

WEIGHTED L^p -BOUNDEDNESS FOR HIGHER ORDER COMMUTATORS OF OSCILLATORY SINGULAR INTEGRALS

Dedicated to Professor Satoru Igari on his sixtieth birthday

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Abstract. In this paper the authors study the weighted L^p -boundedness for higher order commutators of a class of oscillatory singular integrals with rough kernel. The main result in this paper gives a necessary and sufficient condition so that this higher order commutator is bounded on the weighted L^p space with certain weight.

1. Introduction. Let us consider the oscillatory singular integral defined by

$$Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{iP(x,y)} K(x-y) f(y) dy,$$

where $P(x, y)$ is a real polynomial on $\mathbf{R}^n \times \mathbf{R}^n$, and $K(x) = h(|x|)\Omega(x/|x|)|x|^{-n}$ with $h(r) \in \text{BV}(\mathbf{R}_+)$, where $\text{BV}(\mathbf{R}_+)$ denotes the class of functions of bounded variation on \mathbf{R}_+ . In 1987, Ricci and Stein [7] proved that T is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, if $K \in C^1(\mathbf{R}^n \setminus \{0\})$ and $h \equiv 1$. In 1992, Lu and Zhang [6] improved the result of Ricci and Stein and showed that T is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, provided $\Omega \in L^q(S^{n-1})$, $1 < q \leq \infty$ and $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$, where S^{n-1} denotes the unit sphere in \mathbf{R}^n . Moreover, the authors of [6] gave a necessary and sufficient condition so that T is bounded on $L^p(\mathbf{R}^n)$. Recently, the above result in [6] was extended by Jiang and Lu [5] to the case of $\Omega \in L \log^+ L(S^{n-1})$. The purpose of this paper is to study the weighted L^p -boundedness for higher order commutators formed by T and a function in $\text{BMO}(\mathbf{R}_+)$. If we restrict ourselves to the case where $P(x, y)$ is a nontrivial polynomial, then we shall get a criterion on weighted L^p -boundedness for the higher order commutators mentioned above.

Let us first give some definitions.

DEFINITION 1. Let $b(r) \in L_{\text{loc}}(\mathbf{R}_+)$. We say $b(r) \in \text{BMO}(\mathbf{R}_+)$, if

$$\|b\|_{*,+} = \sup_{I \subset \mathbf{R}_+} \frac{1}{|I|} \int_I |b(r) - b_I| dr < \infty,$$

where $b_I = |I|^{-1} \int_I b(r) dr$.

DEFINITION 2. Suppose that $\omega(r) \geq 0$ and $\omega \in L_{\text{loc}}(\mathbf{R}_+)$. For $1 < p < \infty$, we say $\omega \in A_p(\mathbf{R}_+)$, if there is a $C > 0$ such that for any $I \subset \mathbf{R}_+$,

$$\left(\frac{1}{|I|} \int_I \omega(r) dr \right) \left(\frac{1}{|I|} \int_I \omega(r)^{-1/(p-1)} dr \right)^{p-1} \leq C < \infty.$$

Moreover, if there is a $C > 0$ such that

$$\omega^*(r) \leq C\omega(r) \quad \text{a.e. } r \in \mathbf{R}_+,$$

then we say $\omega \in A_1(\mathbf{R}_+)$, where ω^* denotes the Hardy-Littlewood maximal function of ω defined by

$$\omega^*(t) = \sup_{t \in I \subset \mathbf{R}_+} \frac{1}{|I|} \int_I \omega(r) dr.$$

DEFINITION 3. For $1 < p < \infty$, we denote

$$\widetilde{A}_p(\mathbf{R}_+) = \{ \omega : \omega \geq 0, \omega \in L_{\text{loc}}(\mathbf{R}_+) \text{ and } \omega^2 \in A_p(\mathbf{R}_+) \}.$$

Now, we may formulate our results as follows:

THEOREM 1. Let $1 < p < \infty$, $\Omega \in L \log^+ L(S^{n-1})$, homogeneous of degree zero, $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$, $h(|x|) \in \text{BV}(\mathbf{R}_+)$, $b(x) = b(|x|) \in \text{BMO}(\mathbf{R}_+)$, and $\omega(x) = \omega(|x|) \in \widetilde{A}_p(\mathbf{R}_+)$. If the operator

$$\bar{T}f(x) = \text{p.v.} \int \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy$$

is bounded on $L^p(\omega)$, then for any $m \in \mathbf{Z}_+$ and any real polynomial $P(x, y)$ on $\mathbf{R}^n \times \mathbf{R}^n$, the higher order commutator

$$T_b^m f(x) = \text{p.v.} \int e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy$$

is also bounded on $L^p(\omega)$.

The following theorem is the main result of this paper.

