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A DIFFERENTIAL GEOMETRIC PROPERTY OF BIG LINE BUNDLES

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Abstract. A holomorphic line bundle over a compact complex manifold is shown to be big if it has a singular Hermitian metric whose curvature current is smooth on the complement of some proper analytic subset, strictly positive on some tubular neighborhood of the analytic subset, and satisfies a condition on its integral. In particular, we obtain a sufficient condition for a compact complex manifold to be a Moishezon space.

1. Introduction. In this paper, we consider sufficient conditions for a singular Hermitian line bundle (L, h) over a compact complex space X to be big and, consequently, for X to be a Moishezon space. A holomorphic line bundle L is said to be big if $\dim \Phi_{|L^{\otimes \nu}|}(X) = \dim X$ for some $\nu \in \boldsymbol{N}$, where $\Phi_{|L^{\otimes \nu}|}$ is the meromorphic map to some $\boldsymbol{P}^N$ by means of the global sections of $L^{\otimes \nu}$. A reduced and irreducible compact complex space X is said to be a Moishezon space if the transcendence degree of the meromorphic function field of X over a complex number filed $\boldsymbol{C}$ is equal to the dimension of X. By Moishezon [Mo], X is a Moishezon space if and only if there exists a bimeromorphic holomorphic map from a projective manifold to X. Our problem arose from an attempt to generalize the following theorem due to Kodaira [Ko]: A compact complex manifold is projective algebraic if and only if there exists a positive line bundle on it. There are several works in this direction; [De], [G-R], [Ri], [Si 1], [Si 2], [Ji 1], [Ji 2], [J-S] and so on. The former five works are related to *smooth* Hermitian metrics, and their theorems are motivated by the conjecture of Grauert and Riemenschneider: A compact complex manifold admits a smooth Hermitian holomorphic line boundle whose curvature form is positive definite on a dense subset of it, then it is Moishezon. However, it is not enough to characterize Moishezon spaces by smooth metrics as is mentioned below. Let X be a non-projective Moishezon manifold. Then there exists a proper modification $\pi: \tilde{X} \to X$ from a projective manifold. By Kodaira, $\tilde{X}$ carries a smooth integral Kähler form $\tilde{\omega}$. Then the push-forward $\pi_* \tilde{\omega}$ is an integral Kähler *current* which is smooth on the complement of some proper analytic subset of X. However, X does not have a smooth Kähler *form* (by [Mo], Kähler and Moishezon imply projectivity). On the other hand, [Ji 1], [Ji 2], [J-S] are related to *singular* Hermitian metrics. Ji and Shiffman [J-S] proved the conjecture of Shiffman: A compact complex manifold

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is Moishezon if and only if there exists an integral Kähler current on it. In our Main Theorem, which is a generalization of Demailly's theorem [De], the curvature current of a line bundle may have singularities and negative parts.

Let M be an n-dimensional complex manifold and L a holomorphic line bundle over M with a smooth Hermitian metric h. We denote the curvature form of h by $c(L, h):=\sqrt{-1}(2\pi)^{-1}\bar{\partial}\partial\log h$. Set $M(q, L):=\{m\in M;\ c(L, h)$ has q negative eigenvalues and $n-q$ positive eigenvalues at $m\}$ for each $q=0, 1, \dots, n$, and put $M(\leq q, L):=M(0, L)\cup M(1, L)\cup\cdots\cup M(q, L)$. Our goal is the following:

MAIN THEOREM. *Let X be an n-dimensional reduced and irreducible compact complex space and let L be a holomorphic line bundle over X with a singular Hermitian metric h. Assume that the curvature current $c(L, h)$ is smooth on the complement of some proper analytic subset $Z\subset X$ and that $c(L, h)$ is strictly positive on some tubular neighborhood B of Z. Then $\int_{X_{\mathrm{reg}}(\leq 1,L)}c(L, h)^n$ exists, where X_{reg} is the smooth locus of X. If $\int_{X_{\mathrm{reg}}(\leq 1,L)}c(L, h)^n>0$, then L is big, and, in particular, X is a Moishezon space.*

The proof is based on some lemmas; the decomposition principle, Demailly's generalization of Weyl's formula for the asymptotic spectrum and the absence of essential spectrum. As a corollary, we reprove a characterization of Moishezon manifolds.

COROLLARY. *Lex X be a compact complex manifold. Then X is Moishezon if and only if X has an integral Kähler current which is smooth on the complement of some proper analytic subset of X.*

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2. Notation and preliminaries. In this section, we recall classical terminologies on complex spaces and Hermitian manifolds (the reader is referred to [Fu], [G-H], [Ve] and so on).

2.1. Differential forms and currents on complex spaces. In this paragraph, we let X be an n-dimensional paracompact complex space.

