r)^{1/2}}\cdot\frac{1}{r^{(a+1/2)/2}}\cdot\exp\left(-\frac{1}{2}\frac{1+r}{1-r}x^2\right) \\ &\le \frac{1}{(1-r)^{1/2}}\cdot\frac{1}{r^{(a+1/2)/2}}\cdot\exp(-C_3/r). \end{aligned} \tag{4.11}$$

Estimating $|m_3-x|$ gives

$$\left|\frac{1-r}{2r^{1/2}x}-x\right| \le C((1-r)/r)^{1/2}$$

so, combining (4.11) and (4.12) proves that $I_3 \le C$.

We are now ready to prove the main theorem concerning heat-diffusion integrals for considered expansions.

THEOREM 4.3. *Let* $a \ge -1/2$, $1 \le p \le \infty$ *and* $f \in L^p(0, \infty)$. *Let* $g(t, x)$ *denote its heat-diffusion integral* (4.2) *and* $f^*(x)$ *the maximal function* (1.1) *with* μ *being the Lebesgue measure on* $(0, \infty)$. *Then*

$$(4.13) \qquad |g(t, x)| \leq C \exp(-2at) f^*(x)$$

if $-1/2 \leq a < 1/2$, *and*

$$(4.14) \qquad |g(t, x)| \leq C \exp(-t) f^*(x)$$

if $a \geq 1/2$. *Consequently, for* $a \geq 0$ *the associated maximal function is of weak type* (1, 1), *i.e.*

$$\mu\left(\left\{x : \sup_{t>0} |g(t, x)| > \lambda\right\}\right) \leq \frac{C}{\lambda} \int_0^\infty |f| d\mu .$$

For $-1/2 \leq a < 0$ (4.13) *implies only that the local maximal function, with the supremum over* t *limited above to* $0 < t < 1$, *is of weak type* (1, 1). *This implies that* $\lim_{t \to 0} g(t, x) = f(x)$ *almost everywhere. Moreover*

$$(4.15) \qquad \|g(t, x)\|_p \leq C \exp(-t(2a+2)) \|f\|_p$$

which gives $\|g(t, x) - f(x)\|_p \to 0$ *as* $t \to 0$, *if* $1 \leq p < \infty$.

PROOF. Lemmas 4.2 and 1.1 immediately imply (4.13) and (4.14) if we make use from the identity

$$P(t, x, y) = \exp(-t(2a+2)) R(e^{-4t}, x, y) .$$

Minkowski's integral inequality then gives (4.15). As a dense linear subspace in $L^p(0, \infty)$ we choose the space of even polynomials multiplied by $\exp(-x^2/2)x^{a+1/2}$. The density is an easy consequence of Lemma 1.3 with $d = (a - 1/p - 1/2)/2$. The proof of Theorem 4.3 is now complete.

We now define the Poisson integral $f(t, x)$, $t > 0$, for any function $f(x)$ in $L^p(0, \infty)$, by

$$(4.16) \qquad f(t, x) = \sum_0^\infty \exp(-t\lambda_n^{1/2}) d_n \varphi_n^a(x) ,$$

where $\sum d_n \varphi_n^a(x)$ is the expansion of $f(x)$. The result for the Poisson integrals is given in the following corollary.

COROLLARY 4.4. *Let* $f \in L^p(0, \infty)$, $1 \leq p \leq \infty$, $a \geq -1/2$ *and* $f(t, x)$ *be its Poisson integral given by* (4.16). *Then* $f(t, x)$ *is a* C^∞-*function satisfying the differential equation*

$$\left(L_x + \frac{\partial^2}{\partial t^2}\right) f(t, x) = 0 .$$

Furthermore, if $a \geq 0$ *the conclusions of Theorem* 4.3 *are valid with* $g(t, x)$ *replaced by* $f(t, x)$ *and the exponentials in* (4.13), (4.14) *and* (4.15) *dropped. If* $-1/2 \leq a < 0$ *the results of Theorem* 4.3 *imply only a modified version of* (4.15) *and the* L^p-*convergence.*

REMARK 4.5. Thangavelu [T1] proved that for $a \geq 1/2$, the Cesàro means $\sigma_n^\alpha f$ of order $\alpha > 1/6$ for expansion with respect to φ_n^a converge to f both in L^p-norm and almost everywhere. This clearly implies the convergence results in Theorem 4.3 in case $a \geq 1/2$.

Added on October 20, 1993. The author has recently become aware of the paper
M. Horváth: Some saturation theorems for classical orthogonal expansions, II, Acta Math. Hung. 58 (1991), 157–191, [MR 93g: 42020]
that partly overlaps with Section 3 of the present paper. In particular, since both are modeled on Muckenhoupt's technique, arguments leading to the proof of the inequality (3.8) and Lemma 3.2 in our paper are more or less identical with those from the proofs of Lemmas 3 and 4 of Horváth's paper.

REFERENCES

[A-H] R. ASKEY AND I. I. HIRSCHMAN, Mean summability for ultraspherical polynomials, Math. Scand. 12 (1963), 167–177.

[Dł] J. DŁUGOSZ, Almost every convergence of some summability methods for Laguerre series, Studia Math. 82 (1985), 199–209.

[GR] I. S. GRADSHTEYN AND I. M. RYZHIK, Table of integrals, series and products, Academic Press, New York, 1980.

[Ma] C. MARKETT, Mean Cesàro summability of Laguerre expansions and norm estimates with shifted parameter, Anal. Math. 8 (1982), 19–37.

[Mu 1] B. MUCKENHOUPT, Poisson integrals for Hermite and Laguerre expansions, Trans. Amer. Math. Soc. 139 (1969), 231–242.

[Mu 2] B. MUCKENHOUPT, Mean convergence of Hermite and Laguerre series II, Trans. Amer. Math. Soc. 147 (1970), 433–460.

[Po] H. POLLARD, The mean convergence of orthogonal series II, Trans. Amer. Math. Soc. 63 (1948), 355–367.

[St 1] K. STEMPAK, Almost everywhere summability of Laguerre series, Studia Math. 100 (1991), 129–147.

[St 2] K. STEMPAK, Mean summability methods for Laguerre series, Trans. Amer. Math. Soc. 322 (1990), 671–690.

[Sz] G. SZEGÖ, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., Vol. 23, fourth edition, Providence, R. I., 1975.

[T1] S. THANGAVELU, Summability of Laguerre expansions, Anal. Math. 16 (1990), 303–315.

[T2] S. THANGAVELU, On almost everywhere convergence of Hermite and Laguerre expansions, Colloq. Math. 61 (1990), 21–34.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW
PL. GRUNWALDZKI 2/4
50–384 WROCŁAW
POLAND