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ON THE GEOGRAPHY OF THREEFOLDS

(WITH AN APPENDIX BY MEI-CHU CHANG)

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Abstract. By the covering of some toric varieties we construct a family of minimal smooth threefolds of general type which gives some effective results on the geography problem.

1. Introduction. The name "geography" introduced by Persson [P2] is used to describe the distribution of the Chern numbers of algebraic manifolds of general type. In general the Chern numbers of algebraic manifolds of general type satisfy some inequalities. The geography problem asks whether all tuples of the numbers satisfying all the numerical relations are the Chern numbers of some algebraic manifold of general type. For the surface geography Persson [P2], Xiao [X], Chen [C], etc. have done a lot of work.

Let X be a threefold of general type. Since the minimal models of X may not be unique and may not be smooth, we have some difficulties to talk about the geography problem. We cannot speak of the Chern numbers unless we fix a unique model in each birational equivalence class. In this paper we restrict ourselves to certain threefolds of general type with some smooth minimal model. The known inequalities (equalities) of the Chern numbers, when K_X is ample, are (cf. Hunt [H]):

$$c_1^3(X)<0\,,\quad c_1c_2(X)<0\,,\quad c_1c_2\equiv 0 \pmod{24}\,,\quad 3c_1^3(X)\geq 8c_1c_2(X)\,.$$

We can ask a crude geography problem as Sommese [S] did for surfaces: what is the closure of $[c_1^3 : c_1c_2 : c_3]$ in $\boldsymbol{P}^2(\boldsymbol{Q})$ for the smooth minimal threefolds of general type?

Hunt [H] obtained a partial answer to this question by Kummer coverings: for any pair of rational numbers (α, β) in the two triangles ABC and DEF with $A=(12/11, 1/11)$, $B=(6/5, 3/5)$, $C=(14/11, 19/33)$, $D=(1, -2/5)$, $E=(1, 2/3)$, $F=(32/29, 55/87)$, there exists a minimal threefold of general type such that $c_1^3/c_1c_2=\alpha$, $c_3/c_1c_2=\beta$.

In this paper we use the double covers of $\boldsymbol{P}^1\times \boldsymbol{F}_n$ branched along some specially chosen configuration of divisors to construct some smooth minimal threefolds of general type, where $\boldsymbol{F}_n$ is the Hirzebruch rational surface. Then by base changes over their two natural fibrations we get the following theorem which much improves the result quoted above.

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THEOREM 3.2. *Let* $(n, a, b, r)\in\{(n, a, b, r)\in \boldsymbol{Z}^{+4} \mid n+a\geq 3,\ r\geq 3,\ 0\leq b\leq [2a/3]\}\setminus \{(0, 3, 2, r) \mid r\in \boldsymbol{Z}^{+}\}$. *Then for any pair of rational numbers* (α, β) *in the triangles ABC with*

$$A=\left(\frac{-2n-2b-rb+2rn+2ra}{4rn+8ra-8rb}, \frac{4n+2a+b}{6n+12a-12b}\right),$$

$$B=\left(\frac{2n+2a-b}{2n+4a-4b}, \frac{4n+10a-11b}{6n+12a-12b}\right), \quad C=\left(\frac{1}{2}, \frac{5}{6}\right),$$

there exists a minimal smooth complex threefold X of general type such that $c_1^3(X)/c_1c_2(X)=\alpha$, $c_3(X)/c_1c_2(X)=\beta$.

NOTATION AND CONVENTIONS. The threefolds considered in this paper are complex projective threefolds.

If X is a smooth (canonical) algebraic variety, then K_X denotes a canonical divisor of X, $\chi(\mathcal{O}_X)$ denotes the Euler-Poincare characteristic of the structure sheaf, $c_i(X)$ denotes the i-th Chern class and $\chi_{\mathrm{top}}(X)$ denotes the topological Euler-Poincare characteristic.

$\boldsymbol{F}_n$ denotes the Hirzebruch rational surface, while S_∞ denotes its section such that $S_\infty^2=-n$.

$\equiv$ denotes the linear equivalence.

$[a]$ is the greatest integer not exceeding a for $a\in \boldsymbol{R}$.

$\boldsymbol{Z}^+$ denotes the set of non-negative integers.

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2. Some technical lemmas. In this section we will give some technical lemmas. First we give some lemmas about the double covers between threefolds, whose proofs are similar to those in the surface case. See [P2], [P3] for the surface case.

A double cover between two algebraic varieties $\pi: X\to Y$ is a finite morphism of degree two. We assume that Y is a smooth variety hereafter. Then it is well known (cf. [P1]) that π is determined by a pair (B, δ) (called the double cover data or the building data): an even effective divisor B and an invertible sheaf δ such that $B\in|\delta^{\otimes 2}|$. X is normal if and only if B is reduced; X is smooth if and only if B is smooth. In the following formulas and computations we will use $\mathcal{O}(-B/2)$ instead of δ^{-1} since they are numerically equivalent by $B\in|\delta^{\otimes 2}|$.