(1) We follow the definitions of smooth differential forms and currents by Fujiki [Fu]. The sheaf $\mathscr{A}_X^r$ (resp. $\mathscr{A}_X^{p,q}$) of germs of C^∞-r-forms (resp. C^∞-(p, q)-forms) with direct sum decomposition $\mathscr{A}_X^r=\bigoplus_{p+q=r}\mathscr{A}_X^{p,q}$ and the differentials $d\colon\mathscr{A}_X^r\to\mathscr{A}_X^{r+1}$ (resp. $\partial\colon\mathscr{A}_X^{p,q}\to\mathscr{A}_X^{p+1,q}$ and $\bar{\partial}\colon\mathscr{A}_X^{p,q}\to\mathscr{A}_X^{p,q+1}$) with $d=\partial+\bar{\partial}$ is locally defined as follows and globally defined by gluing them.

When X is a subspace of a domain V in $C^l=C^l(z_1, \dots, z_l)$ with the ideal sheaf $\mathscr{I}=\mathscr{I}_X$. Then we define $\mathscr{A}_X=\mathscr{A}_X^0$ by $\mathscr{A}_X=\mathscr{A}_V/(\mathscr{I}+\bar{\mathscr{I}})\mathscr{A}_V$, where $\bar{\mathscr{I}}=\{\bar{f};\ f\in\mathscr{I}\}$, $\bar{f}$ being the complex conjugate of f. Next define the $\mathscr{A}_V$-submodule $\mathscr{A}'_X$ of $\mathscr{A}_V^1$ by

$$\mathscr{A}'_X=\sum\mathscr{I}\mathscr{A}_V dz_\alpha+\sum\bar{\mathscr{I}}\mathscr{A}_V d\bar{z}_\beta+\mathscr{A}_V d\mathscr{I}+\mathscr{A}_V d\bar{\mathscr{I}}\ ,$$

where $\mathscr{A}_V d\mathscr{J} = \{\sum h_\gamma dg_\gamma ; h_\gamma \in \mathscr{A}_V \text{ and } g_\gamma \in \mathscr{J}\}$ and similarly for $\mathscr{A}_V d\bar{\mathscr{J}}$. Then put

$$\mathscr{A}^r_X = \mathscr{A}^r_V / (\mathscr{A}'_X \wedge \mathscr{A}^{r-1}_X)$$

for $r \geq 1$. These naturally form as $\mathscr{A}_X$-graded algebra $\mathscr{A}^\bullet_Y$. Further, define the $\mathscr{A}_V$-submodules $\mathscr{A}^{p,q}_X$ $(p+q=r)$ of $\mathscr{A}^r_X$ by $\mathscr{A}^{p,q}_{X,x} = \{\psi \in \mathscr{A}^r_{X,x}$; there exists a $\tilde{\psi} \in \mathscr{A}^{p,q}_{V,x}$ inducing $\psi\}$. Then it is immediate to see that we have a direct sum decomposition $\mathscr{A}^r_X = \bigoplus_{p+q=r} \mathscr{A}^{p,q}_X$. Moreover, the usual differetial d (resp. ∂ and $\bar{\partial}$) on $\mathscr{A}^r_V$ (resp. $\mathscr{A}^{p,q}_V$) induces the one on $\mathscr{A}^r_X$ (resp. $\mathscr{A}^{p,q}_X$) with $d = \partial + \bar{\partial}$. On the other hand, the natural complex conjugation on $\mathscr{A}^r_V$ induces a $\boldsymbol{C}$-antilinear involution on $\mathscr{A}^r_X$. In particular, we can define the real form on X as those left fixed by this involution. Morphisms of complex spaces $f: X \to Y$ and $g: Y \to Z$ induce a natural pull-back homomorphism $f^*: \mathscr{A}^\bullet_Y \to \mathscr{A}^\bullet_X$ and satisfy $f^* \circ g^* = (g \circ f)^*$.

(2) We let C^r_X denote the space of smooth r-forms on X with compact support. Then the convergence in C^r_X (the so-called C^∞-topology) is defined as follows. Take a locally finite open covering $\mathscr{U} = \{U_\alpha\}$ of X with an embedding $j_\alpha: U_\alpha \to V_\alpha$ for each α, where V_α is a domain in some $\boldsymbol{C}^{l_\alpha}$. Let $\{\rho_\alpha\}$ be a smooth partition of unity subordinate to the covering $\mathscr{U}$. Taking $\mathscr{U}$ suitably, we may assume that the support of ρ_α is compact for each α. Then we say that a sequence $\{\phi_m\}_{m \in \boldsymbol{N}}$ $(\phi_m \in C^r_X)$ converges to $\phi_0 \in C^r_X$, if the support of ϕ_m are contained in a fixed compact set K and for each α there exists a compact set K_α in V_α and a representatives $\tilde{\phi}_{m\alpha}$ of $\rho_\alpha \phi_m$ $(m \geq 0)$ with support contained in K_α such that $\tilde{\phi}_{m\alpha} \to \tilde{\phi}_{0\alpha}$ uniformly on K_α together with all the derivatives of their coefficients. One can see that this definition is independent of the choice of $\mathscr{U}$ and $\{\rho_\alpha\}$ as above.