THEOREM 2. Let $1 < p < \infty$. If Ω , h , b , m and ω are as in Theorem 1, then the following three statements are equivalent:

- (i) If $P(x, y)$ is a nontrivial polynomial (i.e., $P(x, y)$ does not take the form of $P_1(x) + P_2(y)$ (see [6])), then T_b^m is bounded on $L^p(\omega)$.
- (ii) If a nontrivial polynomial $P(x, y)$ satisfies

$$(1.1) \quad P(x, y) = P(x-t, y-t) + R_1(x, t) + R_2(y, t), \quad t \in \mathbf{R}^n,$$

where R_1 and R_2 are real polynomials, then T_b^m is bounded on $L^p(\omega)$.

- (iii) The truncated operator

$$\bar{T}_{b,0}^m f(x) = \text{p.v.} \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x)-b(y)]^m f(y) dy$$

is bounded on $L^p(\omega)$.

In proving Theorems 1 and 2, the operator M_Ω defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y) f(y)| dy,$$

a variant of the Hardy-Littlewood maximal function associated with $\Omega \in L^1(S^{n-1})$, shall play a key role.

In 1993, Duoandikoetxea [3] gave a weighted result for M_Ω :

THEOREM A. *Let $1 < p < \infty$ and $\omega(x) = \omega(|x|) = v_1(|x|)v_2(|x|)^{1-p}$, where either $v_i \in A_1(\mathbf{R}_+)$ and is decreasing or $v_i^2 \in A_1(\mathbf{R}_+)$, $i = 1, 2$. Then M_Ω is bounded on $L^p(\omega)$ and*

$$\|M_\Omega f\|_{p,\omega} \leq C \|\Omega\|_{L^1(S^{n-1})} \|f\|_{p,\omega}.$$

In proving Theorem 2, we shall use the following weighted L^p -boundedness of $M_{\Omega,b}^m$, a maximal operator related to higher order commutators, defined by

$$M_{\Omega,b}^m f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)| |b(x)-b(y)|^m |f(y)| dy,$$

where $b \in \text{BMO}(\mathbf{R}_+)$.

THEOREM 3. *Let $1 < p < \infty$ and $\Omega \in L^1(S^{n-1})$, homogeneous of degree zero, $b(x) = b(|x|) \in \text{BMO}(\mathbf{R}_+)$, $m \in \mathbf{Z}_+$ and $\omega(x) = \omega(|x|) \in \tilde{A}_p(\mathbf{R}_+)$. Then $M_{\Omega,b}^m$ is bounded on $L^p(\omega)$ and*

$$\|M_{\Omega,b}^m f\|_{p,\omega} \leq C \|\Omega\|_{L^1(S^{n-1})} \|f\|_{p,\omega}.$$

2. Some results on $\tilde{A}_p(\mathbf{R}_+)$.

LEMMA 1. *If $1 < p < \infty$, then the weights in $\tilde{A}_p(\mathbf{R}_+)$ have the following properties:*

- (i) $\tilde{A}_p(\mathbf{R}_+) \subset A_p(\mathbf{R}_+)$.
- (ii) For any $\omega(r) \in \tilde{A}_p(\mathbf{R}_+)$, there are weights v_1, v_2 such that $\omega = v_1 \cdot v_2^{1-p}$ and $v_1^2, v_2^2 \in A_1(\mathbf{R}_+)$.
- (iii) For any $\omega(r) \in \tilde{A}_p(\mathbf{R}_+)$, there exists an $\varepsilon > 0$ so that $\omega^{1+\varepsilon} \in \tilde{A}_p(\mathbf{R}_+)$.
- (iv) For any $\omega(r) \in \tilde{A}_p(\mathbf{R}_+)$, there exists an $\varepsilon > 0$ so that $p - \varepsilon > 1$ and $\omega \in \tilde{A}_{p-\varepsilon}(\mathbf{R}_+)$.

The above facts can be easily deduced from the definition of $\tilde{A}_p(\mathbf{R}_+)$ and corresponding properties of $A_p(\mathbf{R}_+)$. We omit the details here.

REMARK 1. By (ii) in Lemma 1 and Theorem A, we see that if $\omega \in \tilde{A}_p(\mathbf{R}_+)$, $1 < p < \infty$, then M_Ω is bounded on $L^p(\omega)$ and

$$\|M_{\Omega}f\|_{p,\omega} \leq C \|\Omega\|_{L^1(S^{n-1})} \|f\|_{p,\omega}.$$

LEMMA 2. Let $1 < p < \infty$. If $b(r) \in \text{BMO}(\mathbf{R}_+)$, then there is a $\lambda > 0$ such that $e^{\lambda b(r)} \in \widetilde{A}_p(\mathbf{R}_+)$.

PROOF. From the John-Nirenberg inequality for BMO and the reverse Hölder inequality for the weights in $A_p(\mathbf{R}_+)$, it follows that there is a $\lambda_0 > 0$ such that $e^{\lambda_0 b(r)} \in A_p(\mathbf{R}_+)$ (see, e.g., [4]). Now we take $\lambda = \lambda_0/2$. Then $(e^{\lambda b})^2 \in A_p(\mathbf{R}_+)$, i.e., $e^{\lambda b} \in \widetilde{A}_p(\mathbf{R}_+)$ by the definition of $\widetilde{A}_p(\mathbf{R}_+)$.