If B is smooth, then (cf. [P1]) we have $\pi_*\mathcal{O}_X=\mathcal{O}_Y\oplus\delta^{-1}$ and $K_X=\pi^*(K_Y+B/2)$, so we obtain the following:

LEMMA 2.1 (cf. [P3]). *Let* $\pi: X\to Y$ *be a double cover between two smooth threefolds with building data* (B, δ). *Then*

(1) $K_X^3 = 2(K_Y + B/2)^3$,
(2) $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \chi(\mathcal{O}_Y(-B/2))$,
(3) $c_3(X) = 2c_3(Y) - \chi_{\mathrm{top}}(B)$.

If B is reduced but singular, then X is singular over the singularities of B. Similarly to the surface case, we have a not necessarily unique resolution based on the embedded resolution of $B \subset Y$ as follows:

LEMMA 2.2 (cf. [P2]). *Let $\pi: X \to Y$ be a double cover between two projective threefolds with building data (B, δ) with B reduced. Then there exists a birational morphism $\rho: \tilde{Y} \to Y$ which is based on a minimal embedded resolution of $B \subset Y$, a smooth even effective divisor $\tilde{B} \subset \tilde{Y}$ and an invertible sheaf $\tilde{\delta} \in \mathrm{Pic}(\tilde{Y})$ such that*

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "\tilde{\pi}"'] & \cdots \arrow[r] & X_1 \arrow[r, "\phi_1"] \arrow[d] & X \arrow[d, "\pi"] \\ \tilde{Y} \arrow[r] & \cdots \arrow[r] & Y_1 \arrow[r, "\rho_1"] & Y. \end{tikzcd}$$

(1) $B = \rho(\tilde{B})$, $\delta = \rho(\tilde{\delta})$;

(2) $(\tilde{B}, \tilde{\delta})$ *is a double cover data, and if $\tilde{\pi}: \tilde{X} \to \tilde{Y}$ is the corresponding cover, $\tilde{X}$ gives a resolution of X. (We will also call any such $\tilde{X}$ the "canonical" resolution of X.)*

For the proof of the lemma, notice that $B \subset Y$ has an embedded resolution such that its corresponding reduced even inverse image (cf. [P3] for the definition) is a divisor with only normal crossing. Then one extra blow-up (if two smooth irreducible components intersect along a smooth curve) or three extra blow-ups (if three smooth irreducible components intersect at a point) will make the corresponding reduced even inverse image smooth, which is what we wanted. The resolution is not unique due to the nonuniqueness of the minimal embedded resolution of $B \subset Y$.

The "canonical" resolution in Lemma 2.2 may produce exceptional divisors on $\tilde{X}$. In the following we will restrict ourselves to the exceptional divisors of the first kind whose contractions give a smooth variety. Now we give the definitions and a criterion for the exceptional divisors of the first kind of types (I) and (II), respectively.

Type (I): $E \subset X$ is a divisor on X, $E \cong \boldsymbol{P}^2$, such that $\mathcal{O}_E(E) = \mathcal{O}_{\boldsymbol{P}^2}(-1)$. E can be contracted to a smooth point;

Type (II): $E \subset X$ is a minimal ruled surface over a curve with its fiber by f. If $Ef = -1$, then E can be contracted to a smooth curve.

LEMMA 2.3 (cf. [P2]). *Let $\pi: X \to Y$ be a double cover between two smooth threefolds. Then the exceptional divisors of the first kind on X occur in the following ways:*

(1) *The pull-backs of the exceptional divisors of the first kind of the same type on Y which are disjoint from the branch locus. These always come in pairs;*

(2) *The reduced part of the pull-back of one component F of the branch locus. For type* (I) *F is isomorphic to $\boldsymbol{P}^2$ such that $\mathcal{O}_F(F) = \mathcal{O}_{\boldsymbol{P}^2}(-2)$; for type* (II) *$F$ is a minimal*

smooth ruled surface with $Fg=-2$, *g being the fibre of the ruling of F.*

(3) *The pull-backs of the exceptional divisors of the first kind of the same type on Y which intersect the branch locus.*

PROOF. The lists given above are exceptional divisors of the corresponding type, so we only need to prove that they exhaust all the possibilities.

Let $\sigma: X \to X$ be the involution induced by π and $E \subset X$ an exceptional divisor. Then either $\sigma(E)=E$ or $\sigma(E)=E' \neq E$. Let $\sigma(E) \neq E$. Then $\pi(E)=\pi(E')=F$ is disjoint from the branch locus. Otherwise the inverse image of $\pi(E)$ will consist of only one component, which contradicts the fact that E and E' are two components of $\pi^{-1}(F)$. Hence F, E and E' are exceptional divisors of the same type, which is (1).

Now suppose $\sigma(E)=E$. If π is ramified along E then $\pi(E)=F$ is one component of the branch locus, which is (2). If π is not ramified along E, F is not any component of the branch locus, we get (3). In the third case F will intersect the branch locus and the exceptional divisors do not come in pairs.

The third case does not occur in the sequel, since in our discussion the exceptional divisors of the first kind on the base threefolds are either one component of or disjoint from the branch locus.

