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MODULI OF PAIRS AND GENERALIZED THETA DIVISORS

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Abstract. We construct the moduli space of pairs consisting of a vector bundle together with a vector space of global sections on a fixed algebraic curve over an algebraically closed field of characteristic zero. The infinitesimal deformations of such a pair are shown to be parametrized by the first hypercohomology of a natural complex of sheaves of vector spaces on the base curve. We then apply these results to obtain desingularizations of theta divisors in moduli spaces of semistable vector bundles.

Introduction. The Brill-Noether loci W_d^r in the Jacobian of a smooth curve and the related objects G_d^r have been extensively studied [ACGH] and are known to enjoy several interesting properties. In this paper we construct analogous varieties $G_{n,d}^r$ related to the Brill-Noether loci $W_{n,d}^r$ in the moduli space of vector bundles on a curve. We prove that in certain special cases these varieties $G_{n,d}^r$ are smooth and consider some applications in the study of generalized theta divisors.

During the final stages of writing this paper, we came across the work of Thaddeus [Th]. Our results in the rank two case partly coincide with some of his results. For large degrees, the moduli of pairs has been constructed by Bertram [B] using methods of Gieseker. We thank Professor Le Potier for showing interest in our work and for telling us about his theory [P] of coherent systems, part of which is, in a sense, a generalization of the moduli of pairs to varieties of arbitrary dimension.

The contents of the paper are as follows. In §1, the definition of α-semistability is given and its immediate consequences are observed. The next two sections are devoted to the construction of $G_{n,d}^r$. The fourth section is an infinitesimal study of our objects. In the final section we consider some consequences of this infinitesimal study with special interest in the generalized theta divisor. We prove (cf. Theorem 5.5) that $G_{n,n(g-1)}^0$ is a desingularization of the generalized theta divisor Θ_n in $\mathcal{U}_X(n, n(g-1))$. Moreover, we show that for a generic curve this desingularization is an isomorphism precisely on the complement of the singular locus of Θ_2.

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NOTATION.

- k is an algebraically closed field of characteristic zero.
- If X and Y are k-schemes, then p is the second projection $X\times Y\to Y$ and q is the first projection $X\times Y\to X$. Sometimes we denote p by p_Y and q by p_X.
- $\mathrm{Grass}(a, m)$ is the Grassmanian of a-dimensional quotients of k^m.
- For a vector bundle E, $\mathrm{Grass}(a, E)$ is the a-th Grassmanization of E.
- $\mathcal{U}_X(n, d)$ and $\mathcal{U}_X^s(n, d)$ denote the moduli spaces of semistable and stable vector bundles respectively.
- $V \overset{s}{\sim} V'$ stands for S-equivalence of vector bundles V and V'.
- $\mathrm{Sing}(X)$ denotes the singular locus of X.
- If E is a vector bundle over a smooth projective curve X, then $\deg(E)$ and $\mathrm{rk}(E)$ denote the degree and rank of E respectively.

1. Preliminaries. In this section we briefly go through the notions of α-semistability and S-equivalence of α-semistable pairs. We state at the outset that all schemes considered are algebraic and defined over algebraically closed fields k of characteristic zero.

Let X be an irreducible, smooth projective curve of genus $g\geq 2$. If E is an algebraic vector bundle on X, we let $\mu(E)$ denote the slope, $\deg(E)/\mathrm{rk}(E)$, of E.

DEFINITION 1.1. (i) A pair (E, Λ) of type (n, d, r) on X consists of a vectot bundle E of rank n and degree d on X and an r-dimensional subspace Λ of $H^0(E)$.

(ii) A morphism $f:(E, \Lambda)\to(F, \Pi)$ of pairs is a homomorphism $f: E\to F$ of vector bundles which takes Λ to Π.

(iii)
- A subpair of a pair (E, Λ) is a pair (F, Π) such that $F\subset E$ and $\Pi\subset\Lambda$.
- A quotient pair of (E, Λ) is a pair (G, Σ) together with a surjective homomorphism $f: E\to G$ such that $\Sigma=H^0(f)(\Lambda)$.

REMARK 1.2. Let (E, Λ) be a pair and $F\subset E$ a subbundle of E. If we let $\Lambda_F=H^0(F)\cap\Lambda$, then (F, Λ_F) becomes a subpair. In what follows we reserve the notation λ_F for $\dim\Lambda_F$.

DEFINITION 1.3. Fix a rational number $\alpha\geq 0$. We say a pair (E, Λ) is α-semistable (resp. α-stable) if for every subpair (F, Π) of (E, Λ),

$$\mu(F)+\alpha\cdot\frac{\dim\Pi}{\mathrm{rk}(F)}\leq\mu(E)+\alpha\frac{\dim\Lambda}{\mathrm{rk}\,E}\quad(\text{resp. } <).$$

We write

$$\mu_\alpha(E, \Lambda):=\mu(E)+\alpha\frac{\dim\Lambda}{\mathrm{rk}\,E}.$$

REMARK 1.4. (i) Note that the α-semistability condition is the same as

$$\mu(F)+\alpha\lambda_F\le\mu(E)+\alpha\frac{\dim\Lambda}{\operatorname{rk}E}\qquad\text{for all}\quad F\subseteq E\,.$$

(ii) It is not hard to verify that for 'small' α (in fact, $\alpha<1/\lambda r$ where $\lambda=\dim\Lambda$ and $r=\operatorname{rk}E$) α-semistability (resp. α-stability) of (E,Λ) is equivalent to

- E is semistable; and
- We have

$$\frac{\dim\Pi}{\operatorname{rk}F}\le\frac{\dim\Lambda}{\operatorname{rk}E},\quad(\text{resp.}\ <)$$

for all subpairs (F,Π) of (E,Λ) with $0\ne F\ne E$ and $\mu(F)=\mu(E)$.

This is a recast (cf. [B]) of the condition of stability given in [B-D] for the case $\dim\Lambda=1$. *Whenever we say that a pair (E,Λ) is 'semistable' we mean that (E,Λ) is α-semistable for an α which is small in the above snese.*

(iii) For α small, the following are true.

- E is stable $\Rightarrow$ (E,Λ) is α-semistable for all $\Lambda\subset H^0(E)$.
- (E,Λ) α-semistable $\Rightarrow$ E is semistable.
- Further, if $(n,r)=1$, then (E,Λ) is α-semistable $\Leftrightarrow$ (E,Λ) is α-stable.

(iv) We say that a pair (E,Λ) is simple if the only endomorphisms of the pair are scalars. It is not hard to see that every α-stable pair is simple (cf. [M-S]).

(v) If (E,Λ) is a pair of type $(2,d,1)$ then the above notion of α-semistability of (E,Λ) coincides with that in [Th].

Fix a point $p\in X$. This point defines an ample invertible sheaf $\mathcal{O}_X(1)$ on X of degree 1. Let m_0 be any positive integer. If (E,Λ) is a pair of type (n,d,r) we get a new pair $(E(m_0),\bar{\Lambda})$ of type (n,nm_0+d,r), where $E(m_0):=E\otimes\mathcal{O}_X(1)^{\otimes m_0}$, and $\bar{\Lambda}$ is the image of Λ under the canonical inclusion $H^0(E)\subseteq H^0(E(m_0))$.

LEMMA 1.5. *A pair (E,Λ) is α-semistable (resp. α-stable) if and only if $(E(m_0),\bar{\Lambda})$ is α-semistable (resp. α-stable).*

PROOF. This follows at once from the definitions once we notice that if $W\subseteq E(m_0)$, then

$$\dim(H^0(W(-m_0)\cap\Lambda)=\dim(H^0(W)\cap\bar{\Lambda})\,.$$

DEFINITION 1.6. Let α and μ be fixed positive rational numbers. We let $S_\alpha(\mu)$ denote the category of α-semistable pairs (E,Λ) such that $\mu_\alpha(E,\Lambda)=\mu$.

PROPOSITION 1.7. *The category $S_\alpha(\mu)$ is abelian in which simple objects are α-stable pairs.*

PROOF. The only thing to be verified is that if f is a morphism of pairs in $S_\alpha(\mu)$, f is of constant rank at every point of X. The verification of this proceeds as in

Proposition 1.15 of [M-S].

COROLLARY 1.8. *If* (E, Λ) *is a* α*-semistable pair, then there exists a filtration* (*the Jordan-Hölder filtration*):

$$0=(E_0, \Lambda_0)\subset(E_1, \Lambda_1)\subset \cdots \subset(E_r, \Lambda_r)=(E, \Lambda)$$

such that

- $\mu_\alpha(E_i/E_{i-1}, \Lambda_i/\Lambda_{i-1})=\mu_\alpha(E, \Lambda)$.
- $(E_i/E_{i-1}, \Lambda_i/\Lambda_{i-1})$ *is an* α*-stable pair for all* i.

Moreover, the isomorphism class of the pair

$$\mathrm{gr}_\alpha(E, \Lambda):=\bigoplus(E_i/E_{i-1}, \Lambda_i/\Lambda_{i-1})$$

is independent of the filtration.

DEFINITION 1.9. Two pairs (E, Λ) and (E', Λ') are said to be S-equivalent if

$$\mathrm{gr}_\alpha(E, \Lambda)\simeq \mathrm{gr}_\alpha(E', \Lambda').$$

We write $(E, \Lambda) \overset{s}{\sim} (E', \Lambda')$ if (E, Λ) and (E', Λ') are S-equivalent.

