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A TYPE OF UNIQUENESS OF SOLUTIONS FOR THE DIRICHLET PROBLEM ON A CYLINDER

Dedicated to Professor Satoru Igari on his sixtieth birthday

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Abstract. The aim of this paper is to prove a type of uniqueness for the Dirichlet problem on a cylinder the special case of which is a strip in the plane. By defining generalized Poisson integrals with certain continuous functions on the boundary of a cylinder, we shall investigate the difference between them and harmonic functions having the same boundary value. Given any continuous function on the boundary of a cylinder, we shall also give a harmonic function with that function as the boundary value.

1. Introduction. Let $\boldsymbol{R}$ be the set of all real numbers. The boundary and the closure of a set S in the n-dimensional Euclidean space $\boldsymbol{R}^n$ $(n \geq 2)$ are denoted by ∂S and $\bar{S}$, respectively. Given a domain $G \subset \boldsymbol{R}^n$ and a continuous function g on ∂G, we say that h is a solution of the Dirichlet problem on G with g, if h is harmonic in G and

$$\lim_{P \in G, P \to Q} h(P) = g(Q)$$

for every $Q \in \partial G$. If G is a bounded domain and g is a bounded function on ∂G, then the existence of a solution of the Dirichlet problem and its uniqueness is completely known (see, e.g., [8, Theorem 5.21]). When G is the typical unbounded domain

$$\boldsymbol{T}_n = \{(X, y) \in \boldsymbol{R}^n ; X \in \boldsymbol{R}^{n-1}, y > 0\} ,$$

the solution of the Dirichlet problem on $\boldsymbol{T}_n$ with a continuous function on $\partial \boldsymbol{T}_n$ was given by using the (generalized) Poisson integral in Armitage [1], Finkelstein and Scheinberg [5] and Gardiner [6], etc. But the uniqueness of solutions was not much considered until Siegel [11] picked up this problem. Helms [9, p. 42 and p. 158] states that even if $g(X)$ is a bounded continuous function on $\partial \boldsymbol{T}_n$, the solution of the Dirichlet problem on $\boldsymbol{T}_n$ with g is not unique and to obtain the unique solution $H(P)$ $(P = (X, y) \in \boldsymbol{T}_n)$ we must specify the behavior of $H(P)$ as $y \to \infty$. After Siegel gave a type of uniqueness of solutions, Yoshida [16] proved the same result under less restricted conditions. All these results were extended in Yoshida and Miyamoto [17] to the case where G is a cone. Since $\boldsymbol{T}_n$ is regarded as a special cone, we can say that a cone is one of typical unbounded domains.

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There is another typical unbounded domain which is a cylinder

$$\Gamma_n(D) = D \times \boldsymbol{R}$$

with a bounded domain $D \subset \boldsymbol{R}^{n-1}$. The existence and the uniqueness of solutions of the Dirichlet problem on $\Gamma_n(D)$ with a continuous function on $\partial\Gamma_n(D)$ are worth inquiry. In this direction, Yoshida [15] proved the following Theorem A. To state it we need some preliminaries.

Consider the Dirichlet problem

$$\begin{aligned} (\Delta_{n-1} + \lambda) f &= 0 \quad \text{in } D \\ f &= 0 \quad \text{on } \partial D \end{aligned} \tag{1.1}$$

for a bounded domain $D \subset \boldsymbol{R}^{n-1}$ ($n \geq 2$), where $\Delta_1 = d^2/dx^2$. Let $\lambda(D, 1)$ be the least positive eigenvalue of (1.1) and $f_1^D(X)$ the normalized eigenfunction corresponding to $\lambda(D, 1)$. In order to make the subsequent consideration simpler, we put a strong assumption on D throughout this paper: If $n \geq 3$, then D is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ in $\boldsymbol{R}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (for example, see Gilberg and Trudinger [7, pp. 88–89] for the definition of $C^{2,\alpha}$-domains). Let $G_{\Gamma_n(D)}(P_1, P_2)$ be the Green function of $\Gamma_n(D)$ $(P_1, P_2 \in \Gamma_n(D))$ and $\partial G_{\Gamma_n(D)}(P, Q)/\partial v$ the differentiation at $Q \in \partial\Gamma_n(D)$ along the inward normal into $\Gamma_n(D)$ $(P \in \Gamma_n(D))$.

Given a function $F(X, y)$ on $\Gamma_n(D)$, we denote by $N(F)(y)$ the function of y defined by the integral

$$\int_D F(X, y) f_1^D(X) dX ,$$

where dX denotes the $(n-1)$-dimensional volume element. We write

$$\mu_0(N(F)) = \lim_{y \to \infty} \exp(-\sqrt{\lambda(D, 1)}y) N(F)(y)$$

and

$$\eta_0(N(F)) = \lim_{y \to -\infty} \exp(\sqrt{\lambda(D, 1)}y) N(F)(y) ,$$

if they exist.

THEOREM A (Yoshida [15, Theorem 6]). *Let $g(Q)$ be a continuous function on $\partial\Gamma_n(D)$ satisfying*

$$\int_{-\infty}^{\infty} \exp(-\sqrt{\lambda(D, 1)}\,|y|)\left(\int_{\partial D} |g(X, y)|\, d\sigma_X\right) dy < \infty , \tag{1.2}$$

where $d\sigma_X$ is the surface area element of ∂D at X and if $n = 2$ and $D = (\gamma, \delta)$, then

$$\int_{\partial D} |g(X, y)| d\sigma_X = |g(\gamma, y)| + |g(\delta, y)| .$$

Then the Poisson integral

$$PI_g(P) = c_n^{-1} \int_{\partial \Gamma_n(D)} g(Q) \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) d\sigma_Q$$

is a solution of the Dirichlet problem on $\Gamma_n(D)$ *with* g, *where*

$$c_n = \begin{cases} 2\pi & (n=2) \\ (n-2)s_n & (n \geq 3) \ (s_n \text{ is the surface area of the unit sphere } \boldsymbol{S}^{n-1}) \end{cases}$$

and $d\sigma_Q$ *is the surface area element on* $\partial\Gamma_n(D)$ *at* Q. *Let* $h(P)$ *be any solution of the Dirichlet problem on* $\Gamma_n(D)$ *with* g. *Then all of the limits* $\mu_0(N(h))$, $\eta_0(N(h))$ $(-\infty < \mu_0(N(h)), \eta_0(N(h)) \leq \infty)$, $\mu_0(N(|h|))$ *and* $\eta_0(N(|h|))$ $(0 \leq \mu_0(N(|h|)), \eta_0(N(|h|)) \leq \infty)$ *exist, and if*

(1.3) $$\mu_0(N(|h|)) < \infty \quad \text{and} \quad \eta_0(N(|h|)) < \infty ,$$

then

$$h(P) = PI_g(P) + (\mu_0(N(h)) \exp(\sqrt{\lambda(D, 1)}y) + \eta_0(N(h)) \exp(-\sqrt{\lambda(D, 1)}y)) f_1^D(X)$$

for any $P = (X, y) \in \Gamma_n(D)$.

This Theorem A shows that under the conditions (1.2) and (1.3) the existence and a type of uniqueness of solutions for the Dirichlet problem on $\Gamma_n(D)$ can be proved, respectively.

If $n=2$, then $\Gamma_n(D)$ is a strip. The strip $\Gamma_2((0, \pi))$ with $D=(0, \pi)$ is simply denoted by Γ_2. With respect to the Dirichlet problem on Γ_2, Widder obtained:

THEOREM B (Widder [13, Theorems 1 and 3]). *If* $g_i(t)$ $(i=1, 2)$ *is a continuous function on* $\boldsymbol{R}$ *satisfying*

(1.4) $$\int_{-\infty}^{\infty} |g_i(t)| \exp(-|t|) dt < \infty ,$$

then

$$H(\Gamma_2; g_1, g_2)(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(x, t-y) g_1(t) dt + \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\pi - x, t-y) g_2(t) dt$$

$$\left(P(x, y) = \frac{\sin x}{\cosh y - \cos x} \right)$$

is a harmonic function in Γ_2 *and a continuous function on* $\overline{\Gamma_2}$ *shuch that*

$$H(\Gamma_2; g_1, g_2)(0, y) = g_1(y) \quad \text{and} \quad H(\Gamma_2; g_1, g_2)(\pi, y) = g_2(y) \qquad (-\infty < y < \infty) .$$

If $h(x, y)$ *is a harmonic function in* Γ_2 *and a continuous function on* $\overline{\Gamma_2}$ *such that*

$$h(0, y)=g_1(y), \quad h(\pi, y)=g_2(y) \qquad (-\infty<y<\infty)$$

and

$$\int_0^{\pi} |h(x, y)|\, dx=o(e^{|y|}) \qquad (|y| \to \infty),$$

then

$$h(x, y)=H(\Gamma_2; g_1, g_2)(x, y) \qquad on \quad \overline{\Gamma_2}.$$

Though by a conformal mapping a strip is reduced to $\boldsymbol{T}_2$ which was treated in [17] as a special case, it may be of interest to treat this case independently as a special case of cylinders.

In this paper, the first parts of Theorems A and B will be extended by defining generalized Poisson integrals with continuous functions under less restricted conditions than (1.2) and (1.4) (Theorem 1 and Corollary 1). We shall also prove that for any continuous function g on $\partial\Gamma_n(D)$ there is a solution of the Dirichlet problem on $\Gamma_n(D)$ with g (Theorem 2 and Corollary 2). The results (Theorem 3 and Corollary 3) which generalize the second parts of Theorems A and B will be connected with a type of uniqueness of solutions for the Dirichlet problem on $\Gamma_n(D)$.