In the following we give two base change lemmas which will be used in the next section.

LEMMA 2.4 (cf. [H]). *Let $f_i: X \to C_i$ $(i=1, 2)$ be two fibrations of a threefold X over the curves C_i with $g(C_i)>0$, S_i the corresponding general fiber of f_i, such that both $f_2: S_1 \to C_2$ and $f_1: S_2 \to C_1$ are fibrations. Then for any pair of rational numbers (α, β) in the triangle ABC with*

$$A=\left(\frac{c_1^3(X)}{c_1c_2(X)}, \frac{c_3(X)}{c_1c_2(X)}\right),$$

$$B=\left(\frac{3c_1^2(S_1)}{c_1^2(S_1)+c_2(S_1)}, \frac{c_2(S_1)}{c_1^2(S_1)+c_2(S_1)}\right),$$

$$C=\left(\frac{3c_1^2(S_2)}{c_1^2(S_2)+c_2(S_2)}, \frac{c_2(S_2)}{c_1^2(S_2)+c_2(S_2)}\right),$$

there exists a threefold $\tilde{X}$ such that $c_1^3(\tilde{X})/c_1c_2(\tilde{X})=\alpha$, $c_3(\tilde{X})/c_1c_2(\tilde{X})=\beta$, and $\tilde{X}$ is smooth minimal if X is.

If X has two fibrations over $\boldsymbol{P}^1$, then we have:

LEMMA 2.5. *Let $f_1, f_2: X \to \boldsymbol{P}^1$ be two fibrations of the smooth threefold X in Lemma 2.4, and F_i the general fibre of $S_i \to \boldsymbol{P}^1$. Then there exists a smooth threefold $\tilde{X}$ such that $\tilde{X}$ has two fibrations over elliptic curves and*

$$c_1^3(\tilde{X})=4c_1^3(X)-24c_1^2(S_1)-24c_1^2(S_2)+96\chi_{\mathrm{top}}(F_2),$$

$$c_1c_2(\tilde{X})=4c_1c_2(X)-8(c_1^2(S_1)+c_2(S_1))-8(c_1^2(S_2)+c_2(S_2))+48\chi_{\rm top}(F_2)\,,$$
$$c_3(\tilde{X})=4c_3(X)-8c_2(S_1)-8c_2(S_2)+16\chi_{\rm top}(F_2)\,.$$

Let $\tilde{S}_i$ $(i=1,2)$ *be the general fibers of* $\tilde{X}$. *Then*

$$c_1^2(\tilde{S}_i)=2c_1^2(S_i)-8\chi_{\rm top}(F_i)\,,$$
$$c_2(\tilde{S}_i)=2c_2(S_i)-4\chi_{\rm top}(F_i)\,,\qquad (i=1,2)\,.$$

PROOF. Let $E_i\to\boldsymbol{P}^1$ $(i=1,2)$ be double covers such that they do not ramify at the critical points of f_i and E_i's elliptic curves. Then $(X\times_{\boldsymbol{P}^1}E_1)\times_{\boldsymbol{P}^1}E_2$ is what we wanted. The invariants can be calculated in a standard way.

REMARK 2.6. Instead of the elliptic curves we can use any two curves E_i $(i=1,2)$ of genera >0 to get a general form of Lemma 2.5, which involves the ramification degrees and degrees of the base changes besides the above parameters. For the proof of our main theorem, however, Lemma 2.5 is sufficient.

3. Construction. In this section we will construct a family of smooth minimal threefolds of general type with two fibrations to get some partial results on the geography problem.

First we will give some remarks about the canonical singularities: By the "canonical" resolution of double cover, if the branch locus has some singularities such that the resulting variety has the corresponding canonical singularities and their resolutions up to terminal singularities are really resolutions (cf. [R]), then as in the surface case the adjunction formula of the canonical divisor does not involve the exceptional divisors, and we can regard the branch locus as smooth when calculating K^3 and $\chi(\mathcal{O})$ by the Riemann-Roch theorem. Therefore if the branch locus has singularities of the type $xy=0$ or $xyz=0$, one can check easily that K^3 and $\chi(\mathcal{O})$ can be calculated as if the branch locus is smooth. The topological Euler characteristic can be calculated by some standard formula.

Now let V be the toric variety $\boldsymbol{P}^1\times\boldsymbol{F}_n$. Then by [O], $K_V^3=-48$, $\chi(\mathcal{O}_V)=1$, $p_g(V)=0$, $\chi_{\rm top}(V)=8$, and $\mathrm{Pic}(V)$ is freely generated by $E_4:=\mathrm{pt}\times\boldsymbol{F}_n$, $E_5:=\boldsymbol{P}^1\times S_\infty$ and $E_6:=\boldsymbol{P}^1\times\boldsymbol{P}^1$. Moreover,

$$E_4^3=E_5^3=E_6^3=E_4^2E_5=E_4^2E_6=E_5^2E_6=0\,,$$
$$E_6^2E_4=E_6^2E_5=0\,,\quad E_4E_5E_6=1\,,\quad E_4E_5^2=-n\,,$$
$$K_V=-2E_4-2E_5-(n+2)E_6\,,\quad \chi_{\rm top}(V)=8\,,$$
$$c_2(V)=E_5^2+4E_4E_5+(2n+4)E_4E_6+(n+4)E_5E_6\,.$$