REMARK 1.10. (i) It is not hard to see that

$$(E, \Lambda) \overset{s}{\sim} (E'\Lambda') \Leftrightarrow (E(m_0), \bar{\Lambda}) \overset{s}{\sim} (E'(m_0), \bar{\Lambda}').$$

(ii) The category $S_\alpha(\mu)$ is bounded. The proof of this is similar to Lemma 5.2 of [N]. So we may choose $m_0 \gg 0$ such that $H^1(E(m_0))=0$ and $H^0(E(m_0))$ generates $E(m_0)$ for all $(E, \Lambda)\in S_\alpha(\mu)$.

We fix an m_0, chosen as in Remark 1.10 (ii) above, for the rest of the paper. Now we go on to the definition of a family of pairs parameterized by a scheme and the moduli functor for pairs of a fixed type.

DEFINITION 1.11 (cf. [ACGH]). (i) By a family of pairs (E_T, Λ_T) of type (n, d, r) on X, parameterized by a scheme T, we mean:

- A vector bundle E_T on $X\times T$ such that $\mathrm{rk}\, E_t=n$, $\deg E_t=d$ for all $t\in T(k)$; and
- A locally free subsheaf Λ_T of $p_* E_T$ of rank r such that the canonical map $\Lambda_T\otimes k(t)\to H^0(E_t)$ is injective for all $t\in T(k)$.

(ii) Two families of pairs (E_T, Λ_T) and (E'_T, Λ'_T) are said to be equivalent if there exists a line boundle $\mathscr{L}_T$ on T such that $E'_T\simeq E_T\otimes p^*\mathscr{L}_T$ and this isomorphism takes Λ'_T to $\Lambda_T\otimes\mathscr{L}_T$. If (E_T, Λ_T) and (E'_T, Λ'_T) are equivalent, we write $(E_T, \Lambda_T)\sim(E'_T, \Lambda'_T)$.

REMARK 1.12. If (E_T, Λ_T) and (E'_T, Λ'_T) are families of pairs such that $(E_t, \Lambda_t)\simeq(E'_t, \Lambda'_t)$ for all $t\in T(k)$ and if (E_t, Λ_t) is α-stable for all $t\in T(k)$, then $(E_T, \Lambda_T)\sim(E'_T, \Lambda'_T)$.

With this we define the moduli functor for pairs of type (n, d, r) as

$$\mathscr{G}^r_{n,d}: (\text{Schemes}) \to (\text{Sets})$$

defined by

$$\mathscr{G}^r_{n,d}(T) := \{\text{equivalence classes of families of pairs } (E_t, \Lambda_t) \text{ of type } (n, d, r+1) \text{ such that } (E_t, \Lambda_t) \text{ is semistable for all } t \in T(k)\}\,.$$

See Remark 1.4 (ii) regarding semistability of a pair.

2. **Construction of moduli spaces.** In this section we construct the moduli spaces of semistable pairs of type (n, d, r) for $d \gg 0$. Recall (cf. Remark 1.4 (ii)) that 'semistability' of a pair (E, Λ) means α-semistability for small α. Note that (E, Λ) is semistable $\Rightarrow E$ is semistable. (See Remark 1.4 (iii).)

For the sake of brevity, set $d_0 := nm_0 + d$ (recall that m_0 is fixed and $m_0 \gg 0$). Let $N := \chi(E(m_0))$ where E is a vector bundle of rank n, degree d, and χ denotes the Euler characteristic. Also set $H := H^0(\mathcal{O}_X^N)$.

Let P be the Hilbert polynomial of vector bundles of rank n, degree d_0 on X, that is, $P(m) = d_0 + n(m - g + 1)$. Let Q stand for the Quot-scheme of coherent quotients of $H \otimes \mathcal{O}_X$ with fixed Hilbert polynomial P. Let $\mathscr{V}$ be the universal quotient on $X \times Q$. For $q \in Q(k)$, we let V_q denote the restriction of $\mathscr{V}$ to $X \times \{q\}$. Define

$R := \{q \in Q :$
- V_q is locally free of rank n and degree d_0
- The canonical map $H \otimes \mathcal{O}_X \to H^0(V_q)$ is an isomorphism$\}$.

It is well known (cf. [S-1]) that R is a smooth, irreducible, open subscheme of Q. Let R^{ss} (resp. R^s) denote the open subscheme of R consisting of $q \in R$ such that V_q is semistable (resp. stable). Notice that $p_* \mathscr{V}$ is actually locally free as $\deg V_q = d_0 \gg 0$. Now set

$$\tilde{R} := \mathrm{Grass}(N - r, p_* \mathscr{V})\,.$$

The second hypothesis in the definition of R shows that $p_* \mathscr{V}$ is in fact free on R and

$$\tilde{R} := R \times \mathrm{Grass}(N - r, N)\,.$$

So $\tilde{R}$ is smooth and irreducible.

PROPOSITION 2.1. (i) *Let $\tau : \tilde{R} \to R$ be the natural projection. Then the pair $((1 \times \tau)^* \mathscr{V}, \Lambda)$, Λ being the tautological subbundle of $\tau^* p_* \mathscr{V} \cong p_*(1 \times \tau)^* \mathscr{V}$, gives a family of pairs of type (n, d_0, r) on X parameterized by $\tilde{R}$. We denote this pair by $(\mathscr{V}_{\tilde{R}}, \Lambda_{\tilde{R}})$.*

(ii) *$\tilde{R}$ enjoys the following 'local universality' property:*

Let (E_T, Λ_T) be any family of pairs of type (n, d_0, r) parameterized by a scheme T. Then given $t \in T(k)$, there exists an open set S containing t and a morphism $f : S \to \tilde{R}$ such that

$$(E_S, \Lambda_S) \sim f^*(\mathscr{V}_{\tilde{R}}, \Lambda_{\tilde{R}}),$$

where (E_S, Λ_S) denotes the pair (E_T, Λ_T) restricted to the open subscheme S of T.

PROOF. (i) is obvious. (ii) follows from the fact that the Grassmann functor is representable and from the local universality property of R (cf. [S-1]).

The natural action of $\mathrm{Aut}(H) \simeq GL(N)$ on R lifts to an action on $\tilde{R}$. If we let $\tilde{R}^{ss}$ (resp. $\tilde{R}^s$) denote the subset of $\tilde{R}$ consisting of semistable pairs (resp. stable pairs), it is easily seen that $\tilde{R}^{ss}$ and $\tilde{R}^s$ are invariant under the action of $GL(N)$. It is not difficult to prove:

PROPOSITION 2.2. (i) *The action of $GL(N)$ on $\tilde{R}$ goes down to an action of $PGL(N)$, and $PGL(N)$ acts freely on $\tilde{R}^s$.*

(ii) *Two points in $\tilde{R}$ lie in the same $PGL(N)$ orbit if and only if the corresponding pairs are isomorphic.*

(iii) *The orbit closure equivalence relation (cf. [S-2]) in $\tilde{R}^{ss}$ is the same as S-equivalence of semistable pairs.*

Now let M be a positive integer and let Z denote the M-fold product of $\mathrm{Grass}(n, N)$. There is a natural linearization of the diagonal action of $PGL(N)$ on Z (cf. [M-2], [S-2]). Let Z^{ss} (resp. Z^s) denote the open subschemes of Z consisting of semistable (resp. stable) points with respect to this linearization. If we fix a sequence $\{x_1, \ldots, x_M\}$ of M points in X, we get a natural $PGL(N)$-equivariant morphism γ: $R^s \to Z$ given by $\gamma(q) := ((V_q)_{x_1}, \ldots, (V_q)_{x_M})$, where $(V_q)_{x_i}$ denotes the fibre of the vector bundle V_q at x_i and hence is canonically a quotient of H.

THEOREM 2.3 (cf. [S-2], [N]). *There exists a positive integer M_0 such that whenever $M \geq M_0$, we can find a sequence $\{x_1, \ldots, x_M\}$ of M points in X for which the associated morphism γ satisfies the following properties:*

(i) $\gamma^{-1}(Z^{ss}) = R^{ss}$

(ii) $\gamma^{-1}(Z^s) = R^s$

(iii) $\gamma: R^{ss} \to Z^{ss}$ *is a closed embedding.*

Fix M_0 as given by Theorem 2.3 above. Let $M \geq M_0$. Set $\tilde{Z} := Z \times \mathrm{Grass}(N-r, N)$. and $\bar{\gamma} := \gamma \times \mathrm{id}: \tilde{R} \to \tilde{Z}$. In what follows we fix a positive rational number ε. Also μ_0 will denote the constant d_0/n.

Give $\tilde{Z}$ the polarization $(1, \ldots, 1, \varepsilon)$. Let $\tilde{Z}^{ss}$ (resp. $\tilde{Z}^s$) denote the open subscheme of semistable (resp. stable) points with respect to this polarization. For any point $(\phi_1, \ldots, \phi_M, \psi) \in \tilde{Z}$, and any proper non-zero subspace V of H define

$$\rho(V) = \frac{1}{M \cdot (\dim V)} \sum_{i=1}^{M} \dim V_i - \frac{n}{N}; \quad \text{and}$$

$$\theta(V)=\rho(V)-\frac{\varepsilon}{NM(\dim V)}[N(\dim V-\dim \bar{V})-n\dim V]$$

$$=\rho(V)-\frac{\varepsilon}{NM(\dim V)}[N\dim(V\cap\ker\psi)-n\dim V],$$

where $V_i=\phi_i(V)$ and $\bar{V}:=\psi(V)$.

It is not difficult to extract the following numerical criterion for semistability in $\tilde{Z}$ with respect to the polarization $(1,\dots,1,\varepsilon)$ (cf. [M-S]).