(3) Next we define the space D^r_X of r-currents on X as the vector space of complex-valued continuous linear functionals on C^{2n-r}_X with the C^∞-topology. The differential $d: D^r_X \to D^{r+1}_X$ is defined by $dT(\phi) = (-1)^{r+1} T(d\phi)$ for $T \in D^r_X$ and $\phi \in C^{2n-r-1}_X$. By gluing them, we can define the sheaf $\mathscr{D}^r_X$ of germs of r-currents on X and $d: \mathscr{D}^r_X \to \mathscr{D}^{r+1}_X$. We also denote by $C^{p,q}_X$ the space of smooth (p, q)-forms on X with compact support. The C^∞-topology of $C^{p,q}_X$, the space $D^{p,q}_X$ of (p, q)-currents, the sheaves $\mathscr{D}^{p,q}_X$ and $\partial: D^{p,q}_X \to D^{p+1,q}_X$, $\partial: \mathscr{D}^{p,q}_X \to \mathscr{D}^{p+1,q}_X$, $\bar{\partial}: D^{p,q}_X \to D^{p,q+1}_X$, $\bar{\partial}: \mathscr{D}^{p,q}_X \to \mathscr{D}^{p,q+1}_X$ with $d = \partial + \bar{\partial}$ are also defined as above and as in the case of usual complex manifolds.

(4) By (1) and (3) as above, we get complexes of sheaves on X

$$(\mathscr{A}^*_X, d): \mathscr{A}^0_X \to \mathscr{A}^1_X \to \mathscr{A}^2_X \to \cdots$$

and

$$(\mathscr{D}^*_X, d): \mathscr{D}^0_X \to \mathscr{D}^1_X \to \mathscr{D}^2_X \to \cdots .$$

Note that the sheaves $\mathscr{A}^r_X$ and $\mathscr{D}^r_X$ $(r \geq 0)$ are fine, but in general, $(\mathscr{A}^*_X, d)$ and $(\mathscr{D}^*_X, d)$ are not resolutions of $\boldsymbol{C}$ (or $\boldsymbol{R}$) on X (cf. [B-H]). There are natural homomorphisms of complexes of sheaves

$$Z \to R \to C \to (\mathscr{A}_X^*, d) \to (\mathscr{D}_X^*, d)$$

which induce homomorphisms of hypercohomology groups

$$H^*(X, Z) \to H^*(X, R) \to \boldsymbol{H}^*(X, \mathscr{A}_X^*) \to \boldsymbol{H}^*(X, \mathscr{D}_X^*)\,.$$

By the fineness of $\mathscr{A}_X^r$ and $\mathscr{D}_X^r$ $(r \geq 0)$, the canonical edge homomorphisms $H^*(\Gamma(X, \mathscr{A}_X^*)) \to \boldsymbol{H}^*(X, \mathscr{A}_X^*)$ and $H^*(\Gamma(X, \mathscr{D}_X^*)) \to \boldsymbol{H}^*(X, \mathscr{D}_X^*)$ are isomorphisms.

(5) The *singular support* of a current $T \in D_X^{p,q}$ is defined as the smallest subset S of X such that T is a smooth form on $X - S$.

(6) A real C^∞-(p, p)-form ξ on X is *strictly positive* (resp. *semipositive*) if there exists an open covering $\mathscr{U} = \{U_\alpha\}$ of X with, for each α, an embedding $j_\alpha\colon U_\alpha \to V_\alpha$ of U_α into a subdomain V_α in C^{l_α} and a C^∞ strictly positive (resp. semipositive) (p, p)-form ξ_α on V_α in the usual sense such that $j_\alpha^* \xi_\alpha = \xi|_{U_\alpha}$.

(7) A (p, p)-current T is *real* if $T = \bar{T}$ in the sense that $\bar{T}(\phi) = T(\bar{\phi})$ for all $\phi \in C_X^{n-p, n-p}$, and a real current T is *positive* when $(\sqrt{-1})^{p^2} T(\psi \wedge \bar{\psi}) \geq 0$ for all $\psi \in C_X^{n-p, 0}$.

(8) A real (p, p)-current T on X is *strictly positive* if there exists a strictly positive C^∞-(p, p)-form ω^p on X such that $T - \omega^p$ is a positive current on X. T is said to be *strictly positive* at a point $x \in X$ if there exists a neighborhood U of x such that $T|_U$ is a strictly positive current on U.

(9) A real $(1, 1)$-current ω on X is said to be a *Kähler current* (cf. [J-S]) if it is d-closed and strictly positive on X. A d-closed $(1, 1)$-current or a d-closed C^∞-$(1, 1)$-form is said to be *integral* if its hypercohomology class is in the image of $H^2(X, Z)$ under the map in (4).

