INTERFACIAL PROGRESSIVE WATER WAVES
—A SINGULARITY-THEORETIC VIEW

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Abstract. Interfacial water waves of permanent profile between two fluids of different densities are considered. We will show that interfacial waves are generalizations of surface waves, which have been studied extensively in both mathematical and physical papers. The purpose of the present paper is to give a mathematical explanation for numerical results on bifurcations of surface waves by Shōji and ourselves. A hypothesis of degeneracy plays a key role in the present analysis. In fact, we showed in an early paper that a certain degenerate bifurcation point, if it is assumed to be present, can elucidate the complicated bifurcation structure of the surface waves by Shōji. However, in previous papers, we proved unexpectedly that any degenerate bifurcation point does not appear if we vary the depth of the flow. So, the idea of degeneracy has not been physically substantiated in the category of surface waves. In this paper we prove that such a degenerate bifurcation point actually exists when we vary the ratio of the propagation speeds between the upper and lower fluids. Consequently the complicated structure of the surface waves can be explained by regarding the surface waves as special cases of the interfacial waves.

1. Introduction. We consider progressive water waves on the interface of two fluids with different densities. By definition, progressive waves move at a constant speed and do not change their profiles during the motion. Therefore the profiles are stationary when we observe them in a suitably moving coordinate system. The interface is a free boundary to be sought. We employ the following terminology:

- *surface wave* means a free boundary and the associated fluid motion in which the fluid below the free boundary is taken into account but the fluid motion above the free boundary is neglected;
- *interfacial wave* means a free boundary and the associated fluid motion in which both the motions above and below the interface are taken into account.

The present paper has two purposes. The first of them is to reformulate the problem of the interfacial waves by modifying Kotchine [13]. We will show that our reformulated interfacial wave problem contains the surface wave problem as a special case. We actually shows that our formulation contains a new parameter: $b = m_u c_u^2/(m_l c_l^2)$, where m_u and m_l are the mass densities of the upper and lower fluids, respectively, and c_u and c_l are mean speeds of the upper and lower fluids, respectively. When the parameter b is equal to zero, our formulation reduces to the surface wave problem given in [16].

Our motivation for this generalization is to establish a mathematical theory of the bifurcation of capillary-gravity *surface* waves. The reader might wonder why the interfacial waves should be treated when our purpose is to understand the surface waves. We will show that the generalization from the surface to interfacial waves is natural from the singularity-theoretic viewpoint. The generalization or more specifically the additional parameter b is needed in the sense that a degenerate bifurcation point which was conjectured to exist in earlier papers [19], [21] is proved to exist by introducing the parameter b.

The second purpose of the present paper is to prove that such a degenerate bifurcation point exists when we vary the ratio of the propagation speeds between upper and lower fluids. In this way, the degenerate bifurcation hypothesis is substantiated.

We now explain the situation stated above in more quantitative way. The numerical computation in [24] is concerned mainly with the capillary-gravity surface waves of infinite depth. The problem involves two parameters which are related with the gravity constant and the surface tension coefficient, respectively. Namely, *the problem of the surface wave of infinite depth is a bifurcation problem of two parameters.* Two parameters ensure generically the existence of double bifurcation points, i.e., the point from which two branches of different modes bifurcates. Wilton [26] was the first to consider one of the double bifurcation points and he found mixed mode waves. Nowadays, we have a general theory of bifurcation at double bifurcation points like [5], [7], [8], [9] and can apply it to the problem of the surface waves. This theory predicts qualitative bifurcation diagrams in a rigorous way. The computation in [24], however, revealed that the actual bifurcation diagrams are more complicated than what are guaranteed by the generic bifurcation theory in [5], [7], as was pointed out by [19]. There, it was also pointed out that a certain degeneracy hypothesis (which will be introduced later in §5) well explains the diagrams of what is called mode $(1, 2)$ in [24]. Namely, if we admit the existence of a certain degenerate bifurcation point, then an abstract theorem in [6], [15], [17] is applicable and its abstract bifurcation diagrams reproduce almost all of those in [24] as far as waves of mode $(1, 2)$ are concerned. Therefore, in order to obtain a complete understanding of the phenomena, it is natural to consider the problem with an additional parameter, since we can expect the existence of a kind of degeneracy only when we have an additional parameter.

We guessed at the beginning that the required parameter is the ratio between the wave length and the depth of the fluid. However, the numerical computation showed that the numerical bifurcation diagrams with varying depth have no qualitative difference (cf. [24]). This is natural since we have the following fact proved in [19], [21]: for any value of the depth, the bifurcation equation does not degenerate in the way as is required by [19]. Thus we must consider the surface wave problem in a situation where it is embedded in an enlarged problem with additional new parameter(s). This is the motivation for the present study of interfacial waves.

This paper is composed of eight sections. In §2 we introduce an equation originally

due to Levi-Civita [12] and Kotchine [13], which is a fundamental equation for the progressive water waves. In §3 we give a functional-analytic setting of the problem and in §4 we consider bifurcation equations around a double bifurcation points. There, a simplification by means of its O(2)-equivariance is introduced. The presence of O(2)-equivariance in the surface wave problem was first recognized by [2], [16]. Its existence and usefulness in the interfacial wave problem will be explained here. The bifurcation equations are classified by a pair of two positive integers, which we call a mode. We consider only the cases of mode (1, 2). In §5, we study the bifurcation equation at the double bifurcation point of mode (1, 2). The analysis there helps us to prove in §6 that the bifurcation equation has degenerate bifurcation points as was required by [19]. We prove in §7 that there is no triple bifurcation point, which seems to have its own interest. Finally we consider a degenerate bifurcation point of different kind, which may serve as a guide to further numerical researches.

2. The fundamental equation. We consider progressive water waves on the interface of two fluids of different densities. Both of the fluids are assumed to be incompressible and inviscid. The flows are assumed to be two-dimensional and irrotational. In a reference frame moving at the propagating speed of the interface, we take (x, y) coordinate system with x horizontally to the right and y vertically upwards. The two layers of the fluids are bounded by two flat, parallel, and horizontal walls placed on $y = -h_l$ and $y = h_u$. We assume that h_l and h_u are positive constants. We let $y = h(x)$ represent the interface, which is stationary in our coordinate system. We further assume that the wave profile is periodic in x with a period, say L, and that the wave profile is symmetric with respect to the y-axis. By the periodicity assumption, it suffices to consider the fluid in the rectangle

$$R \equiv \{(x, y); -L/2 < x < L/2, -h_l < y < h_u\} .$$

By the assumptions above, the fluid motion is described by the velocity potential and the stream function. We denote by

$$f_l = U_l + iV_l, \quad \text{and} \quad f_u = U_u + iV_u$$

the complex potentials of the lower and upper flows, respectively. Here U_l and U_u are the velocity potentials and V_l and V_u are the stream functions. Then the problem is to find a period L, a wave profile function $y = h(x)$ and complex potentials such that f_l and f_u are analytic functions of $z \equiv x + iy$ in $-h_l < y < h(x)$ and $h(x) < y < h_u$, respectively. They satisfy the following (1)–(7):

(1) $$U_l\left(\pm\frac{L}{2}, y\right) = \pm\frac{c_l L}{2}, \quad \text{on} \quad -h_l < y < h(\pm L/2), \text{ respectively},$$

(2) $$U_u\left(\pm\frac{L}{2}, y\right) = \pm\frac{c_u L}{2}, \quad \text{on} \quad h(\pm L/2) < y < h_u, \text{ respectively},$$

(3) $$V_l = V_u = 0 \qquad \text{on} \quad y = h(x) \,,$$

(4) $$V_l = -a_l \qquad \text{on} \quad y = -h_l \,,$$

(5) $$V_u = a_u \qquad \text{on} \quad y = h_u \,,$$

(6) $$\lim_{y \uparrow h(x)} \left[\frac{1}{2} \left| \frac{df_l}{dz} \right|^2 + gy + \frac{p_l}{m_l} \right] = \text{constant} \,,$$

(7) $$\lim_{y \downarrow h(x)} \left[\frac{1}{2} \left| \frac{df_u}{dz} \right|^2 + gy + \frac{p_u}{m_u} \right] = \text{constant} \,,$$

where p_l and p_u are the pressures, m_u and m_l are the mass densities of the upper and lower fluids, respectively. g is the gravity acceleration. c_l, c_u, a_l and a_u are constants satisfying $a_l c_l > 0$ and $a_u c_u > 0$. Physical meanings of (1)–(7) are well-known; see, for instance, Crapper [4], Kotchine [13], or Zeidler [27]. The reason that a_l and c_l (a_u and c_u) have the same sign is explained in [21]. We remark that the stream function V, which is defined as $V = V_l$ for $y < h(x)$ and $V = V_u$ for $y > h(x)$, is continuous in the rectangle R. The complex velocities df_u/dz and df_l/dz, may, however, have a jump discontinuity at the interface. The discontinuity may appear in the tangential component but both of the normal components of the complex velocities are zero at the interface by (3).