PROPOSITION 2.4. *A point $(\phi_1,\dots,\phi_M,\psi)\in\tilde{Z}$ is semistable (resp. stable) if and only if for every proper non-zero subspace V of H, we have $\theta(V)\geq 0$ (resp. >0).*

We now come to the main ingredient in the construction of the moduli space:

PROPOSITION 2.5. *There exists an integer $M_1\geq M_0$ such that for all $M\geq M_1$, the morphism $\tilde{\gamma}$: $\tilde{R}\to\tilde{Z}$ satisfies the following:* (i) $\tilde{\gamma}^{-1}(\tilde{Z}^{ss})=\tilde{R}^{ss}$, *and* (ii) $\tilde{\gamma}^{-1}(Z^s)=\tilde{R}^s$.

PROOF. We first prove $\tilde{\gamma}^{-1}(\tilde{Z}^{ss})\subseteq\tilde{R}^{ss}$. Let $(q,\psi)\in\tilde{\gamma}^{-1}(\tilde{Z}^{ss})$. Suppose that $(q,\psi)\notin\tilde{R}^{ss}$. Then there exists a subpair (G,Π) of $(V_q,\ker\psi)$, such that $0\neq G\neq V_q$, $\mu(G)=\mu_0$ and $\mu(G,\Pi)>r/n$. Let $V=H^0(G)$. It is then not difficult to see that (cf. [N, pp. 155–157]) V is a proper non-zero subspace of H, $\rho(V)=0$ and $N\cdot\mathrm{rk}(G)=n\cdot\dim V$. Therefore,

$$\theta(V)<\frac{\varepsilon}{NM\dim V}[n\mu(G,\Pi)\dim V-N\cdot\mathrm{rk}(G)\cdot\mu(G,\Pi)]=0,$$

a contradiction. (See Proposition 2.4.) This proves $\tilde{\gamma}^{-1}(\tilde{Z}^{ss})\subseteq\tilde{R}^{ss}$.

Conversely, let $(q,\psi)\in\tilde{R}^{ss}$ and let $(\phi_1,\dots,\phi_M,\psi)$ denote the point $\tilde{\gamma}(q,\psi)$. Fix a subspace V of H. Let G denote the subbundle generated by V. Set $\Pi=H^0(G)\cap\ker\psi$. We have to consider now two cases:

Case 1. $\mu(G)=\mu_0$ and $V=H^0(G)$. In this case, arguing as above, we see that $\rho(V)=0$ and $N\cdot\mathrm{rk}(G)=n\cdot\dim V$. Therefore,

$$\theta(V)\geq\frac{\varepsilon}{NM(\dim V)}[n\mu(G,\Pi)\dim V-N\cdot\mathrm{rk}(G)\mu(G,\Pi)]=0.$$

Case 2. Either $\mu(G)<\mu$ or $V\neq H^0(G)$. In this case it can be verified that

$$\rho(V)>\frac{1}{n^2}-\frac{d(G)}{N}.$$

(See [N, pp. 153–155].) Hence

$$\theta(V)>\frac{1}{N^2}-\frac{1}{M}\left[\deg(G)+\frac{\varepsilon}{N\dim V}(N\dim\Pi-r\dim V)\right].$$

But $\deg(G)\leq\mathrm{rk}(G)\mu_0\leq n\mu_0=d_0$. Hence

$$\deg G+\frac{\varepsilon}{N(\dim V)}(N\dim \Pi - r\dim V)\leq d_0+\frac{\varepsilon r(N-1)}{N}=:C\,.$$

Thus

$$\theta(V)>\frac{1}{N^2}-\frac{C}{M}\,.$$

So $\theta(V)>0$, for $M\gg 0$. This shows that $\theta(V)\geq 0$ in both Case 1 and Case 2. Namely, $(\phi_1,\dots,\phi_M,\psi)$ is semistable. This proves $\tilde{R}^{ss}\subseteq\tilde{\gamma}^{-1}(\tilde{Z}^{ss})$. The proof of (ii) is similar and is omitted.

COROLLARY 2.6. *Let (E_T, Λ_T) be a family of pairs of type (n, d, r) on X parameterized by a scheme T. Let T^{ss} (resp. T^s) denote the set of $t\in T$ such that (E_t, Λ_t) is semistable (resp. stable). Then T^{ss} (resp. T^s) is open in T.*

PROOF. By Theorem 2.3, $\tilde{R}^{ss}$ (resp. $\tilde{R}^s$) is open in $\tilde{R}$. Now use the local universality property of $\tilde{R}$ to finish the proof.

We are now in a position to complete the construction of the moduli space of semistable pairs of type (n, d_0, r). First we have:

PROPOSITION 2.7 (cf. [S-3]). *Let $\tilde{Z}$ and Z be as above and let $\pi\colon \tilde{Z}\to Z$ be the natural projection. Then there exists a positive number ε_0 such that for any $0<\varepsilon\leq\varepsilon_0$, we have*

$$\pi^{-1}(Z^s)\subseteq\tilde{Z}^s \quad \text{and} \quad \pi(\tilde{Z}^{ss})\subseteq Z^{ss}\,.$$

(Recall that the constant ε figures in the polarization $(1,\dots,1,\varepsilon)$ on $\tilde{Z}$ with respect to which we take semistable points.)

PROOF. This can be seen by a direct verification using the numerical criterion of Proposition 2.4.

Let $P(n, d, r)$ denote the set of S-equivalence classes of semistable pairs of type (n, d, r) (see §1). Then we have:

THEOREM 2.8. (i) *There exists a structure of an irreducible, normal projective algebraic variety on $P(n, d_0, r)$. We denote it by G^{r-1}_{n,d_0}. Further, G^{r-1}_{n,d_0} has the property that given any family of semistable pairs of type (n, d_0, r) on X parameterized by a scheme T, the natural map $\eta\colon T\to G^{r-1}_{n,d_0}$ sending t to the S equivalence class $\eta(t)$ of (E_t, Λ_t) is a morphism.*

(ii) *If $(n, r)=1$, we have a universal family of stable pairs on G^{r-1}_{n,d_0} denoted by $(\mathscr{E}_P, \Lambda_P)$. Also G^{r-1}_{n,d_0} is smooth in this case.*

(iii) *There is a canonical surjective morphism $\tau\colon G^{r-1}_{n,d_0}\to\mathscr{U}_X(n, d_0)$. Furthermore τ is a projective bundle over the open subscheme consisting of stable bundles.*

REMARK 2.9. Note that Theorem 2.8 (ii) can be interpreted as: The scheme G^{r-1}_{n,d_0} represents the functor $\mathscr{G}^{r-1}_{n,d_0}$ if $(n, r)=1$.

PROOF OF THEOREM 2.8. (i) Consider the morphism $\tilde{\gamma}:=\gamma\times\mathrm{id}: \tilde{R}^{ss}\to\tilde{Z}^{ss}$. The image of $\tilde{\gamma}$ is actually contained in, as $\varepsilon<\varepsilon_0$, the open subset $U:=Z^{ss}\times\mathrm{Grass}(N-r, N)$ of $\tilde{Z}$. Let Y be the image of $\tilde{\gamma}$ and $\bar{Y}$ the closure of Y. Now the proof consists in showing $\bar{Y}^{ss}=\tilde{\gamma}(\tilde{R}^{ss})$ and $\bar{Y}^{s}=\tilde{\gamma}(\tilde{R}^{s})$. To see this first observe that

$$\bar{Y}^{ss}=\bar{Y}\cap\tilde{Z}^{ss} \quad\text{and}\quad \bar{Y}^{ss}\supseteq\tilde{\gamma}(\tilde{R}^{ss}).$$

To prove the other inclusion, let $z\in\bar{Y}^{ss}$. Since $\varepsilon<\varepsilon_0$, by Proposition 2.6, $z\in U$. But $Y\subset U$, hence $z\in\bar{Y}\cap U$. But as γ: $R^{ss}\to Z^{ss}$ is proper, $z\in Y$. Let $z=\tilde{\gamma}(q, \psi)$. Since $z\in\tilde{Z}^{ss}$, $(q, \psi)\in\tilde{\gamma}^{-1}(\tilde{Z}^{ss})=\tilde{R}^{ss}$ (see Proposition 2.5). That is $z\in\tilde{\gamma}(\tilde{R}^{ss})$. This proves the claim.

It follows now, (cf. [N, Lemma 3.12]), that there is a good quotient of $\tilde{R}^{ss}$ by $PGL(N)$ and $P(n, d, r)$, as a set, is in bijection with the k-valued points of the quotient scheme. We denote the quotient scheme by G^{r-1}_{n,d_0}. That G^{r-1}_{n,d_0} is irreducible and normal follows from the fact that $\tilde{R}^{ss}$ is irreducible and smooth (cf. [M-2]). The coarse moduli property mentioned in (i) now follows immediately from the local universality property of $\tilde{R}^{ss}$ (see Proposition 2.1) and Proposition 2.2 (iii).