2. Statements of results. We denote the non-decreasing sequence of positive eigenvalues of (1.1) by $\{\lambda(D, k)\}_{k=1}^{\infty}$. In this expression we write $\lambda(D, k)$ the same number of times as the dimension of the corresponding eigenspace. When the normalized eigenfunction corresponding to $\lambda(D, k)$ is denoted by f_k^D, the set of sequential eigenfunctions corresponding to the same value of $\lambda(D, k)$ in the sequence $\{f_k^D\}_{k=1}^{\infty}$ makes an orthonormal basis for the eigenspace of the eigenvalue $\lambda(D, k)$. We can also say that for each $D\subset \boldsymbol{R}^{n-1}$ there is a sequence $\{k_i\}$ of positive integers such that $k_1=1$, $\lambda(D, k_i)<\lambda(D, k_{i+1})$

$$\lambda(D, k_i)=\lambda(D, k_i+1)=\lambda(D, k_i+2)=\cdots=\lambda(D, k_{i+1}-1)$$

and $\{f_{k_i}^D, f_{k_i+1}^D, \ldots, f_{k_{i+1}-1}^D\}$ is an orthonormal basis for the eigenspace of the eigenvalue $\lambda(D, k_i)$ $(i=1, 2, 3, \ldots)$. It is well known that $k_2=2$ and $f_1^D(X)>0$ for any $X\in D$ (see Courant and Hilbert [3, p. 451 and p. 458]). With respect to $\{k_i\}$, the following Example (2) shows that even in the case where D is an open disk in $\boldsymbol{R}^2$, not the simplest case $k_i=i$ $(i=1, 2, 3, \ldots)$, but more complicated cases can appear. When D has sufficiently smooth boundary, we know that

$$\lambda(D, k)\sim A(D, n)k^{2/(n-1)} \qquad (k\to\infty)$$

and

$$\sum_{\lambda(D,k)\leq x} \{f_k^D(X)\}^2 \sim B(D, n)x^{(n-1)/2} \qquad (x \to \infty)$$

uniformly with respect to $X \in D$, where $A(D, n)$ and $B(D, n)$ are both constants depending on D and n (see, e.g., Weyl [12] and Carleman [2]). Hence there exist two positive constants M_1, M_2 such that

$$M_1 k^{2/(n-1)} \leq \lambda(D, k) \qquad (k=1, 2, 3, \ldots) \tag{2.1}$$

and

$$|f_k^D(X)| \leq M_2 k^{1/2} \qquad (X \in D, k=1, 2, 3, \ldots) . \tag{2.2}$$

We remark that both

$$\exp(\sqrt{\lambda(D, k)}y) f_k^D(X) \quad \text{and} \quad \exp(-\sqrt{\lambda(D, k)}y) f_k^D(X) \qquad (k=1, 2, 3, \ldots)$$

are harmonic on $\Gamma_n(D)$ and vanish continuously on $\partial\Gamma_n(D)$.

For a domain D and the sequence $\{k_i\}$ mentioned above, by $I(D, k_i)$ we denote the set of all positive integers less than k_i $(i=1, 2, 3, \ldots)$. Even if $I(D, k_1)=\varnothing$, the summation over $I(D, k_1)$ of any function $S(k)$ of a variable k will be used to mean

$$\sum_{k \in I(D,k_1)} S(k)=0 .$$

EXAMPLES. (1) Let $D=(0, \pi)$. Then (1.1) is reduced to finding solutions $f(x)$ $(0 \leq x \leq \pi)$ such that

$$\frac{d^2 f(x)}{dx^2} + \lambda f(x)=0 \qquad (0<x<\pi)$$

and

$$f(0)=f(\pi)=0 .$$

It is easy to see that $k_i=i$, $\lambda(D, k)=k^2$ and $f_k^D(x)=\sqrt{2/\pi}\sin kx$ $(k=1, 2, 3, \ldots)$.

(2) *Let* $D=\{(x, y)\in \boldsymbol{R}^2;\ x^2+y^2<1\}$. Let $\{\alpha_{n,m}\}_{m=1}^{\infty}$ be an increasing sequence of positive real numbers $\alpha_{n,m}$ such that

$$J_n(\alpha_{n,m})=0 \qquad (n=0, 1, 2, \ldots) ,$$

where $J_n(z)$ is the Bessel function of order n. If the spherical coordinates $x=r\cos\theta$, $y=r\sin\theta$ $(0\leq r<1, 0\leq\theta<2\pi)$ are introduced, then $J_n(\alpha_{n,m}r)\cos n\theta$ and $J_n(\alpha_{n,m}r)\sin n\theta$ $(n\neq 0, m=1, 2, 3, \ldots)$ are two eigenfunctions coresponding to the eigenvalue $\lambda=\alpha_{n,m}^2$ (see Courant and Hilbert [3]). Since we do not know how the zeros of the Bessel functions distribute, we cannot explicitly determine the sequence $\{k_i\}$ with respect to this D.

The Fourier coefficient

$$\int_D F(X) f_k^D(X) dX$$

of a function $F(X)$ on D with respect to the orthonormal sequence $\{f_k^D(X)\}$ is denoted by $c(F, k)$, if it exists. Now we shall define generalized Poisson kernels. Let l and m be two non-negative integers. For two points $P=(X, y)\in\Gamma_n(D)$, $Q=(X^*, y^*)\in\partial\Gamma_n(D)$, we put

(2.3) $\bar{V}(\Gamma_n(D), l)(P, Q)$

$$=\sum_{k\in I(D,k_{l+1})} \exp(\sqrt{\lambda(D, k)})c((H_{X^*})_1, k) f_k^D(X)\exp(\sqrt{\lambda(D, k)}y)\exp(-\sqrt{\lambda(D, k)}y^*)$$

and

(2.4) $\underline{V}(\Gamma_n(D), m)(P, Q)$

$$=\sum_{k\in I(D,k_{m+1})} \exp(\sqrt{\lambda(D, k)})c((H_{X^*})_1, k) f_k^D(X)\exp(-\sqrt{\lambda(D, k)}y)\exp(\sqrt{\lambda(D, k)}y^*),$$

where

$$(H_{X^*})_1(X)=c_n^{-1}\frac{\partial}{\partial v}G_{\Gamma_n(D)}((X, 1), (X^*, 0)).$$

We remark that $\bar{V}(\Gamma_n(D), l)(P, Q)$ and $\underline{V}(\Gamma_n(D), m)(P, Q)$ are two harmonic functions of $P\in\Gamma_n(D)$ for any fixed $Q\in\partial\Gamma_n(D)$. We introduce two functions of $P\in\Gamma_n(D)$ and $Q=(X^*, y^*)\in\partial\Gamma_n(D)$

$$\bar{W}(\Gamma_n(D), l)(P, Q)=\begin{cases}\bar{V}(\Gamma_n(D), l)(P, Q) & (y^*\geq 0)\\ 0 & (y^*<0)\end{cases}$$

and

$$\underline{W}(\Gamma_n(D), m)(P, Q)=\begin{cases}\underline{V}(\Gamma_n(D), m)(P, Q) & (y^*\leq 0)\\ 0 & (y^*>0).\end{cases}$$

The Poisson kernel $K(\Gamma_n(D), l, m)(P, Q)$ with respect to $\Gamma_n(D)$ is defined by

$$K(\Gamma_n(D), l, m)(P, Q)=c_n^{-1}\frac{\partial}{\partial v}G_{\Gamma_n(D)}(P, Q)-\bar{W}(\Gamma_n(D), l)(P, Q)-\underline{W}(\Gamma_n(D), m)(P, Q).$$

We note

$$K(\Gamma_n(D), 0, 0)(P, Q)=c_n^{-1}\frac{\partial}{\partial v}G_{\Gamma_n(D)}(P, Q).$$

Let p, q be two non-negative integers and $I(y)$ a function on $\boldsymbol{R}$. The finite or infinite limits

$$\lim_{y\to\infty} \exp(-\sqrt{\lambda(D, k_{p+1})}y)I(y) \quad \text{and} \quad \lim_{y\to-\infty} \exp(\sqrt{\lambda(D, k_{q+1})}y)I(y)$$

are denoted by $\mu_p(I)$ and $\eta_q(I)$, respectively, when they exist.

THEOREM 1. *Let l, m be two non-negative integers and $g(Q)=g(X^*, y^*)$ a continuous function on $\partial\Gamma_n(D)$ satisfying*

$$\int^{\infty} \exp(-\sqrt{\lambda(D, k_{l+1})}y^*)\left(\int_{\partial D} |g(X^*, y^*)|\, d\sigma_{X^*}\right)dy^* < \infty \tag{2.5}$$

and

$$\int_{-\infty} \exp(\sqrt{\lambda(D, k_{m+1})}y^*)\left(\int_{\partial D} |g(X^*, y^*)|\, d\sigma_{X^*}\right)dy^* < \infty\,.$$

Then

$$H(\Gamma_n(D), l, m; g)(P) = \int_{\partial\Gamma_n(D)} g(Q)K(\Gamma_n(D), l, m)(P, Q)d\sigma_Q$$

is a solution of the Dirichlet problem on $\Gamma_n(D)$ with g satisfying

$$\mu_l(N(|H(\Gamma_n(D), l, m; g)|)) = \eta_m(N(|H(\Gamma_n(D), l, m; g)|)) = 0\,. \tag{2.6}$$

If $n=2$ and $D=(0, \pi)$, then we immediately obtain the following Corollary 1 which generalizes Theorem B.