The above formulation involves p_l and p_u, hence it is not closed in h, f_l and f_u only. A closed formulation is obtained as follows: At the interface, we assume the Laplace relation

(8) $$p_l = p_u + TK \,,$$

where T is the surface tension coefficient and K is the curvature of the boundary $y = h(x)$. K is represented as

(9) $$K = -\left(\frac{h'}{\sqrt{1 + h'^2}} \right)' ,$$

where the primes mean the differentiation with respect to x. We combine (6) and (7) with (8) to obtain

(10) $$\frac{1}{2} \left(\left| \frac{df_l}{dz} \right|^2 - \frac{m_u}{m_l} \left| \frac{df_u}{dz} \right|^2 \right) + g \left(1 - \frac{m_u}{m_l} \right) h(x) + \frac{T}{m_l} K = \text{constant} \,,$$

which holds on $y = h(x)$. Accordingly we can formulate the problem as to find a period L, a wave profile function $y = h(x)$ and complex potentials such that f_l and f_u are analytic functions of $z \equiv x + iy$ in $-h_l < y < h(x)$ and $h(x) < y < h_u$, respectively, and satisfy (1)–(5) and (10).

We now rewrite this problem in a more convenient form as Kotchine [13] did.

The aim of the remaining part of the present section is to show that the problem above is equivalent to the following one:

Find 2π-periodic functions $\theta = \theta(\sigma)$ and $S = S(\sigma)$ such that

$$(11) \quad \frac{1}{2}\frac{d}{d\sigma}(\Gamma_l^2 e^{2H\theta} - b\Gamma_u^2 e^{-2(\tilde{H}\tilde{\theta})\circ\tilde{S}}) - \frac{p}{\Gamma_l}e^{-H\theta}\sin(\theta) + q\Gamma_l\frac{d}{d\sigma}\left(e^{H\theta}\frac{d\theta}{d\sigma}\right) = 0$$

$$(0 \le \sigma < 2\pi),$$

$$(12) \quad \Gamma_l\frac{d\tilde{S}(\sigma)}{d\sigma} = \Gamma_u\exp(-(\tilde{H}\tilde{\theta})\circ\tilde{S} - H\theta) \qquad (0 \le \sigma < 2\pi),$$

where

$$\tilde{S}(\sigma) = \sigma + S(\sigma), \qquad \tilde{\theta} = \theta \circ \tilde{S}^{-1}$$

and

$$(13) \quad \Gamma_l = \frac{1}{2\pi}\int_0^{2\pi} e^{-(H\theta)(\sigma)}\cos\theta(\sigma)\,d\sigma, \qquad \Gamma_u = \frac{1}{2\pi}\int_0^{2\pi} e^{(\tilde{H}\tilde{\theta})(\sigma)}\cos\tilde{\theta}(\sigma)\,d\sigma.$$

H and $\tilde{H}$ are linear operators defined through the Fourier series as follows:

$$(14) \quad H\left(\sum_{n=1}^{\infty}(a_n\sin n\sigma + b_n\cos n\sigma)\right) = \sum_{n=1}^{\infty}\frac{1+r_l^{2n}}{1-r_l^{2n}}(-a_n\cos n\sigma + b_n\sin n\sigma).$$

$$(15) \quad \tilde{H}\left(\sum_{n=1}^{\infty}(a_n\sin n\sigma + b_n\cos n\sigma)\right) = \sum_{n=1}^{\infty}\frac{1+r_u^{-2n}}{1-r_u^{-2n}}(-a_n\cos n\sigma + b_n\sin n\sigma).$$

Note that the problem involves five dimensionless parameters b, p, q, r_l and r_u. These parameters are defined by

$$(16) \quad b = \frac{m_u c_u^2}{m_l c_l^2}, \qquad p = \frac{gL(m_l - m_u)}{2\pi c_l^2 m_l}, \qquad q = \frac{2\pi T}{m_l c_l^2 L},$$

and

$$r_l = \exp\left(-\frac{2\pi a_l}{c_l L}\right), \qquad r_u = \exp\left(\frac{2\pi a_u}{c_u L}\right).$$

Note that $0 \le r_l < 1 < r_u \le +\infty$. Since θ and S are 2π-periodic, we regard them as functions on the circle S^1. In this formulation, S is tacitly assumed not to be too large so that $\tilde{S}$ is an isomorphism from S^1 onto itself.

REMARK. If $b = 0$, then (11) contains unknown θ only and the equation with $b = 0$ represents the problem of surface wave, which was considered in [16], [24]. Our formulation (11)–(16) is due to [13] but there is a slight modification. Hence we think it is worthwhile to give a complete derivation of (11)–(16).

Derivation of (11)–(16):   We define real-valued functions θ_l, θ_u, τ_l and τ_u by

$$(17) \qquad \theta_l + i\tau_l = i\log\left(\frac{1}{c_l}\frac{df_l}{dz}\right) \qquad (|x| \le L/2,\ -h_l \le y \le h(x)),$$

and

$$(18) \qquad \theta_u + i\tau_u = i\log\left(\frac{1}{c_u}\frac{df_u}{dz}\right) \qquad (|x| \le L/2,\ h(x) \le y \le h_u).$$

Following Kotchine [13], we introduce independent variables

$$\zeta_l = \exp\left(-\frac{2\pi i f_l}{c_l L}\right), \qquad \zeta_u = \exp\left(-\frac{2\pi i f_u}{c_u L}\right)$$

and dependent variables

$$\omega_l = \theta_l + i\tau_l, \qquad \omega_u = \theta_u + i\tau_u.$$

We regard ω_l and ω_u as functions of ζ_l and ζ_u, respectively. Note that ζ_l runs in $r_l \le |\zeta_l| \le 1$ as z does in $-h_l \le y \le h(x)$, $|x| \le L/2$. Similarly ζ_u runs in $1 \le |\zeta_u| \le r_u$ when z does in $h(x) \le y \le h_u$, $|x| \le L/2$. Note also that the negative real axis corresponds to $\{x = \pm L/2\}$. Since df_l/dz and df_u/dz are periodic in x, the functions $\omega_l(\zeta_l)$ and $\omega_u(\zeta_u)$ are continuous across the negative realsand axis. Therefore, ω_l and ω_u are analytic in the complete annuli $r_l < |\zeta_l| < 1$ and $1 < |\zeta_u| < r_u$, respectively. The conditions (4), (5) imply that df_l/dz and df_u/dz are real numbers on the circles $|\zeta_l| = r_l$ and $|\zeta_u| = r_u$, respectively. Thus, θ_l and θ_u vanishes at $|\zeta_l| = r_l$ and $|\zeta_u| = r_u$, respectively. By this property of θ_l and θ_u, the analytic functions ω_l and ω_u can be expanded in the Laurent series

$$(19) \qquad \omega_l = i\alpha_0 + \sum_{n=1}^{\infty}[(\beta_n - i\alpha_n)\zeta_l^n - (\beta_n + i\alpha_n)r_l^{2n}\zeta_l^{-n}] \qquad \text{in}\ \ r_l < |\zeta_l| < 1,$$

$$(20) \qquad \omega_u = ia_0 + \sum_{n=1}^{\infty}[(b_n + ia_n)\zeta_u^{-n} - (b_n - ia_n)r_u^{-2n}\zeta_u^n] \qquad \text{in}\ \ 1 < |\zeta_u| < r_u,$$

where a_n, b_n, α_n and β_n are real constants.

If we assume the smoothness of the interface, then each of these expansions can be continued up to the boundary $|\zeta_l| = |\zeta_u| = 1$. Let (ρ, σ) be the polar coordinates for ζ_l, i.e., $\zeta_l = \rho e^{i\sigma}$. Similarly we define $\zeta_u = \rho' e^{i\sigma'}$. Expansions of the form (19), (20) imply that

$$(21) \qquad \tau_l(1, \sigma) = H(\theta_l(1, \sigma)) + \alpha_0, \qquad \tau_u(1, \sigma') = -\tilde{H}(\theta_u(1, \sigma')) + a_0.$$

On the other hand, the equations (17) and (18) are written as

$$(22) \qquad \frac{dz}{d\zeta_l} = -\frac{L}{2\pi i \zeta_l}e^{i\omega_l}, \qquad \frac{dz}{d\zeta_u} = -\frac{L}{2\pi i \zeta_u}e^{i\omega_u}.$$

On the interface, we have $\zeta_l = e^{i\sigma}$ and $\zeta_u = e^{i\sigma'}$. Thus, (22) yields

$$(23) \qquad \frac{dx}{d\sigma} + i\frac{dy}{d\sigma} = -\frac{L}{2\pi}(e^{-\tau_l(1,\sigma)}\cos\theta_l(1,\sigma) + ie^{-\tau_l(1,\sigma)}\sin\theta_l(1,\sigma))$$

and

$$(24) \qquad \frac{dx}{d\sigma'} + i\frac{dy}{d\sigma'} = -\frac{L}{2\pi}(e^{-\tau_u(1,\sigma')}\cos\theta_u(1,\sigma') + ie^{-\tau_u(1,\sigma')}\sin\theta_u(1,\sigma')) .$$

