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COMPLEX STRUCTURES ON PARTIAL COMPACTIFICATIONS OF ARITHMETIC QUOTIENTS OF CLASSIFYING SPACES OF HODGE STRUCTURES

Dedicated to Professor Phillip A. Griffiths

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Abstract. We construct partial compactifications of arithmetic quotients of the classifying spaces of polarized Hodge structures of general weight by adding the restrictions of the 'tamest' nilpotent orbits to the invariant cycles, and introduce complex structures on them. We prove holomorphic extendability of period maps from a punctured disc whose monodromy logarithm satisfies a certain property. We also examine some geometric examples which can be settled within the present framework.

Introduction. In this paper, we try to introduce a complex structure on a partial compactification of an arithmetic quotient D/Γ of the classifying space of polarized Hodge structures of general weight w under the restriction (0.1) below.

In the classical case where D is a symmetric domain of Hermitian type (cf. (1.8)), we have Satake compactifications, Baily-Borel compactifications, toroidal compactifications by Mumford et al. of D/Γ, which carry not only complex structures but algebraic structures. On the other hand, beyond the classical case, we know very little about these problems. As far as the author knows, the only work in this direction is the one due to Cattani and Kaplan [CK1], who constructed partial compactifications of D/Γ in the case $w=2$, which are Hausdorff topological spaces. In contrast to the classical case, the essential difference lies in the fact that the isotropy subgroup I of G with $D \simeq G/I$ is compact but *not maximal.*

As a first attempt, we restrict ourselves throughout this paper to those boundary points which appear as the limits of nilpotent orbits (N, F) whose nilpotent endomorphism N satisfies the condition

$$N^2=0\,, \qquad \dim \operatorname{Im} N=\begin{cases} 1 & \text{if } w \text{ is odd}\,, \\ 2 & \text{if } w \text{ is even}\,. \end{cases} \tag{0.1}$$

This condition (equivalent to (2.1) and (2.2)) implies that the monodromy cone has

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dimension one (cf. (2.5.i)), which postpones the combinatorial complexity (or interest) of the toroidal method as a problem in the future. Even under the restriction (0.1), however, we encounter new phenomena such as: nilpotent orbits with a common N are divided into several types when the weight w is even and ≥ 4 (see (3.4), (3.5)); partial compactifications are not known to be locally compact; the loci $\mathcal{N}$ of nilpotent orbits in (3.14) are the counterparts of the classical objects, on which topologies and complex structures are the expected ones, whereas off these loci the situation is not easy (cf. also [Sc, (6.17)] and the comment just before it).

We construct our partial compactification $\overline{D/\Gamma}$ by the method of torus embedding in [AMRT]. Namely, we first prepare several pieces $\tilde{D}_{W,N}$ in (3.16) by adding suitable boundary components to partial quotients $D/\exp(\boldsymbol{Z}N)$, which have a natural topology and a complex structure. We then introduce a Satake topology on them, and finally patching up their full quotients along suitable Satake open sets to make up $\overline{D/\Gamma}$. The $\overline{D/\Gamma}$ thus constructed is Hausdorff in the induced Satake topology and carries the induced complex structure (Theorem (4.17) and Construction (4.19)). Our boundary points consist of the full data of gradedly polarized mixed Hodge structures on the space $W_w := \operatorname{Ker} N$ of invariant cycles, whereas the boundary points in [CK1] lose their extension data, which play an important role in the interaction between Hodge theory and geometry. Our partial compactification $\overline{D/\Gamma}$ is characterized by the property that they contain exactly those boundary points with which every period map from a punctured disc with its monodromy logarithm N satisfying (0.1) can be extended holomorphically (Theorem (5.1)). Hence we can now talk about the differentials at boundary points of these extended period maps.

In §1 we recall several facts and definitions, which will be used later. In §2 we construct a line bundle $L(W, N)$ containing $D/\exp(\boldsymbol{Z}N)$ by the technique of torus embedding, and in §3 we define boundary components $B(W, p, N)$ and embed them into the zero section of the line bundles $L(W, N)$ to make up our pieces $\tilde{D}_{W,N}$. §4 is devoted to the construction of a partial compactification $\overline{D/\Gamma}$ from the pieces $\tilde{D}_{W,N}$ and to introduce a complex structure on it. In §5 were examine holomorphic extendability of period maps. The examples in §6 suggest an interesting interplay between the present result and geometry, which is our motivation.

The present work owes much to Schmid [Sc].

We leave the following problems open: comparison of the Satake topology and the natural topology on $\tilde{D}_{W,N} \subset L(W, N)$; removal of the restriction (0.1) and generalization to the case of higher dimensional monodromy cones.

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1. Preliminaries. We recall first the definition of a (polarized) Hodge structure of weight w. Fix a free $\boldsymbol{Z}$-module $H_{\boldsymbol{Z}}$ of finite rank. Set $H_{\boldsymbol{Q}} := \boldsymbol{Q} \otimes H_{\boldsymbol{Z}}$, $H = H_{\boldsymbol{R}} := \boldsymbol{R} \otimes H_{\boldsymbol{Z}}$ and $H_{\boldsymbol{C}} := \boldsymbol{C} \otimes H_{\boldsymbol{Z}}$, whose complex conjugation is denoted by σ. Let w be an integer. A *Hodge structure of weight* w on $H_{\boldsymbol{C}}$ is a decomposition

$$H_{\boldsymbol{C}} = \bigoplus_{p+q=w} H^{p,q} \quad \text{with} \quad \sigma H^{p,q} = H^{q,p}. \tag{1.1}$$

$F^p := \bigoplus_{p' \geq p} H^{p',q'}$ is called a *Hodge filtration*, and $H^{p,q}$ is recovered by $H^{p,q} = F^p \cap \sigma F^q$. The integers

$$h^{p,q} := \dim H^{p,q} \tag{1.2}$$

are called the *Hodge numbers*.

A *polarization* S for a Hodge structure (1.1) of weight w is a non-degenerate bilinear form on $H_{\boldsymbol{Q}}$, symmetric if w is even and skew-symmetric if w is odd, such that its $\boldsymbol{C}$-bilinear extension, denoted also by S, satisfies

$$\begin{aligned} &S(H^{p,q}, \sigma H^{p',q'}) = 0 && \text{unless} \quad (p,q) = (p',q'), \\ &i^{p-q} S(v, \sigma v) > 0 && \text{for all} \quad 0 \neq v \in H^{p,q}. \end{aligned} \tag{1.3}$$

For fixed S and $\{h^{p,q}\}$, the classifying space D for Hodge structures and its 'compact dual' $\check{D}$ are defined by

$$\begin{aligned} \check{D} &:= \{\{H^{p,q}\} \mid \text{Hodge structure on } H_{\boldsymbol{C}} \text{ with } \dim H^{p,q} = h^{p,q}, \\ &\qquad\qquad \text{satisfying the first condition in (1.3)}\}, \\ D &:= \{\{H^{p,q}\} \in \check{D} \mid \text{satisfying also the second condition in (1.3)}\}. \end{aligned} \tag{1.4}$$

These are homogeneous spaces under the natural actions of the groups

$$G_{\boldsymbol{C}} := \mathrm{Aut}(H_{\boldsymbol{C}}, S), \qquad G = G_{\boldsymbol{R}} := \{g \in G_{\boldsymbol{C}} \mid g H_{\boldsymbol{R}} = H_{\boldsymbol{R}}\}, \tag{1.5}$$

respectively. Taking a reference point $r \in D$, one obtains identifications

$$\check{D} \simeq G_{\boldsymbol{C}}/I_{\boldsymbol{C},r}, \qquad D \simeq G/I_r, \tag{1.6}$$

where $I_{\boldsymbol{C},r}$ and I_r are the isotropy subgroups of $G_{\boldsymbol{C}}$ and of G at $r \in D$, respectively. It is a direct consequence of the definition that

$$G \simeq \begin{cases} O(k, 2h), \\ Sp(2h, \boldsymbol{R}), \end{cases} \quad I_r \simeq \begin{cases} U(h^{w,0}) \times \cdots \times U(h^{t+1,t-1}) \times O(h^{t,t}) & \text{if} \quad w = 2t, \\ U(h^{w,0}) \times \cdots \times U(h^{t+1,t}) & \text{if} \quad w = 2t+1, \end{cases} \tag{1.7}$$

where $k := \sum_{|j| \leq [t/2]} h^{t+2j,t-2j}$ and $h := (\dim H - k)/2$ if $w = 2t$, and $h := \dim H/2$ if $w = 2t+1$. It is an important observation that I_r is compact, but not maximal compact in general. Hence D is a symmetric domain of Hermitian type if and only if one of the following is satisfied:

$$\begin{aligned} &w = 2t+1, \ h^{p,q} = 0 \ \text{unless} \ p = t+1, \ t; \\ &w = 2t, \ h^{p,q} = 1 \ \text{for} \ p = t+1, \ t-1, \ h^{t,t} \ \text{is arbitrary}, \ h^{p,q} = 0 \ \text{otherwise}; \ \text{or} \\ &w = 2t, \ h^{p,q} = 1 \ \text{for} \ p = t+a, \ t+a-1, \ t-a+1, \ t-a \ \text{for some} \ a \geq 2, \\ &\qquad h^{p,q} = 0 \ \text{otherwise}. \end{aligned} \tag{1.8}$$

We denote

$$\Gamma := \{g \in G \mid gH_{\mathbf{Z}} = H_{\mathbf{Z}}\} . \tag{1.9}$$

Then Γ acts on D properly discontinuously because the isotropy subgroup I_r is compact and Γ is discrete in G.

A reference Hodge structure $r = \{H_r^{p,q}\} \in D$ induces a Hodge structure of weight 0 on the Lie algebra $\mathfrak{g}_{\boldsymbol{C}} := \operatorname{Lie} G_{\boldsymbol{C}}$ by

$$\mathfrak{g}_{\boldsymbol{C}}^{s,-s} := \{X \in \mathfrak{g}_{\boldsymbol{C}} \mid XH_r^{p,q} \subset H_r^{p+s,q-s} \text{ for all } p, q\} . \tag{1.10}$$

One can define the associated Cartan involution θ_r on $\mathfrak{g} := \operatorname{Lie} G$ induced by

$$\theta_r(X) := \sum_s (-1)^s X^{s,-s} \qquad \text{for} \quad X = \sum_s X^{s,-s} \in \mathfrak{g}_{\boldsymbol{C}} = \bigoplus_s \mathfrak{g}_{\boldsymbol{C}}^{s,-s} . \tag{1.11}$$

We take the standard generators for the Lie algebras $\mathfrak{sl}_2(\boldsymbol{R})$ and $\mathfrak{su}(1, 1)$ which are related by the Cayley transformation $\operatorname{Ad}(c_1)$, where

$$c_1 := \exp\left(\frac{\pi i}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{1.12}$$

as follows:

$$\begin{array}{lccc} \mathfrak{sl}_2(\boldsymbol{R}) \ni & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ \operatorname{Ad}(c_1) \downarrow & \downarrow & \downarrow & \downarrow \\ \mathfrak{su}(1, 1) \ni & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, & \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}. \end{array} \tag{1.13}$$

REMARK (1.14). $i \in \mathfrak{h} :=$ (the upper half plane) $\simeq SL_2(\boldsymbol{R})/U(1)$ corresponds to a Hodge structure

$$\boldsymbol{C}^2 = H_i^{1,0} \oplus H_i^{0,1} \qquad \text{with} \quad \boldsymbol{H}^{1,0} = \boldsymbol{C}\begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{polarized by} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The canonical decomposition of $\mathfrak{g}_{1\boldsymbol{C}} := \mathfrak{sl}_2(\boldsymbol{C})$ by the standard 'H-element' (cf., e.g., [Sa2, II. §7])

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

coincides with the Hodge structure induced by $i \in \mathfrak{h}$:

$$\mathfrak{g}_{1\boldsymbol{C}} = \mathfrak{g}_{1\boldsymbol{C}}^{1,-1} + \mathfrak{g}_{1\boldsymbol{C}}^{0,0} + \mathfrak{g}_{1\boldsymbol{C}}^{-1,1} = \mathfrak{p}_- + \mathfrak{k}_{\boldsymbol{C}} + \mathfrak{p}_+ = \boldsymbol{C}\frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} + \boldsymbol{C}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \boldsymbol{C}\frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}.$$

From now on, we assume that $w > 0$ and all Hodge structures of weight w satisfy

$H^{p,q}=0$ unless $p, q \geq 0$.

