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GROWTH AND THE SPECTRUM OF THE LAPLACIAN OF AN INFINITE GRAPH*

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Abstract. We give an upper bound for the infimum of the essential spectrum of the combinatorial Laplacian on an infinite graph in terms of the exponential growth of the graph.

1. Introduction and the statement of results. Let M be a smooth, complete, non-compact Riemannian manifold and Δ the Laplacian on $L^2(M)$. We denote by λ_0 the infimum of the spectrum of Δ and by λ_0^{ess} the infimum of the essential spectrum of Δ. We clearly have $\lambda_0 \leq \lambda_0^{\mathrm{ess}}$. $\lambda_0^{\mathrm{ess}}(M)=\lim_K \lambda_0(M-K)$ where K runs through all compact subsets of M.

Brooks [B1] gave an upper bound for λ_0^{ess} using the exponential growth $\mu(M)$ of M. Pick a point $x_0 \in M$ and let $B(r)$ be the ball of radius r at x_0. Let $V(r)$ denote the volume of $B(r)$. Put $\mu=\limsup_r\{(\log V(r))/r\}$. Then μ is independent of the choice of x_0. He showed $\lambda_0^{\mathrm{ess}} \leq \mu^2/4$ if the volume of M is infinite.

The objective of this paper is not only to give a proper discrete analog of Brooks' result, but also to have a better understanding of his somewhat mysterious proof. Discrete analog of other results on Riemannian manifolds can be found, for example, in [K], [Fo], [DK].

Let $G(V,E)$ be a locally finite, infinite graph. The Laplacian of G is given by

$$\Delta f(x)=\frac{1}{m(x)}\sum_{x\sim y}(f(x)-f(y)), \qquad x \in V,$$

where $x \sim y$ for $x, y \in V$ means that x and y are connected by an edge and $m(x)$ is the number of the edges at x. The domain of Δ is

$$L^2(V)=\left\{f \,\middle|\, (f,f)=\sum_{x\in V} m(x)f^2(x)<\infty\right\}.$$

We make G into a metric space by the path metric assigning 1 to every edge of G. Pick a point $x_0 \in V$ and denote by $B(r)$ the ball of radius r at x_0. Put $V(r)=\sum_{x\in B(r)} m(x)$. The *exponential growth* of G is defined by

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$$\mu(G)=\limsup_{r\to\infty}\frac{1}{r}\log V(r)\,.$$

It is easy to see that μ is independent of the choice of x_0, and $0\le\mu\le\infty$. Let $\underline{\lambda}^{\mathrm{ess}}$ be the infimum of the essential spectrum of Δ, and $\bar{\lambda}^{\mathrm{ess}}$ the supremum. Then, for an infinite graph,

$$0\le\underline{\lambda}^{\mathrm{ess}}\le 1\le\bar{\lambda}^{\mathrm{ess}}\le 2\,,$$

as we show in the next section.

Dodziuk and Karp [DK, Prop. 1.8 and Prop. 1.18] showed, among many inequalities,

$$\underline{\lambda}^{\mathrm{ess}}\le\min\left\{\frac{2(1-\exp(\mu/2))^2}{\exp(\mu)}\,,\ 1-\frac{1}{\exp(\mu)}\right\}.$$

Ohno and Urakawa [OU] showed

$$\underline{\lambda}^{\mathrm{ess}}\le\frac{\mu^2}{8}\exp(\mu)\,.$$

Though Ohno and Urakawa's upper bound grows exponentially, it is better than Dodziuk and Karp's for a smaller μ. See the Figure. An interesting corollary immediately follows from their results: if $\mu=0$, which is called *subexponential*, then $\underline{\lambda}^{\mathrm{ess}}=0$. Note that there is an infinite graph G with $\underline{\lambda}^{\mathrm{ess}}=0$ and $\mu>0$. The Cayley graph of a solvable group with exponential growth gives such an example [DK, Prop. 1.6].