These are the parameterizations of the interface. So, these two equations represent the same curve in the (x, y) plane: in particular, we must have $x(2\pi) - x(0) = -L$ in either representation. This condition determines α_0 and a_0:

$$(25) \qquad e^{\alpha_0} = \frac{1}{2\pi}\int_0^{2\pi} e^{-H(\theta_l(1,\sigma))}\cos\theta_l(e^{i\sigma})\,d\sigma, \qquad e^{a_0} = \frac{1}{2\pi}\int_0^{2\pi} e^{\tilde{H}(\theta_u(1,\sigma'))}\cos\theta_u(e^{i\sigma'})\,d\sigma' .$$

We note that

$$\theta_l(1,\sigma) = \theta_u(1,\sigma') \qquad \text{if} \quad z(1,\sigma) = z(1,\sigma') ,$$

which comes from the fact that the velocities on either side of the interface are tangent to the interface. We now define θ by

$$\theta(\sigma) = \theta_l(1,\sigma) .$$

Then we define Γ_l and Γ_u as in (13). By (21), (22) and (25), we have

$$\frac{d\sigma'}{d\sigma} = \frac{\Gamma_u}{\Gamma_l}\exp\{-H(\theta_l(1,\sigma)) - \tilde{H}(\theta_u(1,\sigma'))\} ,$$

which is just (12).

We now rewrite (10) by means of $\theta(\sigma)$. On the interface, we have $V = 0$. Therefore, (17) and (18) give

$$\left|\frac{df_l}{dz}\right|^2 = c_l^2 e^{2\tau_l}, \qquad \left|\frac{df_u}{dz}\right|^2 = c_l^2 e^{2\tau_l}, \qquad \frac{\partial}{\partial x} = -\frac{2\pi e^{\tau_l}}{L\cos\theta_l}\frac{\partial}{\partial\sigma} .$$

By the last formula and (9), we easily obtain the following expression for the curvature:

$$K = \frac{2\pi e^{\tau_l}}{L}\frac{\partial\theta_l}{\partial\sigma} .$$

We now differentiate (10) with respect to σ. Then we get

$$\frac{1}{2}\frac{\partial}{\partial\sigma}(e^{2\tau_l} - be^{2\tau_u}) - pe^{-\tau_l}\sin\theta_l + q\frac{\partial}{\partial\sigma}\left(e^{\tau_l}\frac{\partial\theta_l}{\partial\sigma}\right) = 0 \qquad \text{on} \quad \rho = 1 ,$$

where $\tau_u = \tau_u(1,\sigma')$. We now obtain (11) by (25).

3. Function spaces. We apply the mathematical bifurcation theory of Golubitsky and Schaeffer [7], [8], [9]. To this end we use a mathematical setting with function spaces. Namely, for a non-negative integer k, we define Hilbert spaces X_k by

$$X_k = \left\{ f = \sum_{n=1}^{\infty} (\alpha_n \sin n\sigma + \beta_n \cos n\sigma) \,\middle|\, \sum_{n=1}^{\infty} n^{2k}(|\alpha_n|^2 + |\beta_n|^2) < \infty \right\}$$

or we can symbolically write $X_k = H^k(S^1)/\mathbf{R}$, where S^1 means the circle. X_k is equipped with the norm

$$\|f\|_k = \left\{ \sum_{n=1}^{\infty} n^{2k}(\alpha_n^2 + \beta_n^2) \right\}^{1/2}.$$

We let $D = X_2 \times \{S \in X_2 ; \|S\|_2 < \sqrt{6}/\pi\}$.

LEMMA 1. *If $S \in X_2$ satisfies $\|S\|_2 < \sqrt{6}/\pi$, then $\tilde{S}(\sigma) \equiv \sigma + S(\sigma)$ has an inverse and $I - \tilde{S}^{-1}$ belongs to X_2, where I is the identity operator.*

PROOF. By the Sobolev embedding theorem, X_2 is continuously embedded into $C^1(S^1)$. So, S is continuously differentiable. Suppose that $S = \sum_{n=1}^{\infty} (\alpha_n \sin n\sigma + \beta_n \cos n\sigma)$ is the Fourier expansion of S. Then,

$$\left| \frac{dS}{d\sigma} \right| \le \sum_{n=1}^{\infty} n(|\alpha_n| + |\beta_n|) \le \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \left(\sum_{n=1}^{\infty} n^4(\alpha_n^2 + \beta_n^2) \right) < 1$$

for all σ, since $\|S\|_2 < \sqrt{6}/\pi$. Hence $\sigma \mapsto \sigma + S(\sigma)$ is a strictly increasing function. This guarantees that $\tilde{S}$ is a C^1-diffeomorphism of the circle S^1. Set $\sigma + T(\sigma) = \tilde{S}^{-1}(\sigma)$. Differentiating $\sigma = \tilde{S}(\sigma + T(\sigma))$ twice with respect to σ, we have

$$T''(\sigma)\tilde{S}'(\sigma + T(\sigma)) + \tilde{S}''(\sigma + T(\sigma))(1 + T'(\sigma))^2 = 0 .$$

It is now easy to show that $T = -I + \tilde{S}^{-1}$ actually belongs to X_2. $\square$

For a given $\eta \in [0, 1)$ we define an operator H_η by

$$(26) \qquad H_\eta \left(\sum_{n=1}^{\infty} (a_n \sin n\sigma + b_n \cos n\sigma) \right) = \sum_{n=1}^{\infty} \frac{1 + \eta^{2n}}{1 - \eta^{2n}} (-a_n \cos n\sigma + b_n \sin n\sigma) .$$

We then define a nonlinear operator $F: \mathbf{R}^3 \times [0, 1)^2 \times D \to X_0 \times X_1$ by $F = (F_1, F_2)$, where

$$(27) \qquad F_1(b, p, q, \eta_1, \eta_2 ; \theta, S) = \frac{1}{2} \frac{d}{d\sigma} \left(\Gamma_l^2 e^{2H_{\eta_1}\theta} - b\Gamma_u^2 e^{-2(H_{\eta_2}\tilde{\theta}) \circ \tilde{S}} + 2q\Gamma_l e^{H_\eta \theta} \frac{d\theta}{d\sigma} \right)$$

$$- \frac{p}{\Gamma_l} e^{-H_{\eta_1}\theta} \sin(\theta)$$

and

$$(28) \qquad F_2(b, p, q, \eta_1, \eta_2; \theta, S) = \Gamma_l \exp([H_{\eta_2} \tilde{\theta}] \circ \tilde{S}) \cos(\theta) \frac{d\tilde{S}}{d\sigma} - \Gamma_u \exp(-H_{\eta_1} \theta) \cos(\theta),$$

with $\tilde{S}(\sigma) = \sigma + S(\sigma)$, $\tilde{\theta} = \theta \circ \tilde{S}^{-1}$ and

$$(29) \qquad \Gamma_l = \frac{1}{2\pi} \int_0^{2\pi} e^{-H_{\eta_1}\theta(\sigma)} \cos\theta(\sigma) \, d\sigma, \qquad \Gamma_u = \frac{1}{2\pi} \int_0^{2\pi} e^{H_{\eta_2}\tilde{\theta}(\sigma)} \cos\tilde{\theta}(\sigma) \, d\sigma.$$

Note that $\eta_l = r_l$ and $\eta_2 = r_u^{-1}$ in the notation of the previous section.

Thus, our task is to determine the zeros of F. Clearly, $\theta \equiv 0$ and $S(\sigma) \equiv 0$ solve $F = 0$ for any value of the parameters b, p, q, η_1 and η_2. We remark that $\theta \equiv 0$ corresponds to the case where $h(x)$ is constant, so the corresponding interface is flat (see (23)).

We now prove:

LEMMA 2. *The mapping F is a well-defined, smooth mapping from $\mathbf{R}^3 \times [0, 1)^2 \times D$ into $X_0 \times X_1$ and satisfies $F(b, p, q, \eta_1, \eta_2; 0, 0) \equiv 0$.*

PROOF. We prove that (i) F is a smooth mapping into $L^2(S^1) \times H^1(S^1)$ and then prove that (ii) the range of F is actually in $X_0 \times X_1$.