LEMMA 3. For $1 < p < \infty$ and $\lambda > 0$, there exists an $\eta = \eta(\lambda, p) > 0$ such that if $b(r) \in \text{BMO}(\mathbf{R}_+)$ and $\|b\|_{*,+} < \eta$, then $e^{\lambda b(r)} \in \widetilde{A}_p(\mathbf{R}_+)$.

PROOF. If we take $\eta_0 = \min\{c/\lambda, c(p-1)/\lambda\}$, where c is the absolute constant in the John-Nirenberg inequality, then when $\|b\|_{*,+} < \eta_0$ we have $e^{\lambda b(r)} \in A_p(\mathbf{R}_+)$ (see [4]). Now we let $\eta = \eta_0/2$. Obviously, if $\|b\|_{*,+} < \eta$, i.e., $\|2b\|_{*,+} < \eta_0$, then

$$e^{2\lambda b(r)} \in A_p(\mathbf{R}_+).$$

By the definition of $\widetilde{A}_p(\mathbf{R}_+)$, we have $e^{\lambda b(r)} \in \widetilde{A}_p(\mathbf{R}_+)$.

3. Proof of Theorem 3. Let us first give the proof of Theorem 3 by induction on m . By Theorem A, we see that Theorem 3 holds for $m=0$. Now we assume that the conclusion of Theorem 3 holds for $m-1$, and prove the conclusion for m . Since $\omega \in \widetilde{A}_p(\mathbf{R}_+)$, we can choose an $\varepsilon > 0$ so that $\omega^{1+\varepsilon} \in \widetilde{A}_p$ by Lemma 1. Then by the assumption of induction, $M_{\Omega,b}^{m-1}$ is bounded on $L^p(\omega^{1+\varepsilon})$ and

$$(3.1) \quad \|M_{\Omega,b}^{m-1}\varphi\|_{p,\omega^{1+\varepsilon}} \leq C_1 \|\Omega\|_{L^1(S^{n-1})} \|\varphi\|_{p,\omega^{1+\varepsilon}}, \quad \text{for } \varphi \in L^p(\omega^{1+\varepsilon}).$$

On the other hand, by taking $\lambda = p(1+\varepsilon)/\varepsilon$ and Lemma 3, we see that there exists an $\eta > 0$ such that

$$e^{pb(1+\varepsilon)/\varepsilon} \in \widetilde{A}_p(\mathbf{R}_+), \quad \text{if } \|b\|_{*,+} < \eta.$$

Since $b \in \text{BMO}$ implies that $tb \in \text{BMO}$ for $|t| \leq 1$ with a smaller BMO norm, we have

$$(3.2) \quad e^{tpb(1+\varepsilon)/\varepsilon} \in \widetilde{A}_p(\mathbf{R}_+), \quad \text{for } |t| \leq 1.$$

Without loss of generality we may assume that $\|b\|_{*,+} < \eta$. Indeed, otherwise we take $0 < \delta_0 < \eta$ and set $a(x) = \delta_0 b(x)/\|b\|_{*,+}$. Thus, $\|a\|_{*,+} = \delta_0 < \eta$ and

$$M_{\Omega,b}^m f(x) = \left(\frac{\|b\|_{*,+}}{\delta_0} \right)^m M_{\Omega,a}^m f(x).$$

Therefore, it suffices to consider $M_{\Omega,a}^m$. By the assumption of induction and (3.2), we see that for any $\theta \in [0, 2\pi]$ and $\varphi \in L^p(e^{pb(1+\varepsilon)\cos\theta/\varepsilon})$,

$$(3.3) \quad \|M_{\Omega,b}^{m-1}\varphi\|_{p, e^{pb(1+\varepsilon)\cos\theta/\varepsilon}} \leq C_2 \|\Omega\|_{L^1(S^{n-1})} \|\varphi\|_{p, e^{pb(1+\varepsilon)\cos\theta/\varepsilon}},$$

where C_2 depends on p , b and ω , but not on θ . Applying the Stein-Weiss interpolation theorem (see [8] or [2]) between (3.1) and (3.3), we obtain that for any $\theta \in [0, 2\pi]$ and $\varphi \in L^p(\omega e^{pb\cos\theta})$,

$$(3.4) \quad \|M_{\Omega,b}^{m-1}\varphi\|_{p, \omega e^{pb\cos\theta}} \leq C \|\Omega\|_{L^1(S^{n-1})} \|\varphi\|_{p, \omega e^{pb\cos\theta}},$$

where $C = \max\{C_1, C_2\}$ and depends only on p , b and ω , but not on θ . In the following, we shall use the equality

$$(3.5) \quad b(x) - b(y) = \frac{1}{2\pi} \int_0^{2\pi} e^{e^{i\theta}[b(x)-b(y)]} e^{-i\theta} d\theta.$$