DEFINITION (1.15) (cf. [Sc, p. 258]). An SL_2-representation $\rho : SL_2(\boldsymbol{R}) \to G$ is horizontal at $r=\{H_r^{p,q}\} \in D$ if

$$\rho_*\left(\frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}\right) \in \mathfrak{g}_{\boldsymbol{C}}^{-1,1}.$$

When this is the case, we call the pair (ρ, r) an SL_2-orbit.

REMARK (1.16). Clearly, (ρ, r) is an SL_2-orbit if and only if $\rho_* : \mathfrak{sl}_2(\boldsymbol{R}) \to \mathfrak{g}$ is a morphism of Hodge structures of type (0, 0) with respect to the Hodge structures induced by $i \in \mathfrak{h}$ and $r \in D$, respectively. A horizontal SL_2-representation ρ induces an equivariant horizontal map $\tilde{\rho} : \boldsymbol{P}^1 \to \check{D}$ with $\tilde{\rho}(i)=r$:

$$\begin{array}{ccc} SL_2(\boldsymbol{C}) & \xrightarrow{\rho} & G_{\boldsymbol{C}} \\ \downarrow & & \downarrow \\ \boldsymbol{P}^1 & \xrightarrow{\tilde{\rho}} & \check{D}. \end{array}$$

This is a generalization to the present context of the notion of '(H_1)-homomorphism' in the case of symmetric domains of Hermitian type (cf., e.g., [Sa2, II. (8.5), III. §1]).

Let (σ, r) be an SL_2-orbit and $\tilde{\rho} : \boldsymbol{P}^1 \to \check{D}$ the associated horizontal equivariant map. We set

$$(1.17) \qquad Y := \rho_*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad N_+ := \rho_*\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N_- := \rho_*\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad c := \rho(c_1).$$

We denote by $H(Y; \lambda)$ the λ-eigenspace of the action of Y on H, and set

$$(1.18) \qquad W(Y)_{w-j} := \bigoplus_{\lambda \geq j} H(Y; \lambda).$$

LEMMA (1.19). *Let (ρ, r) be an SL_2-orbit. Then, in the above notation, $\lim_{\mathrm{Im} z \to \infty} \exp(-zN_+) \cdot \tilde{\rho}(z) = c^{-1} \cdot r \in \check{D}$. The corresponding filtration, denoted by F_0, together with $W(Y)$, determines the limiting S-polarized split mixed Hodge structure.*

PROOF. We have

$$\tilde{\rho}(z) = \tilde{\rho}(i+(z-i)) = \tilde{\rho}\left(\exp\left((z-i)\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\cdot i\right) = \exp((z-i)\cdot N_+)\cdot r,$$

$$\exp(-zN_+)\cdot\tilde{\rho}(z) = \exp(-iN_+)\cdot r = \tilde{\rho}\left(\exp\left(-i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\cdot i\right) = \tilde{\rho}(0).$$

On the other hand, $c^{-1}\cdot r = \tilde{\rho}(c_1^{-1}\cdot i) = \tilde{\rho}(0)$.

The second assertion follows from [Sc, (6.16)]. (N, L in [Sc, (6.16)] correspond

to N_+, N_- in our present notation, respectively.) □

We recall a definition in [CK1, (2.19)], [U, (3.3)]:

DEFINITION (1.20). A pair $(Y, r) \in \mathfrak{g} \times D$ is admissible if there exists an SL_2-orbit (ρ, r) such that Y is as in (1.17). Y is said to be admissible if (Y, r) is an admissible pair for some $r \in D$.

Note that giving an admissible pair (Y, r) is equivalent to giving an SL_2-orbit (ρ, r), and an admissible element is characterized numerically (for more details, see [CK1], [U]).

2. Line bundles $L(W, N)$. Let W_{w-1} be a subspace of $H_{\boldsymbol{Q}}$ defined over $\boldsymbol{Q}$ which is isotropic with respect to S, i.e., $S(u, v) = 0$ for all $u, v \in W_{w-1}$. We assume throughout this paper that

$$\dim W_{w-1} = \begin{cases} 1 & \text{if } w \text{ is odd}, \\ 2 & \text{if } w \text{ is even}. \end{cases} \tag{2.1}$$

Let W_w be the annihilator of W_{w-1} in $H_{\boldsymbol{Q}}$ with respect to S. Then we have a filtration W of $H_{\boldsymbol{Q}}$:

$$0 \subset W_{w-1} \subset W_w \subset W_{w+1} := H_{\boldsymbol{Q}}. \tag{2.2}$$

By abuse of notation, we also use W for the filtrations induced on $H = H_{\boldsymbol{R}}$, $H_{\boldsymbol{C}}$ if it does not lead to any confusion.

We define the following subgroups of G:

$$\begin{aligned} &N(W) := \{g \in G \mid gW_j = W_j \text{ for all } j\}, \\ &U(W) : \text{the unipotent radical of } N(W), \\ &C(W) : \text{the center of } U(W). \end{aligned} \tag{2.3}$$

The induced sub- and sub-quotient groups of Γ are denoted by

$$\begin{gathered} \Gamma_W := \Gamma \cap N(W), \quad U(W)_{\boldsymbol{Z}} := \Gamma \cap U(W), \quad C(W)_{\boldsymbol{Z}} := \Gamma \cap C(W), \\ \bar{\Gamma}_W := \Gamma_W / C(W)_{\boldsymbol{Z}}. \end{gathered} \tag{2.4}$$

We denote

$$D(W) := C(W)_{\boldsymbol{C}} \cdot D.$$

LEMMA (2.5). (i) $\dim C(W) = 1$.

(ii) *$C(W)$ is a normal subgroup of $N(W)$, and* $\mathrm{Ad}(g)X = (g \mid W_{w-1})^2 X$ *if w is odd and* $\mathrm{Ad}(g)X = \det(g \mid W_{w-1})X$ *if w is even for $g \in N(W)$, $X \in \mathrm{Lie}\, C(W)$.*

(iii) *$C(W)_{\boldsymbol{C}}$ acts on $D(W)$ freely.*

PROOF. Since we assume (2.1), (i) is obvious in the case of odd w. In order to

examine (i) in the case of even w, we choose a $\boldsymbol{Q}$-basis of $H_{\boldsymbol{Q}}$ adapted to the filtration W so that the polarization form S is represented by a matrix $S=\text{antidiagonal}(J, *, J)$, where $J:=\text{diagonal}(1, -1)$ of rank 2. In this basis, any $X \in \text{Lie}\, C(W)$ is represented by a matrix

$$X=\begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{where} \quad A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is a } 2\times 2 \text{ matrix}.$$

From ${}^tXS+SX=0$, we can readily derive $a=d=0$, $b=c$. This completes the proof of (i).

By using the above basis, (ii) can also be verified easily.

Let $r \in D$ be a reference point and let N be a basis of $\text{Lie}\, C(W)$. Since N is nilpotent, $v: \boldsymbol{C} \simeq C(W)_{\boldsymbol{C}} \to D(W) \subset \check{D}$, sending z to $\exp(zN)\cdot r$, is an algebraic morphism. v is not a constant map, because the isotropy subgroup I_r of G at r is compact hence does not contain a unipotent subgroup $C(W) \simeq \boldsymbol{R}$. It follows that v is quasi-finite. If $v(z_1)=v(z_2)$, $z_1, z_2 \in \boldsymbol{C}$, then $\exp((z_1-z_2)N)\cdot r=r$ and so $\boldsymbol{Z}(z_1-z_2) \subset v^{-1}(r)$, which occurs only if $z_1=z_2$. □

DEFINITION (2.6). In the case of odd w, $N \in \text{Lie}\, C(W)$ is positive if $S(N^{-1}\cdot, \cdot)>0$ on W_{w-1}.

We denote by $\mathfrak{o}(W)$ the set of orientations of $\text{Lie}\, C(W)$ consisting of

$$\begin{cases} \text{the positive generator of } \text{Lie}\, C(W)_{\boldsymbol{Z}} & \text{if } w \text{ is odd}, \\ \text{the two generators of } \text{Lie}\, C(W)_{\boldsymbol{Z}} & \text{if } w \text{ is even}. \end{cases}$$

We define

$$\begin{aligned} &N(W)^+ := \{g \in N(W) \mid \text{Ad}(g)N \in \boldsymbol{R}_{>0}N,\ N \in \mathfrak{o}(W)\}, \\ &\Gamma_W^+ := \Gamma \cap N(W)^+, \quad \bar{\Gamma}_W^+ := \Gamma_W^+/C(W)_{\boldsymbol{Z}}. \end{aligned} \tag{2.7}$$

Note that, by (2.5.ii), we see that

$$[N(W): N(W)^+]=\begin{cases} 1 & \text{if } w \text{ is odd}, \\ 2 & \text{if } w \text{ is even}. \end{cases}$$

By Lemma (2.5.iii), the quotient

$$D(W)' := D(W)/C(W)_{\boldsymbol{C}}$$

is a complex manifold and that the principal $C(W)_{\boldsymbol{C}}$-bundle $D(W) \to D(W)'$ is a complex affine bundle. Starting from this affine bundle, we shall construct a complex line bundle $L(W, N) \to D(W)'$ in the following way. Take a quotient bundle

$$D(W)/C(W)_{\boldsymbol{Z}} \to D(W)'. \tag{2.8}$$

Set $T(W) := C(W)_{\boldsymbol{C}}/C(W)_{\boldsymbol{Z}}$. According to a choice of $N \in \mathfrak{o}(W)$, we have an identification $T(W) \xrightarrow{\sim} \boldsymbol{C}^*$, $\exp(zN) \mapsto \exp(2\pi i z)$. Let $\boldsymbol{C}^* \subset \boldsymbol{C}$ be the natural embedding. We denote by

$$\pi : L(W, N) := (D(W)/C(W)_{\mathbf{Z}}) \times^{\mathbf{C}^*} \mathbf{C} \to D(W)' \tag{2.9}$$

the complex line bundle associated to the principal $\mathbf{C}^*$-bundle (2.8).

PROPOSITION (2.10). *The action of $\bar{\Gamma}_W^+$ on the $\mathbf{C}^*$-bundle* (2.8) *extends to the action on the complex line bundle* (2.9), *which commutes with the action of $T(W)$. $\bar{\Gamma}_W^+$ acts properly discontinuously on $D(W)'$ and hence on $L(W, N)$.*

PROOF. The first part follows easily from (2.5.ii).

In order to prove the second part, we use the $\mathbf{C}^*$-bundle (2.8). Given a compact subset $A' \subset D(W)'$. Put $A := \pi^{-1}(A')$. Take a neighborhood V_a of $a \in A \cap (D/C(W)_{\mathbf{Z}})$ such that the closure $\bar{V}_a$ is compact and is contained in $D/C(W)_{\mathbf{Z}}$. Then $\{\pi(V_a) \mid a \in A \cap (D/C(W)_{\mathbf{Z}})\}$ is an open covering of A' and so we can choose a finite subset $\{\pi(V_{a_i}) \mid 1 \leq i \leq n\}$ which covers A'. Set $V := \bigcup_{1 \leq i \leq n} N(W)^1 \cdot \bar{V}_{a_i}$, where

$$N(W)^1 := \mathrm{Ker}(\mathrm{Ad} : N(W) \to \mathrm{Aut}(\mathrm{Lie}\, C(W))) . \tag{2.11}$$

Then, by construction, we see that $V \subset D/C(W)_{\mathbf{Z}}$, $\pi(V) \supset A'$ and that the restriction $\pi : V \to D(W)'$ is a proper map. Since $\Gamma_W^+ \subset N(W)^1$ whose action preserves the fiber coordinate of (2.9), we see that

$$\{\gamma \in \bar{\Gamma}_W^+ \mid \gamma \cdot A' \cap A' \neq \emptyset\} = \{\gamma \in \bar{\Gamma}_W^+ \mid \gamma \cdot (A \cap V) \cap (A \cap V) \neq \emptyset\} .$$

The latter set is finite because the action of $\bar{\Gamma}_W^+$ on $D/C(W)_{\mathbf{Z}}$ is properly discontinuous and $A \cap V \subset D/C(W)_{\mathbf{Z}}$ is a compact subset. This proves that $\bar{\Gamma}_W^+$ acts on $D(W)'$ properly discontinuously. The assertion on the action on $L(W, N)$ follows from this easily. □

3. Boundary components $B(W, p, N)$. Let $\{h^{p,q}\}$ be a set of Hodge numbers in (1.2). For a filtration W in (2.2), we set

$$n_\lambda := \dim \mathrm{gr}_{w-\lambda}^W . \tag{3.1}$$

We recall a definition in [U, (2.15)]:

DEFINITION (3.2). A set $p = \{p_\lambda^{a,b}\}$ of non-negative integers is called a set of primitive Hodge numbers belonging to $\{h^{p,q}, n_\lambda\}$ if it satisfies the following conditions.

