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THE SPECTRAL TYPE OF THE STAIRCASE TRANSFORMATION

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Abstract. We show that a certain Riesz-product type measure is singular. This proves the singularity of the spectral measures of a certain ergodic transformation, known as the staircase.

Introduction. The staircase transformation is an example of a "rank one" transformation whose properties have been of interest in ergodic theory recently [Adams], [Adams, Friedman], [Choksi, Nadkarni]. Here we prove that it has singular spectrum. To be more precise, this means that the maximal spectral type of the induced unitary operator is singular with respect to the Lebesgue measure on the circle. We refer the reader to [Choksi, Nadkarni, (example 2)] for the definition of the staircase transformation and for other background information. It is shown there that the problem reduces to proving the singularity of a specific measure μ, which is defined as follows. Let h_n, $n=1, 2, \cdots$ be the integers defined inductively by

$$h_1=1\,, \quad h_{n+1}=nh_n+(1+2+3+\cdots+n)\,.$$

Define trigonometric polynomials $P_n(z)$, where $z=e^{i\theta}$, $\theta\in[0, 2\pi)$, by

$$P_n(z)=\frac{1}{\sqrt{n}}(1+z^{h_n+1}+z^{2h_n+1+2}+z^{3h_n+1+2+3}+\cdots+z^{(n-1)h_n+n(n-1)/2})\,.$$

If λ denotes the normalized Lebesgue measure on $[0, 2\pi)$ the measures $\prod_{n=1}^{N}|P_n|^2d\lambda$ turn out to have weak* limit $d\mu$. The purpose of this paper is to prove that $\mu\perp\lambda$.

THEOREM. $\mu\perp\lambda$.

For this theorem, the reader will not need to know the ergodic theory background. Only the definitions of the polynomials P_n are really used. The overall method of the proof is based on [Bourgain]. Then some specific properties of the above polynomials are needed to make the method work in this case.

The proof actually gives more than the statement of the theorem. It gives the same result for other "staircase constructions". By this we mean that one can have polynomials P_{n_j}, of the above type, but with h_n replaced by h_j where $h_{j+1}=n_jh_j+(1+2+\cdots+n_j)$.

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Some mild conditions on n_j seem to be required however. For example, the proof of Proposition 9 requires that

$$n_j^5/h_j \to 0 .$$

As this stage we have not attempted to optimize the proof for such other staircase constructions. (Recently, the condition on n_j has been removed by F. L. Nazarov (unpublished).)

I would like to thank Reem Yassawi and David Clark for their work on Propositions 9 and 10b) respectively.

The Proof of the Theorem. In the following, all 1-norms and integrals are taken with respect to normalized Lebesgue measure on the circle.

PROPOSITION 1. *It suffices to show that*

$$\inf\{\|P_{n_1}\cdots P_{n_k}\|_1 :\ k\in N,\ n_1<\cdots<n_k\}=0 .$$

PROPOSITION 2. *Fix k, $n_1<\cdots<n_k$, and let $Q=P_{n_1}\cdots P_{n_k}$. Then*

$$\limsup_{n\to\infty}\int|QP_n|\le\int|Q|-c_1\left(\liminf_{m\to\infty}\int|Q|\,||P_m|^2-1|\right)^2$$

where $c_1>0$ is an absolute constant.

PROPOSITION 3. *Let $w\ge 0$ be any continuous function on the circle T. Then*

$$\liminf_{m\to\infty}\int w||P_m|^2-1|\ge c_2\int w$$

where $c_2>0$ is an absolute constant.