In fact, let $g(z) = e^{z[b(x)-b(y)]}$, $z \in \mathbb{C}$. Then by the analyticity of $g(z)$ on \mathbb{C} and the Cauchy integration formula, we have

$$b(x) - b(y) = g'(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{g(z)}{|z|^2} dz.$$

And this is just (3.5). Moreover, if we denote $g_\theta(x) = f(x)e^{-b(x)\cos\theta}$ for any $\theta \in [0, 2\pi]$, then it follows from $f \in L^p(\omega)$ that

$$(3.6) \quad g_\theta \in L^p(\omega e^{pb\cos\theta}) \quad \text{and} \quad \|g_\theta\|_{p, \omega e^{pb\cos\theta}} = \|f\|_{p, \omega}.$$

Hence, by (3.5) and (3.6), we have

$$\begin{aligned} M_{\Omega,b}^m f(x) &= \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)| |b(x)-b(y)|^m |f(y)| dy \\ &= \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)| |b(x)-b(y)|^{m-1} \\ &\quad \cdot \left| \frac{1}{2\pi} \int_0^{2\pi} e^{e^{i\theta}[b(x)-b(y)]} e^{-i\theta} d\theta \right| |f(y)| dy \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)| |b(x)-b(y)|^{m-1} \\ &\quad \cdot |f(y)e^{-b(y)\cos\theta}| dy e^{b(x)\cos\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} M_{\Omega,b}^{m-1}(g_\theta)(x) \cdot e^{b(x)\cos\theta} d\theta. \end{aligned}$$

Using Minkowski's inequality, (3.4), (3.6) and the above, we get

$$\|M_{\Omega,b}^m f\|_{p, \omega} \leq \left(\int_{\mathbb{R}^n} \left| \frac{1}{2\pi} \int_0^{2\pi} M_{\Omega,b}^{m-1}(g_\theta)(x) e^{b(x)\cos\theta} d\theta \right|^p \omega(x) dx \right)^{1/p}$$

$$\begin{aligned}
&\leq \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{\mathbf{R}^n} \left[M_{\Omega, b}^{m-1}(g_\theta)(x) \right]^p \omega(x) e^{pb(x) \cos \theta} dx \right)^{1/p} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \|M_{\Omega, b}^{m-1}(g_\theta)\|_{p, \omega e^{pb \cos \theta}} d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} C \cdot \|\Omega\|_{L^1(S^{n-1})} \|g_\theta\|_{p, \omega e^{pb \cos \theta}} d\theta \\
&= C \|\Omega\|_{L^1(S^{n-1})} \|f\|_{p, \omega}.
\end{aligned}$$

This finishes the proof of Theorem 3.

4. Some lemmas. Before proving Theorems 1 and 2, we give some lemmas. Let

$$Gf(x) = \text{p.v.} \int_{\mathbf{R}^n} K(x, y) f(y) dy$$

and

$$G_0 f(x) = \text{p.v.} \int_{|x-y| < 1} K(x, y) f(y) dy.$$

LEMMA 4. Let $1 \leq p < \infty$ and $\omega(x) \geq 0$. If G_0 is bounded on $L^p(\omega)$, then the inequality

$$(4.1) \quad \left(\int_{|x-t| < \varepsilon} |G_0 f(x)|^p \omega(x) dx \right)^{1/p} \leq C_\varepsilon \left(\int_{|y-t| < 1+\varepsilon} |f(y)|^p \omega(y) dy \right)^{1/p}$$

holds for any $\varepsilon > 0$, where C_ε is independent of t and f . Conversely, if (4.1) holds for certain $\varepsilon > 0$, then G_0 is bounded on $L^p(\omega)$.

See [5] for the proof.

LEMMA 5. Let $1 < p < \infty$, $b(x) = b(|x|) \in \text{BMO}(\mathbf{R}_+)$, $m \in \mathbf{Z}_+$, $\omega(x) = \omega(|x|) \in \widetilde{A}_p(\mathbf{R}_+)$. If $\Omega \in L^1(S^{n-1})$, homogeneous of degree zero, $|K(x, y)| \leq C |\Omega(x-y)| |b(x) - b(y)|^m |x-y|^{-n}$, and G is bounded on $L^p(\omega)$, then so is G_0 .

PROOF. By Lemma 4, it will suffice to prove

$$\int_{|x-t| < 1/4} |G_0 f(x)|^p \omega(x) dx \leq C \int_{|y-t| < 5/4} |f(y)|^p \omega(y) dy, \quad t \in \mathbf{R}^n.$$

Now, we split f into three parts $f = f_1 + f_2 + f_3$ for given t , where

$$f_1(y) = f(y) \chi_{(|y-t| < 1/2)}(y),$$

$$f_2(y) = f(y) \chi_{(1/2 \leq |y-t| < 5/4)}(y),$$

and

$$f_3(y) = f(y)\chi_{(|y-t| \geq 5/4)}(y).$$

Note that $|x-t| < 1/4$ and $|y-t| < 1/2$ imply $|x-y| < 1$. Thus, we have

$$G_0 f_1(x) = G f_1(x), \quad |x-t| < 1/4.$$

Since G is bounded on $L^p(\omega)$, we get

$$\begin{aligned} \int_{|x-t| < 1/4} |G_0 f_1(x)|^p \omega(x) dx &= \int_{|x-t| < 1/4} |G f_1(x)|^p \omega(x) dx \\ &\leq C \int_{|y-t| < 1/2} |f(y)|^p \omega(y) dy. \end{aligned}$$