(0) The indices a, b and λ are non-negative integrers satisfying $a + b = w - \lambda$.

(i) $\sum_{a+b=w-\lambda} p_\lambda^{a,b} = n_\lambda - n_{\lambda+2}$ for all λ.

(ii) $p_\lambda^{b,a} = p_\lambda^{a,b}$ for all a, b, λ.

(iii) $h^{s,t} = h^{s+1,t-1} - \sum_{0 \leq \lambda \leq t-1} p_\lambda^{s+1,t-1-\lambda} + \sum_{0 \leq \lambda \leq s} p_\lambda^{s-\lambda,t}$ for all s, t with $s + t = w$.

Under the assumption (2.1), only the following sets of primitive Hodge numbers are possible.

(3.3) Case $w = 2t + 1$. The possibility is unique .

$$p_1^{a,b}=\begin{cases}1 & \text{if } a=t\,,\\ 0 & \text{otherwise}\,.\end{cases}\qquad p_0^{a,b}=\begin{cases}h^{a,b}-1 & \text{if } a=t+1,t\,,\\ h^{a,b} & \text{otherwise}\,.\end{cases}$$

Case $w=2t$. There are $t+1$ possible cases.

(3.4) For each $s=w, w-1,\dots, t+1$,

$$p_1^{a,b}=\begin{cases}1 & \text{if } a=s, w-s-1\,,\\ 0 & \text{otherwise}\,.\end{cases}\qquad p_0^{a,b}=\begin{cases}h^{a,b}-1 & \text{if } a=s+1, s, w-s, w-s-1\,,\\ h^{a,b} & \text{otherwise}\,.\end{cases}$$

(3.5) For $s=t$,

$$p_1^{a,b}=\begin{cases}1 & \text{if } a=t, t-1\,,\\ 0 & \text{otherwise}\,.\end{cases}\qquad p_0^{a,b}=\begin{cases}h^{a,b}-1 & \text{if } a=t+1, t-1\,,\\ h^{a,b}-2 & \text{if } a=t\,,\\ h^{a,b} & \text{otherwise}\,.\end{cases}$$

DEFINITION (3.6). Given a filtration W in (2.2) satisfying (2.1), a set $p=\{p_\lambda^{a,b}\}$ of primitive Hodge numbers belonging to $\{h^{p,q}, n_\lambda\}$ and an orientation $N\in\mathfrak{o}(W)$ in (2.6) the corresponding boundary component $B(W, p, N)$ is the classifying space of gradedly polarized mixed Hodge structures on $W_wH_{\boldsymbol{C}}$ with Hodge type $\{p_0^{a,b}\}$ (resp. $\{p_1^{a,b}\}$) and polarization form S (resp. $S(N^{-1}\cdot,\cdot)$) on gr_w^W (resp. W_{w-1}).

PROPOSITION (3.7). *There is an $N(W)^+$-equivariant embedding $B(W, p, N)\hookrightarrow D(W)'$.*

PROOF. We shall first construct a map $\varphi\colon B\to D'$, where $B=B(W, p, N)$, $D'=D(W)'$. Let $F\in B$. In the present case, the weight length is one, hence we have the Hodge-Deligne decomposition

$$W_wH_{\boldsymbol{C}}=\bigoplus P_\lambda^{a,b}\,,\qquad P_\lambda^{a,b}:=F^a\cap\sigma F^b\cap W_{w-\lambda}H_{\boldsymbol{C}}\,,$$

where the summation is taken over $a+b=w-\lambda$, $\lambda=0, 1$. We want to extend this to an S-polarized split mixed Hodge structure on $H_{\boldsymbol{C}}$ uniquely up to $C(W)_{\boldsymbol{C}}$-action. Setting $P_{0,\boldsymbol{C}}:=\bigoplus_{a,b}P_0^{a,b}$, we have a splitting over $\boldsymbol{R}$ of $W_{w-1}H_{\boldsymbol{C}}\subset W_wH_{\boldsymbol{C}}$. In case $w=2t+1$, our assertion follows immediately from the fact that $P_{-1,\boldsymbol{C}}:=P_{-1}^{t+1,t+1}$ should be perpendicular to $P_{0,\boldsymbol{C}}$ with respect to S. Similarly, in case $w=2t$, $P_{-1,\boldsymbol{C}}:=P_{-1}^{s+1,w-s}+P_{-1}^{w-s,s+1}$ is distinguished up to $C(W)_{\boldsymbol{C}}$-action by the same condition, where s is the integer satisfying $p_1^{s,w-s-1}=1$ in the given set of primitive Hodge numbers. Moreover, the summands $P_{-1}^{a+1,w-a}(a=s, w-s-1)$ are distinguished up to $C(W)_{\boldsymbol{C}}$-action by the condition that $P_{-1}^{a+1,w-a}$ should be perpendicular to $P_1^{a,w-a-1}+P_{-1}^{a+1,w-a}$ with respect to S. Now let $P_{-1}^{t+1,t+1}$ in case $w=2t+1$ and $P_{-1}^{s+1,w-s}$, $P_{-1}^{w-s,s+1}$ in case $w=2t$ be representatives in the above constructions. These data determine a splitting $P_1\oplus P_0\oplus P_{-1}$ over $\boldsymbol{R}$ of the filtration $W_{w-1}\subset W_w\subset W_{w+1}$, where $P_1:=W_{w-1}$, $P_\lambda:=P_{\lambda,\boldsymbol{C}}\cap H$ $(\lambda=0, -1)$. This, in turn, determines a real semi-simple element $Y\in\mathfrak{g}$ so that P_λ is the λ-eigenspace of Y. Since $[Y, N]=2N$ by construction, we have a representation $\rho: SL_2(\boldsymbol{R})\to G$ (not necessarily rational) such that

$$\rho_*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Y, \qquad \rho_*\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N.$$

Transforming the $P_\lambda^{a,b}$ by the Cayley element $c=\rho(c_1)$ in (1.17), we get the Hodge-$(Z, X_\pm)$-decomposition $\bigoplus Q_\lambda^{a,b+\lambda} := \bigoplus cP_\lambda^{a,b}$. Then we know that $H^{p,q} := \bigoplus_\lambda Q_\lambda^{p,q}$ determines an element $r \in D$ where ρ is horizontal (see [CK2, (2.18) and its proof] and [U, (3.4) and its proof]). We now define a map

$$\varphi : B \to D' \quad \text{by} \quad F \mapsto C(W)_{\boldsymbol{C}} \cdot r. \tag{3.8}$$

Next we define a map

$$\psi : \varphi(B) \to B \quad \text{by} \quad C(W)_{\boldsymbol{C}} \cdot F \mapsto F \cap W_w H_{\boldsymbol{C}}. \tag{3.9}$$

This is well-defined. Indeed, since $W_w = \operatorname{Ker} N$, we see that $g|W_w$ is the identity for any $g \in C(W)_{\boldsymbol{C}}$ and so

$$g \cdot F \cap W_w H_{\boldsymbol{C}} = g(F \cap W_w H_{\boldsymbol{C}}) = F \cap W_w H_{\boldsymbol{C}}.$$

We claim now that $\psi\varphi$ is the identity. Indeed, let $F \in B$ and F_0 the Hodge filtration associated to the S-polarized split mixed Hodge structure $\{P_\lambda^{a,b} | a+b=w-\lambda, \lambda=1, 0, -1\}$ constructed above. Then the filtration F_r corresponding to $r \in D$ is $F_r = cF_0$ by definition. On the other hand, $cF_0 = \exp(iN) \cdot F_0$ as in the proof of Lemma (1.19). Hence $F_r \cap W_w H_{\boldsymbol{C}} = \exp(iN) \cdot F_0 \cap W_w H_{\boldsymbol{C}} = F$.

It is obvious that ψ is $N(W)^+$-equivariant, whence so is φ. □

Let (ρ, r) be an SL_2-orbit, Y in (1.17), and $W = W(Y)$ in (1.18). We assume that W is defined over $\boldsymbol{Q}$. We denote

$$G_Y^+ := \{g \in N(W)^+ | \operatorname{Ad}(g)Y = Y\}. \tag{3.10}$$

In the notation of (2.9), we set

$$\tilde{r} := (r \bmod C(W)_{\boldsymbol{Z}}) \in L(W, N), \qquad b := \pi(\tilde{r}) \in D(W)', \tag{3.11}$$

where $N \in \mathfrak{o}(W)$ with $N_+ = aN$, $a > 0$. Then, as in the proof of (1.19), we have $cF_0 = F_r = \exp(iN_+) \cdot F_0$, and hence $b \in B(W, p, N)$ under the identification of (3.7).

PROPOSITION (3.12). *In the above situation, we have the following.*

(i) *The orbits $G_Y^+ \cdot b \subset N(W)^+ \cdot b = B \subset D(W)'$ are complex submanifolds, where $B = B(W, p, N)$.*

(ii) *$((C(W) \rtimes G_Y^+) \cdot r)^\sim \to G_Y^+ \cdot b$ and $(N(W)^+ \cdot r)^\sim \to B$ are punctured disc bundles contained in the line bundle $L(W, N)$, which are the family of all SL_2-orbits corresponding to the triple (Y, p, N) and the family of all nilpotent orbits corresponding to the triple (W, p, N), respectively.*

(iii) *$N(W)^+ \cdot r$ is open in D if and only if D is a Hermitian symmetric domain.*

PROOF. We first claim that

$$\begin{aligned}
&\dim_{\boldsymbol{R}} N(W)^+/I_r \cap N(W)^+ = 2\dim_{\boldsymbol{C}} N(W)_{\boldsymbol{C}}/I_{\boldsymbol{C},r} \cap N(W)_{\boldsymbol{C}}\,,\\
&\dim_{\boldsymbol{R}}(C(W)\rtimes G_Y^+)/I_r \cap (C(W)\rtimes G_Y^+)\\
&\qquad = 2\dim_{\boldsymbol{C}}(C(W)_{\boldsymbol{C}}\rtimes G_{Y,\boldsymbol{C}})/I_{\boldsymbol{C},r} \cap (C(W)_{\boldsymbol{C}}\rtimes G_{Y,\boldsymbol{C}})\,,
\end{aligned} \tag{3.13}$$

where I_r and $I_{\boldsymbol{C},r}$ are the isotropy subgroups at r of G and $G_{\boldsymbol{C}}$, respectively. (3.13) can be verified easily by the dimension count of the corresponding Lie algebras using bases of $H_{\boldsymbol{C}}$ adapted to the mixed Hodge-$(Y, N_\pm)$-decomposition of (ρ, r) (cf. [U, §2]). Hence we leave it to the reader. Similarly, we can verify easily that $N(W)^+$ acts on B transitively and so we omit this verification. (3.13) shows that the orbit $N(W)^+\cdot r$ (resp. $(C(W)\rtimes G_Y^+)\cdot r$) is open in $N(W)_{\boldsymbol{C}}\cdot r$ (resp. $(C(W)_{\boldsymbol{C}}\rtimes G_{Y,\boldsymbol{C}})\cdot r$) in the Hausdorff topology and the latter is a closed complex submanifold of $\check{D}=G_{\boldsymbol{C}}\cdot r$. Hence the former induces a complex submanifold $(N(W)^+\cdot r)^\sim$ (resp. $((C(W)\rtimes G_Y^+)\cdot r)^\sim$) of $D(W)/C(W)_{\boldsymbol{Z}}$. From this we know that the interior of the closure of $(N(W)^+\cdot r)^\sim$ (resp. $((C(W)\rtimes G_Y^+)\cdot r)^\sim$) in $L(W, N)$, denoted by

$$\mathscr{N}(W, p, N) \qquad (\text{resp. } \mathscr{S}(Y, p, N))\,, \tag{3.14}$$

is a complex submanifold and so its intersection with the zero section of the line bundle (2.9) is a complex submanifold of the zero section. Via the projection, we get the assertion (i).