(10) Let $\pi\colon L \to X$ be a holomorphic line bundle over X. A *singular Hermitian metric* h on L is a map $h\colon L \to [-\infty, +\infty]$ which is given in any local trivialization $\tau\colon \pi^{-1}(U) \xrightarrow{\sim} U \times C$ by $h(v) = |\tau(v)| e^{-\psi_U(\pi(v))}$ for $v \in \pi^{-1}(U)$, where $\psi_U \in L^1_{\mathrm{loc}}(U)$. The *curvature current* of (L, h) is the d-closed $(1, 1)$-current $c(L, h)$ given by $c(L, h) = \sqrt{-1}\pi^{-1}\partial\bar{\partial}\psi_U$ on U, which is independent of the choice of the local trivialization.

(11) We give a fundamental example of positive curvature current. Let $\Delta = \{z \in C; |z| < r < 1\}$. The trivial line bundle $\mathcal{O} \to \Delta$ has a singular Hermitian metric $|z|^{-\alpha} = \exp(-\alpha \log|z|)$ for $\alpha \in R$, and the curvature current is $\alpha\sqrt{-1}\pi^{-1}\partial\bar{\partial}\log|z|$ (as a distribution). Then it is positive if and only if $\alpha \geq 0$.

2.2. Hermitian geometry. Let (M, ω) be an n-dimensional complete Hermitian manifold without boundary and E a holomorphic vector bundle over M with a smooth Hermitian metric h (we will use the Hermitian metric on M and the associated fundamental $(1, 1)$-form interchangeably). For an open subset $\Omega \subset M$, we denote by $C_\Omega^{p,q}(E)$ the space of E-valued smooth (p, q)-forms with compact support in Ω. The length of $f \in C_\Omega^{p,q}(E)$ with respect to ω and h is denoted by $|f|$ $(= |f|_{\omega,h}$ to be precise, but from now on we will omit indices ω, h, Ω, E and so on if there is no fear of confusion). Let dv be the volume form on M with respect to ω and set

$$\|f\|:=\left(\int_{\Omega}|f|^2 dv\right)^{1/2},$$

which is the usual L^2-norm with respect to ω and h. The L^2-norm $\|f\|$ determines a Hermitian inner product in $C_{\Omega}^{p,q}(E)$, which we denote by (f,g). Let $\mathscr{L}_{\Omega}^{p,q}(E)$ be the Hilbert space completion of $C_{\Omega}^{p,q}(E)$ with respect to the above norm. We define the Dirichlet form $Q_{\Omega}^{p,q}$ to be the densely defined quadratic form on $\mathscr{L}_{\Omega}^{p,q}(E)$ obtained by taking the form closure (cf. [R-S]) of the form

$$C_{\Omega}^{p,q}(E)\ni f\to\|\bar\partial f\|^2+\|\vartheta_h f\|^2,$$

where ϑ_h is the formal adjoint of $\bar\partial$. When Ω is a smoothly bounded relatively compact domain, the Rellich lemma (cf. [We]) implies that $Q_{\Omega}^{p,q}$ has discrete spectrum.

Let $K\subset M$ be a compact manifold with boundary and $\dim K=\dim M$ or $K=\varnothing$. When $\Omega=M-K$, there is a second way of defining $Q_{M-K}^{p,q}$, namely, as the $\bar\partial$-Neumann form: this second way turns out to be equivalent to the first due to the completeness of ω (roughly speaking, the $\bar\partial$-Neumann boundary conditions get pushed to infinity). More precisely, we replace

$$\bar\partial\colon C_{M-K}^{p,q}\to C_{M-K}^{p,q+1}$$

by its graph closure to get a closed densely defined operator $\bar\partial\colon \mathscr{L}_{M-K}^{p,q}\to\mathscr{L}_{M-K}^{p,q+1}$ (which acts in the sense of distributions). We let $\bar\partial_h^*\colon \mathscr{L}_{M-K}^{p,q}\to\mathscr{L}_{M-K}^{p,q-1}$ to be the Hilbert space adjoint of $\bar\partial$. $\bar\partial_h^*$ is also closed and densely defined. Then we define $Q_{M-K}^{p,q}$ as follows:

$$Q_{M-K}^{p,q}(f)=\|\bar\partial f\|^2+\|\bar\partial_h^* f\|^2,$$

$$\operatorname{Dom} Q_{M-K}^{p,q}=\operatorname{Dom}\bar\partial\cap\operatorname{Dom}\bar\partial_h^*\cap\mathscr{L}_{M-K}^{p,q}(E).$$

For the equivalence of this definition with the one given above, see [Ve, Theorem 1.1]; by the completeness of ω, we show that $C_{M-K}^{p,q}(E)$ is dense in $\operatorname{Dom} Q_{M-K}^{p,q}$ relative to the graph norm $f\mapsto(\|f\|^2+\|\bar\partial f\|^2+\|\bar\partial_h^* f\|^2)^{1/2}$. Associated to the form $Q_{M-K}^{p,q}$, the selfadjoint densely defined operator $\square_{M-K}$, the Laplace-Beltrami operator, is defined by

$$\square_{M-K} f=\bar\partial\bar\partial_h^* f+\bar\partial_h^*\bar\partial f,$$

$$\operatorname{Dom}\square_{M-K}=\{f\in\operatorname{Dom}\bar\partial_h^*\cap\operatorname{Dom}\bar\partial,\ \bar\partial f\in\operatorname{Dom}\bar\partial_h^*\text{ and }\bar\partial_h^* f\in\operatorname{Dom}\bar\partial\}.$$