Let us fix two non-negative integers a and b such that $b\leq[2a/3]$. Persson [P2] constructed three bisections $S_i\in|2S_\infty+(b+2n)\boldsymbol{P}^1|\subset\boldsymbol{F}_n$ such that $S_1+S_2+S_3$ has exactly $2n+2b$ infinitely near triple points $\{A_1,\cdots,A_{2n+2b}\}$ and no other singular points. Choose $2a-3b$ fibers $\boldsymbol{P}^1_{q_i}$ in $\boldsymbol{F}_n$ not passing through the $2n+2b$ points and let

$$\begin{aligned}B=&p_1\times\boldsymbol{F}_n+p_2\times\boldsymbol{F}_n+\cdots+p_{2r}\times\boldsymbol{F}_n\\&+\boldsymbol{P}^1\times S_1+\boldsymbol{P}^1\times S_2+\boldsymbol{P}^1\times S_3\\&+\boldsymbol{P}^1\times\boldsymbol{P}^1_{q_1}+\boldsymbol{P}^1\times\boldsymbol{P}^1_{q_2}+\cdots+\boldsymbol{P}^1\times\boldsymbol{P}^1_{q_{2a-3b}}.\end{aligned}$$

Then $B\equiv 2rE_4+6E_5+(6n+2a)E_6$ has $2n+2b$ triple lines $\boldsymbol{P}^1\times A_i$ $(1\le i\le 2n+2b)$ and some double lines as its singularities. Moreover

$$\begin{aligned}&B^3=72(3rn+2ar)\,,\\&KB^2=-72n-48a-16ar-24rn-48r\,,\\&K^2B=16r+24n+16a+48\,,\\&c_2B=8r+12n+8a+24\,.\end{aligned}$$

Let W be the double cover of V branched along B. Then W is singular. We will give its "canonical" resolution as follows.

Let $\pi_1\colon V_1\to V$ be the blow-up of V along all the $2n+2b$ triple lines $\boldsymbol{P}^1\times A_i\cong\boldsymbol{P}^1$ $(1\le i\le 2n+2b)$. Since all their normal bundles are isomorphic to $\mathcal{O}_{\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1}$, the exceptional divisor E_1 consists of mutually disjoint $2n+2b$ copies of $\boldsymbol{P}^1\times\boldsymbol{P}^1$. Now

$$\begin{aligned}&K_{V_1}=\pi_1^*K_V+E_1\,,\\&c_2(V_1)=\pi_1^*c_2(V)-2f_1-E_1^2\,,\\&\pi_1^*E_5E_1=\pi_1^*E_6E_1=0\,,\\&\pi_1^*E_4^2E_1=0\,,\quad \pi_1^*E_4E_1^2=-(2n+2b)\,,\\&E_1^3=0\,,\quad E_1f_1=-(2n+2b)\,,\\&\pi_1^*E_4f_1=\pi_1^*E_5f_1=\pi_1^*E_6f_1=\pi_1^*c_2(V)f_1=0\,,\end{aligned}$$

and

$$\begin{aligned}&\pi_1^*(\boldsymbol{P}^1\times S_i)=(\boldsymbol{P}^1\times S_i)^-+E_1 && (1\le i\le 3)\,,\\&\pi_1^*(p_i\times\boldsymbol{F}_n)=(p_i\times\boldsymbol{F}_n)^- && (1\le i\le 2n+2b)\,,\\&\pi_1^*(\boldsymbol{P}^1\times\boldsymbol{P}^1_{q_i})=(\boldsymbol{P}^1\times\boldsymbol{P}^1_{q_i})^- && (1\le i\le 2a-3b)\,,\end{aligned}$$

where D^- represents the strict transform of D and f_1 is the fiber of E_1. Hence $(p_i\times\boldsymbol{F}_n)^-$ is the blow-up of $p_i\times\boldsymbol{F}_n$ centred at the $2n+2b$ points $p_i\times A_j$, $(\boldsymbol{P}^1\times\boldsymbol{P}^1_{q_i})^-\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$ and $(\boldsymbol{P}^1\times S_i)^-\cong\boldsymbol{P}^1\times S_i$. Therefore the resulting branch locus B_1 is

$$B_1=\sum(\boldsymbol{P}^1\times S_i)^-+\sum(p_i\times\boldsymbol{F}_n)^-+\sum(\boldsymbol{P}^1\times\boldsymbol{P}^1_{qi})^-+E_1\,.$$

It has $2n+2b$ quadruple lines $(P^1\times S_i)^-\cap E_1$. All their normal bundles are isomorphic to $\mathcal{O}_{\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1}$.