We will show the following:

THEOREM 1. *If G is an infinite graph, then*

$$\underline{\lambda}^{\mathrm{ess}}\le 1-\frac{2\exp(\mu/2)}{1+\exp(\mu)}\,,$$

where we define the right hand side to be 1 *in the case $\mu=\infty$.*

REMARK. (1) For $\mu\ge 0$, we have

$$1-\frac{2\exp(\mu/2)}{1+\exp(\mu)}\le\frac{1}{8}\mu^2\exp(\mu)\,,$$

and

$$1-\frac{2\exp(\mu/2)}{1+\exp(\mu)}\le\min\left\{\frac{2(1-\exp(\mu/2))^2}{\exp(\mu)}\,,\ 1-\frac{1}{\exp(\mu)}\right\}.$$

The equality holds if and only if $\mu=0$ for both of the inequalities. See the Figure.

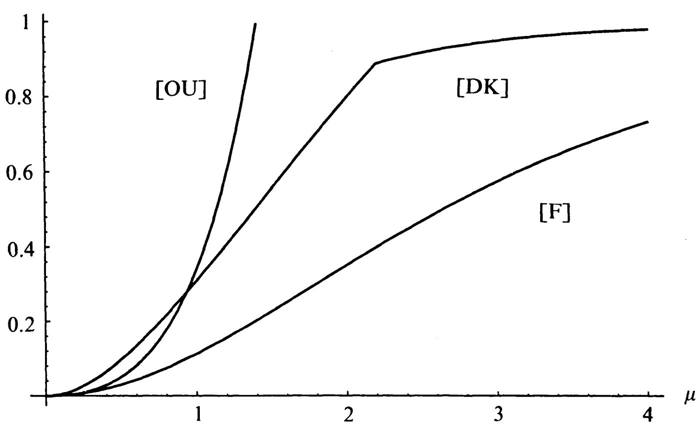

FIGURE.

(2) Our upper bound is best possible in the sense that the equality holds for regular trees. See the example below.

(3) From Theorem 1, $\underline{\lambda}^{\mathrm{ess}}=1$ implies $\mu=\infty$, but there exists an infinite graph such that $\mu=\infty$ and $\underline{\lambda}^{\mathrm{ess}}<1$.

EXAMPLE (cf. [B2], [Su], [OU]). Let T_d be a d-regular tree. Then $\mu(T_d)=\log(d-1)$, $\underline{\lambda}^{\mathrm{ess}}=1-2\sqrt{d-1}/d$, $\overline{\lambda}^{\mathrm{ess}}=1+2\sqrt{d-1}/d$ and $\alpha_\infty=(d-2)/d$. Therefore the equality holds in Theorem 1 for regular trees. Note that in Brooks' result [B1] for Riemannian manifolds, we have $\lambda_0^{\mathrm{ess}}\leq\mu^2/4$, with the equality holding for a simply connected hyperbolic space H^n in any dimension $n\geq 2$.

The upper bound in Theorem 1 is a proper analog of Brooks' result for Riemannian manifolds, though they have different forms. One reason for this is that the Laplacian for a Riemannian manifold is an unbounded operator, while the combinatorial one for a graph is bounded. The value 1 for the combinatorial Laplacian corresponds to the value $+\infty$ for the Riemannian Laplacian. Another supporting evidence for this interpretation can be found in [F2].

If G has no closed loop with odd length, we call it *bipartite* (cf. [DK]). It is known that a bipartite graph satisfies $2=\underline{\lambda}^{\mathrm{ess}}+\overline{\lambda}^{\mathrm{ess}}$. Hence we have:

COROLLARY 1. *If G is an infinite bipartite graph, then*

$$1+\frac{2\exp(\mu/2)}{1+\exp(\mu)}\leq\overline{\lambda}^{\mathrm{ess}},$$

where we define the left hand side to be 1 *in the case* $\mu=\infty$.