By the assumption on $K(x, y)$, we have

$$\begin{aligned} |G_0 f_2(x)| &\leq \int_{1/4 < |x-y| < 3/2} \frac{C|\Omega(x-y)|}{|x-y|^n} |b(x) - b(y)|^m |f_2(y)| dy \\ &\leq C M_{\Omega, b}^m f_2(x). \end{aligned}$$

Thus, it follows from Theorem 3 and the above that

$$\left(\int_{|x-t| < 1/4} |G_0 f_2(x)|^p \omega(x) dx \right)^{1/p} \leq C \left(\int_{|y-t| < 5/4} |f(y)|^p \omega(y) dy \right)^{1/p}.$$

Finally, we notice that $|x-t| < 1/4$ and $|y-t| > 5/4$ imply $|x-y| > 1$. Thus, $G_0 f_3(x) = 0$ if $|x-t| < 1/4$. This completes the proof of Lemma 5.

LEMMA 6. Let $1 < p < \infty$, $h(r) \in L^\infty(\mathbf{R}_+)$ and $\omega(x) = \omega(|x|) \in \widetilde{A}_p(\mathbf{R}_+)$. If $\Omega \in L^1(S^{n-1})$, homogeneous of degree zero and \bar{T} , defined in Theorem 1, is bounded on $L^p(\omega)$, then for any $b(x) = b(|x|) \in \text{BMO}(\mathbf{R}_+)$, $m \in \mathbf{Z}_+$ and any real polynomial $P(x, y)$, the operator

$$T_{b,0}^m f(x) = \text{p.v.} \int_{|x-y| < 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy$$

is bounded on $L^p(\omega)$.

PROOF. By Lemma 5 for $m=0$, we see that the truncated operator of \bar{T} defined by

$$\bar{T}_0 f(x) = \text{p.v.} \int_{|x-y| < 1} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy$$

is bounded on $L^p(\omega)$. By Theorem A (or Remark 1) and Lemma 3 in [5], we see that Lemma 6 holds for $m=0$. On the other hand, it follows from Lemmas 1–3 that the results on commutators of linear operators given in [1] also hold if we use $\text{BMO}(\mathbf{R}_+)$ and $\widetilde{A}_p(\mathbf{R}_+)$ instead of $\text{BMO}(\mathbf{R}^n)$ and $A_p(\mathbf{R}^n)$ respectively. Thus, we see that the m -th commutator of \bar{T} and $b \in \text{BMO}(\mathbf{R}_+)$, defined by

$$\bar{T}_b^m(x) = \text{p.v.} \int \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy,$$

is bounded on $L^p(\omega)$. Using Lemma 5 again, we get that the truncated operator of \bar{T}_b^m , i.e.

$$\bar{T}_{b,0}^m f(x) = \text{p.v.} \int_{|x-y| < 1} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy,$$

is also bounded on $L^p(\omega)$. By the above result, Theorem 3 and Lemma 4, and using the method proving Lemma 3 in [5], one can prove that the truncated operator $T_{b,0}^m$ is bounded on $L^p(\omega)$. We omit the details.

LEMMA 7. *Let $1 < p < \infty$. If Ω , h , m and ω are as in Theorem 1, then the operator*

$$T_{b,\infty}^m f(x) = \int_{|x-y| \geq 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy$$

is bounded on $L^p(\omega)$ for any real nontrivial polynomial $P(x, y)$ on $\mathbf{R}^n \times \mathbf{R}^n$.

PROOF. We split $T_{b,\infty}^m$ as follows:

$$\begin{aligned} T_{b,\infty}^m f(x) &= \int_{|x-y| \geq 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy \\ &= \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \int_{2^{j-1} \leq |x-y| < 2^j} e^{iP(x,y)} \frac{\Omega_k(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy \\ &:= \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} T_{b,j,k}^m f(x), \end{aligned}$$

where

$$\begin{aligned} \Omega_k(x') &= \Omega(x') \chi_{E_k}(x'), \\ E_0 &= \{x' \in S^{n-1} : |\Omega(x')| < 1\}, \end{aligned}$$

and

$$E_k = \{x' \in S^{n-1} : 2^{k-1} \leq |\Omega(x')| < 2^k\}, \quad k \in \mathbf{N}.$$