PROOF OF THE THEOREM. Let α be the infimum in Proposition 1, i.e., $\alpha=\inf\{\|Q\|_1 : Q=P_{n_1}\cdots P_{n_k},\ k\in N,\ n_1<\cdots<n_k\}$. For a fixed $Q=P_{n_1}\cdots P_{n_k}$, Proposition 3 with $w=|Q|$ gives

$$\liminf_{m\to\infty}\int|Q|\,||P_m|^2-1|\ge c_2\|Q\|_1\ge c_2\alpha .$$

Hence Proposition 2 gives

$$\limsup_{n\to\infty}\|QP_n\|_1\le\|Q\|_1-c_1c_2^2\alpha^2 .$$

But the left hand side is bounded below by α since $n>n_k$ as $n\to\infty$ and $QP_n=P_{n_1}\cdots P_{n_k}\cdot P_n$. Hence

$$\alpha\le\|Q\|_1-c_1c_2^2\alpha^2 .$$

Taking the infinum over all Q now gives

$$\alpha \leq \alpha - c_1 c_2^2 \alpha^2 ,$$

and hence $\alpha = 0$ since $c_1 c_2^2 > 0$. Hence $\mu \perp \lambda$ by Proposition 1.

Proof of the Propositions 1, 2 and 3. Propositions 1 and 2 follow from the initial remarks in Bourgain's paper [Bourgain, equations (2.15) and (2.22)]. We remark here that the inequality (2.20) in Bourgain's paper may be interpreted as

$$\limsup_{m \to \infty} \int |Q| |P_m|^2 \leq \int |Q|$$

(where Q is fixed).

The remainder of this paper will consist of the proof of Proposition 3. Before giving the proof, we will explain the main ideas. The main problem is the case $w = 1$. In that special case, we are trying to prove that the L^1 norm of $|P_n|^2 - 1$ is at least an absolute constant. We do this by explicitly computing $|P_n|^2 - 1$ (see Proposition 4) and seeing that it consists of approximately n Dirichlet kernels $S_i(z)/n$, each of order about n. If all of these contributed their full L^1 norms $(\log n)/n$, we would therefore get $n(\log n)/n = \log n$, which cannot be correct because the L^1 norm is at most $1 + 1 = 2$. Therefore, we make the guess that in reality, they each contribute $1/n$, and that this comes from the intervals around the central maximum of the Dirichlet kernels (which also happen to be uniformly distributed around the circle). The proof below then consists of carrying out this estimate. We use the characteristic functions of these central intervals (see Definition 6) to do the calculation.

PROOF OF PROPOSITION 3. We first show that without loss of generality w is a trigonometric polynomial: Suppose that the proposition holds for all trigonometric polynomials $v \geq 0$. Let $w \geq 0$ be a continuous function on the circle and let $\varepsilon > 0$. Then there is a trigonometric polynomial $v \geq 0$ with $\|w - v\|_\infty \leq \varepsilon$ (take $v = K_N * w$ with N large enough, K_N: Fejér kernel). So for each m,

$$\begin{aligned} \int w \big| |P_m|^2 - 1 \big| &= \int v \big| |P_m|^2 - 1 \big| + \int (w - v) \big| |P_m|^2 - 1 \big| \\ &\geq \int v \big| |P_m|^2 - 1 \big| - \|w - v\|_\infty \int \big| |P_m|^2 - 1 \big| \\ &\geq \int v \big| |P_m|^2 - 1 \big| - 2\varepsilon , \end{aligned}$$

since $\int \big| |P_m|^2 - 1 \big| \leq \int (|P_m|^2 + 1) = 2$. Taking lim inf on both sides,

$$\liminf_{m \to \infty} \int w \big| |P_m|^2 - 1 \big| \geq \liminf_{m \to \infty} \int v \big| |P_m|^2 - 1 \big| - 2\varepsilon$$

$$\geq c_2 \int v - 2\varepsilon \geq c_2 \int w - c_2\varepsilon - 2\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, the proposition for $w \in C(T)$ follows. So from now on, $w \geq 0$ is without loss of generality a trigonometric polynomial.

PROPOSITION 4. *Define* $f_n = P_n\bar{P}_n - 1$. *Then*

$$f_n = g_n + \bar{g}_n \qquad \textit{where} ,$$

4a) $$g_n(z) = \frac{1}{n} \sum_{i=1}^{n-1} z^{a_i} S_i(z) , \qquad (z = e^{i\theta})$$

4b) $$S_i(z) = 1 + z^i + z^{2i} + \cdots + z^{(n-1-i)i} = \sum_{j=0}^{n-1-i} (z^i)^j$$

4c) $$a_i = ih_n + \sum_{j=1}^{i} j , \qquad i = 1, \ldots, (n-1) .$$

PROOF. Multiply out $P_n\bar{P}_n$.