(i) is easily seen by the Sobolev embedding theorem and the fact that H_η is a linear isomorphism from X_s onto itself for any s. To prove (ii), we consider, for a given function $\theta \in X_2$, the analytic function

$$\Omega(\zeta) = \sum_{n=1}^{+\infty} \left[\frac{\beta_n - i\alpha_n}{1 - \eta_1^{2n}} \zeta^n - \frac{\beta_n + i\alpha_n}{1 - \eta_1^{2n}} \eta_1^{2n} \zeta^{-n} \right] \qquad (\eta_1 < |\zeta| < 1),$$

where α_n and β_n are the Fourier coefficients of θ, i.e.,

$$\theta(\sigma) = \sum_{n=1}^{+\infty} (\alpha_n \sin n\sigma + \beta_n \cos n\sigma).$$

Since $\theta \in X_2$, Ω is analytic in $\eta_1 < |\zeta| < 1$ and continuous up to the boundary $|\zeta| = 1$. It satisfies

$$\Omega(e^{i\sigma}) = \theta(\sigma) + iH_{\eta_1}\theta(\sigma) \quad \text{and} \quad \text{Re}[\Omega(\eta_1 e^{i\sigma})] \equiv 0.$$

By Cauchy's integral formula, it holds that

$$\int_{|\zeta|=1} \frac{e^{i\Omega}}{\zeta} d\zeta = \int_{|\zeta|=\eta_1} \frac{e^{i\Omega}}{\zeta} d\zeta.$$

Taking the real part and noting that $\text{Re}[\Omega(\eta_1 e^{i\sigma})] \equiv 0$, we obtain

$$\int_0^{2\pi} e^{-H_{\eta_1}\theta} \sin(\theta) \, d\sigma = 0.$$

This shows that $F_1 \in X_0$. On the other hand, the change of variables shows that

$$\int_0^{2\pi} e^{[H_{\eta_2}\tilde{\theta}]\circ\tilde{S}}\cos(\theta)\frac{d\tilde{S}}{d\sigma}d\sigma = \int_0^{2\pi} e^{H_{\eta_2}\tilde{\theta}(\sigma')}\cos(\tilde{\theta}(\sigma'))d\sigma' .$$

This and the definition of Γ_l and Γ_u prove that $F_2 \in X_1$. $\qquad\square$

REMARK. If $r_l = 0$, then $\Gamma_l = 1$ for any θ. Similarly, if $r_u = 0$, then $\Gamma_u = 1$ for any θ. These are proved in the same way as in [16].

4. Bifurcation equation. In this section we compute the bifurcation equation via the Lyapunov-Schmidt method. Then, exploiting its O(2)-equivariance, we rewrite it in a normal form.

We begin with determining the bifurcation set of parameters. For this purpose, we need the Fréchet derivatives:

LEMMA 3. *The first order Fréchet derivatives of F at $\theta \equiv 0$, $S \equiv 0$ are given by*

$$D_\theta F_1(b, p, q, \eta_1, \eta_2; 0, 0)w = \frac{dH_{\eta_1}w}{d\sigma} + b\frac{dH_{\eta_2}w}{d\sigma} - pw + q\frac{d^2w}{d\sigma^2} \qquad (w \in X_2),$$

$$D_S F_1(b, p, q, \eta_1, \eta_2; 0, 0)T = 0 \qquad (T \in X_1),$$

$$D_\theta F_2(b, p, q, \eta_1, \eta_2; 0, 0)w = H_{\eta_1}w + H_{\eta_2}w \qquad (w \in X_2),$$

$$D_S F_2(b, p, q, \eta_1, \eta_2; 0, 0)T = \frac{dT}{d\sigma}(T \in X_1).$$

The proof is straightforward if we note that

(30) $$\Gamma_l = 1 + \frac{1}{4\pi}\int_0^{2\pi}[(H_{\eta_1}\theta)^2 - \theta^2]d\sigma + O(\|\theta\|_2^3),$$

(31) $$\Gamma_u = 1 + \frac{1}{4\pi}\int_0^{2\pi}[(H_{\eta_2}\tilde{\theta})^2 - \tilde{\theta}^2]d\sigma + O(\|\theta\|_2^3),$$

and $e^{-H\theta}\sin(\theta) = \theta - \theta H\theta + O(\|\theta\|^3)$, etc. $\qquad\square$

For notational convenience, we define the following symbols:

$$L(n) = \frac{1 + \eta_1^{2n}}{1 - \eta_1^{2n}}, \qquad U(n) = \frac{1 + \eta_2^{2n}}{1 - \eta_2^{2n}}.$$

LEMMA 4. *The Fréchet derivative of F with respect to (θ, S), which is denoted by*

$$D_{\theta,S}F = \begin{pmatrix} D_\theta F_1 & D_S F_1 \\ D_\theta F_2 & D_S F_2 \end{pmatrix},$$

has a nontrivial null space if and only if there is a positive integer n such that

(32) $$[L(n)+bU(n)]n=p+n^2q .$$

In this case,

$$\Sigma_n \equiv \left(\sin n\sigma, \frac{L(n)+U(n)}{n} \sin n\sigma \right) \quad and \quad \Sigma_n' \equiv \left(\cos n\sigma, \frac{L(n)+U(n)}{n} \cos n\sigma \right)$$

are null vectors: Namely, $D_{\theta,S}F\Sigma_n = D_{\theta,S}F\Sigma_n' = 0.$

This lemma is easily proven by Lemma 3.

Later we will prove that $(b, p_0, q_0, \eta_1, \eta_2; 0, 0)$ which satisfies (32) is actually a bifurcation point. Namely there is a nontrivial solution to $F=0$ in any neighborhood of $(b, p_0, q_0, \eta_1, \eta_2; 0, 0)$ in $R^3 \times [0, 1)^2 \times D$. It is, however, very important to notice that some of the bifurcation points differ from others by the dimension of the null space. In fact, some $(b, p, q, \eta_1, \eta_2)$ satisfies (32) for two distinct n's.

DEFINITION 1. If $(b, p, q, \eta_1, \eta_2)$ satisfies (32) for one and only one n, then we call it a simple bifurcation point of mode n. On the other hand, if $(b, p, q, \eta_1, \eta_2)$ satisfies (32) for two and only two integers n and m $(n \neq m)$, then we call it a double bifurcation point of mode (m, n).

We note that the kernel of $D_{\theta,S}F(b, p, q, \eta_1, \eta_2; 0, 0)$ is a two-dimensional space spanned by Σ_n and Σ_n', if $(b, p, q, \eta_1, \eta_2)$ is a simple bifurcation point. On the other hand, the kernel of $D_{\theta,S}F(b, p, q, \eta_1, \eta_2; 0, 0)$ is a four-dimensional space spanned by Σ_n, Σ_m, Σ_n', and Σ_m', if $(b, p, q, \eta_1, \eta_2)$ is a double bifurcation point. The reader may wonder whether triple bifurcation points exist or not. We actually prove in the last section that there is no triple bifurcation point.

The bifurcation from a simple bifurcation point is literally simple but the bifurcation from the double bifurcation point is rather complicated and many of them are not yet investigated completely. In fact, Shōji [24] computed many bifurcation diagrams which did not seem to appear in the previous literature. Among her computations, we focus in this paper on the bifurcation of mode (1, 2) and show that some degenerate bifurcation equations, which will be introduced later in §5, will explain many of her computations and that the Kotchine equations (27), (28) do contain such a degenerate bifurcation point.

Although our main concern is to study the double bifurcation point of mode (1, 2), it will be useful for later analysis to give a mathematical description of a general double bifurcation point of mode (m, n). Without loss of generality, we assume that $0 < m < n$.

A double bifurcation point of mode (m, n) is characterized by

$$(L(m)+bU(m))m=p+m^2q , \qquad (L(n)+bU(n))n=p+n^2q .$$

We define

$$p_0(b, \eta_1, \eta_2, m, n) = \frac{nm}{n^2-m^2} \{nL(m)-mL(n)+b[nU(m)-mU(n)]\} ,$$

$$q_0(b, \eta_1, \eta_2, m, n) = \frac{mL(m) - nL(n) + b[mU(m) - nU(n)]}{m^2 - n^2}.$$

It is easy to observe that the bifurcation points of mode (m, n) are characterized as $(b, p, q, \eta_1, \eta_2; \theta, S) = (b, p_0(b, \eta_1, \eta_2, m, n), q_0(b, \eta_1, \eta_2, m, n), \eta_1, \eta_2; 0, 0)$. So we have an important fact: *the set of all the double bifurcation points of mode (m, n) is a three-dimensional manifold parameterized by b, η_1 and η_2.* As is mentioned before, we will prove in §7 that there is no triple bifurcation point in the range $0 < b, p, q < \infty$, $\eta_1, \eta_2 \in [0, 1)$. So, all the points on the manifold satisfy (32) for exactly two integers m and n.