(ii) If $(n, r)=1$, every semistable pair is stable. So to give a descent of the universal pair $(\mathscr{V}_{\tilde{R}}, \Lambda_{\tilde{R}})$, it is enough to show that this pair can be modified to get an equivalent pair

$$(\mathscr{V}_{\tilde{R}}\otimes p^*\mathscr{L}, \Lambda_{\tilde{R}}\otimes\mathscr{L}),$$

on which the isotropy k^* acts trivially. Clearly k^* acts as scalars on both $\mathscr{V}_{\tilde{R}}$ and $\Lambda_{\tilde{R}}$. So it is enough to produce a $GL(N)$ line bundle $\mathscr{L}$ on $\tilde{R}$ on which the group k^* acts by the character $t\mapsto t^{-1}$. To this end consider the line bundles $F_0=\det(\Lambda_{\tilde{R}})^*$; and $F_1=\det(p_*\mathscr{V}_{\tilde{R}}(1))\otimes\det(p_*\mathscr{V}_{\tilde{R}})^*$. Now k^* acts by characters $t\mapsto t^{-r}$ and $t\mapsto t^n$ on F_0 and F_1, respectively. If we choose $a, b\in\boldsymbol{Z}$ such that $ar+bn=1$, then $\mathscr{L}=(F_0)^{-a}\otimes(F_1)^{+b}$ will do the trick.

(iii) This follows from the fact that the morphism τ is proper and dominant.

3. Construction of $G^r_{n,d}$ for all degrees. In this section we construct the moduli scheme of semistable pairs of type (n, d, r) for all $d\geq 0$ and show that if $(n, r)=1$, it represents the functor $\mathscr{G}^{r-1}_{n,d}$. All along we keep the notation of §2. The general reference for this section is [ACGH].

On $X\times R$, considered in §2, we have the short exact sequence

$$0\to\mathscr{V}(-m_0)\to\mathscr{V}\to\mathscr{V}/\mathscr{V}(-m_0)\to 0.$$

Applying p_* to this we get the exact sequence

$$0\longrightarrow p_*\mathscr{V}(-m_0)\longrightarrow p_*\mathscr{V}\xrightarrow{\ \phi\ } p_*(\mathscr{V}/\mathscr{V}(-m_0))\longrightarrow R^1p_*\mathscr{V}(-m_0)\longrightarrow 0.$$

Let K^0 and K^1 denote the locally free sheaves $p_*\mathscr{V}$ and $p_*(\mathscr{V}/\mathscr{V}(-m_0))$, (cf. [M-F, p. 19]) of ranks $n(m_0-g+1)+d$ and nm_0, respectively. In what follows we will be crucially concerned with the morphism $\phi\colon K^0\to K^1$ of locally free sheaves.

For the sake of brevity let $l=n(m_0-g+1)+d$ and $h=l-(r+1)$. Consider now the closed, invariant subschemes of R^{ss} (resp. R^s) defined by

$$D^{ss}(n,d,r):=R^{ss}\cap D_\phi(h) \quad \text{and} \quad D^s(n,d,r):=R^s\cap D_\phi(h)\,,$$

where $D_\phi(h)$ stands for the h-th determinantal locus of ϕ in R defined by

$$D_\phi(h):=\mathrm{Spec}(\mathscr{O}_R/\mathscr{I}_\phi(h))\,,$$

where $\mathscr{I}_\phi:=\mathrm{im}\,\Lambda^h\phi^*\colon \mathscr{H}om(\Lambda^hK^1,\Lambda^hK^0)\to\mathscr{O}_R$. Here ϕ^* stands for the dual of ϕ.

It is now easily checked that

$$\mathrm{Supp}\,D^{ss}(n,d,r)=\{q\in R^{ss}\colon h^0(V_q)\geq r+1\}\,.$$

Also a good quotient of $D^{ss}(n,d,r)$ by $PGL(N)$ exists (cf. [N, Prop. 3.12]). Let us denote it by $SW^r_{n,d}$. We note also that, as char.$(k)=0$, $SW^r_{n,d}$ becomes canonically a closed subscheme of $\mathscr{U}_X(n,d)$.

$$\mathrm{Supp}(SW^n_{n,d})=\{[V]\in\mathscr{U}_X(n,d)\colon h^0(V)\geq r+1\}\,.$$

Recall now that $\tilde{R}$ (see §2) is the $(r+1)$-th Grassmannization of the locally free sheaf $p_*\mathscr{V}=K^0$. The map $\tilde{\tau}$ is the projection onto R. Let $\tilde{D}(n,d,r)$ be the canonical blow-up in $\tilde{R}$ of $D(n,d,r)$:

$$\tilde{D}(n,d,r):=\mathrm{Spec}(\mathscr{O}_{\tilde{R}}/\tilde{\mathscr{I}}_\phi)\,,$$

where $\tilde{\mathscr{I}}_\phi:=\mathrm{im}\,\tilde{\phi}^*\colon \mathscr{H}om(\Lambda_{\tilde{R}},\tilde{\tau}^*K^1)\to\mathscr{O}_{\tilde{R}}$. Here $\tilde{\phi}$ is the composite

$$\tilde{\phi}\colon \Lambda_{\tilde{R}}\hookrightarrow \tilde{\tau}^*K^0\xrightarrow{\tilde{\tau}^*(\phi)}\tilde{\tau}^*K^1\,.$$

(Recall that $\Lambda_{\tilde{R}}$ is the tautological subbundle of $\tilde{\tau}^*K^0$).

It can now readily be verified that

$$\mathrm{Supp}\,\tilde{D}(n,d,r)=\{(E,\Lambda)\in\tilde{R}\colon \Lambda\subseteq H^0(E(-m_0))\}\,,$$

where we regard $H^0(E(-m_0))$ as a subspace of $H^0(E)$ via the canonical injection from $H^0(E(-m_0))$ into $H^0(E)$. Hence $\tilde{D}(n,d,r)$ parameterizes canonically a family of pairs of type $(n,d,r+1)$. Note also that $\tilde{D}(n,d,r)$ is a closed, invariant subscheme of $\tilde{R}$. Let

$$\tilde{D}^{ss}(n,d,r)=\tilde{D}(n,d,r)\cap\tilde{R}^{ss} \quad \text{and} \quad \tilde{D}^s(n,d,r)=\tilde{D}(n,d,r)\cap\tilde{R}^s\,.$$

By the property of invariance of the definition of semistability (see Lemma 1.5), on tensoring by a line bundle, $D^{ss}(n,d,r)$ (resp. $D^s(n,d,r)$) consists of pairs $(E(-m_0),\Lambda)$ of type $(n,d,r+1)$ which are semistable (resp. stable). Furthermore we have:

PROPOSITION 3.1. *The scheme $\tilde{D}(n,d,r)$ has the local universality property* (*see* §2) *for any family of pairs of type $(n, d, r+1)$ parameterized by a scheme T.*

PROOF. Let (E_T, Λ_T) be a family of pairs of type $(n, d, r+1)$ on X parameterized by a scheme T. Consider the family $(E_T(m_0), \bar{\Lambda}_T)$ of semistable pairs of type $(n, d_0, r+1)$ (here $\bar{\Lambda}_T$ stands for the canonical image of Λ_T in $p_*E_T(m_0)$). There is a natural morphism $f: U \to \tilde{R}$, where U is an open subset of S, as $(E_T(m_0), \bar{\Lambda}_T)$ is a family of pairs of type $(n, d_0, r+1)$ (see Proposition 2.1 (ii)). But as the set-theoretic image of f is contained in $\tilde{D}(n, d, r)$, we only have to show that f actually factors as a morphism through $\tilde{D}(n, d, r)$. This follows if we know that $\mathscr{I} := \ker f^\# \subset \mathscr{I}_\phi$, where $f^\#$ is the morphism $\mathcal{O}_{\tilde{R}} \to f_*\mathcal{O}_T$ of $\mathcal{O}_{\tilde{R}}$ modules associated to f. To show this note that $\tilde{\phi}: \Lambda_{\tilde{R}} \to \tilde{\tau}^*K^1$ is zero when restricted to $\operatorname{Spec}(\mathcal{O}_{\tilde{R}}/\mathscr{I})$. Now it is not hard to see that $\mathscr{I}_\phi$ is the smallest ideal with this property and hence $\mathscr{I}_\phi \subset \mathscr{I}$. At this stage it is clear, as f factors as a morphism through $\tilde{D}(n, d, r)$, that $f^*(\mathscr{V}_{\tilde{D}}(-m_0), \Lambda_{\tilde{D}}) \sim (E_T, \Lambda_T)$. Here $(\mathscr{V}_{\tilde{D}}(-m_0), \Lambda_{\tilde{D}})$ denotes the pair $(\mathscr{V}_{\tilde{R}}(-m_0), \Lambda_{\tilde{R}})$ restricted to $X \times \tilde{D}$.

REMARK 3.2. (i) It follows from Corollary 2.6 that $\tilde{D}^{ss}$ and $\tilde{D}^s$ are open in $\tilde{D}$.

(ii) It is not hard to see that the orbit closure equivalence relation in $\tilde{D}^{ss}(n, d, r)$ is the same as S-equivalence of pairs of type $(n, d, r+1)$. (See Proposition 2.2 (iii) and Remark 1.10.)

THEOREM 3.3. (i) *The set of S-equivalence classes of semistable pairs of type $(n, d, r+1)$ can be equipped with the structure of a projective k-scheme.*

(ii) *If we denote the moduli scheme of pairs of type $(n, d, r+1)$ by $G^r_{n,d}$, we have:*

- *For any family of semistable pairs (E_T, Λ_T) of type $(n, d, r+1)$ parameterized by T, the natural map $\phi: T \to G^r_{n,d}$ taking t to the S-equivalence class of (E_t, Λ_t) is a morphism of schemes. Furthermore $G^r_{n,d}$ becomes naturally a closed subscheme of G^r_{n,d_0}.*
- *If $(n, r+1) = 1$, then $G^r_{n,d}$ represents the functor $\mathscr{G}^r_{n,d}$.*

PROOF. The existence of a good quotient $G^{r-1}_{n,d}$ of $\tilde{D}^{ss}(n, d, r)$ by $PGL(N)$ follows from [N, Prop. 3.12]. As $\operatorname{char}(k) = 0$, $G^{r-1}_{n,d}$ becomes naturally a closed subscheme of G^{r-1}_{n,d_0}. Now the theorem follows from Theorem 2.7 and Proposition 3.1.