Now the first part of (ii) follows from an observation that, for $g_1\exp(\eta Y)\in N(W)^+ = N(W)^1\rtimes\exp(\boldsymbol{R}Y)$, we have

$$\begin{aligned}
g_1\exp(\eta Y)\cdot r &= g_1\cdot\tilde{\rho}\left(\exp\left(\eta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)\cdot i\right) = g_1\cdot\tilde{\rho}(e^{2\eta}i)\\
&= g_1\cdot\tilde{\rho}\left(\exp\left(i(e^{2\eta}-1)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\cdot i\right) = g_1\exp(i(e^{2\eta}-1)N_+)\cdot r\\
&= \exp(i(e^{2\eta}-1)N_+)g_1\cdot r = \exp(i(e^{2\eta}-1)aN)g_1\cdot r \qquad (\text{see } (3.11))\,,
\end{aligned}$$

and $\exp(2\pi i\cdot i(e^{2\eta}-1)a) = \exp(2\pi(1-e^{2\eta})a) < e^{2\pi a}$ (cf. the identification $T(W)\xrightarrow{\sim}\boldsymbol{C}^*$ in (2.9)). As for the assertion on the families in (ii), it follows from [U, (3.16.iii)], (2.5.ii) and (3.11) that $G_Y^+\cdot r$ is the set of all reference points $g\cdot r$ such that $(Y, g\cdot r)$ is an admissible pair of type p whose associated $\mathfrak{sl}_2$-representation $\mathrm{Ad}(g)\rho_*$ satisfies

$$\mathrm{Ad}(g)\rho_*\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \mathrm{Ad}(g)N_+ = a'N_+ = a'aN$$

for some $a'>0$. Hence

$$\begin{aligned}
&\exp((x+iy)N)g\cdot r = \exp(xN)g\exp(ia'^{-1}yN)\cdot r\\
&\quad = \exp(xN)g\exp(\log((a'a)^{-1}y+1)^{1/2}Y)\cdot r\in(C(W)\rtimes G_Y^+)\cdot r \qquad \text{if} \quad y>-a'a\,.
\end{aligned}$$

Similarly for $g\cdot r\in N(W)^+\cdot r$,

$$\exp((x+iy)N)g\cdot r=\exp(xN)g\exp(\log((a'a)^{-1}y+1)^{1/2}Y)\cdot r \in N(W)^+\cdot r \quad \text{if} \quad y>-a'a\,. \tag{3.15}$$

Let $H_{\boldsymbol{C}}=\bigoplus Q_\lambda^{a,b+\lambda}=\bigoplus P_\lambda^{a,b}$ be the Hodge-$(Z, X_\pm)$-decomposition and the mixed Hodge-$(Y, N_\pm)$-decomposition associated to (ρ, r), respectively (cf. [U, §2]). We see that, for $P':=P_1^{a-1,w-a}+P_{-1}^{a,w-a+1}$,

$$N_+Q_{-1}^{a,w-a}\subset N_+cP'=N_+P'\subset P'=cP'\,.$$

It follows that $N_+F_r^a\subset F_r^{a-1}$ and hence $NF_{g\cdot r}^a\subset F_{g\cdot r}^{a-1}$ by (2.5.ii) and (3.11), where F_r, $F_{g\cdot r}$ are the filtrations corresponding to $r, g\cdot r\in D$, respectively. Therefore $(N, F_{g\cdot r})$ is a nilpotent orbit in the direction of (W, p, N). Conversely, let (N, F), $F\in\check{D}$, be a nilpotent orbit, i.e., $NF^a\subset F^{a-1}$ and $\exp(iyN)\cdot F\in D$ for $y\gg 0$. Then, by [Sc, (6.16)], (W, F) is an S-polarized mixed Hodge structure. If (W, F) has mixed Hodge type p, then this determines a point of B by $F\cap W_wH_{\boldsymbol{C}}$, hence, by (3.7) and the first part of (ii), we have $\exp(iyN)\cdot F\in N(W)^+\cdot r$ for $y\gg 0$. This completes the proof of (ii).

In order to prove (iii), we compute $\dim D-\dim N(W)^+\cdot r$. Let K be a maximal compact subgroup of G containing the isotropy subgroup I_r, $G=RTK$ an Iwasawa decomposition.

Case $w=2t+1$, i.e., (3.3). We see that

$$G=Sp(2h, \boldsymbol{R})\,,\quad K\simeq U(h)\,,\quad I_r\simeq U(h^{w,0})\times\cdots\times U(h^{t+1,t})\,,\quad K_Y^+:=K\cap G_Y^+\simeq U(h-1)\,,$$
$$I_{r,Y}^+:=I_r\cap G_Y^+\simeq U(h^{w,0})\times\cdots\times U(h^{t+2,t-1})\times U(h^{t+1,t}-1)\,.$$

Hence

$$\begin{aligned}\dim D-\dim N(W)^+\cdot r&=\dim G/I_r-\dim N(W)^+/I_{r,Y}^+=\dim K/I_r-\dim K_Y^+/I_{r,Y}^+\\&=h^2-(h-1)^2-(h^{t+1,t})^2+(h^{t+1,t}-1)^2=2(h-h^{t+1,t})\,.\end{aligned}$$

This is zero if and only if $h=h^{t+1,t}$, that is, $K=I_r$.

Case $w=2t$. We see that

$$G=O(k, 2h)\,,\quad K\simeq O(k)\times O(2h)\,,\quad I_r\simeq U(h^{w,0})\times\cdots\times U(h^{t+1,t-1})\times O(h^{t,t})\,,$$
$$K_Y^+\simeq O(k-2)\times O(2h-2)\times SO(2)\,.$$

According to the subcases (3.4), (3.5), $I_{r,Y}^+$ is isomorphic, respectively, to

$$U(h^{w,0})\times\cdots\times U(h^{s+1,w-s-1}-1)\times U(h^{s,w-s}-1)\times\cdots\times U(h^{t+1,t-1})\times O(h^{t,t})\times U(1)\,,$$
$$U(h^{w,0})\times\cdots\times U(h^{t+1,t-1}-1)\times O(h^{t,t}-2)\times U(1)\,.$$

As before, we can compute $\dim D-\dim N(W)^+\cdot r$ to obtain

$$2(k+2h-h^{s+1,w-s-1}-h^{s,w-s}-2)\quad\text{in Case (3.4)}\,,$$
$$2(k+2h-h^{t+1,t-1}-h^{t,t}-1)\quad\text{in Case (3.5)}\,.$$

These are zero if and only if

$$k=2h^{s,w-s}\ (\text{or } 2h^{s+1,w-s-1})=2\,,\quad h=h^{s+1,w-s-1}\ (\text{or } h^{s,w-s})=1 \quad \text{in Case (3.4)}\,,$$
$$k=h^{t,t}\,,\quad h=h^{t+1,t-1}=1 \quad \text{in Case (3.5)}\,.$$

Hence, $\dim D=\dim N(W)^+\cdot r$ if and only if $K=I_r$. □

We denote

$$\tilde{D}_{W,p,N}:=D/C(W)_{\mathbf{Z}}\cup\mathcal{N}(W,p,N)\subset L(W,N)\,,$$
$$\tilde{D}_{W,N}:=\bigcup_p \tilde{D}_{W,p,N}\subset L(W,N)\,,\qquad \tilde{D}:=\bigsqcup_{W,N}\tilde{D}_{W,N}\,, \tag{3.16}$$

where the unions are taken over all sets p of primitive Hodge numbers belonging to $\{n_\lambda, h^{p,q}\}$, all rational S-isotropic filtrations W of $H_{\mathbf{Q}}$ in (2.2) satisfying (2.1) and the orientations $N\in\mathfrak{o}(W)$.

4. Partial compactifications $\overline{D/\Gamma}$. We first recall the partial compactification D^{**}/Γ of Cattani-Kaplan in [CK1] and its generalization into arbitrary weight [U, Appendix] within our present context. Under the assumption (2.1), the disjoint union D^{**} of all *rational boundary components* and the disjoint union D^* of all *rational boundary bundles*, both in the sense of [CK1], coincide and it is defined by

$$D^*:=D\sqcup\left(\bigsqcup_{W,p}F(W,p)\right),\qquad F(W,p):=\left\{\mathrm{gr}^W F\,\middle|\,F\in\bigsqcup_{N\in\mathfrak{o}(W)}B(W,p,N)\right\}, \tag{4.1}$$

where W and p run over all rational S-isotropic filtrations (2.2) of $H_{\mathbf{Q}}$ satisfying the condition (2.1) and all sets of primitive Hodge numbers, respectively, and $B(W,p,N)$ is a boundary component in the sense of (3.6).

In order to introduce the Satake topology on D^*, we choose a maximal $\mathbf{Q}$-split Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ with $\mathfrak{p}\supset\mathfrak{t}$. Let $\Phi\subset\mathfrak{t}^*$ be the $\mathbf{Q}$-root system, $\Phi^+\subset\Phi$ the positive root system with respect to some lexicographical order in $\mathfrak{t}^*$. Let $G=RTK$ be the Iwasawa decomposition, where $R:=\exp(\sum_{\alpha\in\Phi^+}\mathfrak{g}_\alpha)$, $T:=\exp\mathfrak{t}$ and K is the maximal compact subgroup of G with $\mathrm{Lie}\,K=\mathfrak{k}$.

Let $\mathfrak{t}^+:=\{A\in\mathfrak{t}\,|\,\alpha(A)>0 \text{ for all } \alpha\in\Phi^+\}$ be the Weyl chamber. We denote by $\mathscr{A}$ the set of all rational admissible elements A with $\dim H(A;\lambda)=\delta_{\lambda 1}c$ $(\lambda=1,2,\ldots)$ in the closure $\overline{\mathfrak{t}^+}$ of $\mathfrak{t}^+$ in $\mathfrak{t}$, where $c=1$ if w is odd and $c=2$ if w is even. Then we see by construction that $\mathscr{A}$ is finite and is a set of complete representatives of all $G_{\mathbf{Q}}$-conjugacy classes of rational admissible elements A with $\dim H(A;\lambda)=\delta_{\lambda 1}c$ $(\lambda=1,2,\ldots)$. Under the assumption (2.1), $\mathscr{A}$ consists of the single element $Y:=\mathrm{diagonal}(1_c,0,\ldots,0,-1_c)$. Let $W(Y)$ be the weight filtration associated to Y in (1.18). For each set $p=\{p_\lambda^{a,b}\}$ of primitive Hodge numbers, we take a reference point $r_p\in D$ lying over $[K]\in G/K$, via some fixed projection $D\to G/K$, such that (Y,r_p) is an admissible pair of type p (for the definition of 'type p', see [U, (3.3), (3.4)]). This is possible by [U, (3.16.ii)]. We set

$$\tilde{r}_p := (r_p \bmod C(W(Y))_{\mathbf{Z}}) \in L(W(Y), N), \tag{4.2}$$

$$b_p := \pi(\tilde{r}_p) \in B(W(Y), p, N), \quad \bar{b}_p := \mathrm{gr}^{W(Y)}(b_p) \in F(W(Y), p),$$

where N is as in (3.11).