LEMMA 5.   *At the double bifurcation point of mode (m, n), the cokernel of $D_{\theta,S}F(b, p_0, q_0, \eta_1, \eta_2; 0, 0)$ is spanned by*

$$(\sin m\sigma, 0), \quad (\cos m\sigma, 0), \quad (\sin n\sigma, 0), \quad (\cos n\sigma, 0).$$

PROOF.   By Lemma 3, the operator dual to $D_{\theta,S}F$ is represented as

$$D_{\theta,S}F^* = \begin{pmatrix} H_{\eta_1}\dfrac{d}{d\sigma} + bH_{\eta_2}\dfrac{d}{d\sigma} - p + q\dfrac{d^2}{d\sigma^2} & -H_{\eta_1} - H_{\eta_2} \\[2mm] 0 & -\dfrac{d}{d\sigma} \end{pmatrix}.$$

The conclusion is easily derived from this formula.   □

We now prove that F satisfies a certain property which is called O(2)-equivariance and that this property forces F to be of a special simple form (see (37), (38) below). We first define an action of the orthogonal group O(2) on D as follows: Let us recall that O(2) is generated by rotations of angle $\alpha \in [0, 2\pi)$ and a reflection. Accordingly,

$$\gamma_\alpha \theta(\sigma) = \theta(\sigma - \alpha), \quad \gamma_\alpha S(\sigma) = S(\sigma - \alpha), \qquad (0 \le \alpha < 2\pi),$$
$$\gamma_- \theta(\sigma) = -\theta(-\sigma), \quad \gamma_- S(\sigma) = -S(-\sigma)$$

define an action of O(2) on D, where γ_α represents the element of O(2) corresponding to the rotation of angle α and γ_- the reflection. We also define an action of O(2) on the range space of F, i.e., for $(f_1, f_2) \in X_0 \times X_1$ we define

$$\gamma_\alpha f_1(\sigma) = f_1(\sigma - \alpha), \quad \gamma_\alpha f_2(\sigma) = f_2(\sigma - \alpha), \qquad (0 \le \alpha < 2\pi),$$
$$\gamma_- f_1(\sigma) = -f_1(-\sigma), \quad \gamma_- f_2(\sigma) = f_2(-\sigma).$$

Note that γ_- acts on the second variable oddly in the defining domain, while it does evenly on the second variable in the range of F. We now prove:

PROPOSITION 1.   *The mapping $F(b, p, q, \eta_1, \eta_2; \cdot, \cdot): D \to X_0 \times X_1$ is O(2)-equivariant, by which we mean*

$$F(b, p, q, \eta_1, \eta_2; \gamma(\theta, S)) = \gamma F(b, p, q, \eta_1, \eta_2; \theta, S) \qquad (\gamma \in O(2)) \,.$$

PROOF.   Note that the following relations hold in D:

$$H(\gamma_\alpha \theta)(\sigma) = \gamma_\alpha(H\theta)(\sigma) \,, \qquad H(\gamma_- \theta)(\sigma) = (H\theta)(-\sigma) \,,$$
$$(\gamma_\alpha \theta) \circ (\gamma_\alpha \tilde{S})^{-1} = \theta \circ \tilde{S}^{-1} \,, \qquad (\gamma_- \theta) \circ (\gamma_- \tilde{S})^{-1} = \gamma_-(\theta \circ \tilde{S}^{-1}) \,.$$

Here, an obvious action of γ_α and γ_- on $\tilde{S} = \sigma + S(\sigma)$ is assumed. The proposition follows from these.   □

As is emphasized in [8], [9], an equivariance with respect to a group greatly simplifies the bifurcation equation and reduces the labor necessary for the analysis. The effect of O(2)-equivariance is fully discussed in [9]. Following it, we explain the analysis here in our context. Proposition 1 enables us to simplify the bifurcation equation as follows: Let us use the following notation:

- $(b, p_0, q_0, \eta_1, \eta_2; 0, 0)$ is a double bifurcation point of mode (m, n) with $0 < m < n$, parameterized by (b, η_1, η_2);
- $D_{\theta,S}F^\circ$ denotes the Fréchet derivative of F at $(b, p_0, q_0, \eta_1, \eta_2; 0, 0)$ with respect to (θ, S);
- P denotes the L^2-projection from $L^2(S^1)$ onto the four-dimensional subspace spanned by $\sin m\sigma$, $\cos m\sigma$, $\sin n\sigma$ and $\cos n\sigma$;
- Z is the L^2-orthogonal complement of $N(D_{\theta,S}F^\circ)$ in $X_2 \times X_2$, where $N(\ )$ denote the null space ($=$ kernel).

Note that the range of $D_{\theta,S}F^\circ$ is $[(I-P)X_2] \times X_1$. Now we consider the following equation:

(33) $(I-P)F_1(b, p, q, \eta_1, \eta_2; x\Sigma_m + y\Gamma_m + z\Sigma_n + w\Gamma_n + \phi(b, p, q, \eta_1, \eta_2; x, y, z, w)) = 0$,

(34) $F_2(b, p, q, \eta_1, \eta_2; x\Sigma_m + y\Gamma_m + z\Sigma_n + w\Gamma_n + \phi(b, p, q, \eta_1, \eta_2; x, y, z, w)) = 0$.

The equations (33) and (34) uniquely define a Z-valued mapping ϕ in some open set containing $(b, p_0, q_0, \eta_1, \eta_2; 0, 0, 0, 0)$. We define G by

$$G(b, p, q, \eta_1, \eta_2; x, y, z, w) = PF_1(b, p, q, \eta_1, \eta_2; x\Sigma_m + y\Gamma_m + z\Sigma_n + w\Gamma_n$$
$$+ \phi(b, p, q, \eta_1, \eta_2; x, y, z, w)) \,.$$

This mapping G is the bifurcation equation.

By Proposition 1 and the fact that the bifurcation equation inherits the group equivariance from the basic differential equation (cf. [8]), we see that G too has an O(2)-equivariance. To represent this more conveniently, we identify $(x, y, z, w) \in \mathbf{R}^4$ with $(\xi, \zeta) \in \mathbf{C}^2$ in the way that $\xi = x + iy$, $\zeta = z + iw$. Therefore, we can regard G as a mapping defined on (some open subset of) $\mathbf{R}^5 \times \mathbf{C}^2$. Similarly, we can regard G as a mapping taking its value in $\mathbf{C}^2$. Let (G_1, G_2) be the componentwise expression of G in $\mathbf{C}^2$. We now have:

PROPOSITION 2. *The mapping G above is* O(2)-*equivariant in the sense that the following two conditions hold true.*

(35) $\quad G(b, p, q, \eta_1, \eta_2; e^{im\alpha}\xi, e^{in\alpha}\zeta) =$

$\qquad (e^{im\alpha}G_1(b, p, q, \eta_1, \eta_2; \xi, \zeta), e^{in\alpha}G_2(b, p, q, \eta_1, \eta_2; \xi, \zeta)), \qquad (\alpha \in [0, 2\pi))$

(36) $\quad G(b, p, q, \eta_1, \eta_2; \bar{\xi}, \bar{\zeta}) = \overline{(G_1(b, p, q, \eta_1, \eta_2; \xi, \zeta), G_2(b, p, q, \eta_1, \eta_2; \xi, \zeta))}$.

The proof is straightforward by definition.

Proposition 2 forces the mapping G to be of a special form. Let us introduce some symbols, which and the terminology below are borrowed from [7], [8], [9].

REMARK. From now on, we write $f: \mathbf{R} \to \mathbf{R}$, even when the defining domain of f is some small open set of $\mathbf{R}$. Strictly speaking, we consider mapping germs at the origin, although we write them as if they were defined in the whole space.

DEFINITION 2. Let k be a positive integer. We call a function $f: \mathbf{R}^k \times \mathbf{C}^2 \to \mathbf{R}$ O(2)-invariant if $f(a; e^{im\alpha}\xi, e^{in\alpha}\zeta) \equiv f(a; \xi, \zeta)$ and $f(a; \bar{\xi}, \bar{\zeta}) \equiv f(a; \xi, \zeta)$ are satisfied for all $\alpha \in [0, 2\pi)$ and $a \in \mathbf{R}^k$, $\xi, \zeta \in \mathbf{C}$.

The set E of all germs (at the origin) of O(2)-invariant C^∞-functions is a commutative ring with unity. The set $\tilde{E}$ of all the mapping $G: \mathbf{R}^5 \times \mathbf{C}^2 \to \mathbf{C}^2$ satisfying (35), (36) constitutes an E-module. In order to give a simple expression for E and $\tilde{E}$, we need to introduce two positive integers n' and m'. We define them to be coprime positive integers satisfying $n/m = n'/m'$. We now have:

PROPOSITION 3. *Any element* $f \in E$ *is of the form*

$$f(a; \xi, \zeta) = g(a; u, v, s),$$

where g is a C^∞-*function of* $k + 3$ *variables, and u, v, s are defined by*

$$u = |\xi|^2, \qquad v = |\zeta|^2, \qquad s = \text{Re}[\bar{\xi}^{n'}\zeta^{m'}].$$

PROPOSITION 4. *The module* $\tilde{E}$ *is generated over E by the four elements*

$$e_1 = (\xi, 0), \quad e_2 = (0, \bar{\xi}^{n'-1}\zeta^{m'}), \quad e_3 = (0, \zeta), \quad e_4 = (0, \xi^{n'}\bar{\zeta}^{m'-1}).$$

In particular, the mapping G at the bifurcation point of mode (1, 2) *is of the form*

(37) $$G_1 = f_1\xi + f_2\bar{\xi}\zeta,$$

(38) $$G_2 = f_3\zeta + f_4\xi^2,$$

where f_j *are of the form*

$$f_j = f_j(b, p, q, \eta_1, \eta_2; |\xi|^2, |\zeta|^2, \text{Re}[\bar{\xi}^2\zeta]) \qquad (j = 1, 2, 3, 4).$$

The proofs of Propositions 3 and 4 can be found in [15] and [8].