It is well-known that the essential spectrum of a densely defined selfadjoint operator on a Hilbert space is stable under compact operator perturbations. The essential spectrum means the closure of that part of the spectrum not corresponding to discrete eigenvalues with finite multiplicity. In our geometric case we have:

PROPOSITION 2.3 (decomposition principle). *In the notation as above, the Laplace-Beltrami operators $\square_M$ and $\square_{M-K}$ have the same essential spectrum.*

PROOF. If A is a densely defined selfadjoint operator on a Hilbert space, then the essential spectrum $\sigma_{\mathrm{ess}}(A)$ may be defined as the set of $\lambda \in \boldsymbol{R}$ for which there exists a noncompact sequence $\{f_n\}_{n \in \boldsymbol{N}}$ in the domain of A with

$$\|f_n\| = 1 \quad \text{for each} \quad n \in \boldsymbol{N} \quad \text{and} \quad \lim_{n \to \infty} \|(A - \lambda I) f_n\| = 0 .$$

Any part of such a sequence, from which it is impossible to extract a convergent subsequence, is called a characteristic sequence for (A, λ).

Let ϕ be a smooth compactly supported nonnegative function on M which is equal to one on a neighborhood of K. If $\{f_n\}_{n \in \boldsymbol{N}}$ is an orthonormal characteristic sequence for $(\square_M, \lambda)$ for some $\lambda \geq 0$, then we set $g_n = f_{2n} - f_{2n-1}$ $(n \geq 1)$. We see that $\{g_n\}_{n \in \boldsymbol{N}}$ is noncompact and that $\lim_{n \to \infty} \|(\square_M - \lambda I) g_n\| = 0$. The Rellich lemma implies that $\{\phi g_n\}_{n \in \boldsymbol{N}}$ is compact, since ϕ is compactly supported. Moreover, by passing to a subsequence of $\{f_n\}_{n \in \boldsymbol{N}}$, if necessary, we may assume that $g_n \to 0$ in the E-valued first Sobolev space $W^1(U, E)$, where U is a relatively compact neighborhood of the support of ϕ. Then $\lim_{n \to \infty} \|(\square_{M-K} - \lambda I)(1 - \phi) g_n\| = 0$, and consequently

$$\{\tilde{g}_n\}_{n \in \boldsymbol{N}} \quad \text{with} \quad \tilde{g}_n := \frac{(1 - \phi) g_n}{\|(1 - \phi) g_n\|_M}$$

is a characteristic sequence for $(\square_{M-K}, \lambda)$. So $\sigma_{\mathrm{ess}}(\square_M) \subset \sigma_{\mathrm{ess}}(\square_{M-K})$. We trivially have $\sigma_{\mathrm{ess}}(\square_{M-K}) \subset \sigma_{\mathrm{ess}}(\square_M)$. q.e.d.

3. An L^2-Riemann-Roch inequality. In this section, we shall prove an L^2-Riemann-Roch inequality (3.5) which is a key lemma for the proof of our Main Theorem. The ideas are the decomposition principle (2.3), an argument due to Nadel and Tsuji [N-T] and the absence of essential spectrum (3.4).

Let (M, ω) be an n-dimensional complete Hermitian manifold without boundary, L a holomorphic line bundle over M with a smooth Hermitian metric h and $\Omega \subset M$ an open subset.

DEFINITION 3.1. Denote by $N^{p,q}_{\Omega, L^{\otimes \nu}}(\lambda)$ the number of eigenvalues of $Q^{p,q}_{\Omega, L^{\otimes \nu}}$, counted with multiplicity, which are not greater than $\nu\lambda$ (note the factor ν). If Ω is not relatively compact, the $Q^{p,q}_{\Omega, L^{\otimes \nu}}$ need not have discrete spectrum; in that case $N^{p,q}_{\Omega, L^{\otimes \nu}}(\lambda)$ can be defined as the dimension of a certain spectral projection. However due to the following Proposition 3.4, we need not be concerned with this extended definition.