PROPOSITION 5.

5a) $|a_i - a_j| \geq h_n \geq (n-1)!$ *for* $1 \leq i \neq j \leq n-1$ (*and* $a_i \geq h_n \geq (n-1)!$).

5b) $\operatorname{Re} S_i(z) \geq (n-i)/\sqrt{2}$ *whenever* $z = e^{i\theta}$, *and* $|\theta - 2\pi k/i| \leq \pi/4i(n-i)$, $k \in \boldsymbol{Z}$.

PROOF. 5a) $|a_i - a_j| \geq |i-j| h_n \geq h_n$ for $i \neq j$, by 4c). Also, $h_1 = 1$, $h_{n+1} \geq nh_n$ implies by induction that $h_n \geq (n-1)!$

5b) If $k \in \boldsymbol{Z}$ and $|\theta - 2\pi k/i| \leq \pi/4i(n-i)$ then $|ij\theta - 2\pi kj| \leq \pi j/4(n-i) \leq \pi/4$ for all i, j with $1 \leq i \leq n-1$, $0 \leq j \leq n-i$. Thus $\cos(ij\theta) \geq \cos(\pi/4) = 1/\sqrt{2}$ and

$$\operatorname{Re} S_i(z) = \sum_{j=0}^{n-1-i} \operatorname{Re}(z^{ij}) = \sum_{j=0}^{n-1-i} \cos(ij\theta) \geq (n-i)/\sqrt{2} .$$

REMARK. 5b) implies that for

$$\frac{1}{4} n \leq i \leq \frac{3}{4} n , \quad (i \in \boldsymbol{N})$$

we have $\operatorname{Re} S_i(z) \geq n/4\sqrt{2}$ whenever

$$|\theta - 2\pi k/i| \leq \frac{\pi}{4(3n/4)(3n/4)} = 4\pi/9n^2 .$$

6. DEFINITIONS. Let $\gamma = \pi/100n^2$ and let

6a) $$B(\theta) = \chi_E(\theta) , \quad \theta \in \boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z} , \qquad \text{where} \quad E = 2\pi Z + [-\gamma, \gamma] .$$

In other words

$$B(\theta)=\begin{cases}1, & |\theta|\le\gamma\\ 0, & \pi\ge|\theta|>\gamma\end{cases}$$

and $B(\theta+2\pi)=B(\theta)$.

6b) Define

$$B_{i,k}(\theta)=B(\theta-2\pi k/i)\,,\qquad i\in\boldsymbol{N}\,,\quad k\in\boldsymbol{N}\,.$$

6c) Define

$$\varphi_i=\sum_{\substack{(i,k)=1\\1\le k\le i}}B_{i,k}\,,$$

where (i, k) denotes the greatest common divisor of i and k.

6d) Define

$$\varphi_{(n)}(z)=\sum_{n/4\le i\le 3n/4}z^{a_i}\varphi_i(z)\,,\quad z=e^{i\theta}\,.$$

PROPOSITION 7.

7a) $\quad |\varphi_{(n)}(z)|\le 1$ *for all* z *with* $|z|=1$, *and* $n\in\boldsymbol{N}$.

7b) $\quad \liminf_{n\to\infty}\left|\int wf_n\bar{\varphi}_{(n)}\right|\ge c_2\int w$ *for some absolute constant* c_2.