REMARK 3.4. The independence of $D^{ss}(n, d, r)$, hence of $SW^r_{n,d}$, of the choices of m_0 and of the universal bundle $\mathscr{V}$ follows from the fact that $D^{ss}(n, d, r)$ can be described as the $(r+1-d+n(g-1))$-th Fitting ideal of $R^1p_*\mathscr{V}(-m_0)$. See [ACGH] for details. Also it is not difficult to see that $G^r_{n,d}$ is also independent of these choices. For instance, when $(n, r+1) = 1$, it follows from the fact that $G^r_{n,d}$ represents $\mathscr{G}^r_{n,d}$.

4. Infinitesimal deformations of a pair. Fix a pair (E, Λ) on X. In this section we prove that the infinitesimal deformations of (E, Λ) are parameterized by the first hypercohomology of a natural complex of sheaves on X arising from (E, Λ). Further, we show that the local deformation functor of (E, Λ) is formally smooth if the second hypercohomology of this complex vanishes. The idea of parameterizing infinitesimal deformations of certain objects by hypercohomology is due to Welters [W, Prop. 1.2]. See also [B-R].

Let $k[\varepsilon]$ denote the algebra of dual numbers over k and set $T = \operatorname{Spec} k[\varepsilon]$.

DEFINITION 4.1. (i) Let A be a finite dimensional local k-algebra. A deformation over A of a pair (E, Λ) is a triple (E_A, Λ_A, ϕ_A), where (E_A, Λ_A) is a family of pairs on X parameterized by $\operatorname{Spec} A$ and $\phi_A: (E_A \otimes k, \Lambda_A \otimes k) \to (E, \Lambda)$ is an isomorphism of pairs.

(ii) A (linear) infinitesimal deformation of (E, Λ) is a deformation of (E, Λ) over $k[\varepsilon]$.

(iii) Two deformations (E_A, Λ_A, ϕ_A) and $(E'_A, \Lambda'_A, \phi'_A)$ of (E, Λ) over A are said to be equivalent if there exists an isomorphism $\psi: E_A \to E'_A$ of vector bundles over $X \times \operatorname{Spec} A$ such that:

- $(p_A)_* \psi$ carries Λ_A onto Λ'_A, where $p_A: X \times \operatorname{Spec} A \to \operatorname{Spec} A$ is the canonical projection; and
- $\phi'_A \circ (\psi \otimes 1) = \phi_A$.

(iv) The local deformation functor $\mathscr{G}_{(E,\Lambda)}$ of a pair (E, Λ) is the functor

$$(\text{Finite dimensional local } k\text{-algebras}) \to (\text{Sets})$$

which assigns to an algebra A the set of equivalence classes of deformations of (E, Λ) over A.

We denote by $T_{(E,\Lambda)}$ the set $\mathscr{G}_{(E,\Lambda)}(k[\varepsilon])$ of equivalence classes of infinitesimal deformations of (E, Λ). From general arguments (cf. [Sch., Lemma 2.10]) there exists a natural structure of a k-vector space on $T_{(E,\Lambda)}$.

REMARK 4.2. Recall that if A is a local k-algebra and E_A is a vector bundle on $X \times \operatorname{Spec} A$, then giving a subbundle Λ_A of $(p_A)_* E_A$ is equivalent to giving a free A-submodule of $H^0(X \times \operatorname{Spec} A, E_A)$.

Consider a fixed pair (E, Λ) on X. Let Λ_X denote the constant sheaf on X with fibre Λ. Since X is an irreducible topological space, Λ_X is flasque. There is a natural monomorphism $\xi: \Lambda_X \to E$ of sheaves of k-vector spaces on X defined by $\xi_U(s) = s|_U$ for all U open in X and $s \in \Lambda$. We shall identify Λ_X as a subsheaf of E via the monomorphism ξ. Let $\mathscr{G}$ denote the quotient sheaf E/Λ_X. Since Λ_X is flasque, $\Gamma(U, \mathscr{G}) = \Gamma(U, E)/\Lambda$ for all U open in X. Let $p: E \to \mathscr{G}$ denote the canonical projection of sheaves. Define a sheaf homomorphism

$$D: \mathscr{E}nd\, E \to \mathscr{H}om(\Lambda_X, \mathscr{G}) \quad \text{by} \quad (D\phi)(s) = p \circ \phi(s)$$

for all local sections ϕ of $\mathscr{E}nd\, E$ and $s \in \Lambda$. Let $\mathscr{K}^{\bullet} = \mathscr{K}^{\bullet}_{(E,\Lambda)}$ denote the complex

$$0 \longrightarrow \mathscr{E}nd\, E \xrightarrow{D} \mathscr{H}om(\Lambda_X, \mathscr{G}) \longrightarrow 0$$

of sheaves of k-vector spaces on X.

Let us compute the hypercohomology of the complex $\mathscr{K}^{\bullet}$ using Čech cocycles. Choose a finite open covering $\mathscr{U}=\{U_i\}_{i=1}^N$ of X such that each U_i is affine. Let $I=\{1,\dots,N\}$ and if $i_1,\dots,i_n\in I$, denote $U_{i_1,\dots,i_n}=\bigcap_{\alpha=1}^n U_{i_\alpha}$. Using the Čech resolution of the complex $\mathscr{K}^{\bullet}$ with respect to the covering $\mathscr{U}$, we see that the p-th hypercohomology $\boldsymbol{H}^p(\mathscr{K}^{\bullet})\cong Z^p/B^p$ where, denoting the Čech differential by δ,

$$Z^p=\{(A,\phi): A\in C^{p-1}(\mathscr{U},\mathscr{H}om(\Lambda_X,\mathscr{G})),\ \phi\in Z^p(\mathscr{U},\mathscr{E}nd\,E),\ \delta A+(-1)^pD\phi=0\}$$

$$B^p=\{(\delta A+(-1)^{p-1}D\phi,\ \delta\phi): A\in C^{p-2}(\mathscr{U},\mathscr{H}om(\Lambda_X,\mathscr{G}),\ \phi\in C^{p-1}(\mathscr{U},\mathscr{E}nd\,E)\}\ .$$

PROPOSITION 4.3. *There is a natural isomorphism of the first hypercohomology* $\boldsymbol{H}^1(\mathscr{K}^{\bullet})$ *of the complex* $\mathscr{K}^{\bullet}_{(E,\Lambda)}$ *with the vector space* $T_{(E,\Lambda)}$ *of infinitesimal deformations of* (E,Λ).

PROOF. View $\boldsymbol{H}^1(\mathscr{K}^{\bullet})=Z^1/B^1$ as above. Denote $T=\operatorname{Spec}k[\varepsilon]$. Let $z=\{A_i\}\times\{\phi_{ij}\}\in Z^1$. We define an infinitesimal deformation $(E_T^z,\Lambda_T^z,\phi_T^z)$ of (E,Λ) as follows. For each pair $i,j\in I$, define $\theta_{ij}\in\Gamma(U_{ij},\mathscr{A}ut\,p_X^*E)$ by $\theta_{ij}=1+\varepsilon\phi_{ij}$. Then $\theta_{ij}\theta_{jk}=\theta_{ik}$, so we can glue the bundles $p_X^*E|_{U_i\times T}$ by the θ_{ij} to obtain a vector bundle E_T^z on $X\times T$. Let Λ_T^z be the subset of $H^0(X\times T,E_T^z)$ consisting of all sections ξ such that for all $i\in I$, $\xi|_{U_i\times T}$ is of the form $s-\varepsilon t$ where $s\in\Lambda$ and $t\in\Gamma(U_i,E)$ with $p(t)=A_i(s)$. (Recall that p is the projection $E\to\mathscr{G}$.) It is easily checked that Λ_T^z is a $k[\varepsilon]$-submodule of $H^0(X\times T,E_T^z)$. We shall now construct a $k[\varepsilon]$-basis for Λ_T^z. Let $\{s^\alpha\}_{\alpha=1}^k$ be a k-basis for Λ. Fix $\alpha\in\{1,\dots,k\}$. For $i=1,\dots,N$, define $t_i^\alpha\in\Gamma(U_i,E)$ as follows:

- Let $t_1^\alpha\in\Gamma(U_1,t)$ be any section such that $p(t_1^\alpha)=A_1(s^\alpha)$.
- If $i\geq 2$, let $\bar{t}_i^\alpha\in\Gamma(U_i,E)$ be any section such that $p(\bar{t}_i^\alpha)=A_i(s^\alpha)$. Then on U_{i1}, $p(t_1^\alpha-\phi_{i1}(s^\alpha)-\bar{t}_i^\alpha)=0$. Hence $\lambda_i=t_1^\alpha-\phi_{i1}(s^\alpha)-\bar{t}_i^\alpha\in\Gamma(U_{i1},\Lambda_X)=\Lambda$. Define $t_i^\alpha=\bar{t}_i^\alpha+\lambda_i$, $i\geq 2$.