COROLLARY 1. *Let l, m be two non-negative integers and let $g_1(y^*)$, $g_2(y^*)$ be two continuous functions on $\boldsymbol{R}$ satisfying*

$$\int^{\infty} |g_i(y^*)| \exp(-(l+1)y^*)dy^* < \infty \tag{2.7}$$

and

$$\int_{-\infty} |g_i(y^*)| \exp((m+1)y^*)dy^* < \infty \qquad (i=1, 2)\,.$$

Then

$$\begin{aligned} &H(\Gamma_2, l, m; g_1, g_2)(x, y) \\ &= \int_{-\infty}^{\infty} g_1(y^*)K(\Gamma_2, l, m)((x, y), (0, y^*))dy^* + \int_{-\infty}^{\infty} g_2(y^*)K(\Gamma_2, l, m)((x, y), (\pi, y^*))dy^* \end{aligned}$$

is a harmonic function in Γ_2 and a continuous function on $\overline{\Gamma_2}$ such that

$$H(\Gamma_2, l, m; g_1, g_2)(0, y^*) = g_1(y^*)$$

and

$$H(\Gamma_2, l, m; g_1, g_2)(\pi, y^*) = g_2(y^*) \qquad (-\infty < y^* < \infty).$$

To solve the Dirichlet problem on $\Gamma_n(D)$ with any function $g(Q)$ on $\partial\Gamma_n(D)$, we shall define another Poisson kernel. Let $\varphi(t)$ be any positive continuous function of $t \geq 0$ satisfying

$$\varphi(0) = \exp(-\sqrt{\lambda(D, 1)}).$$

For a domain $D \subset \boldsymbol{R}^{n-1}$ and the sequence $\{\lambda(D, k_i)\}$, denote the set

$$\{t \geq 0;\ \exp(-\sqrt{\lambda(D, k_i)}) = \varphi(t)\}$$

by $S(D, \varphi, i)$. Then $0 \in S(D, \varphi, 1)$. When there is an integer N such that $S(D, \varphi, N) \neq \emptyset$ and $S(D, \varphi, N+1) = \emptyset$, denote the set $\{i;\ 1 \leq i \leq N\}$ of integers by $J(D, \varphi)$. Otherwise, denote the set of all positive integers by $J(D, \varphi)$. Let $t(i) = t(D, \varphi, i)$ be the minimum of elements t in $S(D, \varphi, i)$ for each $i \in J(D, \varphi)$. In the former case, we put $t(N+1) = \infty$. Then $t(1) = 0$. We define $\overline{W}(\Gamma_n(D), \varphi)(P, Q)$ $(P \in \Gamma_n(D),\ Q = (X^*, y^*) \in \partial\Gamma_n(D))$ by

$$\overline{W}(\Gamma_n(D), \varphi)(P, Q) = \begin{cases} 0 & (y^* < 0) \\ \overline{V}(\Gamma_n(D), i)(P, Q) & (t(i) \leq y^* < t(i+1),\ i \in J(D, \varphi)). \end{cases}$$

We also define $\underline{W}(\Gamma_n(D), \varphi)(P, Q)$ $(P \in \Gamma_n(D),\ Q = (X^*, y^*) \in \partial\Gamma_n(D))$ by

$$\underline{W}(\Gamma_n(D), \varphi)(P, Q) = \begin{cases} 0 & (y^* > 0) \\ \underline{V}(\Gamma_n(D), i)(P, Q) & (-t(i+1) < y^* \leq -t(i),\ i \in J(D, \varphi)). \end{cases}$$

The Poisson kernel $K(\Gamma_n(D), \varphi)(P, Q)$ $(P \in \Gamma_n(D),\ Q \in \partial\Gamma_n(D))$ is defined by

$$K(\Gamma_n(D), \varphi)(P, Q) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) - \overline{W}(\Gamma_n(D), \varphi)(P, Q) - \underline{W}(\Gamma_n(D), \varphi)(P, Q).$$

Now we have:

THEOREM 2. *Let $g(Q)$ be any continuous function on $\partial\Gamma_n(D)$. Then there is a positive continuous function $\varphi(t)$ of $t \geq 0$ depending on g such that*

$$H(\Gamma_n(D), \varphi; g)(P) = \int_{\partial\Gamma_n(D)} g(Q) K(\Gamma_n(D), \varphi)(P, Q)\, d\sigma_Q$$

is a solution of the Dirichlet problem on $\Gamma_n(D)$ with g.

If we take $n = 2$ and $D = (0, \pi)$ in Theorem 2, we obtain:

COROLLARY 2. *Let $g_1(y^*)$ and $g_2(y^*)$ be two continuous functions on $\boldsymbol{R}$. Then there is a positive continuous function $\varphi(t)$ of $t \geq 0$ depending on g_1 and g_2 such that*

$$H(\Gamma_2, \varphi; g_1, g_2)(x, y)$$
$$= \int_{-\infty}^{\infty} g_1(y^*)K(\Gamma_2, \varphi)((x, y), (0, y^*))dy^* + \int_{-\infty}^{\infty} g_2(y^*)K(\Gamma_2, \varphi)((x, y), (\pi, y^*))dy^*$$

is a harmonic function in Γ_2 *and a continuous function on* $\overline{\Gamma_2}$ *satisfying*

$$H(\Gamma_2, \varphi; g_1, g_2)(0, y^*) = g_1(y^*)$$

and

$$H(\Gamma_2, \varphi; g_1, g_2)(\pi, y^*) = g_2(y^*) \qquad (-\infty < y^* < \infty).$$

THEOREM 3. *Let* l, m *be two non-negative integers and let* p, q *be two positive integers satisfying* $p \geq l$, $q \geq m$. *Let* $g(X^*, y^*)$ *be a continuous function on* $\partial\Gamma_n(D)$ *satisfying* (2.5). *If* $h(X, y)$ *is a solution of the Dirichlet problem on* $\Gamma_n(D)$ *with* g *satisfying*

$$\mu_p(N(h^+)) = 0 \quad \text{and} \quad \eta_q(N(h^+)) = 0, \tag{2.8}$$

then

$$h(X, y) = H(\Gamma_n(D), l, m; g)(P)$$
$$+ \sum_{k \in I(D, k_{p+1})} A_k(h) \exp(\sqrt{\lambda(D, k)}y) f_k^D(X) + \sum_{k \in I(D, k_{q+1})} B_k(h) \exp(-\sqrt{\lambda(D, k)}y) f_k^D(X)$$

for every $P = (X, y) \in \Gamma_n(D)$, *where* $A_k(h)$ $(k = 1, 2, \ldots, k_{p+1} - 1)$ *and* $B_k(h)$ $(k = 1, 2, \ldots, k_{q+1} - 1)$ *are all constants.*

If we take $n = 2$ and $D = (0, \pi)$ in Theorem 3, then we have:

COROLLARY 3. *Let* l, m *be two non-negative integers and let* p, q *be two positive integers satisfying* $p \geq l$, $q \geq m$. *Let* $g_1(y^*)$, $g_2(y^*)$ *be two continuous function on* $\boldsymbol{R}$ *satisfying* (2.7). *If* $h(x, y)$ *is a harmonic function in* Γ_2 *and a continuous function on* $\overline{\Gamma_2}$ *such that*

$$h(0, y^*) = g_1(y^*) \quad \text{and} \quad h(\pi, y^*) = g_2(y^*) \qquad (-\infty < y^* < \infty),$$

and

$$\lim_{y \to \infty} \exp(-(p+1)y) \int_0^\pi h^+(x, y) \sin x dx = \lim_{y \to -\infty} \exp((q+1)y) \int_0^\pi h^+(x, y) \sin x dx = 0,$$

then

$$h(x, y) = H(\Gamma_2, l, m; g_1, g_2)(x, y) + \sum_{k=1}^{p} A_k(h) \exp(ky) \sin kx + \sum_{k=1}^{q} B_k(h) \exp(-ky) \sin kx$$

for every $(x, y) \in \Gamma_2$, *where* $A_k(h)$ $(k = 1, 2, \ldots, p)$ *and* $B_k(h)$ $(k = 1, 2, \ldots, q)$ *are all constants.*

3. Proof of Theorems 1, 2 and 3. Given a domain D on $\boldsymbol{R}^{n-1}$ and an interval $I \subset \boldsymbol{R}$, the sets $\{(X, y) \in \boldsymbol{R}^n; X \in D, y \in I\}$ and $\{(X^*, y) \in \boldsymbol{R}^n; X^* \in \partial D, y \in I\}$ are denoted by $\Gamma_n(D; I)$ and $S_n(D; I)$, respectively. In the following, $S_n(D; (-\infty, \infty))$ $(=\partial\Gamma_n(D))$ will be simply denoted by $S_n(D)$.

LEMMA 1. *Let $h(X, y)$ be a harmonic function in $\Gamma_n(D; (0, \infty))$ vanishing continuously on $S_n(D; (0, \infty))$. For any fixed y, $0<y<\infty$, define the function $h_y(X)$ in D by $h_y(X)= h(X, y)$. Then*

$$\begin{aligned} c(h_y, k) = \{&(\exp(\sqrt{\lambda(D, k)}(y-y_2)) - \exp(\sqrt{\lambda(D, k)}(y_2-y)))c(h_{y_1}, k) \\ &+ (\exp(\sqrt{\lambda(D, k)}(y_1-y)) - \exp(\sqrt{\lambda(D, k)}(y-y_1)))c(h_{y_2}, k)\} \\ &\times \{\exp(\sqrt{\lambda(D, k)}(y_1-y_2)) - \exp(\sqrt{\lambda(D, k)}(y_2-y_1))\}^{-1} \end{aligned}$$

for any given y_1, y_2 $(0<y_1<y_2<\infty)$ and

$$\lim_{y\to\infty} c(h_y, k)\exp(-\sqrt{\lambda(D, k)}y)$$

exists $(k=1, 2, 3, \dots)$.

PROOF. First of all, we note that $h(X, y)$ is continuously differentiable twice on $\{(X, y)\in \boldsymbol{R}^n; X\in \bar{D}, 0<y<\infty\}$ (see Gilbarg and Trudinger [7, p. 105]). Now, by differentiating twice under the integral sign, we have

$$\frac{\partial^2 c(h_y, k)}{\partial y^2} = \int_D \frac{\partial^2 h_y(X)}{\partial y^2} f_k^D(X)dX = -\int_D \Delta_{n-1}h_y(X)f_k^D(X)dX .$$

Hence, if we observe from the formula of Green that

$$\int_D (\Delta_{n-1}h_y(X))f_k^D(X)dX = \int_D h_y(X)(\Delta_{n-1}f_k^D(X))dX ,$$

we see that

$$\frac{\partial^2 c(h_y, k)}{\partial y^2} = \lambda(D, k)c(h_y, k)$$

for any y, $0<y<\infty$. This gives

$$c(h_y, k) = A_k(h)\exp(\sqrt{\lambda(D, k)}y) + B_k(h)\exp(-\sqrt{\lambda(D, k)}y) \qquad (0<y<\infty) ,$$

$A_k(h)$ and $B_k(h)$ being constants independent of y. Since $c(h_y, k)$ takes a value $c(h_{y_i}, k)$ at a point y_i $(i=1, 2)$, the conclusion of Lemma 1 follows immediately.