REMARK. By Corollary 1, for a bipartite graph, $\overline{\lambda}^{\mathrm{ess}}=1$ limplies $\mu=\infty$. Note that there exists an infinite bipartite graph such that $\mu=\infty$ and $\overline{\lambda}^{\mathrm{ess}}<1$. We may ask if there exists an infinite bipartite graph such that $\underline{\lambda}^{\mathrm{ess}}=1<\overline{\lambda}^{\mathrm{ess}}$ or $\underline{\lambda}^{\mathrm{ess}}<1=\overline{\lambda}^{\mathrm{ess}}$.

Let S be a finite set of vartices. Put $\partial S=\{(x,y)\mid x\notin S, y\in S, x\sim y\}$, $L(\partial S)=\sharp\partial S$, and $A(S)=\sum_{x\in S}m(x)$. The *isoperimetric constant* α is given by $\alpha(G)=\inf_S\{L(\partial S)/A(S)\mid\sharp S<\infty\}$. For a finite subset K of V, define $\alpha(G-K)=\inf_S\{L(\partial S)/A(S)\mid\sharp S<\infty, K\cap S=\varnothing\}$. The *isoperimetric constant at infinity* α_∞ is defined by

$$\alpha_\infty=\lim_K\alpha(G-K),$$

where K runs through finite subsets. We have $0\le\alpha\le\alpha_\infty\le 1$. It was shown in [F2] that if G is an infinite graph, then

$$1-\sqrt{1-\alpha_\infty^2}\le\underline{\lambda}^{\mathrm{ess}}\le\alpha_\infty\,,\qquad \overline{\lambda}^{\mathrm{ess}}\le 1+\sqrt{1-\alpha_\infty^2}\,.$$

Combining this with Theorem 1, we have:

COROLLARY 2. *If G is an infinite graph, then*

$$\alpha_\infty\le\frac{\exp(\mu)-1}{\exp(\mu)+1}\,.$$

REMARK. (1) By Corollary 2, if $\alpha_\infty=1$, then $\mu=\infty$. There exists a graph such that $\mu=\infty$ and $\alpha_\infty<1$. It is known that $\alpha_\infty=1$ if and only if $\mathrm{Ess\,Spec}(\Delta)=\{1\}$ (cf. [F2]).

(2) The equalities hold in Corollaries 1 and 2 for regular trees. See the example before.

THEOREM 2. *If G is an infinite graph, then*

$$\underline{\lambda}^{\mathrm{ess}}\le\left(\frac{1-\exp(\mu/2)}{1+\exp(\mu/2)}\right)^2\overline{\lambda}^{\mathrm{ess}}\,.$$

COROLLARY 3. *If G is an infinite graph, then*

$$\underline{\lambda}^{\mathrm{ess}}\le 1-\frac{2\exp(\mu/2)}{1+\exp(\mu)}-C_G\left(\frac{1-\exp(\mu/2)}{1+\exp(\mu/2)}\right)^2,$$

where $C_G=1+2\exp(\mu/2)/(1+\exp(\mu))-\overline{\lambda}^{\mathrm{ess}}$.

REMARK. If $C_G\le 0$, Theorem 1 gives an upper bound for $\underline{\lambda}^{\mathrm{ess}}$ better than or equal to Corollary 3. Note that by Corollary 1, $C_G\le 0$ for bipartite graphs. On the other hand, if $C_G>0$, then Corollary 3 is better than Theorem 1. But the author does not know an example of the graphs such that $C_G>0$.

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2. Definitions. Let $G=G(V,E)$ be a locally finite, infinite graph with the set of vertices V and the set of directed edges E. The Laplacian $\Delta=\Delta_G$ is given by

$$\Delta f(x) = \frac{1}{m(x)} \sum_{x \sim y} (f(x) - f(y)) .$$

Δ is a positive definite, self-adjoint operator on the space $L^2(V)$ of real-valued L^2-functions on V, with its natural L^2-structure:

$$L^2(V) = \left\{ f\colon V \to \boldsymbol{R} \,\middle|\, \sum_{x \in V} m(x) f^2(x) < \infty \right\},$$

$$(f, g) = \sum_{x \in V} m(x) f(x) g(x) .$$

Put

$$L^2(E) = \left\{ \phi : E \to \boldsymbol{R} \,|\, \phi([x, y]) = -\phi([y, x]) ,\ \sum_{e \in E} \phi^2(e) < \infty \right\},$$

$$(\phi, \psi) = \frac{1}{2} \sum_{e \in E} \phi(e) \psi(e) ,$$

where $[x, y]$ is an edge from x to y.