3.2. Demailly's generalization of Weyl's formula for the asymptotic spectrum. When Ω is a smoothly bounded relatively compact domain in M, Demailly [De] has already computed $N^{p,q}_{\Omega, L^{\otimes \nu}}(\lambda)$ asymptotically as $\nu \to \infty$ for $\lambda \in (0, \infty) \setminus$(a countable set). We shall not need the full statement of his result. Rather, we shall be content with the following special case:

For $\lambda > 0$, we have

$$\liminf_{\nu\to\infty} \frac{N^{0,0}_{\Omega,L^{\otimes\nu}}(\lambda)}{\nu^n} \geq \frac{1}{n!}\int_{M(\leq 1,L)\cap\Omega} c(L,h)^n .$$

3.3. The Bochner-Kodaira formula for non-Kähler manifolds. We will use the following Bochner-Kodaira formula for non-Kähler manifolds which was given by Griffiths [Gr, (7, 14)] to show the absence of essential spectrum (3.4). We can write locally $\omega=\sqrt{-1}\sum g_{\alpha\bar\beta}dz^\alpha\wedge \overline{dz^\beta}$. The torsion tensor $T^\alpha_{\beta\gamma}$ is given by

$$T^\alpha_{\beta\gamma}=\sum_\lambda g^{\alpha\bar\lambda}\left(\frac{\partial g_{\gamma\bar\lambda}}{\partial z^\beta}-\frac{\partial g_{\beta\bar\lambda}}{\partial z^\gamma}\right),$$

where $(g^{\alpha\bar\beta})$ is the matrix so that $\sum_k g^{i\bar k}g_{j\bar k}=\delta^i_j$. For any smooth compactly supported $L^{\otimes\nu}$-valued $(0,q)$-form with $q\geq 1$

$$f=\frac{1}{q!}\sum f_{\bar\alpha_1\cdots\bar\alpha_q}\overline{dz^{\alpha_1}}\wedge\cdots\wedge\overline{dz^{\alpha_q}}$$

on M, we have

$$(\square_M f,f)=\|\bar\nabla f\|^2+\nu(\Theta f,f)+(\tilde R f,f),$$

$$(\tilde R f,f)=(\mathrm{Ric}\, f,f)+2\,\mathrm{Re}((\bar\partial T^*+T^*\bar\partial)f,f)-((TT^*+T^*T)f,f),$$

where

(i) $\bar\nabla f$ is the covariant differential in the $(0,1)$-direction,

(ii) $\Theta f=(q!)^{-1}\sum\theta^{\bar\lambda}_{\bar\alpha_1}f_{\bar\lambda\bar\alpha_2\cdots\bar\alpha_q}\overline{dz^{\alpha_1}}\wedge\cdots\wedge\overline{dz^{\alpha_q}}$ with $\theta^{\bar\lambda}_{\bar\alpha_1}$ being the curvature tensor of L with the first index raised,

(iii) $(\mathrm{Ric}\, f,f)$ is defined analogously to $(\Theta f,f)$ with $\theta^{\bar\lambda}_{\bar\alpha_1}$ replaced by the Ricci tensor $R^{\bar\lambda}_{\bar\alpha_1}$ with the first index raised,

(iv) $Tf=((q-1)!)^{-1}\sum\overline{T^{\lambda}_{\alpha_0\alpha_1}}f_{\bar\lambda\bar\alpha_2\cdots\bar\alpha_q}\overline{dz^{\alpha_0}}\wedge\overline{dz^{\alpha_1}}\wedge\cdots\wedge\overline{dz^{\alpha_q}}$,

(v) T^* is the adjoint operator of T.

In the following Propositions 3.4 and 3.5, we assume the following condition $(*)$ on the complete Hermitian manifold (M,ω) and the holomorphic Hermitian line bundle (L,h).

$(*)$ There exists a compact manifold $K\subset M$ with boundary and $\dim K=\dim M$ or $K=\varnothing$ satisfying the following property: there exist $\nu_0\in N$ and a positive constant α such that $\tilde R+\nu\Theta\geq\alpha\nu$ for $\nu\geq\nu_0$ on $M-K$.

PROPOSITION 3.4 (absence of essential spectrum). *Let (M,ω) and (L,h) be as in* 3.3 $(*)$. *Then $Q^{0,0}_{M,L^{\otimes\nu}}$ has no essential spectrum in the interval $(0,\alpha\nu)$ for $\nu\geq\nu_0$.*

PROOF. By the decomposition principle (2.3), it suffices to show that $Q^{0,0}_{M-K,L^{\otimes\nu}}$ has such a property. First we look at $(0,1)$-forms. We claim that $Q^{0,1}_{M-K,L^{\otimes\nu}}(f)\geq\alpha\nu\|f\|^2$ for $\nu\geq\nu_0$ and for all $f\in\mathrm{Dom}\,Q^{0,1}_{M-K,L^{\otimes\nu}}$. By the completeness of ω, it suffices to verify the above inequality for smooth compactly supported f with the Dirichlet boundary condition on ∂K. By the Bochner-Kodaira formula, we have

$$(\square_{M-K}f, f)=\|\bar{\nabla}f\|^2+\nu(\Theta f, f)+(\tilde{R}f, f).$$

Then by the assumption (*), we get the assertion for (0,1)-forms. Next, if $f\in \operatorname{Dom}\square_{M-K}\cap\mathscr{L}^{0,0}_{M-K}(L^{\otimes\nu})$ we have

$$\begin{aligned}(\square_{M-K}f, \square_{M-K}f)&=(\bar{\partial}^*\psi, \bar{\partial}^*\psi) \qquad \text{where} \quad \psi=\bar{\partial}f\\ &=Q^{0,1}_{M-K}(\psi)\\ &\geq\alpha\nu\|\psi\|^2 \qquad \text{for} \quad \nu\geq\nu_0 \text{ by our proof for (0, 1)-forms}\\ &=\alpha\nu(\square_{M-K}f, f) \qquad \text{for} \quad \nu\geq\nu_0.\end{aligned}$$

q.e.d.