Now, if we can prove the following two inequalities:

$$(4.2) \quad \|T_{b,j,k}^m f\|_{p,\omega} \leq C 2^{-\delta j} \|\Omega_k\|_{L^\infty(S^{n-1})} \|f\|_{p,\omega}$$

and

$$(4.3) \quad \|T_{b,j,k}^m f\|_{p,\omega} \leq C \|\Omega_k\|_{L^1(S^{n-1})} \|f\|_{p,\omega},$$

where $j = 1, 2, \dots$, $k = 0, 1, \dots$, and $\delta > 0$ is independent of k , f and Ω , then we shall deduce the conclusion of Lemma 7. Indeed, we choose a positive integer $M > 1/\delta$. Then

$$\begin{aligned}
\|T_{b,\infty}^m f\|_{p,\omega} &\leq \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \|T_{b,j,k}^m f\|_{p,\omega} \\
&= \sum_{j=1}^{\infty} \|T_{b,j,0}^m f\|_{p,\omega} + \sum_{k=1}^{\infty} \sum_{1 \leq j \leq Mk} \|T_{b,j,k}^m f\|_{p,\omega} + \sum_{k=1}^{\infty} \sum_{j > Mk} \|T_{b,j,k}^m f\|_{p,\omega} \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

Using (4.2), we get

$$I_1 \leq \sum_{j=1}^{\infty} C 2^{-\delta j} \|\Omega_0\|_{L^\infty(S^{n-1})} \|f\|_{p,\omega} \leq C \|f\|_{p,\omega}$$

and

$$\begin{aligned}
I_3 &\leq C \sum_{k=1}^{\infty} \sum_{j > Mk} 2^{-\delta j} \|\Omega_k\|_{L^\infty(S^{n-1})} \|f\|_{p,\omega} \\
&\leq C \sum_{k=1}^{\infty} \sum_{j > Mk} 2^{-\delta j} \cdot 2^k \|f\|_{p,\omega} \leq C \|f\|_{p,\omega} \sum_{k=1}^{\infty} 2^{-(M\delta-1)k} \leq C \|f\|_{p,\omega}.
\end{aligned}$$

By (4.3), we have

$$\begin{aligned}
I_2 &\leq C \sum_{k=1}^{\infty} \sum_{1 \leq j \leq Mk} \|\Omega_k\|_{L^1(S^{n-1})} \|f\|_{p,\omega} \\
&\leq C \|f\|_{p,\omega} \sum_{k=1}^{\infty} k 2^k \cdot |E_k| \leq C \|f\|_{p,\omega} \|\Omega\|_{L^{\log+L}(S^{n-1})} \leq C \|f\|_{p,\omega}.
\end{aligned}$$

Thus, we obtain

$$\|T_{b,\infty}^m\|_{p,\omega} \leq C \|f\|_{p,\omega}.$$

This confirms the above assertion. It remains to prove (4.2) and (4.3). Let us first prove (4.3). Since

$$\begin{aligned}
|T_{b,j,k}^m f(x)| &\leq \|h\|_{L^\infty(\mathbf{R}^n)} \int_{2^{j-1} \leq |x-y| < 2^j} \frac{|\Omega_k(x-y)|}{|x-y|^n} |b(x)-b(y)|^m |f(y)| dy \\
&\leq C \cdot M_{\Omega_k,b}^m f(x),
\end{aligned}$$

we get

$$\|T_{b,j,k}^m f\|_{p,\omega} \leq C \|M_{\Omega_k,b}^m f\|_{p,\omega} \leq C \|\Omega_k\|_{L^1(S^{n-1})} \|f\|_{p,\omega}$$

by Theorem 3. This proves (4.3).

Let us now turn to the proof of (4.2). The proof is completed by induction on m . Since $h \in \text{BV}(\mathbf{R}_+)$ and $P(x, y)$ is a real nontrivial polynomial, by a method similar to that in [6], we can prove that there exists an $\eta > 0$ such that

$$(4.4) \quad \|T_{b,j,k}^0 f\|_p \leq C 2^{-\eta j} \|\Omega_k\|_{L^\infty(\mathbb{S}^{n-1})} \|f\|_p,$$

where C is independent of j and k . On the other hand, we have

$$|T_{b,j,k}^0 f(x)| \leq \int_{2^{j-1} \leq |x-y| < 2^j} \frac{|\Omega_k(x-y)| |h(|x-y|)|}{|x-y|^n} |f(y)| dy \leq C \cdot M_{\Omega_k} f(x).$$

From (iii) in Lemma 1 and Remark 1, it follows that

$$\|T_{b,j,k}^0 f\|_{p,\omega^{1+\varepsilon}} \leq C \cdot \|M_{\Omega_k} f\|_{p,\omega^{1+\varepsilon}} \leq C \|\Omega_k\|_{L^1(\mathbb{S}^{n-1})} \|f\|_{p,\omega^{1+\varepsilon}} \leq C \|\Omega_k\|_{L^\infty(\mathbb{S}^{n-1})} \|f\|_{p,\omega^{1+\varepsilon}}.$$