LEMMA 2. *Let $H(X, y)$ be a harmonic function in $\Gamma_n(D; (0, \infty))$ such that $H(X, y)$ vanishes continuously on $S_n(D; (0, \infty))$ and converges uniformly to zero as $y\to\infty$. Then for any non-negative integer j we have*

$$|H(X, y)-\sum_{k\in I(D,k_{j+1})}\exp(\sqrt{\lambda(D, k)}(1-y))c(H_1, k)f_k^D(X)|$$
$$\leq L_1(H)\exp(\sqrt{\lambda(D, k_{j+1})}(1-y)) \qquad (1<y<\infty),$$

where $H_1(X)=H(X, 1)$ and $L_1(H)$ is a constant dependent only on H.

PROOF. Put $H_y(X)=H(X, y)$ for any fixed y $(0<y<\infty)$. We see from Lemma 1 that

$$\begin{aligned}c(H_y, k)=\{&(\exp(\sqrt{\lambda(D, k)}(y-y_2))-\exp(\sqrt{\lambda(D, k)}(y_2-y)))c(H_{y_1}, k)\\ &+(\exp(\sqrt{\lambda(D, k)}(y_1-y))-\exp(\sqrt{\lambda(D, k)}(y-y_1)))c(H_{y_2}, k)\}\\ &\times\{\exp(\sqrt{\lambda(D, k)}(y_1-y_2))-\exp(\sqrt{\lambda(D, k)}(y_2-y_1))\}^{-1}\end{aligned}$$

for any y_1 and y_2 $(0<y_1<y_2<\infty)$. Since $c(H_{y_2}, k)\to 0$ $(y_2\to\infty)$ from the assumption, we obtain

$$c(H_y, k)=\exp(\sqrt{\lambda(D, k)}(y_1-y))c(H_{y_1}, k) \qquad (0<y_1<\infty). \tag{3.1}$$

Here we have from (2.2) that

$$|c(H_{y_1}, k)|\leq\int_D |H_{y_1}(X)f_k^D(X)|\,dX\leq M_2k^{1/2}|D|\max_{X\in D}|H(X, y_1)|, \tag{3.2}$$

where $|D|$ is the volume of D. It follows from (2.1), (2.2), (3.1) and (3.2) that

$$\begin{aligned}&\sum_{k=1}^{\infty}|c(H_y, k)f_k^D(X)|\\ &\leq M_2^2|D|\max_{X\in D}|H(X, y_1)|\sum_{k=1}^{\infty}k\exp(\sqrt{M_1}k^{1/(n-1)}(y_1-y)) \qquad (y_1<y).\end{aligned} \tag{3.3}$$

Hence, if we take a number y_1 satisfying $0<y_1<y$, then we know from (3.3) and the completeness of the orthonormal sequence $\{f_k^D(X)\}$ that

$$\sum_{k=1}^{\infty}c(H_y, k)f_k^D(X)=H(X, y) \tag{3.4}$$

for any $X\in D$.

If we put

$$L_1(H)=M_2^2|D|\max_{X\in D}\left|H\left(X, \frac{1}{2}\right)\right|\sum_{k=1}^{\infty}k\exp\left(-\frac{1}{2}\sqrt{M_1}k^{1/(n-1)}\right)$$

and take $y=1$, $y_1=1/2$ in (3.3), then we obtain from (3.3) that

$$\sum_{k=1}^{\infty}|c(H_1, k)||f_k^D(X)|\leq L_1(H). \tag{3.5}$$

If $1<y<\infty$, then by taking $y_1=1$ in (3.1) we have from (3.4) and (3.5) that

$$\left| H(X, y) - \sum_{k \in I(D, k_{j+1})} \exp(\sqrt{\lambda(D, k)}(1-y)) c(H_1, k) f_k^D(X) \right|$$

$$= \left| H(X, y) - \sum_{k \in I(D, k_{j+1})} c(H_y, k) f_k^D(X) \right|$$

$$= \left| \sum_{k=k_{j+1}}^{\infty} c(H_y, k) f_k^D(X) \right| \leq \sum_{k=k_{j+1}}^{\infty} \exp(\sqrt{\lambda(D, k)}(1-y)) \, | c(H_1, k) f_k^D(X) |$$

$$\leq \exp(\sqrt{\lambda(D, k_{j+1})}(1-y)) \sum_{k=1}^{\infty} | c(H_1, k) f_k^D(X) | \leq L_1(H) \exp(\sqrt{\lambda(D, k_{j+1})}(1-y)),$$

which gives the conclusion.

LEMMA 3. *For a non-negative integer l* (*resp. m*) *we have*

$$\left| c_n^{-1} \frac{\partial}{\partial v} G_{\Gamma_n(D)}(P, Q) - \bar{V}(\Gamma_n(D), l)(P, Q) \right| \leq \bar{L}_1 \exp(-\sqrt{\lambda(D, k_{l+1})}(y^* - y))$$

$$\left(resp. \left| c_n^{-1} \frac{\partial}{\partial v} G_{\Gamma_n(D)}(P, Q) - \underline{V}(\Gamma_n(D), m)(P, Q) \right| \leq \underline{L}_1 \exp(-\sqrt{\lambda(D, k_{m+1})}(y - y^*)) \right)$$

for any $P=(X, y) \in \Gamma_n(D)$ and $Q=(X^*, y^*) \in S_n(D)$ satisfying $y^* - y > 1$ (*resp. $y - y^* > 1$*), *where $\bar{L}_1$* (*resp. $\underline{L}_1$*) *is a constant independent of P and Q.*

PROOF. Since

$$G_{\Gamma_n(D)}((X, y), (X', y')) = G_{\Gamma_n(D)}((X, y-y'), (X', 0)) \qquad ((X, y), (X', y') \in \Gamma_n(D)),$$

it is easy to see that

$$\frac{\partial}{\partial v} G_{\Gamma_n(D)}((X, y), (X^*, y^*)) = \frac{\partial}{\partial v} G_{\Gamma_n(D)}((X, |y - y^*|), (X^*, 0)). \tag{3.6}$$

We remark that

$$H_{X^*}(X, y') = c_n^{-1} \frac{\partial}{\partial v} G_{\Gamma_n(D)}((X, y'), (X^*, 0))$$

is a harmonic function of $(X, y') \in \Gamma_n(D)$ such that H_{X^*} vanishes continuously on $S_n(D) - \{(X^*, 0)\}$ and tends uniformly to zero as $y' \to \infty$ (see [15, p. 394]). If we apply Lemma 2 to $H_{X^*}(X, y')$ and put $y' = y^* - y$ (resp. $y' = y - y^*$),

$$\bar{L}_1 = \exp(\sqrt{\lambda(D, k_{l+1})}) \max_{X^* \in \partial D} L_1(H_{X^*}) \quad (\text{resp. } \underline{L}_1 = \exp(\sqrt{\lambda(D, k_{m+1})}) \max_{X^* \in \partial D} L_1(H_{X^*})),$$

then we obtain the conclusion from (3.6) and (2.3) (resp. (2.4)).

LEMMA 4. *Let $\varphi(t)$ be a positive continuous function of $t \geq 0$ satisfying $\varphi(0) = \exp(-\sqrt{\lambda(D, 1)})$ and put $L'_1 = \max_{X^* \in \partial D} L_1(H_{X^*})$. Then*

$$\left| c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) - \overline{W}(\Gamma_n(D), \varphi)(P, Q) \right| < L'_1 \varphi(y^*)$$

$$\left(\textit{resp.} \left| c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) - \underline{W}(\Gamma_n(D), \varphi)(P, Q) \right| < L'_1 \varphi(-y^*) \right)$$

for any $P = (X, y) \in \Gamma_n(D)$ and $Q = (X^*, y^*) \in S_n(D)$ satisfying

$$y^* > \max(0, y+2) \quad (\textit{resp. } y^* < \min(0, y-2)) . \tag{3.7}$$

PROOF. Take any $P = (X, y) \in \Gamma_n(D)$ and $Q = (X^*, y^*) \in S_n(D)$ satisfying (3.7). Choose an integer $i = i(P, Q) \in J(D, \varphi)$ such that

$$t(i) \leq y^* < t(i+1) \quad (\text{resp. } -t(i+1) < y^* \leq -t(i)) . \tag{3.8}$$

Then

$$\overline{W}(\Gamma_n(D), \varphi)(P, Q) = \overline{V}(\Gamma_n(D), i)(P, Q) .$$

$$(\text{resp. } \underline{W}(\Gamma_n(D), \varphi)(P, Q) = \underline{V}(\Gamma_n(D), i)(P, Q)) .$$

Hence we have from Lemma 3, (3.7) and (3.8) that

$$\left| c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) - \overline{W}(\Gamma_n(D), \varphi)(P, Q) \right|$$

$$\leq L'_1 \exp(-\sqrt{\lambda(D, k_{i+1})}(y^* - y)) < L'_1 \varphi(y^*)$$

$$\left(\text{resp. } \left| c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) - \underline{W}(\Gamma_n(D), \varphi)(P, Q) \right| \right.$$

$$\left. \leq L'_1 \exp(-\sqrt{\lambda(D, k_{i+1})}(y - y^*)) < L'_1 \varphi(-y^*) \right)$$

which is the conclusion.

LEMMA 5. *Let $g(Q)$ be locally integrable and upper semicontinuous on $S_n(D)$. Let $W(P, Q)$ be a function of $P \in \Gamma_n(D)$, $Q \in S_n(D)$ such that for any fixed $P \in \Gamma_n(D)$ the function $W(P, Q)$ of $Q \in S_n(D)$ is a locally integrable function on $S_n(D)$. Put*

$$K(P, Q) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) - W(P, Q) \quad (P \in \Gamma_n(D), Q \in S_n(D)) .$$

Suppose that the following (I) *and* (II) *are satisfied*:

(I) *For any $Q^* \in S_n(D)$ and any $\varepsilon > 0$, there exist a neighbourhood $U(Q^*)$ of Q^* in $\boldsymbol{R}^n$*

and two numbers Y_1^*, Y_2^* $(-\infty < Y_1^* < Y_2^* < \infty)$ *such that*

$$\int_{S_n(D;(Y_2^*,\infty))} |g(Q)K(P,Q)|\,d\sigma_Q < \varepsilon$$

and

$$\int_{S_n(D;(-\infty,Y_1^*))} |g(Q)K(P,Q)|\,d\sigma_Q < \varepsilon$$

for any $P=(X,y)\in\Gamma_n(D)\cap U(Q^*)$.