3.5. An L^2-Riemann-Roch inequality. Let (M, ω) and (L, h) be as in 3.3 (*). Let $\Omega\subset M$ be a smoothly bounded relatively compact domain. For any $\lambda\in(0, \alpha)$, $Q^{0,0}_{M,L^{\otimes\nu}}$ has only discrete spectrum on the interval $(0, \nu\lambda]$ by Proposition 3.4. So we can use the min-max principle (3.6) to get the following first inequality:

$$\begin{aligned}N^{0,0}_{M,L^{\otimes\nu}}(\lambda)&\geq N^{0,0}_{\Omega,L^{\otimes\nu}}(\lambda) \quad \text{by the min-max principle (3.6)}\\ &\geq\frac{\nu^n}{n!}\int_{M(\leq 1,L)\cap\Omega}c(L, h)^n+o(\nu^n) \quad \text{by (3.2)}\end{aligned}$$

for all $\nu\gg 0$. Now let Ω expand toward all of M and let $\lambda\to+0$. Then we have

$$\liminf_{\nu\to\infty}\frac{\dim H^0_{(2)}(M, L^{\otimes\nu})}{\nu^n}\geq\frac{1}{n!}\int_{M(\leq 1,L)}c(L, h)^n,$$

where $H^0_{(2)}(M, L^{\otimes\nu})$ is the vector space of L^2-holomorphic sections of $L^{\otimes\nu}$ with respect to ω and h^ν. The right hand side may not exist, but in that case, $\dim H^0_{(2)}(M, L^{\otimes\nu})=+\infty$. If the right hand side is negative, then the above inequality does not make sense.

PROPOSITION 3.6 (min-max principle, cf. [R-S, vol. IV, p. 76]). *Let A be a self-adjoint operator which is bounded from below, i.e., $A\geq cI$ for some c. Define*

$$\mu_n(A)=\sup_{\varphi_1,\dots,\varphi_{n-1}}U_A(\varphi_1, \dots, \varphi_{n-1}),$$

where

$$U_A(\varphi_1, \dots, \varphi_{n-1})=\inf\{(\psi, A\psi); \psi\in\operatorname{Dom}A, \|\psi\|=1 \textit{ and } \psi\in[\varphi_1, \dots, \varphi_{n-1}]^\perp\}.$$

Then, for each fixed n, either:

(a) *there are at least n eigenvalues (degenerate eigenvalues counted as often as their multiplicty) below the bottom of the essential spectrum (which is defined as $\inf\{\lambda\,|\,\lambda\in\sigma_{\mathrm{ess}}(A)\}$ if $\sigma_{\mathrm{ess}}(A)\neq\varnothing$, and as $+\infty$ if $\sigma_{\mathrm{ess}}(A)=\varnothing$), and $\mu_n(A)$ is the n-th*

eigenvalue counting multiplicity
or

(b) μ_n *is the bottom of the essential spectrum, in which case* $\mu_n = \mu_{n+1} = \mu_{n+2} = \cdots$ *and there are at most* $n-1$ *eigenvalues (counting multiplicity) below* μ_n.

4. Proof of the Main Theorem. We first prove the following:

THEOREM 4.1. *Let X be an n-dimensional compact complex manifold and let L be a holomorphic line bundle over X with a singular Hermitian metric h. Assume that the curvature current c(L, h) is smooth on the complement of a divisor Z with only simple normal crossings, and that c(L, h) is strictly positive on some tubular neighborhood B of Z. Then* $\int_{X(\leq 1,L)} c(L,h)^n$ *exists and*

$$\liminf_{\nu\to\infty} \frac{\dim H^0(X, L^{\otimes\nu})}{\nu^n} \geq \frac{1}{n!}\int_{X(\leq 1,L)} c(L,h)^n .$$

If $\int_{X(\leq 1,L)} c(L,h)^n > 0$, *then L is big, and, in particular, X is a Moishezon space.*