Now define $\xi_i^\alpha=s^\alpha-\varepsilon t_i^\alpha$ in $\Gamma(U_i\times T,p_X^*E)$. For all $i\geq 2$, we have $\theta_{i1}\xi_1^\alpha=\xi_i^\alpha$ on $U_{i1}\times T$. Therefore for all $i,j\in I$, on $U_{ij1}\times T$, we have $\theta_{ij}\xi_j^\alpha=\theta_{i1}\theta_{1j}\xi_j^\alpha=\theta_{i1}\xi_1^\alpha=\xi_i^\alpha$. Since U_{ij1} is a dense subset of U_{ij}, we get $\theta_{ij}\xi_j^\alpha=\xi_i^\alpha$ on the whole of $U_{ij}\times T$. Hence the ξ_i^α patch up via the θ_{ij} to give a global section $\xi^\alpha\in H^0(X^0\times T,E_T^z)$. Clearly $\xi^\alpha\in\Lambda_T^z$. Now one can easily verify that $\{\xi^\alpha\}_{\alpha=1}^k$ is a $k[\varepsilon]$-basis of Λ_T^z. This proves that Λ_T^z is a free $k[\varepsilon]$-submodule of $H^0(X\times T,E_T^z)$ of rank $k=\dim\Lambda$. By Remark 4.2, Λ_T^z defines a subbundle of $(p_T)_*E_T^z$ on T. Define $\phi_T^z\colon E_T^z\otimes k\to E$ to be the identity map. Then $(E_T^z,\Lambda_T^z,\phi_T^z)$ is an infinitesimal deformation of (E,Λ).

We thus obtain a map $\Delta\colon Z^1\to T_{(E,\Lambda)}$ which is easily seen to factor through Z^1/B^1 to give a map $\bar{\Delta}\colon\boldsymbol{H}^1(\mathscr{K}^{\bullet})\to T_{(E,\Lambda)}$. Let us prove that this map is surjective. So, let (E_T,Λ_T,ϕ_T) be an arbitrary infinitesimal deformation of (E,Λ). By choosing the U_i small enough, we may assume that $\phi_T\colon E_T\otimes k\to E$ extends to an isomorphism $\psi_i\colon E_T|_{U_i\times T}\to p_X^*E|_{U_i\times T}$. Now let $\theta_{ij}=\psi_i\circ\psi_j^{-1}\in\Gamma(U_{ij},\mathscr{A}ut\,p_X^*E)$. Since $\theta_{ij}\otimes 1=1$, there exists $\{\phi_{ij}\}\in Z^1(\mathscr{U},\mathscr{E}nd\,E)$ such that $\theta_{ij}=1+\varepsilon\phi_{ij}$. Let $\{\xi^\alpha\}_{\alpha=1}^k$ be a $k[\varepsilon]$-basis of $H^0(T,\Lambda_T)\subset H^0(X\times T,E_T)$. Let $\xi_i^\alpha=\xi^\alpha|_{U_i\times T}$; we can write $\psi_i(\xi_i^\alpha)=s^\alpha-\varepsilon t_i^\alpha$ where $\{s^\alpha\}_{\alpha=1}^k$

is a k-basis of Λ and $t_i^\alpha \in \Gamma(U_i, E)$. Define a k-linear map $A_i: \Lambda \to \Gamma(U_i, \mathscr{G})$ by $A_i(s^\alpha) \doteq p(t_i^\alpha)$. Then $z = \{A_i\} \times \{\phi_{ij}\} \in Z^1$ and it is immediate that (E_T, Λ_T, ϕ_T) is equivalent to $(E_T^z, \Lambda_T^z, \phi_T^z)$. Hence $\bar{\Delta}$ is surjective. In the same manner, we can prove that $\bar{\Delta}$ is injective. This completes the proof of Proposition 4.3.

Having computed the infinitesimal variations of a pair, we now turn our attention to the question of smoothness.

PROPOSITION 4.4. *The local deformation functor $\mathscr{G}_{(E,\Lambda)}$ of a pair (E, Λ) is formally smooth if $\boldsymbol{H}^2(\mathscr{K}^\bullet_{(E,\Lambda)}) = 0$.*

PROOF. Recall that a *small extension* is an epimorphism $\eta: A \to \bar{A}$ of finite dimensional local k-algebras whose kernel is a non-zero principal ideal (t) such that $\boldsymbol{m}_A \cdot t = 0$ where $\boldsymbol{m}_A$ denotes the maximal ideal of A. To show that $\mathscr{G}_{(E,\Lambda)}$ is formally smooth it suffices (cf. [Sch, Remark 2.3]) to check that whenever $\eta: A \to \bar{A}$ is a small extension, $\mathscr{G}_{(E,\Lambda)}(\eta)$: $\mathscr{G}_{(E,\Lambda)}(A) \to \mathscr{G}_{(E,\Lambda)}(\bar{A})$ is surjective. Fix such an extension and let $\ker(\eta) = (t)$. Consider an affine open covering $\mathscr{U}$ of X as before. If R is any k-algebra, denote the open covering $\{U_i \times \operatorname{Spec} R\}$ of $X \times \operatorname{Spec} R$ by $\mathscr{U}_R$. Fix a basis $\{s^\alpha\}_{\alpha=1}^k$ of Λ. Note that giving an element $\theta \in \mathscr{G}_{(E,\Lambda)}(A)$ is equivalent to giving the following data:

- a cocycle $\{\theta_{ij}\} \in Z^1(\mathscr{U}_A, \mathscr{A}ut\, p_X^* E)$ such that $\theta_{ij}|_{U_{ij}} = 1$;
- an A-linearly independent set $\{\xi_i^\alpha\}_{\alpha=1}^k \subseteq \Gamma(U_i \times \operatorname{Spec} A, p_X^* \times E)$ for each $i \in I$ such that $\xi_i^\alpha|_{U_i} = S^\alpha$ and $\theta_{ij}\xi_j^\alpha = \xi_i^\alpha$ on $U_{ij} \times \operatorname{Spec} A$.

Now take an element $\bar{\theta} \in \mathscr{G}_{(E,\Lambda)}(\bar{A})$ and lift the corresponding data $(\{\bar{\theta}_{ij}\}, \{\bar{\xi}_i^\alpha\})$ arbitrarily to $\theta_{ij} \in \Gamma(U_{ij} \times \operatorname{Spec} A, \mathscr{A}ut\, p_X^* E)$ and $\xi_i^\alpha \in \Gamma(U_i \times \operatorname{Spec} A, p_X^* E)$. We clearly have $\theta_{ij}\theta_{jk}\theta_{ik}^{-1} = 1 + t h_{ijk}$ and $\theta_{ij}\xi_j^\alpha - \xi_i^\alpha = t\gamma_{ij}^\alpha$, where $h_{ijk} \in \Gamma(U_{ijk}, \mathscr{E}nd\, E)$ and $\gamma_{ij}^\alpha \in \Gamma(U_{ij}, E)$. Now define $A_{ij} \in \Gamma(U_{ij}, \mathscr{H}om(\Lambda_X, \mathscr{G}))$ by $A_{ij}(s^\alpha) = -p(\gamma_{ij}^\alpha)$, where p is the projection $E \to \mathscr{G}$. We will now show that $z = \{A_{ij}\} \times \{h_{ijk}\} \in Z^2$, i.e., that (i) $\delta\{h_{ijk}\} = 0$; and (ii) $\delta\{A_{ij}\} + \{Dh_{ijk}\} = 0$. To check (i), decompose $\theta_{ij} = 1 + \psi_{ij}$ where $\psi_{ij} \in \boldsymbol{m}_A \cdot \Gamma(U_{ij}, \mathscr{E}nd\, E)$, and using $t \cdot \boldsymbol{m}_A = 0$, check that $\theta_{ij}(1 + t h_{jkl}) = (1 + t h_{jkl})\theta_{ij}$. In other words, $\theta_{ij}\theta_{jk}\theta_{kl}\theta_{jl}^{-1} = \theta_{jk}\theta_{kl}\theta_{jl}^{-1}\theta_{ij}$. From these we get

$$(1 + t h_{jkl})(1 + t h_{ijl})(1 - t h_{ikl}) = (1 + t h_{ijk})\,.$$

Now, observing that $t^2 = 0$ gives (i). Let us now prove (ii). Since multiplication by t is injective, it suffices to check that $B := \delta\{t\gamma_{ij}^\alpha\} - t h_{ijk}(s^\alpha) = 0$. Decompose the section ξ_i^α as $s_i^\alpha - t_i^\alpha$ where $s_i^\alpha \in \Lambda$ and $t_i^\alpha \in \boldsymbol{m}_A \cdot \Gamma(U_i, E)$. Then s_i^α patch up to define global sections $s^\alpha \in \Lambda$, and $t\gamma_{ij}^\alpha = \psi_{ij}(s^\alpha) - \psi_{ij}(t_j^\alpha) + t_i^\alpha - t_j^\alpha$. At this stage note that $t h_{ijk} = \theta_{ij}\theta_{jk} - \theta_{ik} = \delta\{\psi_{ij}\} + \psi_{ij}\psi_{jk}$. Therefore we see that $B = -\psi_{ij}\psi_{jk}(s^\alpha) - \delta\{\psi_{ij}(t_j^\alpha)\}$. But applying ψ_{ij} to $t\gamma_{ij}^\alpha$ we get

$$\psi_{ij}\psi_{jk}(s^\alpha) + \psi_{ij}(t_j^\alpha) = \psi_{ij}\psi_{jk}(t_k^\alpha) + \psi_{ij}(t_k^\alpha)\,.$$