Let $\pi_2\colon V_2\to V_1$ be the blow-up of V_1 along all the quadruple lines. Then the exceptional divisor E_2 also consists of mutually disjoin $2n+2b$ copies of $\boldsymbol{P}^1\times\boldsymbol{P}^1$. Hence

$$
\begin{aligned}
&K_{V_2}=\pi_2^*K_{V_1}+E_2\,,\\
&c_2(V_2)=\pi_2^*\pi_1^*c_2(V)-2\pi_2^*f_1-\pi_2^*E_1^2-E_2^2\,,\\
&E_2\pi_2^*\pi_1^*E_5=E_2\pi_2^*\pi_1^*E_6=0\,,\quad E_2\pi_2^*\pi_1^*E_4^2=0\,,\\
&E_2^3=0\,,\quad E_2f_2=-(2n+2b)\,,\quad E_2^2\pi_2^*\pi_1^*E_4=-(2n+2b)\,,\\
&\pi_2^*\pi_1^*E_4f_2=\pi_2^*\pi_1^*E_5f_2=\pi_2^*\pi_1^*E_6f_2=\pi_2^*\pi_1^*c_2(V)f_2=\pi_2^*E_1f_2=0\,,\\
&\pi_2^*E_1^2E_2=\pi_2^*E_1E_2^2=0\,,\quad \pi_2^*f_1E_2=0
\end{aligned}
$$

and

$$
\begin{aligned}
&\pi_2^*((\boldsymbol{P}^1\times S_i)^-)=(\boldsymbol{P}^1\times S_i)^=+E_2 && (1\le i\le 3)\,,\\
&\pi_2^*((p_i\times \boldsymbol{F}_n)^-)=(p_i\times \boldsymbol{F}_n)^= && (1\le i\le 2n+2b)\,,\\
&\pi_2^*((\boldsymbol{P}^1\times \boldsymbol{P}^1_{q_i})^-)=(\boldsymbol{P}^1\times \boldsymbol{P}^1_{q_i})^- && (1\le i\le 2a-3b)\,,\\
&\pi_2^*E_1=E_1^-+E_2\,,\\
&\pi_2^*f_1=f_1^-+f_2\,,
\end{aligned}
$$

where $D^=$ and D^- represent the strict transforms of D^- and D, respectively, and f_2 is the fibre of E_2. Therefore $(p_i\times\boldsymbol{F}_n)^=$ is the blow-up of $(p_i\times\boldsymbol{F}_n)^-$ centred at $2n+2b$ points, $(\boldsymbol{P}^1\times S_i)^=\cong\boldsymbol{P}^1\times S_i$, $(\boldsymbol{P}^1\times\boldsymbol{P}^1_{q_i})^=\cong\boldsymbol{P}^1\times\boldsymbol{P}^1_{q_i}$ and $E_1^-\cong E_1$.

Now the resulting branch locus is

$$B_2=\sum(\boldsymbol{P}^1\times S_i)^=+\sum(p_i\times\boldsymbol{F}_n)^=+\sum(\boldsymbol{P}^1\times\boldsymbol{P}^1_{q_i})^=+E_1^-\,.$$

It has only double lines, $xy=0$ and $xyz=0$, as its singularities, which give some canonical singularities on the covering variety W_2.

Let $\tilde{V}$ be the blow-up of V_2 such that the resulting branch locus $\tilde{B}$ is smooth, and $\tilde{W}$ the covering variety. Then we have to blow up $2r(2n+2b)+(2a-3b)(2r+6)$ rational curves and $6r$ curves of genus $n+b-1$. $E_1^=$, the strict transform of E_1 in $\tilde{V}$, consists of $2n+2b$ copies of $E_{1i}^=\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$ such that for each copy $E_{1i}^=f_{1i}^==-2$. By Lemma 2.3, the strict transform in $\tilde{W}$ of each $E_{1i}^=$ can be contracted to a smooth rational curve. Denote by W^* the contracted variety of $\tilde{W}$. Then W^* is what we wanted. We will calculate its invariants.

$$
\begin{tikzcd}
 & \tilde{W} \arrow[r] \arrow[dl] \arrow[d, "\tilde{\varphi}"] & W_2 \arrow[r] \arrow[d, "\varphi_2"] & W_1 \arrow[r] \arrow[d, "\varphi_1"] & W \arrow[d, "\varphi"] \\
W^* & \tilde{V} \arrow[r, "\pi_3"] & V_2 \arrow[r, "\pi_2"] & V_1 \arrow[r, "\pi_1"] & V\,.
\end{tikzcd}
$$