The *Satake topology* $\tau^{\Gamma}(D^*)$ on D^* relative to Γ in [CK1] is introduced in the following process (i)–(iii):

(i) An *open Siegel set* subject to the Iwasawa decomposition $G = RTK$ is a subset $\mathfrak{S} := \omega T_\mu K$ of G, where ω is a relatively compact open neighborhood of 1 in R, $\mu > 0$ and $T_\mu := \{t \in T \mid e^\alpha(t) > \mu$ for all $\alpha \in \Phi^+\}$. An *extended Siegel set* in D^* is a subset $\mathfrak{S}^* := \bigcup_p (\mathfrak{S} \cdot r_p \cup (\mathfrak{S} \cap N(W(Y))) \cdot \bar{b}_p)$. For suitable choices of ω and μ, there exists a finite subset E of $G_{\mathbf{Q}}$ satisfying $\Gamma E \mathfrak{S} \cdot r_p = D$ and $\Gamma_{W(Y)}(E \cap N(W(Y)))(\mathfrak{S} \cap N(W(Y))) \cdot \bar{b}_p = F(W(Y), p)$ for all p. Then, as in [CK1, (4.28)], $\Omega^* := E\mathfrak{S}^*$ is a *Γ-fundamental set* in D^*, i.e., satisfies the following two conditions.

$$\Gamma \Omega^* = D^*. \tag{4.3}$$

(4.4) *There exist finitely many $\gamma_\nu \in \Gamma$ such that, if $\gamma \in \Gamma$, $\gamma\Omega^* \cap \Omega^* = \emptyset$, then the actions of γ and γ_ν coincide on $\Omega^* \cap \gamma^{-1}\Omega^*$ for some ν.*

(ii) A topology $\tau(\mathfrak{S}^*)$ on $\mathfrak{S}^*$ is defined so that a basis of open sets is given by open subsets of $\mathfrak{S} \cdot r_p$ $(\subset D)$ in the natural topology together with subsets

$$(U_\lambda V \cdot r_p \cup U \cdot \bar{b}_p) \cap \mathfrak{S}^* \tag{4.5}$$

for all p, where U runs over the pull-backs via the projection $N(W(Y)) \to F(W(Y), p)$, $g \mapsto g \cdot \bar{b}_p$, of all open sets in $F(W(Y), p)$ in the natural topology, λ is a positive real number, $U_\lambda := \{g \in U \mid e^\alpha(g) > \lambda$ for all $\alpha \in \Phi$ with $\alpha(Y) > 0\}$, V runs over neighborhoods of 1 in K. The topology $\tau(\Omega^*)$ on Ω^* is induced from $\tau(\mathfrak{S}^*)$ in the following way: the system of neighborhoods of $x \in \Omega^*$ consists of all subsets $\mathcal{U} \subset \Omega^*$ satisfying the condition that, if $x \in e\mathfrak{S}^*$ with $e \in E$, then $e^{-1}\mathcal{U} \cap \mathfrak{S}^*$ is a $\tau(\mathfrak{S}^*)$-neighborhood of $e^{-1}x \in \mathfrak{S}^*$. Then, as in [CK1, (4.32)], the topology $\tau(\Omega^*)$ has the following property.

(4.6) *$\tau(\Omega^*)$ is Hausdorff and the action of $\gamma \in \Gamma$ is continuous in $\tau(\Omega^*)$ in the following sense: for $x \in \Omega^*$, if $\gamma x \in \Omega^*$, then for any $\tau(\Omega^*)$-neighborhood $\mathcal{U}'$ of γx there exists a $\tau(\Omega^*)$-neighborhood $\mathcal{U}$ of x such that $\gamma\mathcal{U} \cap \Omega^* \subset \mathcal{U}'$; if $\gamma x \notin \Omega^*$, then there exists a $\tau(\Omega^*)$-neighborhood $\mathcal{U}$ of x such that $\gamma\mathcal{U} \cap \Omega^* = \emptyset$.*

(iii) By virtue of (4.3), (4.4) and (4.6), we can apply [Sa1, Theorem 1′] to obtain a Satake topology $\tau^{\Gamma}(D^*)$ (uniquely determined) with the following four properties.

(4.7.1) *$\tau^{\Gamma}(D^*)$ induces $\tau(\Omega^*)$ (and also $\tau(\mathfrak{S}^*)$).*

(4.7.2) *The action of Γ on D^* is continuous.*

(4.7.3) *If $\Gamma x \cap \Gamma x' = \emptyset$ with $x, x' \in D^*$, then there exist $\tau^{\Gamma}(D^*)$-neighborhoods $\mathcal{U}$ of x and $\mathcal{U}'$ of x' such that $\Gamma\mathcal{U} \cap \Gamma\mathcal{U}' = \emptyset$.*

(4.7.4) *For each $x \in D^*$, there exists a fundamental system $\{\mathcal{U}\}$ of $\tau^{\Gamma}(D^*)$-neighborhoods of x such that, for $\gamma \in \Gamma$, we have $\gamma\mathcal{U} = \mathcal{U}$ if $\gamma \in \Gamma_x$, and $\gamma\mathcal{U} \cap \mathcal{U} = \emptyset$ if $\gamma \notin \Gamma_x$,*

where Γ_x is the isotropy subgroup of Γ at x.

[CK1] used a closed Siegel set instead of an open one. In both cases the arguments are parallel. [CK1, §5] showed that the Satake topology $\tau^{\Gamma}(D^*)$ is independent of the choices of t, Φ^+, K, r_p, Γ, $\mathfrak{S}$, E. As Looijenga has pointed out to the author, the induced Satake topology on D^*/Γ is not known to be locally compact in general (there is no proof of it in [CK1, (4.36.i)]).

DEFINITION (4.8). In the notation of (3.16), a Satake topology $\tau(\tilde{D})$ on $\tilde{D}$ is defined in the following way.

(i) Set $B(W(Y), N) := \bigsqcup_p B(W(Y), p, N)$. The topology $\tau(D \sqcup B(W(Y), N))$ coincides with the natural one on D and at a boundary point $x \in B(W(Y), N)$ a fundamental system of neighborhoods is given by

$$U_\lambda V \cdot r_p \sqcup U \cdot b_p \qquad \text{for some} \quad p\,,$$

where U runs over the pull-backs via the projection $N(W(Y))^+ \to B(W(Y), N)$, $g \mapsto g \cdot b_p$, of all neighborhoods of x in $B(W(Y), N)$ in the natural topology, λ is a positive real number, $U_\lambda := \{g \in U \mid e^\alpha(g) > \lambda$ for all $\alpha \in \Phi$ with $\alpha(Y) > 0\}$, V runs over neighborhoods of 1 in K.

(ii) We extend $\tau(D \sqcup B(W(Y), N))$ to the topology $\tau(\bigsqcup_{W, M \in \mathfrak{o}(W)}(D \sqcup B(W, M)))$ so that the action of $G_{\mathbf{Q}}$ is continuous in the latter, where W runs over all rational S-isotropic filtrations (2.2) of $H_{\mathbf{Q}}$ satisfying the condition (2.1).

(iii) $\tau(\tilde{D})$ is the topology induced from $\tau(\bigsqcup_{W,M}(D \sqcup B(W, M)))$ via the natural projections $D \sqcup B(W, M) \to \tilde{D}_{W,M}$ whose restriction to $B(W, M)$ is given by the composite of the embedding (3.7) and the zero section of the line bundle (2.9).

It is easy to see that the Satake topology $\tau(\tilde{D})$ is well-defined, and we can prove as in [CK1, §5] that $\tau(\tilde{D})$ is independent of the choices of t, Φ^+, K, r_p.

LEMMA (4.9). *The restriction of $\tau(\tilde{D})$ to $\mathcal{N}(W, p, M)$ coincides with the natural topology on it for every W, p and M, where $\mathcal{N}(W, p, M)$ is as in* (3.14).

PROOF. The assertion follows immediately by Definition (4.8) and (3.15) for the SL_2-orbit (ρ, r_p) corresponding to the admissible pair (Y, r_p). □

PROBLEM (4.10). Compare the topology $\tau(\tilde{D}_{W,M})$ with the natural one on $\tilde{D}_{W,M} \subset L(W, M)$.

LEMMA (4.11). *The natural map $f\colon \tilde{D} \to D^*/\Gamma$ is continuous in the Satake topologies.*

PROOF. Set $W = W(Y)$ and let N be as in (4.2). By Definition (4.8) and [CK1, (5.7)] and its generalization [U, Appendix], it is enough to show that, in the notation of (3.16), the natural map

$$f_{W,p,N} : \tilde{D}_{W,p,N} \to D^*/C(W)_{\mathbf{Z}} \tag{4.12}$$

is continuous in the Satake topologies for any p.

Obviously $f_{W,p,N}$ is continuous on $D/C(W)_{\mathbf{Z}}$. Let $x \in B(W, p, N)$ and $\bar{x}$ its image in $F(W, p)$. Note that a fundamental system of $\tau(D^*)$-neighborhoods of $\bar{x} \in D^*$ is given by the following sets (cf. [CK1, (4.31)], [Sa1, Proof of Theorem 1′]):

$$\mathscr{U} = \Gamma_{\bar{x}}\left(\bigcup_{g \in \Gamma E,\, g\mathfrak{S}^* \ni \bar{x}} g(\tau(\mathfrak{S}^*)\text{-neighborhood of } g^{-1}\bar{x} \in \mathfrak{S}^*)\right). \tag{4.13}$$

Hence, in order to prove the continuity of $f_{W,p,N}$, it is enough to show that, on $\tilde{D}_{W,p,N}$, the topology $\tau_1(\tilde{D}_{W,p,N})$, similarly defined as the topology $\tau(D^*/C(W)_{\mathbf{Z}})$ on $D^*/C(W)_{\mathbf{Z}}$ induced by $\tau^{\Gamma}(D^*)$, coincides with the topology $\tau(\tilde{D}_{W,p,N})$ induced by $\tau(\tilde{D})$.

We may assume that the Siegel set $\mathfrak{S}$ and a finite subset $E \subset G_{\mathbf{Q}}$ satisfy $C(W)_{\mathbf{Z}}\mathfrak{S} \supset C(W)$ and $\Gamma_W^+(E \cap N(W)^+)(\mathfrak{S} \cap N(W)^+) \cdot b_p = B(W, p, N)$ for all p. Set $\mathfrak{S}_W^+ = \mathfrak{S} \cap N(W)^+$, $r = r_p$ and $b = b_p$. Since $\mathfrak{S}_W^+ \exp(\boldsymbol{R}_{>0} \cdot Y) = \mathfrak{S}_W^+$, $(\mathfrak{S}_W^+ \cdot r)^{\sim} \sqcup \mathfrak{S}_W^+ \cdot b$ is an open subset of $\mathscr{N} = \mathscr{N}(W, p, N)$ in the natural topology. It follows that the topology $\tau_1((\mathfrak{S}_W^+ \cdot r)^{\sim} \sqcup \mathfrak{S}_W^+ \cdot b)$, induced from $\tau_1(\mathfrak{S}_W^+ \cdot r \sqcup \mathfrak{S}_W^+ \cdot b)$ which is similarly defined as $\tau(\mathfrak{S}^*)$, coincides with the natural topology on $(\mathfrak{S}_W^+ \cdot r)^{\sim} \sqcup \mathfrak{S}_W^+ \cdot b \subset \mathscr{N}$. Since the action of $N(W)^+$ on $\mathscr{N}$ is continuous in the natural topology, the topology $\tau_1(\mathscr{N})$, similarly defined as $\tau(D^*/C(W)_{\mathbf{Z}})$, coincides with the natural topology on $\mathscr{N}$ by (4.13). Evidently the multiplication by $g \in N(W)^+$ from the left to $U_\lambda V$ in (4.5) does not have any effect on the neighborhood V of 1 in K. Thus we get $\tau_1(\tilde{D}_{W,p,N}) = \tau(\tilde{D}_{W,p,N})$. □

COROLLARY (4.14). *For any $x \in B(W, p, M)$, there exists a Satake neighborhood $\mathscr{U}_x$ of x in $\check{D}$, which is stable by $C(W)$, such that the Γ-equivalence and the Γ_W^+-equivalence coincide on $\hat{\mathscr{U}}_x$, where $\hat{\mathscr{U}}_x$ is the pull-back of $\mathscr{U}_x$ via $D \to D/C(W)_{\mathbf{Z}} \subset \tilde{D}_{W,p,M}$.*

PROOF. By Lemma (4.11), this follows immediately from (4.7.4). □

LEMMA (4.15). *In the Satake topology, the action of $\bar{\Gamma}_W^+$ on $\tilde{D}_{W,M}$ is properly discontinuous, so that the Γ_W^+-equivalence relation is closed on $\tilde{D}_{W,M}$.*

PROOF. Let $x \in B(W, p, M)$, and $\bar{x} \in F(W, p)$ its image. By Definition (4.8), we may assume $W = W(Y)$ and $M = N$ as in (4.2). Let $\mathscr{U}_{\bar{x}}$ be a Satake neighborhood of $\bar{x} \in D^*$ satisfying the condition (4.7.4). By Lemma (4.11), we can take a Satake neighborhood $\mathscr{U}_x = (U_\lambda V \cdot r_p)^{\sim} \cup U \cdot b_p$ of $x \in \tilde{D}_{W,p,N}$ contained in $f_{W,p,N}^{-1}(\mathscr{U}_{\bar{x}} \bmod C(W)_{\mathbf{Z}})$. By Proposition (2.10), we may assume that $\{\gamma \in \bar{\Gamma}_W^+ \mid \gamma U \cdot b_p \cap U \cdot b_p \neq \emptyset\}$ is finite. Since $F(W, p) \supset B(W, p, N)/U(W)$, where $U(W)$ is in (2.3), we see that the isotropy subgroup $\Gamma_{\bar{x}}$ of Γ at $\bar{x}$ is equal to $U(W)_{\mathbf{Z}} \rtimes \Gamma_x$.