Before getting into the proof of Proposition 7, we note that it immediately gives Proposition 3, since

$$\begin{aligned}\left|\int wf_n\bar{\varphi}_{(n)}\right|&\le\int w|f_n||\varphi_{(n)}|\\ &\le\int w|f_n|\quad\text{by 7a)}\\ &=\int w||P_n|^2-1|\quad\text{by definition}\,.\end{aligned}$$

PROOF OF 7a). By 6d)

$$\begin{aligned}|\varphi_{(n)}(z)|&\le\sum_{n/4\le i\le 3n/4}|\varphi_i(z)|\\ &=\sum_{n/4\le i\le 3n/4}\sum_{\substack{(i,k)=1\\1\le k\le i}}B(\theta-2\pi k/i)\\ &=\sum\chi_{2\pi k/i+[-\gamma,\gamma]}(\theta)\,,\quad\theta\in[-\pi,\pi)\end{aligned}$$

where the last summation ranges over all pairs (i, k) with $(i, k)=1$ (relatively prime), $n/4 \le i \le 3n/4$, $1 \le k \le i$. Therefore it suffices to check that none of the translated intervals $2\pi k/i+[-\gamma, \gamma]$ intersect. That is, we need to show that

7c) $$(2\pi k/i+[-\gamma, \gamma]) \cap (2\pi k'/i'+[-\gamma, \gamma])=\varnothing$$

whenever $(i, k) \ne (i', k')$ and (i, k) and (i', k') belong to the range of summation. We have

$$2\pi\left|\frac{k}{i}-\frac{k'}{i'}\right|=2\pi\frac{|ki'-k'i|}{ii'}\ge\frac{2\pi}{ii'}\ge\frac{2\pi}{(3n/4)^2}$$

since $ki'-k'i$ is a nonzero interger. But $\gamma=\pi/100n^2$ is less than one half of the latter estimate. Hence the proof of 7a) is complete. The proof of 7b) requires several more propositions.

PROOF OF 7b). We have

$$\begin{aligned}\int wf_n\bar{\varphi}_{(n)}&=\int w(g_n+\bar{g}_n)\bar{\varphi}_{(n)}\\&=\int w\left(\frac{1}{n}\sum_{i=1}^{n-1}z^{a_i}S_i(z)+\frac{1}{n}\sum_{i=1}^{n-1}z^{-a_i}\overline{S_i(z)}\right)\left(\sum_{n/4\le i\le 3n/4}z^{-a_i}\varphi_i(z)\right)\\&=\frac{1}{n}\int w\sum_{n/4\le i\le 3n/4}S_i(z)\varphi_i(z)+\frac{1}{n}\int w\sum_{i\ne j}z^{a_i-a_j}S_i(z)\varphi_j(z)\\&\equiv \mathrm{I}+\mathrm{II}\end{aligned}$$

where in II the summation is over $i \ne j$ such that $1 \le i \le n-1$, or $-(n-1) \le i \le -1$, $n/4 \le j \le 3n/4$ and we have defined $a_{-i}=-a_i$, $S_{-i}(z)=\overline{S_i(z)}$ for convenience.

PROPOSITION 8. $\liminf_{n\to\infty} \mathrm{Re}(\mathrm{I}) \ge c_2 \int w$ *for some absolute constant* $c_2>0$.

PROPOSITION 9. $\lim_{n\to\infty} |\mathrm{II}|=0$.

REMARK. Clearly, Propositions 8 and 9 imply 7b).

PROOF OF PROPOSITION 8. Fix $n/4 \le i \le 3n/4$. Then

$$\frac{1}{n}\int wS_i\varphi_i=\frac{1}{n}\int wS_i\sum_{(i,k)=1}B_{ik}=\frac{1}{n}\sum_{(i,k)=1}\int wS_iB_{ik}\,.$$

Now $w \ge 0$, $B_{ik} \ge 0$ and

$$\mathrm{Re}\,S_i \ge n/4\sqrt{2}$$

on the support of B_{ik}, which is: $|\theta-2\pi k/i| \le \gamma=\pi/100n^2$ (see Proposition 5b) and Definition 6). Hence

$$\operatorname{Re}\left(\frac{1}{n}\int wS_i\varphi_i\right)=\frac{1}{n}\sum_{(i,k)=1}\int w(\operatorname{Re}S_i)B_{ik}\geq\frac{1}{n}\sum_{(i,k)=1}\int w\left(\frac{n}{4\sqrt{2}}\right)B_{ik}=\frac{1}{4\sqrt{2}}\int w\varphi_i\,.$$