(II) *For any* $Q^*\in S_n(D)$ *and any two numbers* Y_1, Y_2 $(-\infty < Y_1 < Y_2 < \infty)$,

$$\limsup_{P\to Q^*, P\in\Gamma_n(D)} \int_{S_n(D;(Y_1,Y_2))} |g(Q)W(P,Q)|\,d\sigma_Q = 0\,.$$

Then

$$\limsup_{P\to Q^*, P\in\Gamma_n(D)} \int_{S_n(D)} g(Q)K(P,Q)\,d\sigma_Q \le g(Q^*)$$

for any $Q^*\in S_n(D)$.

PROOF. Let $Q^*=(X^*,y^*)$ be any fixed point of $S_n(D)$ and let ε be any positive number. Choose two numbers Y_1^*, Y_2^* $(-\infty < Y_1^* < y^* < Y_2^* < \infty)$ and a neighbourhood $U(Q^*)$ from (I) such that

$$\int_{S_n(D;(Y_2^*,\infty))} |g(Q)K(P,Q)|\,d\sigma_Q < \varepsilon/4\,, \tag{3.9}$$

$$\int_{S_n(D;(-\infty,Y_1^*))} |g(Q)K(P,Q)|\,d\sigma_Q < \varepsilon/4$$

for any $P=(X,y)\in\Gamma_n(D)\cap U(Q^*)$. Let Φ be a continuous function on $S_n(D)$ such that $0\le\Phi\le 1$ and

$$\Phi=\begin{cases} 1 & \text{on} \quad S_n(D;[Y_1^*,Y_2^*]) \\ 0 & \text{on} \quad S_n(D;[Y_2^*+1,\infty))\cup S_n(D;(-\infty,Y_1^*-1])\,. \end{cases}$$

Let $G^j_{\Gamma_n(D)}(P,Q)$ be the Green function of $\Gamma_n(D;(-j,j))$ (j is a positive integer). Since the positive harmonic function $\Pi_j(P,Q)=G_{\Gamma_n(D)}(P,Q)-G^j_{\Gamma_n(D)}(P,Q)$ converges monotonically to 0 on $\Gamma_n(D;(-j,j))$ as $j\to\infty$, we can find an integer j^*, $-j^*<Y_1^*-1$ and $j^*>Y_2^*+1$ such that

$$c_n^{-1}\int_{S_n(D;(Y_1^*-1,Y_2^*+1))} |\Phi(Q)g(Q)|\left|\frac{\partial}{\partial\nu}\Pi_{j^*}(P,Q)\right|d\sigma_Q < \varepsilon/4 \tag{3.10}$$

for any $P=(X,y)\in\Gamma_n(D)\cap U(Q^*)$. Thus we have from (3.9) and (3.10) that

$$(3.11)\qquad \int_{S_n(D)} g(Q)K(P,Q)d\sigma_Q \le c_n^{-1}\int_{S_n(D;(Y_1^*-1,Y_2^*+1))} \Phi(Q)g(Q)\frac{\partial}{\partial\nu}G^{j^*}_{\Gamma_n(D)}(P,Q)d\sigma_Q$$

$$+c_n^{-1}\int_{S_n(D;(Y_1^*-1,Y_2^*+1))}\left|\Phi(Q)g(Q)\frac{\partial}{\partial\nu}\Pi_{j^*}(P,Q)\right|d\sigma_Q$$

$$+\int_{S_n(D;(Y_1^*-1,Y_2^*+1))}|g(Q)W(P,Q)|d\sigma_Q$$

$$+2\int_{S_n(D;(Y_2^*,\infty))}|g(Q)K(P,Q)|\,d\sigma_Q+2\int_{S_n(D;(-\infty,Y_1^*))}|g(Q)K(P,Q)|\,d\sigma_Q$$

$$\le c_n^{-1}\int_{S_n(D;(Y_1^*-1,Y_2^*+1))}\Phi(Q)g(Q)\frac{\partial}{\partial\nu}G^{j^*}_{\Gamma_n(D)}(P,Q)d\sigma_Q$$

$$+\int_{S_n(D;(Y_1^*-1,Y_2^*+1))}|g(Q)W(P,Q)|\,d\sigma_Q+\frac{5}{4}\varepsilon$$

for any $P=(X,y)\in\Gamma_n(D)\cap U(Q^*)$. Consider the Perron-Wiener-Brelot solution $H_V(P;\Gamma_n(D;(-j^*,j^*))$ of the Dirichlet problem on $\Gamma_n(D;(-j^*,j^*))$ with the upper semicontinuous function

$$V(Q)=\begin{cases}\Phi(Q)g(Q) & \text{on}\quad S_n(D;[Y_1^*-1,\ Y_2^*+1])\\ 0 & \text{on}\quad \partial\Gamma_n(D;(-j^*,j^*))-S_n(D;[Y_1^*-1,\ Y_2^*+1])\end{cases}$$

on $\partial\Gamma_n(D;(-j^*,j^*))$. Then we know that

$$c_n^{-1}\int_{S_n(D;(Y_1^*-1,Y_2^*+1))}\Phi(Q)g(Q)\frac{\partial}{\partial\nu}G^{J^*}_{\Gamma_n(D)}(P,Q)d\sigma_Q=H_V(P;\Gamma_n(D;(-j^*,j^*)))$$

(see Dahlberg [4, Theorem 3]) and that

$$\limsup_{P\in\Gamma_n(D),P\to Q^*} H_V(P;\Gamma_n(D;(-j^*,j^*)))\le \limsup_{Q\in S_n(D),Q\to Q^*} V(Q)=g(Q^*)$$

(see, e.g., Helms [9, Lemma 8.20]). Hence we obtain

$$\limsup_{P\in\Gamma_n(D),P\to Q^*} c_n^{-1}\int_{S_n(D;(Y_1^*-1,Y_2^*+1))}\Phi(Q)g(Q)\frac{\partial}{\partial\nu}G^{j^*}_{\Gamma_n(D)}(P,Q)d\sigma_Q\le g(Q^*)\,.$$

With (3.11) and (II) this gives the conclusion.

LEMMA 6 (Miyamoto [10, Theorem 2]). *Let p, q be two positive integers and $h(X,y)$ a harmonic function in $\Gamma_n(D)$ vanishing continuously on $S_n(D)$. If h satisfies*

$$(3.12)\qquad \mu_p(N(h^+))=0 \quad\textit{and}\quad \eta_q(N(h^+))=0\,,$$

then

$$h(X, y)=\sum_{k=1}^{k_{p+1}-1} A_k(h)\exp(\sqrt{\lambda(D, k)}y)f_k^D(X)+\sum_{k=1}^{k_{q+1}-1} B_k(h)\exp(-\sqrt{\lambda(D, k)}y)f_k^D(X)$$

for every $(X, y)\in \Gamma_n(D)$, *where* $A_k(h)$ $(k=1, 2,\dots, k_{p+1}-1)$ *and* $B_k(h)$ $(k=1, 2,\dots, k_{q+1}-1)$ *are all constants.*

PROOF OF THEOREM 1. First of all, we shall show that $H(\Gamma_n(D), l, m; g)(P)$ is a harmonic function on $\Gamma_n(D)$. For any fixed $P=(X, y)\in\Gamma_n(D)$, take two numbers Y_1 and Y_2 satisfying $Y_2>\max(0, y+1)$ and $Y_1<\min(0, y-1)$. Then

$$(3.13)\quad \int_{S_n(D;(Y_2,+\infty))} |g(Q)||K(\Gamma_n(D), l, m)(P, Q)|d\sigma_Q$$
$$=\int_{S_n(D;(Y_2,+\infty))} |g(Q)|\left|c_n^{-1}\frac{\partial}{\partial\nu}G_{\Gamma_n(D)}(P, Q)-\bar{V}(\Gamma_n(D), l)(P, Q)\right|d\sigma_Q$$
$$\leq \bar{L}_1\exp(\sqrt{\lambda(D, k_{l+1})}y)\int_{Y_2}^{\infty}\exp(-\sqrt{\lambda(D, k_{l+1})}y^*)\left(\int_{\partial D}|g(X^*, y^*)|d\sigma_{X^*}\right)dy^*<\infty$$

and

$$(3.14)\quad \int_{S_n(D;(-\infty,Y_1))} |g(Q)||K(\Gamma_n(D), l, m)(P, Q)|d\sigma_Q$$
$$=\int_{S_n(D;(-\infty,Y_1))} |g(Q)|\left|c_n^{-1}\frac{\partial}{\partial\nu}G_{\Gamma_n(D)}(P, Q)-\underline{V}(\Gamma_n(D), m)(P, Q)\right|d\sigma_Q$$
$$\leq \underline{L}_1\exp(-\sqrt{\lambda(D, k_{m+1})}y)\int_{-\infty}^{Y_1}\exp(\sqrt{\lambda(D, k_{m+1})}y^*)\left(\int_{\partial D}|g(X^*, y^*)|d\sigma_{X^*}\right)dy^*<\infty$$

from Lemma 3 and (2.5). Thus $H(\Gamma_n(D), l, m; g)(P)$ is finite for any $P\in\Gamma_n(D)$. Since $K(\Gamma_n(D), l, m; g)(P, Q)$ is a harmonic function of $P\in\Gamma_n(D)$ for any $Q\in S_n(D)$, $H(\Gamma_n(D), l, m; g)(P)$ is also a harmonic function of $P\in\Gamma_n(D)$.