For $\gamma \in U(W)_{\mathbf{Z}}$, We claim that $\gamma\mathscr{U}_x \cap \mathscr{U}_x \neq \emptyset$ if and only if $\gamma U \cdot b_p \cap U \cdot b_p \neq \emptyset$. To see this, notice that $\gamma\mathscr{U}_x \cap \mathscr{U}_x = \emptyset$ is equivalent to

$$\gamma(U_\lambda V \cdot r_p)^{\sim} \cap (U_\lambda V \cdot r_p)^{\sim} \neq \emptyset, \quad \text{or} \quad \gamma U \cdot b_p \cap U \cdot b_p \neq \emptyset.$$

The former implies $\gamma U_\lambda V \cap \delta U_\lambda V I_{r_p} \neq \emptyset$ for some $\delta \in C(W)_{\mathbf{Z}}$, hence, by the uniqueness of the Iwasawa decomposition, we have $\gamma U_\lambda \cap \delta U_\lambda \neq \emptyset$, and so $\gamma U \cdot b_p \cap U \cdot b_p \neq \emptyset$ as

desired. This proves the 'only if' part. The converse is obvious.

Thus we see $\{\gamma \in \bar{\Gamma}_W^+ \mid \gamma \mathscr{U}_x \cap \mathscr{U}_x \neq \emptyset\} = \{\gamma \in \bar{\Gamma}_{\bar{x}} \mid \gamma \mathscr{U}_x \cap \mathscr{U}_x \neq \emptyset\} = \{\gamma \in \bar{\Gamma}_W^+ \mid \gamma U \cdot b_p \cap U \cdot b_p \neq \emptyset\}$, which is finite. □

We shall now construct our partial compactification $\overline{D/\Gamma}$. We denote by $\mathscr{W}$ the set of complete representatives of the $G_{\boldsymbol{Q}}$-orbit of $W(Y)$ modulo the Γ-action, which is a finite set by (4.3). As point sets, we take

$$(4.16)\quad |\overline{D/\Gamma}| := \left(D \sqcup \bigsqcup_{W,\, p,\, M \in \mathfrak{o}(W)} B(W, p, M)\right) \Big/ \Gamma \simeq D/\Gamma \sqcup \bigsqcup_{W \in \mathscr{W},\, p} B(W, p, N)/\bar{\Gamma}_W^+ ,$$

where in the middle term W and p run over all rational S-isotropic filtrations of $H_{\boldsymbol{Q}}$ in (2.2) satisfying the condition (2.1) and all sets of primitive Hodge numbers belonging to $\{h^{p,q}, n_\lambda\}$, respectively, and in the $B(W, p, N)$ on the extreme right hand side N is some element of $\mathfrak{o}(W)$.

$\tilde{D}$ carries the Satake topology $\tau(\tilde{D})$ defined in (4.8) as well as the natural topology, both of which introduce the corresponding quotient topologies on $|\overline{D/\Gamma}|$ via the projection $\tilde{D} \to |\overline{D/\Gamma}|$. We denote by $\overline{D/\Gamma}$ the point set $|\overline{D/\Gamma}|$ together with these two topologies.

THEOREM (4.17). *$\overline{D/\Gamma}$ is Hausdorff in the Satake topology.*

PROOF. Let $\mathscr{E}$ be the graph of the equivalence relation defined by the projection $\tilde{D} \to \overline{D/\Gamma}$. Notice that $\overline{D/\Gamma}$ is Hausdorff in the Satake topology if and only if the graph $\mathscr{E} \subset \tilde{D} \times \tilde{D}$ is Satake closed. To see the Satake closedness of $\mathscr{E}$, it is enough to show the following: if $x_i, y_i \in D$ and $\gamma_i \in \Gamma$ with $y_i = \gamma_i x_i$ satisfy $(x_i \bmod C(W)_{\boldsymbol{Z}}) \to x \in B(W, p, N)$, $(y_i \bmod C(W)_{\boldsymbol{Z}}) \to y \in B(W', p', N')$ in the Satake topology, then $(x, y) \in \mathscr{E}$.

By Lemma (4.11) and the fact that the Satake topology on D^*/Γ is Hausdorff (see [CK1, (4.36.i)], and also [U, Appendix]), the images of x and y in D^*/Γ coincide, hence lie in the same boundary component $F(W, p)/\Gamma_W$ of D^*/Γ. Hence $W' = \delta W$ and $N' = \delta N$ for some $\delta \in \Gamma$ and $p = p'$. Replacing y_i, y by $\delta^{-1} y_i, \delta^{-1} y$, it suffices to prove the assertion in the special case $x, y \in B(W, p, N)$. We consider a diagram:

$$\begin{array}{ccccc} \tilde{D}_{W,p,N} & \xrightarrow{f_{W,p,N}} & D^*/C(W)_{\boldsymbol{Z}} & \longrightarrow & D^*/\Gamma \\ & & \cup & & \cup \\ & & F(W, p) & \longrightarrow & F(W, p)/\Gamma_W . \end{array}$$

Since x, y have the same image in D^*/Γ, their images in $F(W, p) \subset D^*/C(W)_{\boldsymbol{Z}}$ differ by a $\gamma \in \Gamma_W$. Again replacing y_i, y by $\gamma^{-1} y_i, \gamma^{-1} y$, we may assume that x, y have the same image $\bar{x} \in F(W, p) \subset D^*/C(W)_{\boldsymbol{Z}}$. Let $\mathscr{U}_{\bar{x}} \subset D^*$ be a Satake neighborhood of $\bar{x}$ satisfying the condition (4.7.4). Then by Lemma (4.11) $\mathscr{V} := f_{W,p,N}^{-1}(\mathscr{U}_{\bar{x}}/C(W)_{\boldsymbol{Z}})$ is a Satake open subset of $\tilde{D}_{W,p,N}$ containing x, y. Therefore, $x_i, y_i \bmod C(W)_{\boldsymbol{Z}} \in \mathscr{V}$ if $i \gg 0$. In other words, $x_i, y_i \in \mathscr{U}_{\bar{x}} \cap D$ if $i \gg 0$. Now $y_i = \gamma_i x_i$, $\gamma_i \in \Gamma$, so, by the assumption on $\mathscr{U}_{\bar{x}}$, we see $\gamma_i \in \Gamma_{\bar{x}} \subset \Gamma_W^+$ for $i \gg 0$. Hence the assertion follows from Lemma (4.15). □

REMARK (4.18). As we have seen in this section, $\tilde{D}$ (hence $\overline{D/\Gamma}$) carries the two topologies, i.e., the Satake topology $\tau(\tilde{D})$ defined in (4.8) and the natural topology induced by the $\tilde{D}_{W,M} \subset L(W, M)$. These topologies coincide on the loci $\mathcal{N}(W, p, M)$ of the families of the nilpotent orbits (cf. (4.9)), but off these loci we have not yet compared them (cf. (4.10)). So at present we may consider that the topology of $\tilde{D}$ (hence of $\overline{D/\Gamma}$) is the *refinement* of these two topologies.

CONSTRUCTION (4.19). We then introduce a complex structure on $\overline{D/\Gamma}$.

For each $W \in \mathcal{W}$, p and $x \in B(W, p, N)$, we choose a Satake open neighborhood $\mathcal{U}_x$ of $x \in \tilde{D}_{W,N} \subset \tilde{D}$ as in Corollary (4.14), and form a covering

$$\{D/\Gamma, \bar{\Gamma}_W^+ \cdot \mathcal{U}_x/\bar{\Gamma}_W^+ \ (\forall x \in B(W, p, N), \forall W \in \mathcal{W}, \forall p)\} \tag{4.20}$$

of $\overline{D/\Gamma}$. D/Γ has an obvious complex structure and, by Proposition (2.10), $\bar{\Gamma}_W^+ \cdot \mathcal{U}_x/\bar{\Gamma}_W^+$ has the complex structure induced by

$$\bar{\Gamma}_W^+ \cdot \mathcal{U}_x/\bar{\Gamma}_W^+ \subset \tilde{D}_{W,p,N}/\bar{\Gamma}_W^+ \subset L(W, N)/\bar{\Gamma}_W^+ . \tag{4.21}$$

These are patched up by

$$\begin{aligned}
&\bar{\Gamma}_W^+ \cdot \mathcal{U}_x/\bar{\Gamma}_W^+ \overset{\text{open}}{\supset} \Gamma_W^+ \cdot \hat{\mathcal{U}}_x/\Gamma_W^+ \simeq \Gamma \cdot \hat{\mathcal{U}}_x/\Gamma \overset{\text{open}}{\subset} D/\Gamma , \\
&\bar{\Gamma}_W^+ \cdot \mathcal{U}_x/\bar{\Gamma}_W^+ \overset{\text{Satake open}}{\supset} (\bar{\Gamma}_W^+ \cdot \mathcal{U}_x \cap \bar{\Gamma}_W^+ \cdot \mathcal{U}_y)/\bar{\Gamma}_W^+ \overset{\text{Satake open}}{\subset} \bar{\Gamma}_W^+ \cdot \mathcal{U}_y/\bar{\Gamma}_W^+ \\
&\qquad \text{for } x, y \in \bigsqcup_p B(W, p, N) , \quad \text{in } \tilde{D}_{W,N}/\bar{\Gamma}_W^+ , \\
&\bar{\Gamma}_W^+ \cdot \mathcal{U}_x/\bar{\Gamma}_W^+ \overset{\text{open}}{\supset} (\Gamma \cdot \hat{\mathcal{U}}_x \cap \Gamma \cdot \hat{\mathcal{U}}_y)/\Gamma \overset{\text{open}}{\subset} \bar{\Gamma}_{W'}^+ \cdot \mathcal{U}_y/\bar{\Gamma}_{W'}^+ \\
&\qquad \text{for } x \in B(W, p, N) \text{ and } y \in B(W', p', N') \text{ with } N \neq N' , \quad \text{in } \overline{D/\Gamma} .
\end{aligned} \tag{4.22}$$

5. Extension of period maps. Let $\varphi : \Delta^* \to D/\Gamma$ be a *period map*, i.e., a holomorphic map with horizontal local liftings, from the punctured unit disc Δ^*. Let $\mathfrak{h} \to \Delta^*$, $z \mapsto s = \exp(2\pi i z)$, be the universal cover, $\tilde{\varphi} : \mathfrak{h} \to D$ a lifting of φ, $\gamma \in \Gamma$ an element satisfying $\tilde{\varphi}(z+1) = \gamma\tilde{\varphi}(z)$ for all $z \in \mathfrak{h}$, N the logarithm of the unipotent part of γ, and $W(N)[-w]$ the monodromy weight filtration.

THEOREM (5.1). (i) *Any period map $\varphi : \Delta^* \to D/\Gamma$ from the punctured disc with the monodromy weight filtration $W = W(N)[-w]$ satisfying the conditions* (2.1) *and* (2.2) *extends continuously to $\bar{\varphi} : \Delta \to \overline{D/\Gamma}$ in the Satake topology on $\overline{D/\Gamma}$. Moreover, $\bar{\varphi}$ is holomorphic.*

(ii) *For any boundary point $\bar{\xi} \in \overline{D/\Gamma} \setminus D/\Gamma$, there exist a period map $\varphi : \Delta^* \to D/\Gamma$ with the property described in* (i) *and its Satake continuous, holomorphic extension $\bar{\varphi} : \Delta \to \overline{D/\Gamma}$ such that $\bar{\varphi}(0) = \bar{\xi}$.*

PROOF. In order to prove (i), it is enough to show that $\varphi : \Delta^* \to D/\Gamma$ extends

continuously over the puncture both in the Satake topology and in the natural topology on $\overline{D/\Gamma}$.