Summing over i we have that

$$\operatorname{Re}(\mathrm{I})\geq\frac{1}{4\sqrt{2}}\sum_{n/4\leq i\leq 3n/4}\int w\varphi_i\,.$$

We now need the following proposition. It states that a) the numbers k relatively prime to i are "uniformly distributed" as $i\to\infty$, and b) the average number of relatively prime k is at least $\approx n$ when i ranges over $[n/4, 3n/4]$.

PROPOSITION 10. (Recall that φ_i depends implicitly on n)

10a) $$\lim_{\substack{n\to\infty\\(n/4\leq i\leq 3n/4)}}\frac{\int w\varphi_i}{\int\varphi_i}=\int w$$

10b) $$\sum_{n/4\leq i\leq 3n/4}\int\varphi_i\geq c$$

for some absolute constant $c>0$ (*where* n *is large enough*).

PROOF. See the appendix on number theory.

COMPLETION OF THE PROOF OF PROPOSITION 8. Fixing w, we can choose n_0 so that (by 10a))

$$\int w\varphi_i\geq\frac{1}{2}\left(\int w\right)\left(\int\varphi_i\right)\quad\left(\text{whenever } n\geq n_0 \text{ and } \frac{1}{4}n\leq i\leq\frac{3}{4}n\right).$$

Then for all $n\geq n_0$ and large enough so that 10b) holds, we have

$$\begin{aligned}\operatorname{Re}(\mathrm{I})&\geq\frac{1}{4\sqrt{2}}\sum_{n/4\leq i\leq 3n/4}\frac{1}{2}\left(\int w\right)\left(\int\varphi_i\right)\\&\geq\frac{1}{8\sqrt{2}}c\int w\quad\text{by 10b)}\,.\end{aligned}$$

Thus we can take $c_2=c/8\sqrt{2}$.

PROOF OF PROPOSITION 9. Recall that

$$\mathrm{II}=\frac{1}{n}\int w\sum_{i\neq j}z^{a_i-a_j}S_i(z)\varphi_j(z)=\frac{1}{n}\sum_{(i,j,k)}\int wz^{a_i-a_j}S_i(z)B_{jk}(z)$$

where the summation ranges over $i\neq j$, $(j,k)=1$, with $1\leq|i|\leq n-1$, $n/4\leq j\leq 3n/4$ and

$1 \le k \le j$. Also, we have defined $a_{-i} = -a_i$ and $S_{-i}(z) = \overline{S_i(z)}$. In this proof, the fact that j and k are relatively prime will not be used.

Now w is a fixed trigonometric polynomial. Let

$$w(z) = \sum_{-\omega \le \eta \le \omega} \hat{w}(\eta) z^{\eta} .$$

Recall that

$$S_i(z) = \sum_{\alpha=0}^{n-1-|i|} z^{i\alpha}, \qquad 1 \le |i| \le n-1 .$$

Hence $w(z)S_i(z)$ is of the form

$$w(z)S_i(z) = \sum_{|\beta| \le n^2+\omega} C_\beta z^\beta$$

with coefficients C_β bounded by

$$|C_\beta| \le n \cdot \max_{\eta} |\hat{w}(\eta)| \equiv nM .$$

Therefore,

$$\int w z^{a_i - a_j} S_i(z) B_{jk}(z) = \sum_{|\beta| \le n^2+\omega} C_\beta \int z^{a_i - a_j + \beta} B_{jk}(z) = \sum_{|\beta| \le n^2+\omega} C_\beta \hat{B}_{jk}(-a_i + a_j - \beta) .$$