To prove

$$\lim_{P\in\Gamma_n(D), P\to Q^*} H(\Gamma_n(D), l, m; g)(P)=g(Q^*)$$

for any $Q^*\in S_n(D)$, apply Lemma 5 to $g(Q)$ and $-g(Q)$ by putting

$$W(P, Q)=\bar{W}(\Gamma_n(D), l)(P, Q)+\underline{W}(\Gamma_n(D), m)(P, Q),$$

which is locally integrable on $S_n(D)$ for any fixed $P\in\Gamma_n(D)$. Then we shall see that (I) and (II) hold. For any $Q^*=(X^*, y^*)\in S_n(D)$ and any $\varepsilon>0$, take a number δ $(0<\delta<1)$. Then from (2.5), (3.13) and (3.14) we can choose two numbers Y_1^* and Y_2^*, $-\infty<Y_1^*<\min(0, y^*-2)$, $\max(0, y^*+2)<Y_2^*<\infty$ such that for any $P=(X, y)\in\Gamma_n(D)\cap U_\delta(Q^*)$, $U_\delta(Q^*)=\{P\in R^n; |P-Q^*|<\delta\}$,

$$\int_{S_n(D;(Y_2^*,\infty))} |g(Q)K(\Gamma_n(D), l, m)(P, Q)|\, d\sigma_Q < \varepsilon$$

and

$$\int_{S_n(D;(-\infty,Y_1^*))} |g(Q)K(\Gamma_n(D), l, m)(P, Q)|\, d\sigma_Q < \varepsilon ,$$

which is (I) in Lemma 5. To see (II), we only need to observe that for any $Q^* \in S_n(D)$ and any two numbers Y_1, Y_2 $(-\infty < Y_1 < Y_2 < \infty)$

$$\lim_{P \in \Gamma_n(D), P \to Q^*} (\overline{W}(\Gamma_n(D), l)(P, Q) + \underline{W}(\Gamma_n(D), m)(P, Q)) = 0$$

at every $Q \in S_n(D; (Y_1, Y_2))$. This follows from (2.3) and (2.4), because

$$\lim_{X \to X^*, X \in D} f_k^D(X) = 0 \qquad (k = 1, 2, \ldots)$$

as $P = (X, y) \to Q^* = (X^*, y^*) \in S_n(D)$.

We shall proceed to prove (2.6). Consider the inequalities

$$N(|H(\Gamma_n(D), l, m; g^+)|)(y) \leq \overline{I}_1(y) + \overline{I}_2(y) \tag{3.15}$$

and

$$N(|H(\Gamma_n(D), l, m; g^+)|)(y) \leq \underline{I}_1(y) + \underline{I}_2(y) ,$$

where

$$\overline{I}_1(y) = \int_D \left(\int_{S_n(D;(y+1,\infty))} g^+(Q) |K(\Gamma_n(D), l, m)(P, Q)|\, d\sigma_Q \right) f_1^D(X) dX ,$$

$$\overline{I}_2(y) = \int_D \left(\int_{S_n(D;(-\infty,y+1])} g^+(Q) |K(\Gamma_n(D), l, m)(P, Q)|\, d\sigma_Q \right) f_1^D(X) dX ,$$

$$\underline{I}_1(y) = \int_D \left(\int_{S_n(D;(-\infty,y-1))} g^+(Q) |K(\Gamma_n(D), l, m)(P, Q)|\, d\sigma_Q \right) f_1^D(X) dX ,$$

and

$$\underline{I}_2(y) = \int_D \left(\int_{S_n(D;[y-1,\infty))} g^+(Q) |K(\Gamma_n(D), l, m)(P, Q)|\, d\sigma_Q \right) f_1^D(X) dX$$

$$(P = (X, y)) .$$

Let ε be any positive number. From (2.5) we can take a sufficiently large number $\bar{y}_0$ and a sufficiently small number $\underline{y}_0$ such that

$$\int_{y+1}^{+\infty} \exp(-\sqrt{\lambda(D, k_{l+1})}y^*)\left(\int_{\partial D} |g(X^*, y^*)| d\sigma_{X^*}\right)dy^* < \frac{\varepsilon}{2L\bar{L}_1} \qquad (y > \bar{y}_0)$$

$$\int_{-\infty}^{y-1} \exp(\sqrt{\lambda(D, k_{m+1})}y^*)\left(\int_{\partial D} |g(X^*, y^*)| d\sigma_{X^*}\right)dy^* < \frac{\varepsilon}{2L\underline{L}_1} \qquad (y < \underline{y}_0),$$

where $\bar{L}_1$ and $\underline{L}_1$ are two constants in Lemma 3, and

$$L = \int_D f_1^D dX .$$

Then from Lemma 3 we have

$$0 \le \bar{I}_1(y) \le L\bar{L}_1 \exp(\sqrt{\lambda(D, k_{l+1})}y) \int_{y+1}^{\infty} \exp(-\sqrt{\lambda(D, k_{l+1})}y^*)\left(\int_{\partial D} g^+(X^*, y^*) d\sigma_{X^*}\right)dy^*$$

$$< \frac{\varepsilon}{2} \exp(\sqrt{\lambda(D, k_{l+1})}y) \qquad (y > \bar{y}_0)$$

and

$$0 \le \underline{I}_1(y) \le L\bar{L}_1 \exp(-\sqrt{\lambda(D, k_{m+1})}y) \int_{-\infty}^{y-1} \exp(\sqrt{\lambda(D, k_{m+1})}y^*)\left(\int_{\partial D} g^+(X^*, y^*) d\sigma_{X^*}\right)dy^*$$

$$< \frac{\varepsilon}{2} \exp(-\sqrt{\lambda(D, k_{m+1})}y) \qquad (y < \underline{y}_0),$$

which give

$$\mu_l(\bar{I}_1) = \eta_m(\underline{I}_1) = 0 . \tag{3.16}$$

To estimate $\bar{I}_2(y)$ and $\underline{I}_2(y)$, we use the inequalities

$$\bar{I}_2(y) \le \bar{I}_{2,1}(y) + \bar{I}_{2,2}(y) + \bar{I}_{2,3}(y) \qquad (y > -1) \tag{3.17}$$

and

$$\underline{I}_2(y) \le \underline{I}_{2,1}(y) + \underline{I}_{2,2}(y) + \underline{I}_{2,3}(y) \qquad (y < 1),$$

where

$$\bar{I}_{2,1}(y) = c_n^{-1} \int_D \left(\int_{S_n(D;(0,y+1])} g^+(Q) \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) d\sigma_Q\right) f_1^D(X) dX \tag{3.18}$$

$$\bar{I}_{2,2}(y) = \int_D \left(\int_{S_n(D;(0,y+1])} g^+(Q) |\bar{V}(\Gamma_n(D), l)(P, Q)| d\sigma_Q\right) f_1^D(X) dX \qquad (y > -1)$$

$$\bar{I}_{2,3}(y) = \int_D \left(\int_{S_n(D;(-\infty,0])} g^+(Q) \left| c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) - \underline{V}(\Gamma_n(D), m)(P, Q)\right| d\sigma_Q\right) f_1^D(X) dX$$

and

$$\underline{I}_{2,1}(y)=c_n^{-1}\int_D\left(\int_{S_n(D;[y-1,0))} g^+(Q)\frac{\partial}{\partial \nu}G_{\Gamma_n(D)}(P,Q)d\sigma_Q\right)f_1^P(X)dX$$

$$\underline{I}_{2,2}(y)=\int_D\left(\int_{S_n(D;[y-1,0))} g^+(Q)\,|\,\underline{V}(\Gamma_n(D),m)(P,Q)\,|\,d\sigma_Q\right)f_1^P(X)dX \qquad (y<1)$$

$$\underline{I}_{2,3}(y)=\int_D\left(\int_{S_n(D;[0,\infty))} g^+(Q)\left|c_n^{-1}\frac{\partial}{\partial \nu}G_{\Gamma_n(D)}(P,Q)-\bar{V}(\Gamma_n(D),l)(P,Q)\right|d\sigma_Q\right)f_1^P(X)dX\,.$$

First Lemma 3 gives

$$\bar{I}_{2,3}(y)\leq L\underline{L}_1\exp(-\sqrt{\lambda(D,k_{m+1})}y)\int_{-\infty}^{0}\exp(\sqrt{\lambda(D,k_{m+1})}y^*)\left(\int_{\partial D} g^+(X^*,y^*)d\sigma_{X^*}\right)dy^* \qquad (y>1)$$

and

$$\underline{I}_{2,3}(y)\leq L\bar{L}_1\exp(\sqrt{\lambda(D,k_{l+1})}y)\int_{0}^{\infty}\exp(-\sqrt{\lambda(D,k_{l+1})}y^*)\left(\int_{\partial D} g^+(X^*,y^*)d\sigma_{X^*}\right)dy^* \qquad (y<-1)\,.$$

Hence it is evident from (2.5) that

$$\mu_l(\bar{I}_{2,3})=\eta_m(\underline{I}_{2,3})=0\,. \tag{3.19}$$

Next we have from (2.2), (2.3) and (2.4) that if $l\geq 1$, then

$$\bar{I}_{2,2}(y)\leq BLM_2^2\,|\,D\,|\sum_{k\in I(D,k_{l+1})}k\exp(\sqrt{\lambda(D,k)})\exp(\sqrt{\lambda(D,k)}y)\Psi_k(y) \qquad (y>-1)$$

and that if $m\geq 1$, then

$$\underline{I}_{2,2}(y)\leq BLM_2^2\,|\,D\,|\sum_{k\in I(D,k_{m+1})}k\exp(\sqrt{\lambda(D,k)})\exp(-\sqrt{\lambda(D,k)}y)\Phi_k(y) \qquad (y<1)\,,$$

where

$$B=c_n^{-1}\max_{X\in D,X^*\in\partial D}\frac{\partial}{\partial \nu}G_{\Gamma_n(D)}((X,1),(X^*,0))\,, \tag{3.20}$$

$$\Psi_k(y)=\int_0^{y+1}\exp(-\sqrt{\lambda(D,k)}y^*)\left(\int_{\partial D}|\,g(X^*,y^*)\,|\,d\sigma_{X^*}\right)dy^* \qquad (y>-1,\,k\in I(D,k_{l+1}))$$

and

$$\Phi_k(y)=\int_{y-1}^{0}\exp(\sqrt{\lambda(D,k)}y^*)\left(\int_{\partial D}|\,g(X^*,y^*)\,|\,d\sigma_{X^*}\right)dy^* \qquad (y<1,\,k\in I(D,k_{m+1}))\,.$$