Since B_2 has only singularities of types $xy=0$ and $xyz=0$, by Lemma 2.1 and the remark in the second paragraph of this section we get

$$
\begin{aligned}
K_{\tilde W}^3 &= 2(K_{W_2}+B_2/2)^3 \\
&= 2(K_V+B/2)^3-6(r-2)(2n+2b) \\
&= 6(r-2)(3n+2a-4)-6(r-2)(2n+2b)\,,
\end{aligned}
$$

$$
\begin{aligned}
\chi(\mathcal{O}_{\tilde W}) &= \chi(\mathcal{O}_{W_2}) \\
&= \chi(\mathcal{O}_{V_2})+\chi(\mathcal{O}_V(-B/2))+(r-1)(2n+2b) \\
&= 2a+2r+2br+n-2b-2ar-rn-1\,,
\end{aligned}
$$

$$
\begin{aligned}
\chi_{\mathrm{top}}(\tilde W) &= 2\chi_{\mathrm{top}}(\tilde V)-\chi_{\mathrm{top}}(\tilde B) \\
&= -8+40r+(2n+2b)(10-8r)+(2a-3b)(20-4r)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\chi_{\mathrm{top}}(\tilde V) &= 8+(2n+2b)(4-2r)+(2a-3b)(4r+12)+24r\,, \\
\chi_{\mathrm{top}}(\tilde B) &= 24+(2n+2b)(4r-2)+(2a-3b)(12r+4)+8r\,,
\end{aligned}
$$

since in the resolution process from V_2 to $\tilde V$, each $p_i\times \boldsymbol{F}_n$ was blown up, and the topological invariant is affected.

In $\tilde W$ the strict transform E of E_1 can be contracted to $2n+2b$ smooth rational curves, and

$$
\begin{aligned}
K_{\tilde W}^3 &= K_{W^*}^3+3(K_{\tilde W}-E)E^2\,, \\
2E &= \tilde\varphi^*(\pi_3^*(\pi_2^*E_1-E_2)-E_3)\,,
\end{aligned}
$$

where E_3 is the exceptional divisor for the blow-up along the curves $E_1^-\cap(p_i\times \boldsymbol{F}_n)^=$, and

$$
K_{\tilde W}E^2=(2n+2b)(4-r)\,,\quad E^3=0\,.
$$

Hence we get

$$
\begin{aligned}
c_1^3(W^*) &= c_1^3(\tilde W)+3K_{\tilde W}E^2-3E^3 \\
&= 24r+36n+24a+6rb-12rn-12ra-48\,,
\end{aligned}
$$

$$
\begin{aligned}
c_1c_2(W^*) &= c_1c_2(\tilde W)=24\chi(\tilde W) \\
&= 48r+24n+48a+48rb-24rn-48ra-48b-24\,,
\end{aligned}
$$

$$
\begin{aligned}
c_3(W^*) &= c_3(\tilde W)-2(2n+2b) \\
&= 40r+16n+40a-16rn-8ra-4rb-44b-8\,.
\end{aligned}
$$

W^* has two natural fibrations over $\boldsymbol{P}^1$ whose general fibers S_1, S_2 are double covers of the corresponding fibers of V to $\boldsymbol{P}^1$. S_2 is obtained by contracting the $2n+2b$ copies of the (-1)-curves in the corresponding general fiber of $\tilde W$ over $\boldsymbol{P}^1$, so we get by [P2]

$$
\begin{aligned}
c_1^2(S_1) &= 4(r-2)\,, \\
c_2(S_1) &= 4(5r-1)\,,
\end{aligned}
$$

$$c_1^2(S_2)=4n+4a-2b-8\,,$$
$$c_2(S_2)=8n+20a-22b-4\,.$$

THEOREM 3.1. *Let W^* be the contraction of the "canonical" resolution of the double cover of V branched along the divisor B. Then W^* is a smooth minimal threefold of general type which has two natural fibrations over $\boldsymbol{P}^1$ if $n+a>2$, $r>2$, unless $n=0$, $a=3$ and $b=2$. The Chern numbers of W^* and the two general fibers S_1, S_2 are described as above.*

PROOF. We only need to prove that K_{W^*} is nef under the conditions, since by [P2] it is of general type. Now $K_{W_2}=\varphi_2^*(\pi_2^*\pi_1^*(K_V+B/2)-E_2)$, and the first Chow group $A_1(W_2)$ of W_2 is generated (cf. [O]) by $\varphi_2^*(E_4^-E_5^-)$, $\varphi_2^*(E_4^-E_6^-)$, $\varphi_2^*(E_6^-E_5^-)$, $\varphi_2^*(E_4^-E_1^-)$, $\varphi_2^*(E_1^-E_2)$, and $\varphi_2^*(E_4^-E_2)$. Moreover,

$$K_{W_2}\varphi_2^*(E_4^-E_5^-)=n+a-2\,,$$
$$K_{W_2}\varphi_2^*(E_4^-E_6^-)=1\,,$$
$$K_{W_2}\varphi_2^*(E_6^-E_5^-)=r-2\,,$$
$$K_{W_2}\varphi_2^*(E_1^-E_2)=(r-2)(2n+2b)\,,$$
$$K_{W_2}\varphi_2^*(E_4^-E_2)=2n+2b\,,$$
$$K_{W_2}\varphi_2^*(E_4^-E_1^-)=-(2n+2b)\,.$$

The pull-back of the generators of $A_1(W_2)$ to $\tilde{W}$, together with the exceptional divisors of $\tilde{W}\to W_2$, gives a set of generators of $A_1(\tilde{W})$. Contracting the exceptional divisors of the first kind in $\tilde{W}$, we find that the strict transform of $E_4^-E_1^-$ is contracted to $2n+2b$ points in W^*, and the images in W^* of the exceptional divisors of $\tilde{W}\to W_2$ do not intersect K_{W^*}. Hence K_{W^*} is nef as claimed.