As for the first part of (i), the proof is almost analoguous to the one in [CK1], and so we shall write down the proof as far as needed. By the rational version of the SL_2-orbit theorem [Sc, (5.13), (5.19), (5.26)], there exists and SL_2-orbit (ρ, r_p) with ρ defined over $\boldsymbol{Q}$, such that

$$\rho_*\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N\,, \quad \text{together with} \quad Y := \rho_*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfies the property (5.2) below. Choose a maximal $\boldsymbol{Q}$-split Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing Y, and a positive root system $\Phi^+ \subset \mathfrak{t}^*$ for the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$ such that any root α with $\alpha(Y) > 0$ belongs to Φ^+. Set $R := \exp(\sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha)$ and $T := \exp \mathfrak{t}$. Then the centralizer of T in G is a product TL with L $\boldsymbol{Q}$-anisotropic, and $P := RTL$ is a minimal $\boldsymbol{Q}$-parabolic subgroup of G. Let K be the maximal compact subgroup of G corresponding to the Cartan involution θ_{r_p} determined by the reference point r_p as in (1.11). Then $G = PK = RTLK$, and we have the following:

(5.2) There exist functions $r(x, y)$, $t(x, y)$, $l(x, y)$ and $k(x, y)$ defined and real analytic on a domain $\{x + iy \in \mathfrak{h} \mid y > \beta\}$ for some β and taking values in groups R, T, L and K, respectively, such that

(5.2.1) $\tilde{\varphi}(x + iy) = r(x, y)t(x, y)l(x, y)k(x, y) \cdot r_p$, and

(5.2.2) as $y \to +\infty$, the functions

$$r(x, y) \to \exp(xN)r(\infty)\,, \quad \exp(\log(y^{-1/2})Y)t(x, y) \to 1\,, \quad l(x, y) \to 1\,, \quad k(x, y) \to 1\,,$$

converge uniformly in x, where $r(\infty) \in \exp \mathfrak{v}$ with $\mathfrak{v} := \mathrm{Im}(\mathrm{ad}_{\mathfrak{g}} N) \cap \mathrm{Ker}(\mathrm{ad}_{\mathfrak{g}} N)$.

By [CK1, (6.4)], we see $\exp \mathfrak{v} \subset U(W)$. (Since $N^2 = 0$ in the present case, the proof is easier.) φ factors through $\Delta^* \to D/C(W)_{\boldsymbol{Z}}$, denoted also by φ by abuse of notation. Take $M \in \mathfrak{o}(W)$ so that $N = aM$, $a > 0$. We now claim:

(5.3) $\lim_{t \to 0} \varphi(t) = r(\infty) \cdot b_p \in \tilde{D}_{W,p,M}$ (see (3.16)) in the Satake topology, where $b_p \in B(W, p, M)$ is induced from r_p as in (3.11).

In order to reset the situation where we have introduced the Satake topology, we choose a maximal compact subgroup K' of G whose associated Cartan involution acts on $\mathfrak{t}$ as multiplication by -1. Then we have an Iwasawa decomposition $G = RTK'$. As in the proof of [U, (3.16.ii)], there exists $g \in G_Y$ such that $K' = \mathrm{Int}(g)K$. $g \in G_Y$ splits according to the decomposition $G = PK$, hence we may assume moreover $g \in P \cap G_Y$. Set $r'_p := g \cdot r_p \in D$ and $b'_p := g \cdot b_p \in B = B(W, p, M')$, where $M' \in \mathfrak{o}(W)$ is a positive multiple of $\mathrm{Ad}(g)M$. We are thus in the situation after (4.1). Then (5.3) follows if we show:

(5.4) In the notation of (4.8), for the pull-back U' via the projection $N(W)^+ \to B$, $h \mapsto h \cdot b'_p$, of any neighborhood of $\xi' := gr(\infty) \cdot b_p$ in B, any $\lambda > 0$ and any neighborhood V' of 1 in K', there exists $\beta > 0$ such that $g \cdot \tilde{\varphi}(x + iy) \in U'_\lambda V' \cdot r'_p$ for all $y > \beta$ and $|x| < 1$.

Indeed, (5.4) implies $\tilde{\varphi}(x+iy)\in g^{-1}U'_\lambda V'\cdot r'_p$ for all $y>\beta$ and $|x|<1$. It is easy to see that this, in turn, yields $\varphi(t)\in((g^{-1}U')_{\lambda_0\lambda}V'\cdot r'_p)^\sim$ for $0<|t|<e^{-2\pi\beta}$, where $\lambda_0:=\min\{e^\alpha(g^{-1})\,|\,\alpha\in\Phi$ with $\alpha(Y)>0\}$. Since $((g^{-1}U')_{\lambda_0\lambda}V'\cdot r'_p)^\sim\cup(g^{-1}U')\cdot b'_p$ is a Satake neighborhood of $g^{-1}\xi'=r(\infty)\cdot b_p$ in $\tilde{D}_{W,p,M'}$, which can be taken arbitrarily small, we get (5.3).

Now we shall prove (5.4). Set $g=r_0t_0l_0$, $r_0\in R$, $t_0\in T$ and $l_0\in L$. Then, from (5.2.1), $R\lhd P$ and $L\subset K'$, we have

$$g\tilde{\varphi}(x+iy)=r'(x,y)t(x,y)k'(x,y)\cdot r'_p\,,\qquad \text{where}$$

$$r'(x,y):=gr(x,y)g^{-1}r_0(t(x,y)\,l'(x,y))\,r_0^{-1}(t(x,y)l'(x,y))^{-1}\in R\,,$$

$$k'(x,y):=l'(x,y)gk(x,y)g^{-1}\in K'\,,$$

$$l'(x,y):=l_0l(x,y)\,l_0^{-1}\in L\,.$$

It follows from (5.2.2) that, as $y\to+\infty$, these functions converge uniformly in x:

$$l'(x,y)\to 1\,,\quad r'(x,y)\to g\exp(xN)r(\infty)g^{-1}\,,\quad k'(x,y)\to 1\,.$$

Hence there exists $\beta>0$ such that $r'(x,y)t(x,y)\in U'_\lambda$ and $k'(x,y)\in V'$ for all $y>\beta$ and $|x|<1$. (5.4) is proved, and this completes the proof of the first part of (i).

In order to prove the second part of (i), it is enough to show:

(5.5) $\varphi:\Delta^*\to\tilde{D}_{W,p,M}$ extends continuously in the natural topology induced by $\tilde{D}_{W,p,M}\subset L(W,M)$.

Taking a finite cyclic base extension $\Delta'\to\Delta$, $s'\mapsto s=s'^r$, if necessary, we may assume that the monodromy γ is unipotent and $N=\log\gamma$ is the generator M of $\operatorname{Lie}C(W)_{\boldsymbol{Z}}$. Moreover, [B, (7.13.2)] allows us to change the lattice $H_{\boldsymbol{Z}}$, if necessary, so that there exists a basis $\{v_1,\dots,v_n\}$ of $H_{\boldsymbol{Z}}$ adapted to the filtration $W=W(N)[-w]$ and the polarization S (cf. the proof of (2.5)) such that

$$(5.6)\qquad Nv_i=\begin{cases}0 & (1\le i\le n-1)\,,\\ v_1 & (i=n)\,,\end{cases}\qquad Nv_i=\begin{cases}0 & (1\le i\le n-2)\,,\\ v_2 & (i=n-1)\,,\\ v_1 & (i=n)\,,\end{cases}$$

if $w=2t+1$, and if $w=2t$, respectively. Set $\tilde{\psi}(z):=\exp(-zN)\cdot\tilde{\varphi}(z)$. Then $\tilde{\psi}(z+1)=\tilde{\psi}(z)$ hence $\tilde{\psi}:\mathfrak{h}\to\check{D}$ drops to $\psi:\Delta\to\check{D}$, which extends to $\bar{\psi}:\Delta\to\check{D}$ holomorphically by the nilpotent orbit theorem [Sc, (4.9)]. The corresponding Hodge filtration $F^w(\bar{\psi}(s))\subset\cdots\subset F^0(\bar{\psi}(s))=H_{\boldsymbol{C}}$ varies holomorphically in $s\in\Delta$ and is determined by $\{F^p(\bar{\psi}(s))\}_{p\ge t+1}$ because of the first Riemann-Hodge bilinear relation (1.3). Let $\{\psi_1(s),\dots,\psi_f(s)\}$ be a basis of $F^{t+1}(\bar{\psi}(s))$ adapted to the filtration. Set $\psi_j(s)=:\sum_{i=1}^n\psi_{ij}(s)v_i$. Note that the type of the limiting Hodge structure $(W,F(\bar{\psi}(0)))$ in the present situation is one of the cases (3.3), (3.4) and (3.5). Hence, in the case (3.3) $\psi_{nj}(0)=0$ for all $\psi_j(0)\in F^p(\bar{\psi}(0))$ with $w\ge p\ge t+2$ and $\psi_{nj}(0)\ne 0$ for some $\psi_j(0)\in F^{t+1}(\bar{\psi}(0))$, and in the case (3.4), (3.5) the $\boldsymbol{C}$-vector subspace of $\boldsymbol{C}^2$ spanned by $\{(\psi_{n-1,j}(0),\psi_{nj}(0))\,|\,1\le j\le f\}$ is one-dimensional

and $\mathrm{Im}(\psi_{nj}(0)/\psi_{n-1,j}(0))>0$ for any j provided $\psi_{n-1,j}(0)\neq 0$. Let a be the largest integer p such that $F^p(\bar{\psi}(0))$ contains some $\psi_j(0)$ with $\psi_{nj}(0)\neq 0$ in the case (3.3) and $\psi_{n-1,j}(0)\neq 0$ in the cases (3.4), (3.5), and set $f^p:=\dim F^p=\sum_{p'\geq p} h^{p',w-p'}$ and

$$c:=\begin{cases}1 & \text{if } w=2t+1\,,\\ 2 & \text{if } w=2t\,.\end{cases}$$

Then, reordering the part $v_{c+1},\dots,v_{n-c}$ of the basis, if necessary, we may assume that for every $w\geq p\geq a+1$ the following $h^{p,w-p}$ square matrices are invertible at $s=0$:

$$A_p:=(\psi_{ij}(s))\,,\quad \text{where the indices run over } f^{p+1}+1\leq j\leq f^p \text{ and}$$

$$\begin{cases} c+f^{p+1}+1\leq i\leq c+f^p & & \text{if } w\geq p\geq a+1\,,\\ c+f^{p+1}+1\leq i\leq c+f^p-1 & \text{and}\quad i=n-c+1 & \text{if } p=a\,,\\ c+f^{p+1}\leq i\leq c+f^p-1 & & \text{if } a-1\geq p\geq t+1\,.\end{cases}$$

We normalize the basis $\{\psi_1(s),\dots,\psi_f(s)\}$ by descending induction on $p=w,\dots,t+1$ by replacing $(\psi_j(s)\,;f^{p+1}+1\leq j\leq f^p)$ by $(\psi_j(s)\,;f^{p+1}+1\leq j\leq f^p)A_p^{-1}$ and, for each $j'=f^{p+1}+1,\dots,f$, subtracting $\sum_j \delta_{j'j}\psi_{\mu(j),j'}(s)\psi_j(s)$ (the summation is taken over $f^{p+1}+1\leq j\leq f^p$) from $\psi_{j'}(s)$, where

$$\mu(j):=\begin{cases} j+c & \text{if } j\leq f^a-1\,,\\ n-c+1 & \text{if } j=f^a\,,\\ j+c-1 & \text{if } j\geq f^a+1\,.\end{cases}$$

Then the resulting $n\times f$ matrix $(\psi_{ij}(s))$ will be of the form in (5.7) if $w=2t+1$, and if $w=2t$, respectively.