Combining the above with (4.4), and using the Stein-Weiss theorem of interpolation with change of measure [8], we get

$$\|T_{b,j,k}^0 f\|_{p,\omega} \leq C 2^{-\eta_1 j} \|\Omega_k\|_{L^\infty(\mathbb{S}^{n-1})} \|f\|_{p,\omega},$$

where $\eta_1 > 0$ is independent of j , k , f and Ω . This shows that (4.2) holds for $m=0$. We now assume that (4.2) holds for $m-1$, i.e., for any $\varphi \in L^p(\omega)$ with $\omega \in \widetilde{A}_p(\mathbf{R}_+)$ we have

$$(4.5) \quad \|T_{b,j,k}^{m-1} \varphi\|_{p,\omega} \leq C 2^{-\eta_{m-1} j} \|\Omega_k\|_{L^\infty(\mathbb{S}^{n-1})} \|\varphi\|_{p,\omega}.$$

By $\omega \in \widetilde{A}_p(\mathbf{R}_+)$ and Lemma 1, there exists an $\varepsilon > 0$ so that $\omega^{1+\varepsilon} \in \widetilde{A}_p(\mathbf{R}_+)$. Therefore, for any $\varphi \in L^p(\omega^{1+\varepsilon})$,

$$(4.6) \quad \|T_{b,j,k}^{m-1} \varphi\|_{p,\omega^{1+\varepsilon}} \leq C 2^{-\eta'_{m-1} j} \|\Omega_k\|_{L^\infty(\mathbb{S}^{n-1})} \|\varphi\|_{p,\omega^{1+\varepsilon}}.$$

Repeating the proof of Theorem 3, we can obtain the following results: For any $\theta \in [0, 2\pi]$ and $\varphi \in L^p(e^{pb(1+\varepsilon)\cos\theta/\varepsilon})$,

$$(4.7) \quad \|T_{b,j,k}^{m-1} \varphi\|_{p,e^{pb(1+\varepsilon)\cos\theta/\varepsilon}} \leq C 2^{-\eta''_{m-1} j} \|\Omega_k\|_{L^\infty(\mathbb{S}^{n-1})} \|\varphi\|_{p,e^{pb(1+\varepsilon)\cos\theta/\varepsilon}},$$

where C and η''_{m-1} depend on p , b and ω , but not on j , k and θ . By interpolating with change measure between (4.6) and (4.7), we see that for any $\theta \in [0, 2\pi]$ and $\varphi \in L^p(\omega e^{pb\cos\theta})$,

$$(4.8) \quad \|T_{b,j,k}^{m-1} \varphi\|_{p,\omega e^{pb\cos\theta}} \leq C 2^{-\delta j} \|\Omega_k\|_{L^\infty(\mathbb{S}^{n-1})} \|\varphi\|_{p,\omega e^{pb\cos\theta}},$$

where C and $\delta > 0$ are independent of j , k and θ . Moreover, if we let $g_\theta(x) = f(x)e^{-b(x)e^{i\theta}}$, then it is easy to check that for any $\theta \in [0, 2\pi]$,

$$(4.9) \quad g_\theta \in L^p(\omega e^{pb\cos\theta}) \quad \text{and} \quad \|g_\theta\|_{p,\omega e^{pb\cos\theta}} = \|f\|_{p,\omega}.$$

For simplicity, we denote

$$K_{m-1}(x, y) = e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^{m-1} \chi_{[2^{j-1} \leq |x-y| < 2^j]}(x-y).$$

Thus, by (3.5) and the above notation, we have

$$\begin{aligned}
T_{b,j,k}^m f(x) &= \int_{\mathbf{R}^n} K_{m-1}(x, y) [b(x) - b(y)] f(y) dy \\
&= \int_{\mathbf{R}^n} K_{m-1}(x, y) \left(\frac{1}{2\pi} \int_0^{2\pi} e^{e^{i\theta}[b(x)-b(y)]} e^{-i\theta} d\theta \right) f(y) dy \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbf{R}^n} K_{m-1}(x, y) f(y) e^{-b(y)e^{i\theta}} dy \cdot e^{b(x)e^{i\theta}} \cdot e^{-i\theta} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} T_{b,j,k}^{m-1}(g_\theta)(x) \cdot e^{b(x)e^{i\theta}} \cdot e^{-i\theta} d\theta.
\end{aligned}$$

Hence, by Minkowski's inequality, we get

$$\begin{aligned}
\|T_{b,j,k}^m f\|_{p,\omega} &\leq \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{\mathbf{R}^n} \left| T_{b,j,k}^{m-1}(g_\theta)(x) e^{b(x)e^{i\theta}} \right|^p \omega(x) dx \right)^{1/p} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \|T_{b,j,k}^{m-1}(g_\theta)\|_{p,\omega e^{pb \cos \theta}} d\theta.
\end{aligned}$$

From (4.8) and (4.9), it follows that

$$\begin{aligned}
\|T_{b,j,k}^m f\|_{p,\omega} &\leq \frac{1}{2\pi} \int_0^{2\pi} C 2^{-\delta j} \|\Omega_k\|_{L^\infty(\mathbb{S}^{n-1})} \|g_\theta\|_{p,\omega e^{pb \cos \theta}} d\theta \\
&= C 2^{-\delta j} \|\Omega_k\|_{L^\infty(\mathbb{S}^{n-1})} \|f\|_{p,\omega}.
\end{aligned}$$

Thus, we proved (4.2) for m and the proof of Lemma 7 is completed.