Hence $B = -t h_{ijk}(t_k^\alpha)$ which is zero because $t \cdot \boldsymbol{m}_A = 0$. This proves that $z = \{A_{ij}\} \times \{h_{ijk}\} \in Z^2$. Since $\boldsymbol{H}^2(\mathscr{K}^\bullet) = 0$, there exist $\{A_i\} \in C^0(\mathscr{U}, \mathscr{H}om(\Lambda_X, \mathscr{G}))$ and $\{\phi_{ij}\} \in$

$C^1(\mathscr{U}, \mathscr{E}nd\, E)$ such that $\{A_{ij}\}=\delta\{A_i\}-D\{\phi_{ij}\}$ and $\delta\{\phi_{ij}\}=\{h_{ijk}\}$. Define $\tilde{\theta}_{ij}=\theta_{ij}+t\phi_{ij}$ and $\tilde{\xi}_i^\alpha=\xi_i^\alpha-tu_i^\alpha$, where u_i^α is an arbitrary section such that $p(u_i^\alpha)=A_i(s^\alpha)$. Then we have $\tilde{\theta}_{ij}\tilde{\theta}_{jk}=\tilde{\theta}_{ik}$ which is well and good, but the equality $\tilde{\theta}_{ij}\tilde{\xi}_i^\alpha=\xi_i^\alpha$ holds only mod Λ. Yet, as in the proof of Proposition 4.3, we can now perturb each $\tilde{\xi}_i^\alpha$ by an element of Λ to remove the phrase 'mod Λ' and write $\tilde{\theta}_{ij}\tilde{\xi}_j^\alpha=\tilde{\xi}_i^\alpha$. Then the system $\tilde{\theta}=(\{\tilde{\theta}_{ij}\}, \{\tilde{\xi}_i^\alpha\})$ defines an element $\tilde{\theta}\in\mathscr{G}_{(E,\Lambda)}(A)$ which is obviously a lift of $\bar{\theta}$, i.e., $\mathscr{G}_{(E,\Lambda)}(\eta)(\tilde{\theta})=\bar{\theta}$. This proves that $\mathscr{G}_{(E,\Lambda)}(\eta)$ is surjective.

5. Desingularization of the generalized Θ-divisor. In this section we show that the schemes $G_{n,d}^r$ $(0\leq d\leq n(g-1))$ are smooth when $r=0$. We also consider the special case $d=n(g-1)$ in connection with the Θ-divisor in $\mathscr{U}_X(n, n(g-1))$.

Let $(E, \Lambda)\in G_{n,d}^r$. Recall that we have a complex

$$\mathscr{K}^\bullet_{(E,\Lambda)}: 0\to\mathscr{E}nd\, E\to\mathscr{H}om(\Lambda_X, \mathscr{G})\to 0$$

of sheaves of k-vector spaces. This complex fits into a short exact sequence of complexes

$$0\to\mathscr{H}om(\Lambda_X, \mathscr{G})[1]\to\mathscr{K}^\bullet\to\mathscr{E}nd\, E\to 0$$

where for any sheaf $\mathscr{F}$, $\mathscr{F}[1]$ denotes the complex whose first component is $\mathscr{F}$ and all other components are zero. We thus obtain a long exact hypercohomology sequence

$$0\longrightarrow \boldsymbol{H}^0(\mathscr{K}^\bullet)\longrightarrow H^0(\mathscr{E}nd\, E)\xrightarrow{\phi} H^0(\mathscr{H}om(\Lambda_X, \mathscr{G}))\longrightarrow \boldsymbol{H}^1(\mathscr{K}^\bullet)\longrightarrow H^1(\mathscr{E}nd\, E)$$
$$\xrightarrow{\mu} H^1(\mathscr{H}om(\Lambda_X, \mathscr{G}))\longrightarrow \boldsymbol{H}^2(\mathscr{K}^\bullet)\longrightarrow 0\,.$$

At this stage a few observations are in order.

REMARK 5.1. 1. Note that $\boldsymbol{H}^0(\mathscr{K}^\bullet)$ is precisely the vector space of global endomorphisms of E which preserve Λ. Further, by Proposition 4.3, $\boldsymbol{H}^1(\mathscr{K}^\bullet)$ is the vector space $T_{(E,\Lambda)}$ of infinitesimal deformations of (E, Λ).

2. Since Λ_X is a constant sheaf, $H^0(\mathscr{H}om(\Lambda_X, \mathscr{G}))$ is canonically isomorphic to $\mathrm{Hom}(\Lambda, H^0(E)/\Lambda)$. In the above long exact sequence ϕ is the obvious map from $H^0(\mathscr{E}nd\, E)$ to $\mathscr{H}om(\Lambda, H^0(E)/\Lambda)$.

3. If $(n, r+1)=1$, then the scheme $G_{n,d}^r$ represents the functor $\mathscr{G}_{n,d}^r$ as we saw in §3. Therefore by Proposition 4.4, if $\boldsymbol{H}^2(\mathscr{K}^\bullet_{(E,\Lambda)})=0$ for all $(E, \Lambda)\in G_{n,d}^r$, then the scheme $G_{n,d}^r$ is smooth. Notice that $\boldsymbol{H}^2(\mathscr{K}^\bullet_{(E,\Lambda)})$ vanishes if and only if in the above long exact sequence the map μ is surjective.

PROPOSITION 5.2. *Let X be a smooth irreducible projective curve of genus $g\geq 2$ and let $d\geq 0$. Then the scheme $G_{n,d}^0$ is smooth and non-empty.*

PROOF. Fix a pair $(E, \Lambda)\in G_{n,d}^0$. In view of Remark 5.1(3) we need only to check that $\mu: H^1(\mathscr{E}nd\, E)\to H^1(\mathscr{H}om(\Lambda_X, \mathscr{G}))$ is surjective. Note that we have canonical isomorphisms $H^1(E)\cong H^1(\mathscr{G})$ and $H^1(\mathscr{H}om(\Lambda_X, \mathscr{G}))\cong \mathrm{Hom}(\Lambda, H^1(\mathscr{G}))$. The first one is immediate from the exact sequence

$$0\to\Lambda_X\to E\to\mathscr{G}\to 0$$

and the fact that Λ_X is flasque. The second isomorphism comes from the fact that Λ_X is a constant sheaf of vector spaces. Using Serre duality and these identifications, the dual of μ can be thought of as the 'Petri map'

$$\mu^*: H^0(E^*\otimes K)\times\Lambda\to H^0(\mathscr{E}nd\, E\otimes K)$$

where K is the canonical bundle of X. If $\sigma\in H^0(E^*\otimes K)$ and $s\in\Lambda$, then we can think of σ and s as global homomorphisms $\sigma: E\to K$ and $s: \mathscr{O}_X\to E$; the Petri map μ^* is now given by $\mu^*(\sigma, s)=\sigma\otimes s$. Since $\dim\Lambda=1$, it is now clear that μ^* is injective, i.e., μ is surjective. Since the genus $g\geq 2$, there exist stable bundles which admit non-zero global sections (cf. [Su]), so $G^0_{n,d}$ is non-empty. □

In the rest of the section we consider the case $d=n(g-1)$.

LEMMA 5.3. *If X is a smooth curve of genus $g\geq 3$, then the subscheme $SW^0_{n,n(g-1)}$ of $\mathscr{U}_X(n, n(g-1))$ is reduced, irreducible and equals the closure of $W^0_{n,n(g-1)}$ in $\mathscr{U}_X(n,n(g-1))$.*

PROOF. Let us first show that $SW^0_{n,n(g-1)}$ is irreducible. Since $SW^0_{n,n(g-1)}$ is a good quotient of $D^{ss}=D^{ss}(n, n(g-1), 0)$ it suffices to show that D^{ss} is irreducible. By [Su], we know that $W^0_{n,n(g-1)}$ is irreducible, hence so is $D^s=D^s(n, n(g-1), 0)$. We claim that D^{ss} is the closure of D^s in R^{ss}. Surely, since $\bar{D}^s$ is irreducible and of the same dimension as D^{ss}, it is an irreducible component of D^{ss}. Suppose D^{ss} has some other component $T\neq\bar{D}^s$. Since D^{ss} is a determinantal variety in R^{ss} whose expected codimension is 1, the codimension of T in R^{ss} is ≤ 1. But $T\subset R^{ss}\setminus R^s$ and, since the genus $g\geq 3$, the codimension of $R^{ss}\setminus R^s$ in R^{ss} is ≥ 2, a contradiction. Therefore $D^{ss}=\bar{D}^s$ and hence irreducible. This proves that $SW^0_{n,n(g-1)}$ is irreducible. Next note that $W^0_{n,n(g-1)}$ is open in $SW^0_{n,n(g-1)}$ and, by [Su], it is non-empty. Therefore its closure in $\mathscr{U}_X(n, n(g-1))$ must be equal to $SW^0_{n,n(g-1)}$. It remains to show that $SW^0_{n,n(g-1)}$ is reduced. The scheme D^{ss}, being an irreducible determinantal variety of expected codimension, is Cohen-Macaulay (cf. [ACGH]), hence has no embedded components. Further, it is birational to $\tilde{D}^s$ which is smooth by Proposition 5.2. Thus D^{ss} contains a non-empty reduced open subset. Therefore D^{ss} is reduced. Hence its good quotient $SW^0_{n,n(g-1)}$ is also reduced. □

REMARK 5.4. The assumption $d=n(g-1)$ does not seem to be very essential in the above Lemma. Probably it can be proved more generally using results from [Su] together with the fact that the codimension of $R^{ss}\setminus R^s$ in R^{ss} is at least $g-1$. We do not go into this process here. Notice that the lemma also shows that the blow up $\tilde{D}^{ss}(n, n(g-1), 0)$, hence $G^0_{n,n(g-1)}$, is irreducible.