The coboundary operator $L^2(V) \to L^2(E)$ is $df([x, y]) = f(x) - f(y)$ and the adjoint operator δ of d is given by

$$\delta\phi(x) = \frac{1}{m(x)} \sum_{x \sim y} \phi([x, y]) .$$

We have $\Delta f = \delta df$ and $(\Delta f, g) = (df, dg)$ for $f, g \in L^2(V)$.

We denote by $\underline{\lambda}$ the infimum of the spectrum of Δ. It is given by the formula $\underline{\lambda} = \inf_f \{(\Delta f, f)/(f, f)\}$, where f runs over non-zero functions with finite support on V. We denote by $\underline{\lambda}^{\mathrm{ess}}$ the infimum of the essential spectrum of Δ. For a finite subgraph K of G, the Laplacian Δ_{G-K} on $G-K$ with the Dirichlet condition is given by

$$\Delta_{G-K} f(x) = \begin{cases} \Delta_G f(x) & \text{on } G-K \\ 0 & \text{on } K \end{cases}$$

for $f \in L^2(G-K) = \{f \in L^2(G), f|_K = 0\}$. Denote by $\underline{\lambda}(G-K)$ the infimum of the spectrum of Δ_{G-K} and $\overline{\lambda}(G-K)$ the supremum. Then a standard argument in the spectral theory shows that $\underline{\lambda}^{\mathrm{ess}}(G) = \lim_K \underline{\lambda}(G-K)$, $\overline{\lambda}^{\mathrm{ess}}(G) = \lim_K \overline{\lambda}(G-K)$, where K runs over all finite subsets of V.

Let us show $0 \le \underline{\lambda}^{\mathrm{ess}}(G) \le 1 \le \overline{\lambda}^{\mathrm{ess}}(G) \le 2$. Since $0 \le \Delta \le 2$, we have $0 \le \underline{\lambda}^{\mathrm{ess}} \le \overline{\lambda}^{\mathrm{ess}} \le 2$. It is enough to show $\underline{\lambda}(G-K) \le 1 \le \overline{\lambda}(G-K)$ for any finite subgraph K. We write

$$R(f) = \frac{(df, df)}{(f, f)}$$

for f with $(f,f)>0$. For a finite subset F, let χ_F be its characteristic function. Take a point $x\in G-K$. Since $R(\chi_x)=1$ and $\underline{\lambda}(G-K)\leq R(\chi_x)\leq\overline{\lambda}(G-K)$, we have $0\leq\underline{\lambda}^{\mathrm{ess}}\leq 1\leq\overline{\lambda}^{\mathrm{ess}}\leq 2$.

3. Proofs. We prove our result following Brooks' idea in [B1] (see also [OU]). For $x\in V$, let $\rho(x)$ denote the distance from the fixed point $x_0\in V$.

PROPOSITION 1. *If* $\mu<2\alpha$, *then* $(\exp(-\alpha\rho),\exp(-\alpha\rho))<\infty$.

PROOF.

$$\begin{aligned}(\exp(-\alpha\rho),\exp(-\alpha\rho))&=\sum_{x\in V}m(x)\exp(-2\alpha\rho(x))\\&=\sum_{r=0}^{\infty}\{V(r)-V(r-1)\}\exp(-2\alpha r)\\&=(1-\exp(-2\alpha))\sum_{r=0}^{\infty}V(r)\exp(-2\alpha r)<\infty\,.\end{aligned}$$

The sum in the last expression is finite by the definition of μ. Since $\mu<2\alpha$, we can take a number β with $\mu<\beta<2\alpha$. Then $V(r)\leq\exp(\beta r)$ for all sufficiently large r. q.e.d.