We shall later show that

$$\text{(3.21)}\quad \Psi_k(y)=o(\exp(\sqrt{\lambda(D, k_{l+1})}y-\sqrt{\lambda(D, k)}y)) \qquad (y\to\infty)\ (l\geq 1, k\in I(D, k_{l+1}))$$

$$\Phi_k(y)=o(\exp(-\sqrt{\lambda(D, k_{m+1})}y+\sqrt{\lambda(D, k)}y)) \qquad (y\to-\infty)\ (m\geq 1, k\in I(D, k_{m+1}))\,.$$

Hence we can conclude that if $l\geq 1$ and $m\geq 1$, then

$$\text{(3.22)}\qquad \mu_l(\bar{I}_{2,2})=\eta_m(\underline{I}_{2,2})=0$$

which also holds in the case $l=m=0$, because $\bar{I}_{2,2}(y)\equiv\underline{I}_{2,2}(y)\equiv 0$ then. Lastly we can obtain

$$\text{(3.23)}\qquad \mu_l(\bar{I}_{2,1})=\eta_m(\underline{I}_{2,1})=0$$

which will be proved at the end of this proof. We thus obtain from (3.17), (3.19), (3.22) and (3.23) that

$$\text{(3.24)}\qquad \mu_l(\bar{I}_2)=\eta_m(\underline{I}_2)=0\,.$$

We can finally conclude from (3.15), (3.16) and (3.24) that

$$\mu_l(N(|H(\Gamma_n(D), l, m;\, g^+)|))=\eta_m(N(|H(\Gamma_n(D), l, m;\, g^+)|))=0\,.$$

In completely the same way applied to g^-, we also have that

$$\mu_l(N(|H(\Gamma_n(D), l, m;\, g^-)|))=\eta_m(N(|H(\Gamma_n(D), l, m;\, g^-)|))=0\,.$$

Since

$$N(|H(\Gamma_n(D), l, m;\, g)(P)|)\leq N(|H(\Gamma_n(D), l, m;\, g^+)(P)|)+N(|H(\Gamma_n(D), l, m;\, g^-)(P)|)\,,$$

these give the conclusion (2.6).

We shall prove (3.21). We note that $\Psi_k(y)$ (resp. $\Phi_k(y)$) is increasing (resp. decreasing),

$$\int_0^\infty \Psi_k'(y^*)\exp(-\sqrt{\lambda(D, k_{l+1})}y^*+\sqrt{\lambda(D, k)}y^*)dy^*$$
$$=\exp(\sqrt{\lambda(D, k_{l+1})}-\sqrt{\lambda(D, k)})\int_1^\infty \exp(-\sqrt{\lambda(D, k_{l+1})}y^*)$$
$$\times\left(\int_{\partial D}|g(X^*, y^*)|\,d\sigma_{X^*}\right)dy^*$$
$$\left(\text{resp. } \int_{-\infty}^0 \Phi_k'(y^*)\exp(\sqrt{\lambda(D, k_{m+1})}y^*-\sqrt{\lambda(D, k)}y^*)dy^*\right.$$
$$=-\exp(\sqrt{\lambda(D, k_{m+1})}-\sqrt{\lambda(D, k)})\int_{-\infty}^{-1}\exp(\sqrt{\lambda(D, k_{m+1})}y^*)$$
$$\left.\times\left(\int_{\partial D}|g(X^*, y^*)|\,d\sigma_{X^*}\right)dy^*\right)$$

and

$$\begin{aligned}
&\Psi_k(y)\exp(-\sqrt{\lambda(D, k_{l+1})}y+\sqrt{\lambda(D, k)}y)\\
&\quad\le \exp(\sqrt{\lambda(D, k_{l+1})}-\sqrt{\lambda(D, k)})\int_0^{y+1}\exp(-\sqrt{\lambda(D, k_{l+1})}y^*)\\
&\qquad\times\left(\int_{\partial D}|g(X^*, y^*)|\,d\sigma_{X^*}\right)dy^*\le \bar{L}_2\exp(\sqrt{\lambda(D, k_{l+1})}-\sqrt{\lambda(D, k)})
\end{aligned}$$

$$\begin{aligned}
&\Bigg(\text{resp. } \Phi_k(y)\exp(\sqrt{\lambda(D, k_{m+1})}y-\sqrt{\lambda(D, k)}y)\\
&\quad\le \exp(\sqrt{\lambda(D, k_{m+1})}-\sqrt{\lambda(D, k)})\int_{y-1}^{0}\exp(\sqrt{\lambda(D, k_{m+1})}y^*)\\
&\qquad\times\left(\int_{\partial D}|g(X^*, y^*)|\,d\sigma_{X^*}\right)dy^*\le \underline{L}_2\exp(\sqrt{\lambda(D, k_{m+1})}-\sqrt{\lambda(D, k)})\Bigg),
\end{aligned}$$

where

$$\bar{L}_2=\int_0^{\infty}\exp(-\sqrt{\lambda(D, k_{l+1})}y^*)\left(\int_{\partial D}|g(X^*, y^*)|\,d\sigma_{X^*}\right)dy^*$$

$$\left(\text{resp. } \underline{L}_2=\int_{-\infty}^{0}\exp(\sqrt{\lambda(D, k_{m+1})}y^*)\left(\int_{\partial D}|g(X^*, y^*)|\,d\sigma_{X^*}\right)dy^*\right).$$

From these and (2.5) (resp. (2.6)) we see

$$\int_0^{\infty}\Psi_k(y^*)\exp(-\sqrt{\lambda(D, k_{l+1})}y^*+\sqrt{\lambda(D, k)}y^*)dy^*<\infty \tag{3.25}$$

$$\left(\text{resp. } \int_{-\infty}^{0}\Phi_k(y^*)\exp(\sqrt{\lambda(D, k_{m+1})}y^*-\sqrt{\lambda(D, k)}y^*)dy^*<\infty\right)$$

by integration by parts. Since

$$\begin{aligned}
&\Psi_k(y)\exp(-\sqrt{\lambda(D, k_{l+1})}y+\sqrt{\lambda(D, k)}y)\\
&\quad=(\sqrt{\lambda(D, k_{l+1})}-\sqrt{\lambda(D, k)})\Psi_k(y)\int_y^{\infty}\exp(-\sqrt{\lambda(D, k_{l+1})}y^*+\sqrt{\lambda(D, k)}y^*)dy^*\\
&\quad\le(\sqrt{\lambda(D, k_{l+1})}-\sqrt{\lambda(D, k)})\int_y^{\infty}\Psi_k(y^*)\exp(-\sqrt{\lambda(D, k_{l+1})}y^*+\sqrt{\lambda(D, k)}y^*)dy^*
\end{aligned}$$

$$\begin{aligned}
&\Bigg(\text{resp. } \Phi_k(y)\exp(\sqrt{\lambda(D, k_{m+1})}y-\sqrt{\lambda(D, k)}y)\\
&\quad=(\sqrt{\lambda(D, k_{m+1})}-\sqrt{\lambda(D, k)})\Phi_k(y)\int_{-\infty}^{y}\exp(\sqrt{\lambda(D, k_{m+1})}y^*-\sqrt{\lambda(D, k)}y^*)dy^*
\end{aligned}$$

$$\leq(\sqrt{\lambda(D, k_{m+1})}-\sqrt{\lambda(D, k)})\int_{-\infty}^{y}\Phi_k(y^*)\exp(\sqrt{\lambda(D, k_{m+1})}y^*-\sqrt{\lambda(D, k)}y^*)dy^*\Big),$$

(3.25) gives (3.21).

Finally we shall show (3.23). In the following we use the notation in (3.18) and (3.20). First we note that

$$(3.26)\qquad 0\leq\bar{I}_{2,1}(y)\leq N(H(\Gamma_n(D), l, m; g^+))(y)-\bar{I}_1^*(y)+\bar{I}_{2,2}^*(y)+\bar{I}_{2,3}(y)\qquad (y>-1)$$

and

$$0\leq\underline{I}_{2,1}(y)\leq N(H(\Gamma_n(D), l, m; g^+))(y)-\underline{I}_1^*(y)+\underline{I}_{2,2}^*(y)+\underline{I}_{2,3}(y)\qquad (y<1),$$

where

$$\bar{I}_1^*(y)=\int_D\left(\int_{S_n(D;(y+1,\infty))}g^+(Q)K(\Gamma_n(D), l, m)(P, Q)d\sigma_Q\right)f_1^P(X)dX,$$

$$\bar{I}_{2,2}^*(y)=\int_D\left(\int_{S_n(D;(0,y+1])}g^+(Q)\bar{V}(\Gamma_n(D), l)(P, Q)d\sigma_Q\right)f_1^P(X)dX\qquad (y>-1)$$

and

$$\underline{I}_1^*(y)=\int_D\left(\int_{S_n(D;(-\infty,y-1))}g^+(Q)K(\Gamma_n(D), l, m)(P, Q)d\sigma_Q\right)f_1^P(X)dX,$$

$$\underline{I}_{2,2}^*(y)=\int_D\left(\int_{S_n(D;[y-1,0])}g^+(Q)\underline{V}(\Gamma_n(D), m)(P, Q)d\sigma_Q\right)f_1^P(X)dX\qquad (y<1).$$

Since

$$|\bar{I}_1^*(y)|\leq\bar{I}_1(y)\quad\text{and}\quad|\underline{I}_1^*(y)|\leq\underline{I}_1(y),$$

we easily see from (3.16) that

$$(3.27)\qquad \mu_l(|\bar{I}_1^*|)=\eta_m(|\underline{I}_1^*|)=0.$$

Next it follows from the orthonormality of $\{f_k^P(X)\}$ that if $l\geq 1$, then

$$\bar{I}_{2,2}^*(y)\leq BL\exp(\sqrt{\lambda(D, 1)})\exp(\sqrt{\lambda(D, 1)}y)\Psi_1(y)\qquad (y>-1)$$

and that if $m\geq 1$, then

$$\underline{I}_{2,2}^*(y)\leq BL\exp(\sqrt{\lambda(D, 1)})\exp(-\sqrt{\lambda(D, 1)}y)\Phi_1(y)\qquad (y<1).$$