5. Bifurcation equation of mode (1, 2). In this section we compute the Taylor coefficients of the bifurcation equation of mode (1, 2). Then we define the degenerate bifurcation point which was mentioned in §1. We show that a theorem in [15] is applicable to the problem now in question.

The bifurcation equation $G: \mathbf{R}^5 \times \mathbf{C}^2 \to \mathbf{C}^2$ is now written as in (37), (38). We use the theory in [7], [8], [9] in which mapping germs containing one-parameter are considered. Our mapping, however, has five parameters. We thereby pick up an arbitrary point $(b, p_0, q_0, \eta_1, \eta_2)$ which defines a double bifurcation point of mode (1, 2). We then take any smooth curve in $\mathbf{R}^5$ which goes through the point and is transversal to the three dimensional manifold defining the double bifurcation point of mode (1, 2). We take a coordinate λ along this curve such that $\lambda=0$ corresponds to the point $(b, p_0, q_0, \eta_1, \eta_2)$. We regard G as a mapping germ of (λ, ξ, ζ), so we can write as

$$(39) \qquad G_1 = f_1(\lambda; u, v, s)\xi + f_2(\lambda; u, v, s)\bar{\xi}\zeta,$$

$$(40) \qquad G_2 = f_3(\lambda; u, v, s)\zeta + f_4(\lambda; u, v, s)\xi^2.$$

If we show that this mapping is finitely determined and if we compute universal unfoldings, then the equation (35) can be realized by one of the unfolded mappings (cf. [7], [8]). Thus we are led to the analysis of (39), (40).

Since G is a bifurcation equation, all the derivatives of first order vanish at the origin. Accordingly $f_j(0; 0, 0, 0)=0$ $(j=1, 3)$. To be more precise, $f_j(0; 0, 0, 0)=0$ $(j=1, 3)$, whatever the choice of $(b, p_0, q_0, \eta_1, \eta_2)$ may be. In order to go further, we need to compute $f_2(0; 0, 0, 0)$ and $f_4(0; 0, 0, 0)$, which depend implicitly on (b, η_1, η_2). Before going into the details we can make some simple observations.

By the Taylor expansion, we have

$$(41) \qquad G_1 = (\varepsilon\lambda + au + cv + ds + f_5)\xi + (k_1 + d_1\lambda + d_2u + d_3v + d_4s + f_6)\bar{\xi}\zeta,$$

$$(42) \qquad G_2 = (\delta\lambda + \hat{a}u + \hat{c}v + \hat{d}s + f_7)\zeta + (k_2 + e_1\lambda + e_2u + e_3v + e_4s + f_8)\xi^2,$$

where $\varepsilon, \delta, d_j, e_j$ $(j=1, 2, 3, 4)$, $a, c, d, \hat{a}, \hat{c}, \hat{d}, k_1, k_2$ are real constants depending on (b, η_1, η_2). f_j are terms of order ≥ 2. If both k_1 and k_2 are different from zero, we can divide G_1 by f_2 and G_2 by f_4, respectively. The divided mapping is O(2)-equivalent, in the sense of [7], to the original mapping. So, it suffices to analyze

$$(43) \qquad G_1 = (\varepsilon\lambda + au + cv + ds + f_9)\xi + \bar{\xi}\zeta,$$

$$(44) \qquad G_2 = (\delta\lambda + \hat{a}u + \hat{c}v + \hat{d}s + f_{10})\zeta + \xi^2,$$

though the coefficients ε, a, c, etc. are different from those in (41) and (42). f_9 and f_{10} are of order ≥ 2.

The bifurcation equations (43) and (44) were analyzed in [5] and later in [9], [15]; The latter papers made an analysis faithful to the Golubitsky-Schaeffer theory. Although these papers were not concerned with the water wave problem, their analyses

are applicable to the present problem, since theirs and ours are based only on
O(2)-equivariance. Our conclusion in [15] agrees with those in [5], which shows that
the equations (43) and (44) can reproduce only a part of the interesting secondary
bifurcations which were numerically found in Shōji [24]. In order to explain the situation
we use Theorem 3.2 of [15]:

THEOREM 1.   *Assume that* $\varepsilon\delta\hat{c}\neq0$ *and* $\varepsilon\hat{c}\neq\delta(c-\hat{a}/2)$ *in* (43) *and* (44). *Then,* (43) *and*
(44) *are* O(2)-*equivalent in the sense of* [7] *to*

$$(45) \qquad G_1=\left(\frac{\varepsilon}{|\varepsilon|}\lambda+\frac{c-\hat{a}/2}{|\hat{c}|}v\right)\xi+\bar{\xi}\zeta\,,$$

$$(46) \qquad G_2=\left(\frac{\delta}{|\delta|}\lambda+\frac{c}{|\hat{c}|}v\right)\zeta+\xi^2\,.$$

This theorem enables us to easily draw the bifurcation diagram, since it suffices to
compute the roots of the algebraic equations. Let us show this by an example. Shōji
obtained the diagram in Figure 1, which is reproduced from [21]. Here only symmetric
waves are plotted. The symmetry of the waves is represented by $\theta(\sigma)=-\theta(-\sigma)$. This
is equivalent to Im $\xi=0$, Im $\zeta=0$ in (45), (46). The primary pitchfork has two secondary
bifurcation points which are connected by a loop having two turning points. On the
other hand, the non-degenerate bifurcation equations (45) and (46) can reproduce only
a diagram corresponding to (I2-h) of [5], which is the encircled part in Figure 2. So
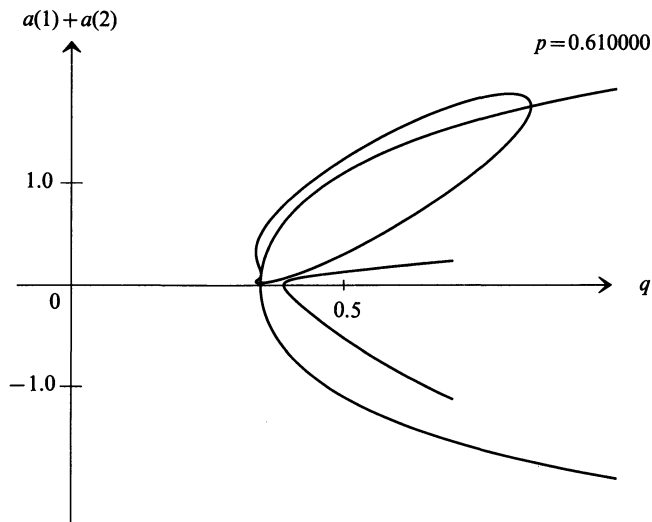the turning points are not captured by (45), (46), let alone the loop in Figure 1. Later,

FIGURE 1.   Bifurcation diagram when $b=1$, $\eta_1=\eta_2=0$ and $p=0.61$. q is taken as
a bifurcation parameter. The y-axis represents the sum of the first two Fourier
coefficients of θ. A big loop having two turning points are observed.
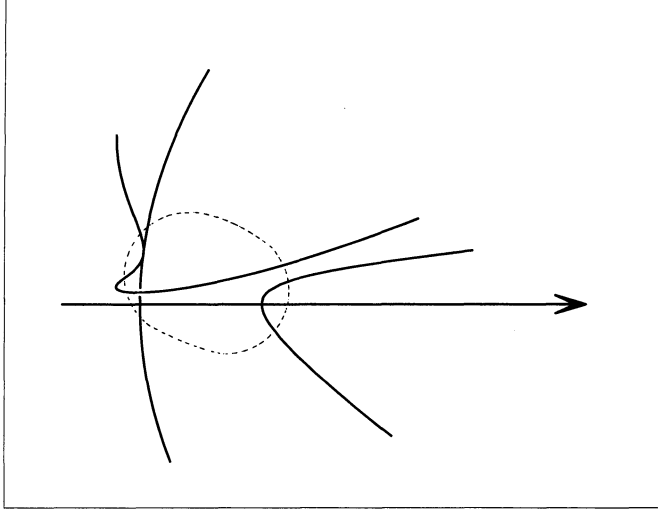
FIGURE 2. A blow-up of Figure 1 near the primary pitchfork. The part enclosed
by a broken line is the diagram which is reproduced by (44) and (45).

Fujii et al. [6] introduced a notion of degeneracy and showed that turning points are reproduced if the degeneracy exists. We pursue their idea but along the Golubitsky-Schaeffer theory (see below).

By these observations, it is natural to study the case where either ε, δ, $\hat{c}$, $\varepsilon\hat{c} - \delta(c - \hat{a}/2)$, k_1, or k_2 vanishes.

The possibility of $\varepsilon = 0$ or $\delta = 0$ will be considered in §8. We do not know if the degeneracy $\hat{c} = 0$ or $\varepsilon\hat{c} = \delta(c - \hat{a}/2)$ happens or not. Instead, we will only consider the possibility of $k_1 = 0$ and/or $k_2 = 0$, since we will later see that the occurrence of $k_1 = 0$ is sufficient for our purpose. We content ourselves that generically $\varepsilon\hat{c} \neq \delta(c - \hat{a}/2)$ and $\hat{c} \neq 0$.