Let Θ_n denote the generalized theta divisor (cf. [D-N]) in the moduli space $\mathscr{U}_X(n, n(g-1))$ and let $\tilde{\Theta}_n$ denote the smooth scheme $G^0_{n,n(g-1)}$. It is clear that $\Theta_n = \bar{W}^0_{n,n(g-1)}$, hence by the above lemma, $\Theta_n = SW^0_{n,n(g-1)}$. Let $\tau: \tilde{\Theta}_n \to \mathscr{U}_X(n, n(g-1))$ be the natural morphism. Then τ factors through Θ_n to give a morphism $\tau: \tilde{\Theta}_n \to \Theta_n$. We now have a smooth birational model of Θ_n:

THEOREM 5.5. *Let X be a smooth curve of genus $g \geq 3$ over an algebraically closed field k of characteristic zero. Then the above morphism $\tau: \tilde{\Theta}_n \to \Theta_n$ is an isomorphism over the open subset $W^0_{n,n(g-1)}[1]$ of Θ_n consisting of stable bundles E such that $h^0(E) = 1$. In other words, $\tilde{\Theta}_n$ is a desingularization of Θ_n.*

PROOF. Clearly the restriction $\tau_0: \tau^{-1}(W^0_{n,n(g-1)}[1]) \to W^0_{n,n(g-1)}[1]$ is proper and bijective. Since $\mathrm{char}(k) = 0$, τ_0 is birational. The morphism $R^s \to \mathscr{U}^s_X(n, n(g-1))$ is smooth because it is a geometric quotient; also $D^s(n, n(g-1), 1)[1]$ is a smooth subscheme of R^s. Therefore its image in $\mathscr{U}^s_X(n, n(g-1))$, namely $W^0_{n,n(g-1)}[1]$ is smooth. Hence by Zariski's Main Theorem, τ_0 is an isosmorphism.

LEMMA 5.6. *If X is a generic smooth curve of genus g $(g > 5)$, then the closed subscheme $SW^1_{2,2g-2}$ of $\mathscr{U}_X(2, 2g-2)$ is irreducible.*

PROOF. By the results of [T], we know that $W^1_{2,2g-2}$ is irreducible for a generic curve and the codimension of $W^1_{2,2g-2}$ in $\mathscr{U}^s_X(2, 2g-2)$ is the expected codimension 4. The same holds for $D^s := D^s(2, 2g-2, 1)$ in R^s. We now claim that $D^{ss} := D^{ss}(2, 2g-2, 1)$ is the closure of D^s in R^{ss}. This follows as in Lemma 5.3, once we notice that the codimension of $R^{ss} \setminus R^s$ in R^{ss} is greater than or equal to $g-1 \geq 5$, and that the codimension of D^{ss} in R^{ss} is not greater than the expected codimension 4. This proves D^{ss} is irreducible and hence $SW^1_{2,2g-2}$ is irreducible.

LEMMA 5.7. *If X is as in Lemma 5.6, then the Θ-divisor Θ_2 in $\mathscr{U}_X(2, 2g-2)$ is normal.*

PROOF. We have already seen that $D^{ss}(2, 2g-2, 0)$ is integral. So to prove the lemma it is enough to show that $D^{ss}(2, 2g-2, 0)$ is normal. As $D^{ss}(2, 2g-2, 0)$ is Cohen-Macaulay, by a theorem of Serre, it is enough to show $\mathrm{Codim}\,\mathrm{Sing}\,D^{ss}(2, 2g-2, 0) \geq 2$. As $D^{ss}(2, 2g-2, 0)$ is a determinantal variety in R^{ss}, we have the inclusion

$$D^{ss}(2, 2g-2, 1) \subseteq \mathrm{Sing}\,D^{ss}(2, 2g-2, 0)\,.$$

On the other hand,

$$\tilde{\tau}: \tilde{D}^{ss}(2, 2g-2, 0) \to D^{ss}(2, 2g-2, 0)$$

is an isomorphism outside $D^{ss}(2, 2g-2, 1)$. But $\tilde{D}^{ss}(2, 2g-2, 0)$ is smooth (see Proposition 5.2). So we have $\mathrm{Sing}\,D^{ss}(2, 2g-2, 0) = D^{ss}(2, 2g-2, 1)$. Recall that the codimension of $D^{ss}(2, 2g-2, 0)$ in R^{ss} is 4. So we get $\mathrm{Codim}\,\mathrm{Sing}\,D^{ss}(2, 2g-2, 1) = 3$. This completes the proof.

PROPOSITION 5.8. *For a generic smooth curve of genus g $(g > 5)$,*

$$\operatorname{Sing}\Theta_2=\Theta_2\setminus W^0_{2,2g-2}[1]\,.$$

Recall that $W^0_{2,2g-2}[1]$ *is the open subscheme consisting of stable bundles with exactly one dimensional space of sections.*

PROOF. Clearly we have a set-theoretic union (disjoint)

$$\Theta_2\setminus W^0_{2,2g-2}[1]=:Y_1\cup Y_2\,,$$

where $Y_1=SW^1_{2,2g-2}$ and

$$Y_2=\{[V]\in\mathscr{U}_X(2,2g-2): V\text{ is semistable but not stable and }h^0(V)=1\}\,.$$

By Theorem 5.5 we see that $\operatorname{Sing}(\Theta_2)\subseteq Y_1\cup Y_2$. We now show that Y_1, $Y_2\subset\operatorname{Sing}\Theta_2$. We first show $Y_1\subset\operatorname{Sing}\Theta_2$. By Lemma 5.6, it is enough to show $W^1_{2,2g-2}\subseteq\operatorname{Sing}\Theta_2$. First note that we have

$$\operatorname{Sing}D^s(2,2g-2,0)=D^s(2,2g-2,1)\,.$$

The required inclusion above now follows readily from the fact that, $W^0_{2,2g-2}$ being a geometric quotient of $D^{ss}(2,2g-2,0)$, the quotient map is smooth.

To show $Y_2\subset\operatorname{Sing}\Theta_2$, we consider the morphism

$$\psi:\mathscr{U}_X(1,g-1)\times\mathscr{U}_X(1,g-1)\to\mathscr{U}_X(2,2g-2)$$

given by $\psi(L,M)=[L\oplus M]$. It is not difficult to see that $Y_2=\psi(\tilde{Y}_2)$, where $\tilde{Y}_2$ is the locally closed set in $\mathscr{U}_X(1,1-g)\times\mathscr{U}_X(1,1-g)$ given by

$$\tilde{Y}_2=(\mathscr{U}_X(1,g-1)\setminus\Theta_1)\times W^0_{1,g-1}[1]\cup W^0_{1,g-1}[1]\times(\mathscr{U}_X(1,g-1)\setminus\Theta_1)\,.$$

(Here Θ_1 is the Θ-divisor in the Jacobian, $\mathscr{U}_X(1,g-1)$, of degree $g-1$ line bundles on X and

$$W^0_{1,g-1}[1]=\{L\in\mathscr{U}_X(1,g-1): h^0(L)=1\}.)$$

Because ψ is a finite morphism, $\dim Y_2=\dim\tilde{Y}_2=2g-1$. Now consider the map $\tau:\tilde{\Theta}_2\to\Theta_2$ and set $U:=\Theta_2\setminus SW^1_{2,2g-2}$. It is clear that U is open in Θ_2 and Y_2 is a closed subset of U. Restricting the morphism τ to $\tau^{-1}(U)\to U$, we see that the locus of points where τ fails to be locally injective is precisely $\tau^{-1}(Y_2)$. Computing the dimension of fibres of τ over Y_2, we see that $\dim\tau^{-1}(Y_2)\le 3g-3$. It now follows that Y_2 is precisely the singular locus of U from the following fact:

Let $f: X\to Y$ *be a proper birational morphism from an n-dimensional smooth variety onto an n-dimensional normal variety. Let*

$$S=\{x\in X: f\text{ is not locally injective at }x\}\,.$$

Then $f(S)=\operatorname{Sing}Y$ *if* $\operatorname{codim}(S)\ge 2$ (cf. [N-R, Lemma 4.4]). This completes the proof.

REMARK 5.9. Note that the above proof shows that the singular locus of Θ_2 splits into two components, that is, $\operatorname{Sing}\Theta_2 = Y_1 \cup \bar{Y}_2$ is a decomposition of $\operatorname{Sing}\Theta_2$ into irreducible components.

Putting Theorem 5.5 and Proposition 5.8 together we get:

THEOREM 5.10. *If X is a generic smooth curve of genus g $(g > 5)$, the desingularization*

$$\tau\colon \tilde{\Theta}_2 \to \Theta_2$$

is an isomorphism precisely outside the singular locus of Θ_2.

REMARK 5.11. More generally Theorem 5.10 is true for $n \geq 3$ and $g \geq 3$. To this end we observe

- The codimension of $R^{ss} \setminus R^s$ in R^{ss} is not less than 5.
- $D^{ss}(n, n(g-1), 1)$ is nonempty (indeed, we can always choose a vector bundle $V = L \oplus M_i$ $(1 \leq i \leq n-1)$ such that $h^0(L) = 1$, $h^0(M_i) = 0$).

These observations together imply that $W^1_{n,n(g-1)}$ is non-empty. Hence by the results of Feinberg [F], $W^1_{n,n(g-1)}$ is irreducible. Now the proof of the theorem in the general case proceeds as above.

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