PROOF. Let $Z = \sum_i Z_i$ be the decomposition into irreducible components. Let ω' be any smooth Hermitian metric on X. Then we can take

$$\omega := \omega'|_{X-Z} - \varepsilon_0\sqrt{-1}\sum_i \partial\bar\partial \log(-\log\|\sigma_i\|_i^2)^2$$

for $0<\varepsilon_0 \ll 1$ as a smooth complete Hermitian metric on $X-Z$ (the so-called generalized Poincaré metric), where σ_i is a holomorphic section of the line bundle $[Z_i]$ which vanishes to first order on Z_i, and $\|\sigma_i\|_i$ is the norm form of a smooth Hermitian metric on $[Z_i]$ such that $\|\sigma_i\|_i < 1$. Let

$$h_\varepsilon := h\prod_i(-\log\|\sigma_i\|_i^2)^\varepsilon \qquad \text{for each} \quad 0<\varepsilon\ll 1$$

be a family of smooth Hermitian metrics of $L|_{X-Z}$. Then $(X-Z, \omega)$ and $(L|_{X-Z}, h_\varepsilon)$ satisfy the condition 3.3 $(*)$ with some $K \subset X-Z$. Then by (3.5), we have

$$\dim H^0_{(2),\varepsilon}(X-Z, L^{\otimes\nu}) \geq \frac{\nu^n}{n!}\int_{X(\leq 1,L)-Z} c(L,h_\varepsilon)^n + o(\nu^n)$$

for all $\nu \gg 0$, where $H^0_{(2),\varepsilon}(X-Z, L^{\otimes\nu})$ is the vector space of L^2-holomorphic sections of $L^{\otimes\nu}|_{X-Z}$ with respect to ω and h_ε^ν. By the positivity of the curvature current on B, the singular Hermitian metric h has a plurisubharmonic weight on a neighborhood of each point of B, i.e. $h = e^{-\varphi}$ (locally), φ being a plurisubharmonic function. By the convexity properties of plurisubharmonic functions, h is bounded from below by a positive constant δ_0 (on each coordinate neighborhood), i.e. $h \geq \delta_0$ a.e. on B. Hence the Poincaré growth of ω gives $H^0_{(2),\varepsilon}(X-Z, L^{\otimes\nu}) \subset H^0(X, L^{\otimes\nu})$. Now letting $\varepsilon \to +0$, we have

$$\dim H^0(X, L^{\otimes \nu}) \geq \frac{\nu^n}{n!} \int_{X(\leq 1,L)} c(L, h)^n + o(\nu^n)$$

for all $\nu \gg 0$, since Z is a set of measure zero. $\int_{X(\leq 1,L)} c(L, h)^n$ is bounded from below, since $c(L, h)$ is positive at infinity B. By the finiteness of $\dim H^0(X, L^{\otimes \nu})$ and the above inequality, $\int_{X(\leq 1,L)} c(L, h)^n$ exists. If $\int_{X(\leq 1,L)} c(L, h)^n > 0$, then L is big by [De, Theorem 5.1]. q.e.d.

PROOF OF THE MAIN THEOREM. There exists a finite sequence of blow-ups

$$\pi : X_l \xrightarrow{\pi_l} X_{l-1} \xrightarrow{\pi_{l-1}} \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

such that

(1) π_i is the blow-up along a nonsingular center Y_{i-1} contained in the singular locus of X_{i-1} for each $i \geq 0$ and

(2) X_l is smooth.

Set $\tilde{Y}_0 \subset X_1$ to be the total transform of Y_0. Then we can construct a smooth Hermitian metric g of the line bundle $[\tilde{Y}_0]^{-1}$ whose curvature form satisfies the following three conditions:

(3) strictly positive along the positive dimensional fibre of π_1,

(4) bounded on X_1, and

(5) $\equiv 0$ on $X_1 -$(some neighborhood of $\tilde{Y}_0$).

We consider X_1, $L_1 := \pi_1^* L^{\otimes k_1} \otimes [\tilde{Y}_0]^{-1}$, $h_1 := \pi_1^* h^{k_1} \otimes g$, $Z_1 := \pi_1^{-1}(Z)$ and a neighborhood B_1 of Z_1 instead of X, L, h, Z and B. They satisfy the properties as in the statement of the Main Theorem for all large $k_1 \in N$, with appropriate choice of B_1. Since the curvature form of g is smooth and bounded, $\int_{X_{\mathrm{reg}}(\leq 1,L)} c(L, h)^n$ exists if and only if $\int_{X_{1\,\mathrm{reg}}(\leq 1,L)} c(L_1, h_1)^n$ exists. We can take k_1 so large that $\int_{X_{\mathrm{reg}}(\leq 1,L)} c(L, h)^n > 0$ implies $\int_{X_{1\,\mathrm{reg}}(\leq 1,L_1)} c(L_1, h_1)^n > 0$. So inductively, we may assume that X is smooth. By the same argument as above, we may assume that Z is a divisor with only simple normal crossings. By Theorem 4.1, the proof is completed.

The Corollary follows from our Main Theorem and the following lemma:

LEMMA 4.2 (cf. [S-S, Lemma 2.36]). *Let M be a complex manifold and η a d-closed integral $(1, 1)$-current of order 0 on M. Then there exists a holomorphic line bundle L on M with a singular Hermitian metric h such that $\eta = c(L, h)$.*

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