Recall that the constants k_1 and k_2 depend on (b, η_1, η_2). So it is natural to ask whether there is a $(b_0, \eta_1^{(0)}, \eta_2^{(0)})$ such that $k_1 = 0$ or $k_2 = 0$. [6], [17], and [19] considered the case where k_1 *vanish while* k_2 *does not.* Recall that they consider only the case $b = 0$. This assumption is the degeneracy assumption which was referred to in the introduction. If this is the case, then we can divide G_2 by f_4. We then obtain (41) and

$$(47) \qquad G_2 = (\delta\lambda + \hat{a}u + \hat{c}v + \hat{d}s + f_{11})\zeta + \xi^2 \,,$$

Theorem 4.1 of [15] shows that the bifurcation equations (41), (47) are O(2)-equivalent (in the sense of [7]) to

$$(48) \qquad G_1 = (\varepsilon\lambda + au + cv + ds)\xi + d_3 v \bar{\xi}\zeta \,,$$

$$(49) \qquad G_2 = (\delta\lambda + \hat{a}u + \hat{c}v)\zeta + \xi^2 \,,$$

where ε, δ, a, c, $\hat{a}$, $\hat{c}$, d_1 and d_2 are nonzero real constants different from those in (41) and (47). Its universal unfolding in the sense of [7] is

$$(50) \qquad G_1 = (\varepsilon\lambda + \alpha + (a + \gamma_1)u + (c + \gamma_2)v + (d_1 + \gamma_3)s)\xi + (\beta + (d_2 + \gamma_4)v)\bar{\xi}\zeta,$$

$$(51) \qquad\qquad\qquad G_2 = (\delta\lambda + \hat{a}u + \hat{c}v)\zeta + \xi^2,$$

where α, β, γ_1, γ_2, γ_3 and γ_4 are unfolding parameters. Among these parameters, α and β are essential and $\gamma_1, \ldots, \gamma_4$ are modal parameters in the sense of Golubitsky and Schaeffer [8].

It was shown that (50) and (51) can reproduce all the bifurcation diagrams of mode $(1, 2)$ in Shōji [24]. For instance, the whole picture of Figure 1 is obtained from (50), (51). Since this is explained in [17], [18], [21], especially in [21], we do not repeat it here. However, we would like to emphasize that the existence of any degenerate bifurcation point has not been proven in these papers.

6. Existence of degenerate bifurcation point.
We have seen that the existence of a degenerate bifurcation equation such that $f_2(0; 0, 0, 0) = 0$ is expected somewhere in the manifold

$$p = p_0(b, \eta_1, \eta_2), \qquad q = q_0(b, \eta_1, \eta_2).$$

The aim of the present section is to show the existence by explicitly computing $f_2(0; 0, 0, 0)$.

In order to derive a concrete expression for $f_2(0; 0, 0, 0)$, we give formulas necessary for the computation of $f_2(0; 0, 0, 0)$.

LEMMA 6.   *The second order Fréchet derivatives of F are*:

$$(52) \qquad D_\theta^2 F_1^\circ(v, w) = 2\frac{d}{d\sigma}(H_{\eta_1}vH_{\eta_1}w) - 2b\frac{d}{d\sigma}(H_{\eta_2}vH_{\eta_2}w) + p_0(wH_{\eta_1}v + vH_{\eta_1}w)$$

$$+ q_0\frac{d}{d\sigma}\left((H_{\eta_1}v)\frac{dw}{d\sigma} + (H_{\eta_1}w)\frac{dv}{d\sigma}\right) \qquad (v, w \in X_2),$$

$$(53) \qquad D_\theta D_S F_1^\circ(w, T) = b\frac{d}{d\sigma}\left\{\frac{dH_{\eta_2}w}{d\sigma}T - H_{\eta_2}\left(\frac{dw}{d\sigma}T\right)\right\} \qquad (w, T \in X_2),$$

$$(54) \qquad\qquad\qquad D_S^2 F_1^\circ \equiv 0.$$

Note that

$$D_\theta D_S F_1^\circ(w, T) = \frac{\partial^2}{\partial t \partial s}F_1(b, p_0, q_0, \eta_1, \eta_2; tw, sT)\bigg|_{t=s=0},$$

which is not symmetric with respect to w and T.

PROOF.   We first consider

$$F_1(b, p, q, \eta_1, \eta_2; \theta, 0) = \frac{1}{2} \frac{d}{d\sigma} \left(\Gamma_l^2 e^{2H_{n_1}\theta} - b\Gamma_u^2 e^{-2H_{n_2}\theta} + q\Gamma_l e^{H_{n_1}\theta} \frac{d\theta}{d\sigma} \right) - \frac{p}{\Gamma_l} e^{-H_{n_1}\theta} \sin\theta$$

(52) is easily derived from this and the equalities (30) and (31). By

$$F_1(b, p, q, \eta_1, \eta_2; 0, S) \equiv 0,$$

we obtain (54).

We now consider

$$D_\theta F_1(b, p, q, \eta_1, \eta_2; 0, S)w = b\frac{d}{d\sigma}[(H_{\eta_2}\tilde{w}) \circ \tilde{S}] + K,$$

where $\tilde{w} = w \circ \tilde{S}(\sigma)^{-1}$ and K is independent of S. Deriving (53) from this is an easy exercise. $\qquad\square$

Using these formulas, we calculate some quantities involving the second order derivatives. The quantities below are needed in the computation of the bifurcation equation.

LEMMA 7.

(55) $$D_\theta^2 F_1^\circ(\sin\sigma, \sin\sigma) = A(1, 1)\sin 2\sigma,$$

(56) $$D_\theta^2 F_1^\circ(\sin\sigma, \sin 2\sigma) = A(1, 2)\sin\sigma + B(1, 2)\sin 3\sigma,$$

(57) $$D_\theta D_S F_1^\circ(\sin\sigma, \sin 2\sigma) = -bU(1)\sin\sigma - \frac{3b}{2}(U(3) - U(1))\sin 3\sigma,$$

(58) $$D_\theta D_S F_1^\circ(\sin 2\sigma, \sin\sigma) = b(U(1) - U(2))\sin\sigma + 3b(U(2) - U(3))\sin 3\sigma,$$

(59) $$D_\theta D_S F_1^\circ(\sin\sigma, \sin\sigma) = b(U(1) - U(2))\sin 2\sigma,$$

where

$$A(1, 1) = 2L(1)L(2) - 4L(1)^2 + 2b[L(1)U(2) - L(1)U(1) + U(1)^2]$$

and

$$A(1, 2) = \frac{L(1)L(2)}{2} - L(1)^2 + b\left[U(1)U(2) + L(1)U(2) - U(1)L(1) + \frac{1}{2}U(1)L(2) \right].$$

The expression for $B(1, 2)$ is unnecessary.

PROOF. Since all computations are elementary, we only prove (55). By (52),

$$D_\theta^2 F_1^\circ(\sin\sigma, \sin\sigma)$$

$$= 2\frac{d}{d\sigma}(L(1)^2 \cos^2\sigma - bU(1)^2 \cos^2\sigma - 2q_0 L(1)\cos^2\sigma) - 2p_0 L(1)\sin\sigma\cos\sigma$$

$$= (-2L(1)^2 + 2bU(1)^2 - p_0 L(1) + 2q_0 L(1))\sin 2\sigma.$$

Since $m = 1$ and $n = 2$, we have

$$p_0 = \frac{1}{3} \{ 4L(1) - 2L(2) + b[4U(1) - 2U(2)] \}$$

and

$$q_0 = \frac{1}{3} \{ 2L(2) - L(1) + b[2U(2) - U(1)] \} \ .$$

Consequently the coefficient of $\sin 2\sigma$ is represented as in the lemma. $\qquad \square$

We now compute the coefficients.