By Definition 6, and recalling that B and B_{jk} are functions on the circle,

$$\hat{B}_{jk}(x) = \hat{B}(x) e^{-i2\pi kx/j} \quad (i = \sqrt{-1}), \quad x \in \boldsymbol{Z}$$

$$|\hat{B}_{jk}(x)| = |\hat{B}(x)|$$

$$= \begin{cases} \gamma/\pi & x = 0 \\ |\sin(x\gamma)|/\pi|x| & x \ne 0, \quad x \in \boldsymbol{Z} . \end{cases}$$

For $x = -a_i + a_j - \beta$ $(i \ne j)$ we have by 5a)

$$|x| \ge |a_i - a_j| - |\beta| \ge (n-1)! - (n^2 + \omega), \qquad 1 \le i \le n-1 .$$

This also holds for $-(n-1) \le i \le -1$ since by definition

$$|a_i - a_j| = |-a_{-i} - a_j| = a_{-i} + a_j \ge 2h_n \ge 2(n-1)!, \quad \left(\frac{1}{4} n \le j \le \frac{3}{4} n\right).$$

Therefore $x \ne 0$ for n large enough, and

$$|\hat{B}_{jk}(x)| \le 2/|x| \le 4/(n-1)!$$

for all $x = -a_i + a_j - \beta$ in the range of the summation. Thus

$$\begin{aligned}\left|\int wz^{a_i-a_j}S_i(z)B_{jk}(z)\right| &\leq [2(n^2+\omega)+1]\cdot\max|C_\beta|\cdot\max|\hat{B}_{jk}(x)| \\ &\leq [2(n^2+\omega)+1]\cdot nM\cdot 4/(n-1)! \\ &\leq cn^3/(n-1)! \quad \text{where } c \text{ does not depend on } n\,.\end{aligned}$$

Hence

$$\begin{aligned}|\mathrm{II}| &\leq \frac{1}{n}\operatorname{card}\{(i,j,k) \text{ in range of } \Sigma\}\cdot cn^3/(n-1)! \\ &\leq \frac{1}{n}\cdot(2n\cdot n\cdot n)\cdot cn^3/(n-1)! \to 0 \qquad \text{as} \quad n\to\infty\,.\end{aligned}$$

This completes the proof of Proposition 9.

Appendix on number theory. Let $x\in[0, 1]$ and define δ_x to be the unit point mass at x. Define measures μ_i, $i=1, 2, \cdots$ by

$$\mu_i=\sum_{\substack{(i,k)=1\\ 1\leq k\leq i}}\delta_{k/i}\,.$$

Thus $\mu_i(A)$ is the number of k such that $k/i\in A$ and k/i is in lowest terms, and $0<k/i\leq 1$.

LEMMA A1. *Fix an interval $I\subset[0, 1]$. Then*

$$\lim_{i\to\infty}\mu_i(I)/\mu_i([0, 1])=|I|\,.$$

PROOF. We first claim that there exist $\alpha<1$ and an integer i_0 such that for all $i\geq i_0$ and for all intervals I, $J\subset[0, 1]$ of equal length,

$$|\mu_i(I)-\mu_i(J)|\leq i^\alpha\,. \tag{1}$$

To see this, let $N_m(I)$ denote the number of integer multiples of m in the (real) interval I, where m is an integer. Clearly $|N_m(I_1)-N_m(I_2)|\leq 1$ for any two intervals of equal length, I_1 and I_2. Let $i\in N$ and let $p_1<\cdots<p_k$ be the distinct prime divisors of i. Then for any interval $I\subset[0, 1]$, $\mu_i(I)$ is the number of integers in the dilated interval iI which are not divisible by $p_1, \ldots, p_k$. Hence

$$\mu_i(I)=N_1(iI)-\sum_\alpha N_{p_\alpha}(iI)+\sum_{\alpha<\beta}N_{p_\alpha p_\beta}(iI)-\cdots$$

$$\mu_i(J)=N_1(iJ)-\sum_\alpha N_{p_\alpha}(iJ)+\sum_{\alpha<\beta}N_{p_\alpha p_\beta}(iJ)-\cdots$$