For $j\in\boldsymbol{N}$, put

$$h_j(x)=\begin{cases}\alpha\rho(x) & \text{if}\quad \rho(x)\leq j\\ 2\alpha j-\alpha\rho(x) & \text{if}\quad \rho(x)>j\end{cases}\tag{1}$$

and

$$f_j=\exp(h_j)\,.\tag{2}$$

PROPOSITION 2. *If* $x\sim y$, *then, for all* j,

$$(f_j(x)-f_j(y))^2\leq\frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}(f_j^2(x)+f_j^2(y))\,.$$

PROOF. Since $x\sim y$, we have $\rho(x)=\rho(y)$ or $|\rho(x)-\rho(y)|=1$. If $\rho(x)=\rho(y)$, then $f_j(x)-f_j(y)=0$, which implies the inequality we want to show. If $|\rho(x)-\rho(y)|=1$, we may assume $\rho(x)=\rho(y)-1$ without loss of generality. We divide our proof into two cases:

Case 1: $\rho(x)\leq j-1$. In this case we have $f_j(y)=\exp(\alpha)f_j(x)$. A straight-forward calculation shows

$$(f_j(x)-f_j(y))^2=\frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}(f_j^2(x)+f_j^2(y))\,.$$

Case 2: $\rho(x)\geq j$. In this case we have $f_j(y)=\exp(-\alpha)f_j(x)$ and we similarly have the same conclusion as in Case 1. q.e.d.

PROPOSITION 3. *For all* j,

$$(df_j, df_j) \leq \frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}(f_j, f_j)\,.$$

PROOF. From Proposition 2,

$$\begin{aligned}(df_j, df_j) &= \frac{1}{2}\sum_{x\sim y}(f_j(x)-f_j(y))^2 \leq \frac{1}{2}\,\frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}\sum_{x\sim y}(f_j^2(x)+f_j^2(y)) \\ &= \frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}\sum_{x\in V} m(x)f_j^2(x) = \frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}(f_j, f_j)\,.\end{aligned}$$

q.e.d.

Let K be a finite subset of V, and put

(3) $$g_j = f_j(1-\chi_K)\,.$$

PROPOSITION 4. *If* $(\exp(-\alpha\rho), \exp(-\alpha\rho)) < \infty$, *then, for all* j,

$$(g_j, g_j) < \infty\,, \qquad \lim_j (g_j, g_j) = \infty\,.$$

PROOF. We have

$$\begin{aligned}(g_j, g_j) &= \sum_{x\in B(j)} m(x)g_j^2(x) + \sum_{x\notin B(j)} m(x)g_j^2(x) \\ &\leq \exp(2\alpha j)V(j) + \exp(4\alpha j)(\exp(-\alpha\rho), \exp(-\alpha\rho)) < \infty\,,\end{aligned}$$

since $g_j(x) \leq \exp(\alpha j)$ for $x\in B(j)$ and $g_j(x) = \exp(2\alpha j)\exp(-\alpha\rho(x))$ for $x\notin B(j)$. Take and fix a number k satisfying $K\subset B(k)$. Then for any j with $j\geq k$, we have $1\leq g_j$ on $B(j)-B(k)$. Thus $(g_j, g_j)\geq V(j)-V(k)$. Since G is infinite, we get $\lim_j(V(j))=\infty$. Therefore, $\lim_j(g_j, g_j)=\infty$. q.e.d.