THEOREM 2. *It holds that*

$$f_2(0; 0, 0, 0) = \frac{L(1)L(2)}{2} - L(1)^2 + bU(1) \left\{ U(1) - \frac{U(2)}{2} \right\}$$

and

$$f_4(0; 0, 0, 0) = 2L(1)L(2) - 4L(1)^2 + 2bU(1)\{2U(1) - U(2)\} \ .$$

PROOF.   We note that

$$f_2(0; 0, 0, 0) = \frac{\partial^2 G_1}{\partial x \partial z}(0; 0, 0) \ .$$

Hence

(60) $\dfrac{\partial G}{\partial x}(\lambda; \xi, \zeta) = PD_\theta F_1(\#)(\sin \sigma + \phi_{1,x}) + [L(1) + U(1)]PD_s F_1(\#)(\sin \sigma + \phi_{2,x}) \ ,$

where $\#$ denotes $(b, p_0, q_0, \eta_1, \eta_2; x\Sigma_1 + y\Gamma_1 + z\Sigma_2 + w\Gamma_2 + \phi)$ and (ϕ_1, ϕ_2) denotes the components of ϕ. Differentiating (60) with respect to z and noting that $\phi_{1,x}^\circ = \phi_{2,x}^\circ = 0$, we obtain

(61) $\dfrac{\partial^2 G}{\partial x \partial z}(0; 0, 0) = PD_\theta F_1^\circ(\phi_{1,xz}^\circ) + (L(1) + U(1))PD_s F_1^\circ(\phi_{2,xz}^\circ)$

$$+ PD_\theta^2 F_1^\circ(\sin \sigma, \sin 2\sigma) + \frac{1}{2}(L(2) + U(2))PD_\theta D_s F_1^\circ(\sin \sigma, \sin 2\sigma)$$

$$+ [L(1) + U(1)]PD_\theta D_s F_1^\circ(\sin 2\sigma, \sin \sigma)$$

$$+ \frac{1}{2}[L(1) + U(1)][L(2) + U(2)]PD_s^2 F_1^\circ(\sin 2\sigma, \sin \sigma) \ .$$

Since ϕ is Z-valued and since DF_1° commutes with P by Lemma 3, the first and second terms on the right hand side vanish. By (54), we obtain

(62) $\dfrac{\partial^2 G}{\partial x \partial z}(0\,;0,0) = PD_\theta^2 F_1^\circ(\sin\sigma,\sin2\sigma) + \dfrac{L(2)+U(2)}{2}\,PD_\theta D_S F_1^\circ(\sin\sigma,\sin2\sigma)$

$$+ [L(1)+U(1)]PD_\theta D_S F_1^\circ(\sin2\sigma,\sin\sigma)\,,$$

which, together with Lemma 7, implies

$$f_2(0\,;0,0,0) = A(1,2) + b\left\{[(U(1)-U(2)][L(1)+U(1)] - \dfrac{1}{2}U(1)[L(2)+U(2)]\right\}$$

$$= \dfrac{L(1)L(2)}{2} - L(1)^2 + bU(1)\left\{U(1) - \dfrac{U(2)}{2}\right\}.$$

The coefficient $f_4(0\,;0,0,0)$ is computed from

$$f_4(0\,;0,0,0) = \dfrac{\partial^2 G_2}{\partial x^2}(0\,;0,0)\,.$$

In a similar way we have

$$\dfrac{\partial^2 G}{\partial x^2}(0\,;0,0) = PD_\theta^2 F_1^\circ(\sin\sigma,\sin\sigma) + 2[L(1)+U(1)]PD_\theta D_S F_1^\circ(\sin\sigma,\sin\sigma)\,.$$

Then Lemma 7 completes the proof.

COROLLARY 1.   *We define a function $\Phi(x)$ for $0 \le x < 1$ by*

$$\Phi(x) = \dfrac{1+4x+x^2}{(1-x)^2}\,.$$

Then $f_2(0\,;0,0,0)$ and $f_4(0\,;0,0,0)$ vanish simultaneously when

(63) $b = \dfrac{\Phi(\eta_1^2)}{\Phi(\eta_2^2)}\,.$

COROLLARY 2.   *If $b=0$, then neither $f_2(0\,;0,0,0)$ nor $f_4(0\,;0,0,0)$ vanish for* $\eta_1, \eta_2 \in [0,1)$.

Concluding remark: Corollary 2 is already known ([19], [21]). However, Corollary 1 is surprising. We found a degenerate bifurcation point but it is actually more degenerate than we expected. Our computation shows that the aspect ratio of the flow, i.e., the depths of the upper and the lower fluid, play little role in the bifurcation from the singularity-theoretic viewpoint. Now that a strongly degenerate bifurcation point was found, it is not surprising in retrospective that there are so many solutions to the interfacial wave problem, which were found numerically. For numerical computations we refer the reader to [10], [14], [22], [23], [25].

When both of the two layers of fluids are infinitely deep, then the degenerate

bifurcation point appears where $b = 1$. In other words, more complicated bifurcation may be possible when *the average energy densities of the two flows are equal or close to each other*.

7. The triple bifurcation point. This section is devoted to proving that there is no triple bifurcation point in $(b, p, q, \eta_1, \eta_2) \in [0, \infty) \times [0, \infty)^2 \times [0, 1)^2$. The next lemma is sufficient for our purpose. This lemma shows that there are triple bifurcation points but they all exist in the unphysical range $b < 0$.

LEMMA 8. *For an arbitrary* $(\eta_1, \eta_2) \in [0, 1)^2$, *there exists one and only one* $(b, p, q) \in$ $\boldsymbol{R}^3$ *satisfying* (32) *with three different positive integers. This unique solution satisfies that* $b < 0$.

PROOF. Let $(\eta_1, \eta_2) \in [0, 1)^2$ be given. Suppose three integers $0 < l < m < n$ are given. Then we look for (b, p, q) such that

$$(64) \qquad \begin{pmatrix} n^2 & 1 & -nU(n) \\ m^2 & 1 & -mU(m) \\ l^2 & 1 & -lU(l) \end{pmatrix} \begin{pmatrix} q \\ p \\ b \end{pmatrix} = \begin{pmatrix} nL(n) \\ mL(m) \\ lL(l) \end{pmatrix} .$$

The determinant of the coefficient matrix is computed to be

$$nU(n)(l^2 - m^2) + mU(m)(n^2 - l^2) + lU(l)(m^2 - n^2) ,$$

which we shall write $g(\eta_2)$. We put $x = \eta_2^2$ and multiply this quantity by $(1 - x^l)(1 - x^m)(1 - x^n)$. Then proving $g(\eta_2) \neq 0$ is equivalent to proving that

$$f(x) \equiv n(l^2 - m^2)(1 + x^n)(1 - x^m)(1 - x^l) + m(n^2 - l^2)(1 + x^m)(1 - x^l)(1 - x^n)$$
$$+ l(m^2 - n^2)(1 + x^l)(1 - x^n)(1 - x^m)$$

is nonzero for all $x \in [0, 1)$. However, this is exactly what was proven in the appendix of [16]. Therefore (64) is uniquely solvable.

Now we eliminate p and q to obtain

$$b = -\frac{g(\eta_1)}{g(\eta_2)} .$$

Since g does not vanish in $[0, 1)$, b is negative. $\qquad \square$

This result is also unexpected in that even *five* parameters cannot give us a triple bifurcation point. However, triple bifurcation points exist in the region $b < 0$. We may discard such points because of its unphysical meaning, but we must note that our problem (11), (12) has an equally valid mathematical meaning for negative b. So such triple bifurcation points may affect the structure of the solutions. In fact, even some surface waves suggest the influence of triple bifurcation point. Zufiria's numerical computations of non-symmetric water waves (cf. [28], [29], [30], [31]) seem to strongly

suggest the existence of a triple bifurcation point of mode (1, 3, 6). Though such a triple bifurcation point was proven not to exist in the surface wave problem, we have proven the existence in the interfacial wave problem with $b < 0$. Since the surface wave problem is embedded in the interfacial wave problem, it might be possible to interpret Zufiria's bifurcations as the effect of the triple bifurcation of interfacial waves.

8. The case where $\varepsilon\delta = 0$. Here we examine if there is a (b, η_1, η_2) which makes ε or δ vanish.

We first note that the tangent space at the point $(b, p_0(1, 2), q_0(1, 2), \eta_1, \eta_2)$, is spanned by the three vectors

(65) $$(3, 4U(1) - 2U(2), 2U(2) - U(1), 0, 0),$$

(66) $$(0, 4\dot{L}(1) - 2\dot{L}(2), 2\dot{L}(2) - \dot{L}(1), 3, 0),$$

(67) $$(0, 4b\dot{U}(1) - 2b\dot{U}(2), 2b\dot{U}(2) - b\dot{U}(1), 0, 3),$$

where the dots mean the differentiation with respect to η_1 or η_2. On the other hand, we have

$$\varepsilon = \frac{\partial^2 G}{\partial \lambda \partial x}(0\,;\,0, 0) = \frac{\partial}{\partial \lambda}[L(1) + bU(1) - p - q].$$

Let $(\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0, \lambda_5^0)$ denote the tangent vector at $\lambda = 0$ of the curve which we have chosen to pick up the parameter (see the beginning of the present section). Then we obtain

$$\varepsilon = U(1)\lambda_1^0 - \lambda_2^0 - \lambda_3^0 + \dot{L}(1)\lambda_4^0 + b\dot{U}(1)\lambda_5^0.$$

By the definition, $(\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0, \lambda_5^0)$ is orthogonal to the tangent vectors (65)–(67). It is easy to see that

(68) $$(U(1), -1, -1, \dot{L}(1), b\dot{U}(1))$$

is orthogonal to the three tangent vectors (65)–(67) above. Therefore ε vanishes if and only if $(\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0, \lambda_5^0)$ is orthogonal to the four vectors (65)–(68).

Similarly, $\delta = 0$ if and only if $(\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0, \lambda_5^0)$ is orthogonal to (65)–(67) and

$$(2U(2), -1, -4, 2\dot{L}(2), 2b\dot{U}(2)).$$

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