which implies

$$|\mu_i(I)-\mu_i(J)|\leq 1+\binom{k}{1}+\binom{k}{2}+\cdots+\binom{k}{k}=2^k,$$

since iI and iJ have the same length. Suppose that i was an odd number. Then $p_1\geq 3$, so $i\geq p_1p_2\cdots p_k\geq 3\cdot 3\cdots 3=3^k$ so $3^k\leq i\Rightarrow 2^k\leq(i)^{\log_3 2}$, and so we can take $\alpha=\log_3 2<1$ and $i_0=3$. If $i=2^tj$ where j is odd, we can modify this reasoning as follows. Let $M_m(I)$ denote the number of *odd* integer multiples of m in the interval I. Then again

$$|M_m(I_1)-M_m(I_2)|\leq 1$$

for all m and all intervals of equal length I_1 and I_2. Let $3\leq p_1<\cdots<p_k$ be the prime divisors of j. Then $\mu_i(I)$ equals the number of odd numbers in iI which are not divisible by $p_1,\ldots,p_k$, hence

$$\mu_i(I)=M_1(iI)-\sum_\alpha M_{p_\alpha}(iI)+\sum_{\alpha<\beta} M_{p_\alpha p_\beta}(iI)-\cdots.$$

So again

$$|\mu_i(I)-\mu_i(J)|\leq 2^k.$$

But $i>j\geq p_1\cdots p_k\geq 3^k$. So again we can take $\alpha=\log_3 2$. Then claim (1) is proved. Next we claim that for any $\beta<1$ there is $c>0$ such that for all i,

$$\mu_i([0,1])\equiv\phi(i)\geq ci^\beta. \tag{2}$$

(Recall that $\phi(i)$ is the Euler function). To see this recall that

$$\phi(i)=i\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_k}\right)$$

where $p_1\cdots p_k$ are the distinct prime divisors of i. But for any $\varepsilon>0$, $1-1/x\geq 1/x^\varepsilon$ when x is large enough, say $x\geq x_0(\varepsilon)$. Thus

$$\begin{aligned}\phi(i)&=i\prod_{p_\alpha<x_0}\left(1-\frac{1}{p_\alpha}\right)\prod_{p_\alpha\geq x_0}\left(1-\frac{1}{p_\alpha}\right)\\ &\geq i\cdot\left(\frac{1}{2}\right)^{x_0}\cdot\prod\frac{1}{p_\alpha^\varepsilon}\geq i\cdot\left(\frac{1}{2}\right)^{x_0}\cdot\frac{1}{i^\varepsilon}=\left(\frac{1}{2}\right)^{x_0}i^{1-\varepsilon}.\end{aligned}$$

So we can let $\varepsilon=1-\beta$ and $c=(1/2)^{x_0(\varepsilon)}$. Now we can prove Lemma A1. Fix $\alpha<\beta<1$ in the claims (1) and (2) above. Let $N\in\mathbf{N}$ and let $I_t=[t/N,(t+1)/N)$, $t=0,1,\ldots,N-1$. Then for all i,

$$N\min_i\mu_i(I_t)\leq\mu_i([0,1])\leq N\max_t\mu_i(I_t).$$

But if i is large enough $\mu_i([0,1])\geq ci^\beta$ and $\max_t\mu_i(I_t)-\min_t\mu_i(I_t)\leq i^\alpha$. Hence

$$\frac{\mu_i([0,1])}{N}-i^\alpha\le\mu_i(I_t)\le\frac{\mu_i([0,1])}{N}+i^\alpha .$$

But $i^\alpha/\mu_i([0,1])\le i^\alpha/ci^\beta\to 0$ as $i\to\infty$. Hence

$$\frac{\mu_i(I_t)}{\mu_i([0,1])}\to\frac{1}{N}=|I_t|\qquad\text{as}\quad i\to\infty .$$

The result for arbitrary intervals $I\subset[0,1]$ follows easily by approximation.