(5.7)
$$\begin{pmatrix} * & \cdots & \cdots & \cdots & * \\ I & 0 & \cdots & 0 & 0 \\ * & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & I & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ * & \cdots & * & I & 0 \\ * & \cdots & \cdots & \cdots & * \\ \vdots & & & & \vdots \\ * & \cdots & \cdots & \cdots & * \\ * & \cdots & * & 0 & 1 \end{pmatrix}
\qquad
\begin{pmatrix} * & \cdots & \cdots & \cdots & \cdots & \cdots & * \\ * & \cdots & \cdots & \cdots & \cdots & \cdots & * \\ I & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ * & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & I & 0 & \ddots & & \vdots \\ \vdots & & * & I & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ * & \cdots & \cdots & * & * & & I \\ * & \cdots & \cdots & \cdots & \cdots & \cdots & * \\ \vdots & & & & & & \vdots \\ * & \cdots & \cdots & \cdots & \cdots & \cdots & * \\ * & * & 0 & 1 & 0 & \cdots & 0 \\ * & * & 0 & * & 0 & \cdots & 0 \end{pmatrix}$$

The entries $\psi_{ij}(s)$ in $*$ of each matrix in (5.7), i.e., with double indices (i, j) running over

$$(5.8)\qquad \begin{cases} 1\le i\le c\,, \quad 1\le j\le f\,; \\ \mu(f^p)+1\le i\le n\,, \quad f^{p+1}+1\le j\le f^p \quad \text{for } p\ge a+1\,; \\ \mu(f^a-1)+1\le i\le n-c\,, \quad f^{a+1}+1\le j\le f^a-1\,; \\ \mu(f^a-1)+1\le i\le n \quad \text{with } i\ne n-c+1\,, \quad j=f^a\,; \\ \mu(f^p)+1\le i\le n-c\,, \quad f^{p+1}+1\le j\le f^p \quad \text{for } p\le a-1\,, \end{cases}$$

yield the affine Plücker coordinates of the filtration $F(\bar{\psi}(s))$, which are holomorphic in $s\in\Delta$. For $j=1,\dots,f$,

$$(5.9)\qquad \tilde{\varphi}_j(z):=\exp(zN)\cdot\psi_j(s)=\sum_{1\le i\le c}(\psi_{ij}(s)+z\psi_{n-i+1,j}(s))v_i+\sum_{i\ge c+1}\psi_{ij}(s)v_i$$

form a basis of $F^{t+1}(\tilde{\varphi}(z))$ adapted to the Hodge filtration $F(\tilde{\varphi}(z))$ corresponding to $\tilde{\varphi}(z)$ and their coefficients with double indices (i, j) in (5.8) are considered as the affine Plücker coordinates of $F(\tilde{\varphi}(z))$. These are still normalized by an affine change of coordinates replacing $\psi_{ij}(s)+\psi_{n-i+1,j}(s)z$ by

$$(5.10)\qquad (\psi_{ij}(s)+\psi_{n-i+1,j}(s)z)-\psi_{n-i+1,j}(s)(\psi_{c,f^a}(s)+z)=\psi_{ij}(s)-\psi_{n-i+1,j}(s)\psi_{c,f^a}(s)$$
$$(1\le i\le c,\ 1\le j\le f^{a+1};\ \text{moreover } (i,j)=(1,f^a) \text{ in case } w=2t)\,.$$

Hence the coordinates of $\varphi(s)\in\tilde{D}_{W,p,M}$ are given by

$$(5.11)\qquad \begin{cases} \exp 2\pi\sqrt{-1}\,(\psi_{c,f^a}(s)+z)=\exp(2\pi\sqrt{-1}\,\psi_{c,f^a}(s))\cdot s\,, \\ \psi_{ij}(s)-\psi_{n-i+1,j}(s)\psi_{c,f^a}(s) \quad ((i,j) \text{ in } (5.10))\,, \\ \psi_{ij}(s) \quad ((i,j) \text{ in } (5.8) \text{ except the above})\,, \end{cases}$$

from which (5.5) follows easily.

In order to prove (ii), we take the lifting $\xi\in B(W, p, M)$ of $\bar{\xi}$ with $W\in\mathscr{W}$. Then, by Proposition (3.12.ii), there exists a point $\tilde{F}\in\mathscr{N}(W, p, M)$ such that $\pi(\tilde{F})=\xi$. Then for some $\beta>0$, $v:\{z\in\boldsymbol{C}\,|\,\operatorname{Im} z>\beta\}\to\mathscr{N}(W, p, M)\subset\tilde{D}_{W,p,M}$, $z\mapsto\exp(zM)\cdot\tilde{F}$, is a holomorphic horizontal map and, by (4.9), $v(z)\to\xi$ as $\operatorname{Im} z\to+\infty$. Hence $\varphi(s):=(\text{projection})\circ v((1/2\pi i)\log s+i\beta)\in D/\Gamma$ is the desired period map. □

REMARK (5.12). In the notation and in the normalized situation in the proof of Theorem (5.1), $\psi(s) \bmod C(W)_{\boldsymbol{Z}}$ and $\varphi(s)\in\tilde{D}_{W,p,M}$ differ by the (i, j)-coordinates with $1\le i\le c$, $1\le j\le f^{a+1}$ and $j=f^a$ in (5.7) and (5.11). The difference between $\bar{\psi}(0) \bmod C(W)_{\boldsymbol{Z}}$ and $\bar{\varphi}(0)$, however, appears only in the (i, j)-coordinates with $1\le i\le c$ and $j=f^a$, because $\psi_{ij}(0)=0$ $(n-c+1\le i\le n, 1\le j\le f^{a+1})$ by observing the type of the limiting mixed Hodge structure $(W, F(\bar{\psi}(0)))$.

6. Examples. We consider, as examples, 'tame' degenerations of surfaces of

general type on the Noether lines, whose canonical images are rational ruled surfaces Σ_d of degree d (i.e., *type* (d) in the terminology in [H, I, p. 363; II, p. 127]). We denote by S_0 and F the section of $\Sigma_d \to \boldsymbol{P}^1$ with $S_0^2 = -d$ and a fiber, respectively.

(I) Let X be a minimal algebraic surface on the Noether line $c_1^2 = 2p_g - 4$, where $c_1 = c_1(X)$ is the first Chern class of X and $p_g = p_g(X)$ is the geometric genus of X. We assume that X is of type (d) in the above sense. Then, by [H, I, Theorem 1.6.iii], such a surface X occurs, via the canonical map, as the minimal resolution of singularities of the double covering of Σ_d branched along a curve $B \in |6S_0 + (p_g + 2 + 3d)F|$ with at most simple singularities, where $p_g \geq \max\{d+4, 2d-2\}$ and $p_g - d$ is even. The p_g of such surfaces range over all integers ≥ 4 and the fundamental groups π_1 are trivial by [H, I, Theorem 3.4].

(II) Let Y be a minimal algebraic surface on the Noether line $c_1^2 = 2p_g - 3$. We assume that Y is of type (d). Then, by [H, II, Theorem (1.3.A)], such a surface Y occurs, via the canonical map, as the contraction of a unique exceptional curve of the first kind on the minimal resolution of singularities of the double covering of Σ_d branched along a curve $C \in |6S_0 + (p_g + 4 + 3d)F|$ where $p_g \geq 2d - 1$ and $p_g - d$ is even. C has two quadruple points x, y, which may be infinitely near, on the same fiber other than simple singularities on the proper transform of C to the blow-up of Σ_d with center x and y. The p_g of such surfaces range over all integers ≥ 4 and the fundamental groups π_1 are trivial by [H, II, Theorem (4.8)].

Between these two series of surfaces, we consider the following two types of degenerations of branch loci.

(II)$\to$(I): The $C_t \in |6S_0 + (p_g + 4 + 3d)F|$ $(t \in \Delta^*)$ on Σ_d, with two quadruple points on a fiber F other than simple singularities, degenerate into $C_0 = B + 2F$, where $B \in |6S_0 + (p_g + 2 + 3d)F|$ has two ordinary double points on the fiber F other than the simple singularities come from that on the C_t.

(I)$\to$(II): The $B_t \in |6S_{0,d} + (p_g + 2 + 3d)F_d|$ $(t \in \Delta^*)$ on Σ_d have one double point P_t at which the two branches have contact number 2 and two ordinary double points A_t, A_t' and possibly other simple singularities. The three points P_t, A_t, A_t' crash one another to make up one triple point P on B_0. P is apart from the minimal section $S_{0,d}$ and each pair of the three branches have contact number 2 at P. B_0 is smooth at the other three intersection points with the fiber F_d containing P. By blowing-up Σ_d at P and contraction the proper transform of F_d, the total transform of B_0 becomes $C + 2F_{d-1}$ with $C \in |6S_{0,d-1} + ((p_g - 1) + 4 + 3(d-1))F_{d-1}|$ on Σ_{d-1}, which has two quadruple points on one fiber other than the simple singularities come from that on B_0.

According to these, we have two types of semi-stable degenerations of surfaces on the Noether lines.

(II)$\to$(I): $g : \mathscr{Y} \to \Delta$ is a semi-stable degeneration whose smooth fibers $Y_t := g^{-1}(t)$ $(t \in \Delta^*)$ are minimal surfaces of type (d) with $c_1(Y_t)^2 = 2p_g(Y_t) - 3$, $p_g(Y_t) \geq 4$. $Y_0 := g^{-1}(0) = X \cup U$, where X is a minimal surface of type (d) with $c_1(X)^2 = c_1(Y_t)^2 - 1 = 2p_g(X) - 4$, $p_g(X) = p_g(Y_t) \geq 4$, and $U \simeq \boldsymbol{P}^2$ intersects X along a smooth conic on $\boldsymbol{P}^2$ hence

$X \cap U$ has self-intersection number -4 on X. We need not extend the base in the semi-stable reduction in this case.

(I)$\to$(II): $f : \mathscr{X} \to \Delta$ is a semi-stable degeneration whose smooth fibers $X_t := f^{-1}(t)$ ($t \in \Delta^*$) are minimal surfaces of type (d) with $c_1(X_t)^2 = 2p_g(X_t) - 4$, $p_g(X_t) \geq 5$. $X_0 := f^{-1}(0) = Y \cup V$, where Y is a minimal surface of type (d) with $c_1(Y)^2 = c_1(X_t)^2 - 1 = 2p_g(Y) - 3$, $p_g(Y) = p_g(X_t) - 1 \geq 4$, V is a rational surface, and $Y \cap V$ is a smooth elliptic curve with self-intersection number -1 on Y, hence is the exceptional curve of the minimal resolution of a simple elliptic singularity of type $\tilde{E}_8$. We need to take a ramified double covering of the base in the semi-stable reduction in this case.

Thus two series of smooth families of surfaces with (p_g, c_1^2) on the Noether lines in question are connected by the above 'tame' degenerations:

$$\begin{array}{lccccccccccc}
\text{(II):} & (4,5) & & (5,7) & & \cdots & & (p, 2p-3) & & (p+1, 2p-1) & & \cdots \\
 & \downarrow & \nwarrow & \downarrow & \nwarrow & \cdots & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & \cdots \\
\text{(I):} & (4,4) & & (5,6) & & \cdots & & (p, 2p-4) & & (p+1, 2p-2) & & \cdots .
\end{array}$$

REMARK (6.1). Ashikaga and Konno [AK] showed that degenerations of the above type are observed widely in the geography of surfaces of general type.

For the above semi-stable degenerations, we observe the Clemens-Schmid sequences (cf. [C]) and the Mayer-Vietoris sequences

$$H^2(\mathscr{Y}, \mathscr{Y} - Y_0) \longrightarrow H^2(Y_0) \longrightarrow H^2(Y_t) \xrightarrow{N} H^2(Y_t),$$

$$H^1(X \cap U) \longrightarrow H^2(Y_0) \longrightarrow H^2(X) \oplus H^2(U),$$

$$H^2(\mathscr{X}, \mathscr{X} - X_0) \xrightarrow{\alpha} H^2(X_0) \longrightarrow H^2(X_t) \xrightarrow{N} H^2(X_t),$$

$$H^1(Y) \oplus H^1(V) \longrightarrow H^1(Y \cap V) \xrightarrow{\beta} H^2(X_0) \longrightarrow H^2(Y) \oplus H^2(V).$$

Since $H^1(X \cap U) = 0$, we see that $H^2(Y_0)$ carries a Hodge structure of pure weight 2 so that the monodromy weight filtration $W = W(N)[-2]$ is trivial, i.e., $0 = W_1 \subset W_2 = H^2(Y_t)$. As for the second family, since $H^1(Y) \oplus H^1(V) = 0$ and $H^2(\mathscr{X}, \mathscr{X} - X_0) \simeq H_4(X_0) \xrightarrow{\sim} H_4(Y) \oplus H_4(V)$, we see that β is injective and $\operatorname{Im}\beta \cap \operatorname{Im}\alpha = 0$ in $H^2(X_0)$ for a reason of weight. It follows that $W_0 = 0$ and $W_1 \xleftarrow{\sim} H^1(Y \cap V)$. Hence W satisfies the conditions (2.1), (2.2) and we can apply Theorem (5.1.i) to the period map $\varphi : \Delta^* \to D/\Gamma$ associated to the variation of Hodge structure of weight 2 arising from the smooth family $f : \mathscr{X} - X_0 \to \Delta^*$ and obtain the holomorphic extension $\bar{\varphi} : \Delta \to \overline{D/\Gamma}$. Thus we can discuss the differential $d\bar{\varphi}(0)$ of $\bar{\varphi}$ at $0 \in \Delta$.

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