By Lemmas 2.4 and 2.5 we get:

THEOREM 3.2.[1] *Let $(n,a,b,r)\in\{(n,a,b,r)\in\boldsymbol{Z}^{+4}\,|\,n+a\geq 3,\ r\geq 3,\ 0\leq b\leq[2a/3]\}\setminus\{(0,3,2,r)\,|\,r\in\boldsymbol{Z}^+\}$. Then for any pair of rational numbers (α,β) in the triangle ABC with*

$$A=\left(\frac{-2n-2b-rb+2rn+2ra}{4rn+8ra-8rb},\ \frac{4n+2a+b}{6n+12a-12b}\right),$$
$$B=\left(\frac{2n+2a-b}{2n+4a-4b},\ \frac{4n+10a-11b}{6n+12a-12b}\right),\quad C=\left(\frac{1}{2},\ \frac{5}{6}\right),$$

there exists a minimal (smooth) threefold X of general type such that $c_1^3(X)/c_1c_2(X)=\alpha$, and $c_3(X)/c_1c_2(X)=\beta$.

Now we will give some examples to cover the area in $\boldsymbol{C}^2(c_1^3/c_1c_2,\ c_3/c_1c_2)$. We can get more by evaluating (n,a,b,r):

If $n=b=0$, then $B=C$, and the triangle degenerates to a line segment;

[1] See the appendix by Mei-Chu Chang in this volume.

If $(n, a, b)=(0, 3, 1)$, $r\geq 3$, then $A(r)=((5r-2)/16r, 7/24)$, $B=(5/8, 19/24)$, $C=(1/2, 5/6)$;

If $(n, a, b)=(0, 4, 1)$, $r\geq 3$, then $A(r)=((7r-2)/24r, 1/4)$, $B=(7/12, 29/36)$, $C=(1/2, 5/6)$;

If $(n, a, b)=(0, 4, 2)$, $r\geq 3$, then $A(r)=((3r-2)/8r, 5/12)$, $B=(3/4, 3/4)$, $C=(1/2, 5/6)$;

If $n>2$, $a=b=0$, $r\geq 3$, then $A(r)=((r-1)/2r, 2/3)$, $B=(1, 2/3)$, $C=(1/2, 5/6)$.

REMARK 3.3. As pointed out by the referees, by Theorem 3.1 and some direct calculations we can get some partial results on the general geography problem as follows: for any three tuples of integers (x, y, z), if there exists an integer $r\geq 3$ such that

(1) $(6r^2-4r-2)x-(r^2+2r-11)y-(3r^2-15r+12)z+24r^3-168r^2-120r+72=0$,

(2) $72r^2+(2x-3z+72)r-10x+6z-432\geq 0$,

(3) $24r^2-(2x+3z-120)r-22x+528\geq 0$,

(4) $12r^2+(3z-4x+168)r+4x-9z-276\geq 0$,

then there exists a smooth minimal threefold X of general type such that $c_1^3(X)=x$, $c_1c_2(X)=y$, $c_3(X)=z$.

The proof is straightforward: (1) is a result of the expression for the invariants of W^* by solving r, while (2), (3) and (4) are the results of $b\geq 0$, $n+a\geq 3$ and $b\leq 2a/3-1$, respectively. We use $b\leq 2a/3-1$ instead of $b\leq[2a/3]$, which is easier and excludes the case $(n, a, b)=(0, 3, 2)$.

For a given (x, y, z) it is easy to show whether there exists an integer r satisfying the above conditions or not.

Combining Theorem 3.1 and Remark 2.6, we can get the same kind of results as in the Remark 3.3, but all these seem to be too vague to be useful, so we omit it.

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Appendix by Mei-Chu CHANG*

In this appendix we give a "global" statement of Liu's theorem.

Let $L=(1/4, 1/6)$, $M=(1/3, 2/3)$, $C=(1/2, 5/6)$, *and* $N=(1, 2/3)$ *be four points on the xy-plane, and let* $\mathscr{E}$ *be the curve connecting L and N with defining equations*

$$y=\begin{cases} 2x-1/3 & 1/4\leq x\leq 2/7 \\ (1/3)(4\sqrt{14x}-8x-5) & 2/7\leq x\leq 7/8 \\ 2/3 & 7/8\leq x\leq 1\,. \end{cases} \tag{1}$$

Let R be the region bounded by the line segments $\overline{LM}$, $\overline{MC}$, $\overline{CN}$ *and the curve* $\mathscr{E}$. *Then for any rational point* (x, y) *in R, there exists a minimal smooth complex threefold X of general type such that* $c_1^3(X)/c_1c_2(X)=x$ *and* $c_3(X)/c_1c_2(X)=y$.

(See the chart attached.)