PROOF OF PROPOSITION 10a). For any interval $I\subset[0,1]$, we have (integrating with respect to normalized Lebesgue measure on T)

$$\int\chi_{2\pi I}\varphi_i\le\mu_i(I)\cdot 2\gamma/2\pi\quad\text{and}\quad\int\chi_{2\pi I}\varphi_i\ge(\mu_i(I)-2)2\gamma/2\pi$$

since the intervals $2\pi k/i+[-\gamma,\gamma]$, $(k,i)=1$, $1\le k\le i$ are disjoint and at most two of them contain end points of $2\pi I$. Also

$$\int\varphi_i=\mu_i([0,1])\cdot 2\gamma/2\pi .$$

Hence

$$\frac{\int\chi_{2\pi I}\varphi_i}{\int\varphi_i}\to|I|$$

as $n\to\infty$, $n/4\le i\le 3n/4$, since $\mu_i([0,1])\to\infty$.

In 10a) the w is continuous, so uniformly continuous. Hence 10a) follows. In other words, we have shown that the weak* limit of $\varphi_i d\theta/\int\varphi_i d\theta$ as $n\to\infty$ is the Lebesgue measure.

PROOF OF PROPOSITION 10b).

$$\sum_{n/4\le i\le 3n/4}\int\varphi_i=\sum_{n/4\le i\le 3n/4}\mu_i([0,1])\cdot 2\gamma/2\pi=\frac{1}{100n^2}\sum_{n/4\le i\le 3n/4}\phi(i)$$

where ϕ is the Euler function. But

$$\phi(i)=i\left(1-\frac{1}{p_1(i)}\right)\left(1-\frac{1}{p_2(i)}\right)\cdots$$

where $p_1(i)<p_2(i)<\cdots$ are the distinct prime divisors of i. Also, the arithmetic-geometric mean inequality gives (letting $N=\operatorname{card}\{i\in N,\ n/4\le i\le 3n/4\}$):

$$
\begin{aligned}
\frac{1}{N}\prod_{n/4\le i\le 3n/4}\phi(i) &\ge \Bigl(\sum_{n/4\le i\le 3n/4}\phi(i)\Bigr)^{1/N}\\
&\ge \Bigl(\prod_{n/4\le i\le 3n/4}\frac{n}{4}\Bigl(1-\frac{1}{p_1(i)}\Bigr)\Bigl(1-\frac{1}{p_2(i)}\Bigr)\cdots\Bigr)^{1/N}\\
&\ge \frac{n}{4}\Bigl(\prod_{p=2}^{n}\Bigl(1-\frac{1}{p}\Bigr)^{\#\{i:\,p|i,\,n/4\le i\le 3n/4\}}\Bigr)^{1/N}\\
&\ge \frac{n}{4}\Bigl(\prod_{p=2}^{n}\Bigl(1-\frac{1}{p}\Bigr)^{N/p+1}\Bigr)^{1/N}\ge\frac{n}{4}\Bigl(\prod_{p=2}^{n}\Bigl(1-\frac{1}{p}\Bigr)^{1/p}\Bigr)\cdot\Bigl(\frac{1}{2}\Bigr)^{n/N}\\
&\ge \frac{n}{4}\Bigl(\prod_{p=2}^{\infty}\Bigl(1-\frac{c}{p^2}\Bigr)\Bigr)\cdot\Bigl(\frac{1}{2}\Bigr)^{3}\ge c'n
\end{aligned}
$$

where c and $c'>0$ are absolute constants. Hence the proof of 10b) is complete.

REMARKS. 10b) is a weakened form of the following fact, whose proof was shown to me by D. Clark:

10c) For any $\varepsilon>0$, there is a $\delta>0$ such that

$$\#\Bigl\{i:\frac{1}{4}n\le i\le\frac{3}{4}n,\ \phi(i)\ge(1-\varepsilon)i\Bigr\}\ge\delta n$$

whenever n is large enough.

The proof of Clark is a generalization of [Hardy & Wright, Thm#330]. Also, it is possible to prove the result of this paper using 10c) instead of 10a) and 10b). This was in fact the strategy in a preliminary form of the proof.

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