Hence (3.21) with $k=1$ gives that

$$(3.28)\qquad \limsup_{y\to\infty}\exp(-\sqrt{\lambda(D, k_{l+1})}y)\bar{I}_{2,2}^*(y)\leq 0$$

and

$$\limsup_{y\to-\infty} \exp(\sqrt{\lambda(D, k_{m+1})}y)\underline{I}^*_{2,2}(y) \le 0 ,$$

which also hold in the case $l=m=0$, because $\bar{I}^*_{2,2}(y)\equiv\underline{I}^*_{2,2}(y)\equiv 0$ then. If we can show that

$$\limsup_{y\to\infty} \exp(-\sqrt{\lambda(D, k_{l+1})}y)N(H(\Gamma_n(D), l, m; g^+))(y) \le 0 \tag{3.29}$$

and

$$\limsup_{y\to-\infty} \exp(\sqrt{\lambda(D, k_{m+1})}y)N(H(\Gamma_n(D), l, m; g^+))(y) \le 0 ,$$

then we finally conclude from (3.19), (3.26), (3.27) and (3.28) that

$$\limsup_{y\to\infty} \exp(-\sqrt{\lambda(D, k_{l+1})}y)\bar{I}_{2,1}(y) = 0$$

and

$$\limsup_{y\to-\infty} \exp(\sqrt{\lambda(D, k_{m+1})}y)\underline{I}_{2,1}(y) = 0 ,$$

which give (3.23).

To prove (3.29), recall that $-H(\Gamma_n(D), l, m; g^+)(P)$ is also a harmonic function on $\Gamma_n(D)$ satisfying

$$\lim_{P\in\Gamma_n(D), P\to Q^*} -H(\Gamma_n(D), l, m; g^+)(P) = -g^+(Q^*) \le 0$$

for every $Q^*\in S_n(D)$. Hence from [14, Theorem 7.2] we know that

$$-\infty < \eta_0(N(-H(\Gamma_n(D), l, m; g^+))) \le \infty, \quad -\infty < \mu_0(N(-H(\Gamma_n(D), l, m; g^+))) \le \infty .$$

Thus we obtain that if $l\ge 1$, then

$$\limsup_{y\to\infty} \exp(-\sqrt{\lambda(D, k_{l+1})}y)N(H(\Gamma_n(D), l, m; g^+))(y) \le 0$$

and if $m\ge 1$, then

$$\limsup_{y\to-\infty} \exp(\sqrt{\lambda(D, k_{m+1})}y)N(H(\Gamma_n(D), l, m; g^+))(y) \le 0 .$$

If $l=0$, then we have

$$\begin{aligned} N(H(\Gamma_n(D), l, m; g^+))(y) &\le c_n^{-1}\int_D\left(\int_{S_n(D)} \bar{g}^+(Q)\frac{\partial}{\partial\nu}G_{\Gamma_n(D)}(P, Q)d\sigma_Q\right)f_1^D(X)dX + \bar{I}_{2,3}(y) \\ &= N(H(\Gamma_n(D), l, m; \bar{g}^+))(y) + \bar{I}_{2,3}(y) , \end{aligned}$$

where

$$\bar{g}^+(Q)=\bar{g}^+(X^*, y^*)=\begin{cases} g^+(X^*, y^*) & (y^* \geq 0) \\ 0 & (-\infty < y^* < 0)\,. \end{cases}$$

If $m=0$, then we have

$$N(H(\Gamma_n(D), l, m; g^+))(y) \leq c_n^{-1} \int_D \left(\int_{S_n(D)} \underline{g}^+(Q) \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) d\sigma_Q \right) f_1^D(X) dX + \underline{I}_{2,3}(y)$$
$$= N(H(\Gamma_n(D), l, m; \underline{g}^+))(y) + \underline{I}_{2,3}(y)\,,$$

where

$$\underline{g}^+(Q)=\underline{g}^+(X^*, y^*)=\begin{cases} g^+(X^*, y^*) & (-\infty < y^* \leq 0) \\ 0 & (y^* > 0)\,. \end{cases}$$

Since

$$\mu_0(N(H(\Gamma_n(D), 0, m; \bar{g}^+))) = \eta_0(N(H(\Gamma_n(D), l, 0; \underline{g}^+))) = 0$$

from the cylindrical version of [15, Lemma 3], (3.19) and this also give

$$\limsup_{y\to\infty} \exp(-\sqrt{\lambda(D, 1)}y) N(H(\Gamma_n(D), 0, m; g^+))(y) \leq 0$$

$$\limsup_{y\to-\infty} \exp(\sqrt{\lambda(D, 1)}y) N(H(\Gamma_n(D), l, 0; g^+))(y) \leq 0\,.$$

Thus we can obtain (3.29) for any non-negative integers l and m.

PROOF OF THEOREM 2. Take a positive continuous function $\varphi(t)$ $(t \geq 0)$ such that

$$\varphi(0) = \exp(-\sqrt{\lambda(D, 1)})\,,$$

$$\varphi(|y^*|)\left(\int_{\partial D} |g(X^*, y^*)| d\sigma_{X^*} \right) \leq L_2 (1+|y^*|)^{-2} \qquad (-\infty < y^* < \infty)\,,$$

where

$$L_2 = \exp(-\sqrt{\lambda(D, 1)}) \int_{\partial D} |g(X^*, 0)| d\sigma_{X^*}\,.$$

For any fixed $P=(X, y) \in \Gamma_n(D)$, choose two numbers Y_1 and Y_2, $Y_2 > \max(0, y+2)$, $Y_1 < \min(0, y-2)$. Then we see from Lemma 4 that

$$\text{(3.30)} \qquad \int_{S_n(D;(Y_2,\infty))} |g(Q) K(\Gamma_n(D), \varphi)(P, Q)| d\sigma_Q$$
$$\leq L_1' \int_{Y_2}^{\infty} \left(\int_{\partial D} |g(X^*, y^*)| d\sigma_{X^*} \right) \varphi(y^*) dy^* \leq L_1' L_2 \int_{Y_2}^{\infty} (1+y^*)^{-2} dy^* < \infty$$

and

$$\int_{S_n(D;(-\infty,Y_1))} |g(Q)K(\Gamma_n(D), \varphi)(P, Q)|\, d\sigma_Q$$
$$\leq L'_1 \int_{-\infty}^{Y_1} \left(\int_{\partial D} |g(X^*, y^*)|\, d\sigma_{X^*} \right) \varphi(-y^*) dy^* \leq L'_1 L_2 \int_{-\infty}^{Y_1} (1-y^*)^{-2} dy^* < \infty .$$

It is evident that

$$\int_{S_n(D;(Y_1,Y_2))} |g(Q)K(\Gamma_n(D), \varphi)(P, Q)|\, d\sigma_Q < \infty .$$

These give that

$$\int_{S_n(D)} |g(Q)K(\Gamma_n(D), \varphi)(P, Q)|\, d\sigma_Q < \infty .$$

To see that $H(\Gamma_n(D), \varphi; g)(P)$ is harmonic in $\Gamma_n(D)$, we remark that $H(\Gamma_n(D), \varphi; g)(P)$ satisfies the local mean-value property by Fubini's theorem.

Finally we shall show

$$\lim_{P \in \Gamma_n(D), P \to Q^*} H(\Gamma_n(D), \varphi; g)(P) = g(Q^*) \tag{3.31}$$

for any $Q^* \in S_n(D)$. Put

$$W(P, Q) = \overline{W}(\Gamma_n(D), \varphi)(P, Q) + \underline{W}(\Gamma_n(D), \varphi)(P, Q)$$

in Lemma 5, which is a locally integrable function of $Q \in S_n(D)$ for any fixed $P \in \Gamma_n(D)$. Then we can see from (3.30) in the same way as in the proof of Theorem 1 that both (I) and (II) are satisfied. Thus Lemma 5 applied to $g(Q)$ and $-g(Q)$ gives (3.31).

Proof of Theorem 3. From Theorem 1, we have the solution $H(\Gamma_n(D), l, m; g)(P)$ of the Dirichlet problem on $\Gamma_n(D)$ with g satisfying (2.6). Consider the function $h - H(\Gamma_n(D), l, m; g)$. Then it follows that this is harmonic in $\Gamma_n(D)$ and vanishes continuously on $S_n(D)$. Since

$$0 \leq \{h - H(\Gamma_n(D), l, m; g)\}^+(P) \leq h^+(P) + \{H(\Gamma_n(D), l, m; g)\}^-(P)$$

for any $P \in \Gamma_n(D)$ and

$$\mu_l(N(\{H(\Gamma_n(D), l, m; g)\}^-)) = \eta_m(N(\{H(\Gamma_n(D), l, m; g)\}^-)) = 0$$

from (2.6), (2.8) gives that

$$\mu_p(N(\{h - H(\Gamma_n(D), l, m; g)\}^+)) = \eta_q(N(\{h - H(\Gamma_n(D), l, m; g)\}^+)) = 0 .$$

From Lemma 6 we see that

$$h(P)-H(\Gamma_n(D), l, m; g)(P)$$
$$=\sum_{k\in I(D,k_{p+1})} A_k(h)\exp(\sqrt{\lambda(D,k)}y)f_k^D(X)+\sum_{k\in I(D,k_{q+1})} B_k(h)\exp(-\sqrt{\lambda(D,k)}y)f_k^D(X)$$

($A_k(h)$ $(k=1,2,\ldots,k_{p+1}-1)$ and $B_k(h)$ $(k=1,2,\ldots,k_{q+1}-1)$ are all constants) for every $P=(X,y)\in\Gamma_n(D)$, which is the conclusion of Theorem 3.

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