5. Proofs of Theorems 1 and 2. Theorem 1 can be directly deduced from Lemmas 6 and 7. Let us now give the proof of Theorem 2.

(i) \Rightarrow (ii). This step is obvious.

(ii) \Rightarrow (iii). Set

$$\begin{aligned}
T_b^m f(x) &= \text{p.v.} \int_{|x-y|<1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy \\
&\quad + \int_{|x-y|\geq 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy \\
&:= T_{b,0}^m f(x) + T_{b,\infty}^m f(x).
\end{aligned}$$

From Lemma 7, it follows that $T_{b,\infty}^m$ is bounded on $L^p(\omega)$. So $T_{b,0}^m$ is a bounded operator on $L^p(\omega)$. We take a $t \in \mathbf{R}^n$. For $|x-t|<1$, we have

$$T_{b,0}^m f(x) = T_{b,0}^m [f(\cdot) \chi_{B(t,2)}](x).$$

Thus, by Lemma 4, we have

$$(5.1) \quad \left(\int_{|x-t|<1} |T_{b,0}^m f(x)|^p \omega(x) dx \right)^{1/p} \leq C \left(\int_{|y-t|<2} |f(y)|^p \omega(y) dy \right)^{1/p},$$

where C is independent of t and f . By (1.1), we write

$$\bar{T}_{b,0}^m f(x) = e^{-iR_1(x,t)} \text{p.v.} \int_{|x-y|<1} e^{iP(x,y)} K_m(x,y) f(y) e^{-iP(x-t,y-t)} e^{-iR_2(y,t)} dy$$

for $t \in \mathbf{R}^n$, where

$$K_m(x,y) = \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m.$$

Express $e^{-iP(x-t,y-t)}$ into the Taylor series:

$$\begin{aligned} e^{-iP(x-t,y-t)} &= \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} [P(x-t, y-t)]^k = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \left[\sum_{\alpha, \beta} a_{\alpha, \beta} (x-t)^\alpha (y-t)^\beta \right]^k \\ &= \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \sum_{\mu, \nu} b_{\mu, \nu} (x-t)^\mu (y-t)^\nu. \end{aligned}$$

Thus, we have

$$\begin{aligned} \left(\int_{|x-t|<1} |\bar{T}_{b,0}^m f(x)|^p \omega(x) dx \right)^{1/p} &\leq \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\mu, \nu} |b_{\mu, \nu}| \cdot \left(\int_{|x-t|<1} |(x-t)^\mu \right. \\ &\quad \cdot \left. \int_{|x-y|<1} \exp(iP(x,y)) K_m(x,y) f(y) \exp(-iR_2(y,t)) (y-t)^\nu dy \right|^p \omega(x) dx \Big)^{1/p} \\ &\leq \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\mu, \nu} |b_{\mu, \nu}| |\xi^\mu| \left(\int_{|x-t|<1} \left| T_{b,0}^m [\exp(-iR_2(\cdot, t)) f(\cdot) (\cdot - t)^\nu](x) \right|^p \omega(x) dx \right)^{1/p}, \end{aligned}$$

where $\xi = (1, 1, \dots, 1)$. By (5.1), we obtain

$$\begin{aligned} &\left(\int_{|x-t|<1} |\bar{T}_{b,0}^m f(x)|^p \omega(x) dx \right)^{1/p} \\ &\leq C \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\mu, \nu} |b_{\mu, \nu}| |\xi^\mu| \left(\int_{|y-t|<2} |f(y)(y-t)^\nu|^p \omega(y) dy \right)^{1/p} \\ &\leq C \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\mu, \nu} |b_{\mu, \nu}| |\xi^\mu \cdot \eta^\nu| \left(\int_{|y-t|<2} |f(y)|^p \omega(y) dy \right)^{1/p} \\ &\leq C \sum_{k=0}^{\infty} \frac{1}{k!} \left[\sum_{\alpha, \beta} |a_{\alpha, \beta}| |\xi^\alpha \cdot \eta^\beta|^k \right] \left(\int_{|y-t|<2} |f(y)|^p \omega(y) dy \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
&= C \exp\left(\sum_{\alpha, \beta} |a_{\alpha, \beta}| |\xi^\alpha \cdot \eta^\beta|\right) \left(\int_{|y-t|<2} |f(y)|^p \omega(y) dy\right)^{1/p} \\
&\leq C \left(\int_{|y-t|<2} |f(y)|^p \omega(y) dy\right)^{1/p},
\end{aligned}$$

where $\eta = (2, 2, \dots, 2)$. By Lemma 4, we see that the above implies (iii).

(iii) \Rightarrow (i). This step is just a direct result of Lemmas 6 and 7. This completes the proof of Theorem 2.

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