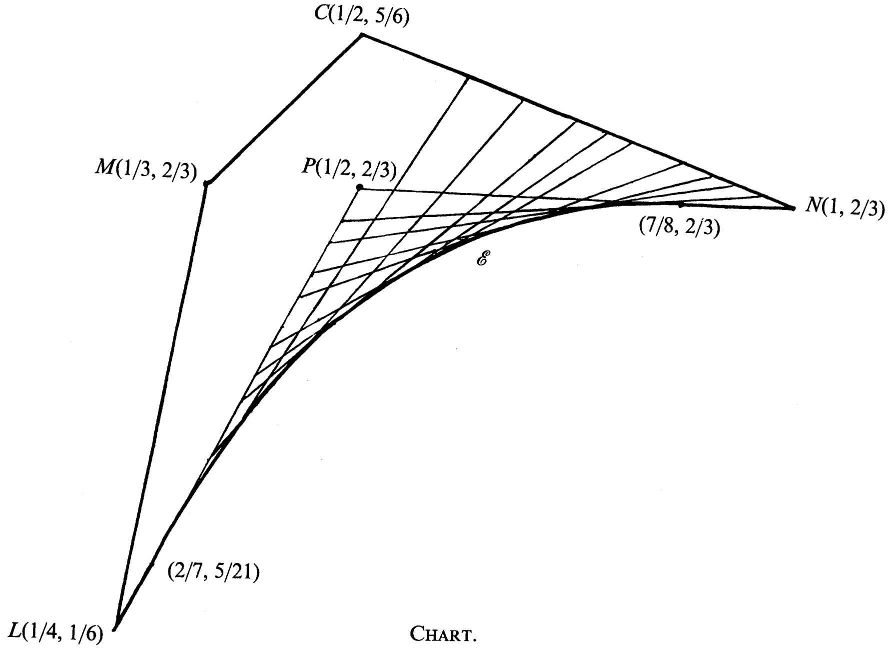

CHART.

* Partially supported by NSF Grant DMS-9304580.

What we need to show is that R is the union of the triangles ABC with

$$A=\left(\frac{-2n-2b-rb+2rn+2ra}{4rn+8ra-8rb},\ \frac{4n+2a+b}{6n+12a-12b}\right),$$

$$B=\left(\frac{2n+2a-b}{2n+4a-4b},\ \frac{4n+10a-11b}{6n+12a-12b}\right),$$

$$C=\left(\frac{1}{2},\ \frac{5}{6}\right),$$

where n, a, b, r are positive integers with $r\geq 3$, $n+a\geq 3$ and satisfying

$$0\leq b\leq [2a/3] \quad \text{and} \quad (n, a, b)\neq (0, 3, 2)\,. \tag{2}$$

First we change variables. Let

$$h=\frac{n+b}{n+2a-2b}\,. \tag{3}$$

CLAIM. *Any rational number between* 0 *and* 1 *can be written as* $(n+b)/(n+2a-2b)$ *with* n, a, b *satisfying* (2).

PROOF. Let p/q be such a rational number. We take

$$a=q-p+3\,,\quad b=2\,,\quad \text{and}\quad n=2p-2\,.$$

□

Expressing the coordinates of points A, B in terms of h and r, we have

$$\begin{aligned} A=A_{h,r}:&=\left(\frac{1}{4}+\frac{h}{4}-\frac{h}{2r},\ \frac{1}{6}+\frac{h}{2}\right),\\ B=B_h:&=\left(\frac{1}{2}+\frac{h}{2},\ \frac{5}{6}-\frac{h}{6}\right). \end{aligned} \tag{4}$$

Let $P(1/2, 2/3)$ and $D_r=(1/2-1/2r, 2/3)$. Then D_r moves from M to P along $\overline{MP}$ (note that $D_3=M$ and $D_\infty=P$). B_h moves from C to N along $\overline{CN}$. With r fixed, $A_{h,r}$ moves from L to D_r along $\overline{LD_r}$.

It is easy to see that, for h fixed, the line $\overline{A_{h,\infty}B_h}$ has the smallest slope among all $\overline{A_{h,r}B_h}$. In other words, $\mathscr{E}$ is the oscillating curve of the family $\{\overline{A_{h,\infty}B_h}\}_h$, and we are working on the family of lines connecting $\overline{LP}$ and $\overline{CN}$.

To find the defining equation of $\mathscr{E}$, we let $\varphi(x, y, h)=0$ be the equation of the line $\overline{A_hB_h}$, with

$$A_h=\left(\frac{1}{4}+\frac{h}{4},\ \frac{1}{6}+\frac{h}{2}\right)$$

and B_h as in (4). Here

$$\varphi(x, y, h) = \left(y - \frac{5}{6} + \frac{h}{6}\right)\left(\frac{1}{4} + \frac{h}{4}\right) - \frac{2}{3}(1-h)\left(x - \frac{1}{2} - \frac{h}{2}\right).$$

Then

$$\frac{\partial\varphi}{\partial h} = \frac{1}{12}(8x + 3y - 7h - 2).$$

Note that the line LP intersects $\overline{CN}$ at a point where $h = 1/7$.

Eliminating the variable h in the system $\{\varphi = 0, \partial\varphi/\partial h = 0\}$, we get the defining equation of the oscillating curve $\mathscr{E}$.

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