PROPOSITION 5. *Suppose a number* k *satisfies* $K\subset B(k)$. *Then for all* j,

$$(dg_j, dg_j) \leq C_1(k) + \frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}(g_j, g_j)\,,$$

where

$$C_1(k) = \exp(2\alpha(k+1))V(k+1)\left\{1+\frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}\right\}.$$

PROOF. Since $g_j = f_j$ on $G-B(k)$ and $0\leq g_j\leq f_j\leq \exp(\alpha(k+1))$ on $B(k+1)$,

$$(f_j, f_j) \leq (g_j, g_j) + \exp(2\alpha(k+1))V(k+1)\,.$$

By Proposition 3, we have

$$\begin{aligned}(dg_j, dg_j) &\leq V(k+1) \max_{x\in B(k+1)} \{g_j^2(x)\} + (df_j, df_j)\\ &\leq \exp(2\alpha(k+1))V(k+1) + \frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}(f_j, f_j)\\ &\leq \exp(2\alpha(k+1))V(k+1)\left\{1+\frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}\right\} + \frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}(g_j, g_j)\,.\end{aligned}$$

q.e.d.

PROOF OF THEOREM 1. Since $\underline{\lambda}^{\mathrm{ess}} \leq 1$, we may assume $\mu < \infty$. Suppose

$$\frac{(1-\exp(\mu/2))^2}{1+\exp(\mu)} < \underline{\lambda}^{\mathrm{ess}}(G)\,.$$

Then there exists a finite subset K with

$$\frac{(1-\exp(\mu/2))^2}{1+\exp(\mu)} < \underline{\lambda}(G-K)\,.$$

We fix K and define g_j applying this K to (3). Since $(1-\exp(\mu/2))^2/(1+\exp(\mu))$ is a monotone increasing function of μ, we can choose α in such a way that

$$\mu < 2\alpha\,, \qquad \frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)} < \underline{\lambda}(G-K)\,.$$

From Proposition 1, we have $(\exp(-\alpha\rho), \exp(-\alpha\rho)) < \infty$, and then from Proposition 4, we have $(g_j, g_j) < \infty$ and $\lim_j (g_j, g_j) = \infty$. From Proposition 5,

$$\frac{(dg_j, dg_j)}{(g_j, g_j)} \leq \frac{C_1(k)}{(g_j, g_j)} + \frac{(1-\exp(\alpha))^2}{1+\exp(2\alpha)}\,.$$

Since $(1-\exp(\alpha))^2/(1+\exp(2\alpha)) < \underline{\lambda}(G-K)$ and $\lim_j (g_j, g_j) = \infty$, we have

$$\frac{(dg_j, dg_j)}{(g_j, g_j)} < \underline{\lambda}(G-K)$$

for all sufficiently large j, which contradicts the definition of $\underline{\lambda}(G-K)$. Therefore we have

$$\underline{\lambda}^{\mathrm{ess}}(G) \leq \frac{(1-\exp(\mu/2))^2}{1+\exp(\mu)} = 1 - \frac{2\exp(\mu/2)}{1+\exp(\mu)}\,,$$

and obtain Theorem 1. q.e.d.

PROOF of COROLLARY 1. It is immediate since Theorem 1 and $\underline{\lambda}^{\mathrm{ess}} + \overline{\lambda}^{\mathrm{ess}} = 2$.

q.e.d.

PROOF OF COROLLARY 2. By $1-\sqrt{1-\alpha_\infty^2} \leq \underline{\lambda}^{\mathrm{ess}}$, we have $1-\sqrt{1-\alpha_\infty^2} \leq 1-2\exp(\mu/2)/(1+\exp(\mu))$, which yields

$$\alpha_\infty \leq \frac{\exp(\mu)-1}{\exp(\mu)+1}.$$

q.e.d.

Put

$$p_j(x)=f_j(x)(-1)^{\rho(x)}. \tag{4}$$

PROPOSITION 6. *If G is an infinite graph, then for all j,*

$$\frac{(df_j, df_j)}{(f_j, f_j)}=\left(\frac{1-\exp(\alpha)}{1+\exp(\alpha)}\right)^2 \frac{(dp_j, dp_j)}{(p_j, p_j)}.$$

PROOF. We put

$$C_2(\alpha)=\left(\frac{1-\exp(\alpha)}{1+\exp(\alpha)}\right)^2.$$

Obviously $(p_j, p_j)=(f_j, f_j)$. Thus we show $(df_j, df_j)=C_2(\alpha)(dp_j, dp_j)$ by showing the following: if $x \sim y$, then $(f_j(x)-f_j(y))^2=C_2(\alpha)(p_j(x)-p_j(y))^2$. Since $x \sim y$, we have $|\rho(x)-\rho(y)| \leq 1$. If $\rho(x)=\rho(y)$, then $p_j(x)=p_j(y)$ and $f_j(x)=f_j(y)$, which implies the equality we need to show. If $|\rho(x)-\rho(y)|=1$, we may assume $\rho(x)=\rho(y)-1$ without loss of generality as in the proof of Proposition 2. We divide our proof into two cases:

Case 1: $\rho(x) \leq j-1$. We have $f_j(y)=\exp(\alpha)f_j(x)$, $p_j(y)=-\exp(\alpha)p_j(x)$, and $f_j(x)=\pm p_j(x)$. Whichever the sign, direct computation shows the claim.

Case 2: $\rho(x) \geq j$. Similarly to Case 2 of Proposition 2, we have the claim. q.e.d.

PROOF OF THEOREM 2. Let $\varepsilon>0$. We can take a finite subgraph K such that

$$\underline{\lambda}^{\mathrm{ess}}(G)-\varepsilon \leq \underline{\lambda}(G-K), \qquad \bar{\lambda}(G-K) \leq \bar{\lambda}^{\mathrm{ess}}(G)+\varepsilon.$$

Take α to satisfy $\mu<2\alpha$ and define g_j and q_j by (1), …, (4),

$$g_j=f_j(1-\chi_K), \qquad q_j=p_j(1-\chi_K).$$

Since $g_j \in L^2(G-K)$, we have $\underline{\lambda}(G-K) \leq R(g_j)$. From the construction of f_j and g_j, we have $R(g_j) \leq R(f_j)+\varepsilon$ for large j. Thus $\underline{\lambda}(G-K)-\varepsilon \leq R(f_j)$ for large j. Similarly, we have $R(p_j) \leq \bar{\lambda}(G-K)+\varepsilon$ for large j. Therefore by Proposition 6,

$$\begin{aligned}\underline{\lambda}^{\mathrm{ess}}(G)-2\varepsilon &\leq \underline{\lambda}(G-K)-\varepsilon \leq R(f_j)=C_2(\alpha)R(p_j)\\ &\leq C_2(\alpha)(\bar{\lambda}(G-K)+\varepsilon) \leq C_2(\alpha)(\bar{\lambda}^{\mathrm{ess}}(G)+2\varepsilon).\end{aligned}$$

Since ε can be chosen arbitrarily small, we have $\underline{\lambda}^{\mathrm{ess}}(G) \leq C_2(\alpha)\bar{\lambda}^{\mathrm{ess}}(G)$. Hence we have $\underline{\lambda}^{\mathrm{ess}}(G) \leq C_2(\mu/2)\bar{\lambda}^{\mathrm{ess}}(G)$ since α can be chosen arbitrarily close to $\mu/2$. q.e.d.

PROOF OF COROLLARY 3. By Theorem 2,

$$\begin{aligned}\underline{\lambda}^{\mathrm{ess}} &\leq \left(\frac{1-\exp(\mu/2)}{1+\exp(\mu/2)}\right)^2 \overline{\lambda}^{\mathrm{ess}} \\ &= \left(\frac{1-\exp(\mu/2)}{1+\exp(\mu/2)}\right)^2 \left(1+\frac{2\exp(\mu/2)}{1+\exp(\mu)}-C_G\right) \\ &= 1-\frac{2\exp(\mu/2)}{1+\exp(\mu)}-C_G\left(\frac{1-\exp(\mu/2)}{1+\exp(\mu/2)}\right)^2 .\end{aligned}$$

q.e.d.

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