Tôhoku Math. J.
49 (1997), 185–202

ON SMOOTH $SO_0(p, q)$-ACTIONS ON S^{p+q-1}, II

FUICHI UCHIDA*

(Received November 6, 1995, revised March 14, 1996)

Abstract. Smooth actions of non-compact semi-simple Lie groups have been considered by Asoh, Mukoyama and others, in the case where the actions restricted to the maximal compact subgroup have codimension-one principal orbits. In this paper, we consider such actions on the $(p+1)$-sphere for Lorentz group of type $(p, 2)$.

Introduction. Consider the standard $\boldsymbol{SO}(p) \times \boldsymbol{SO}(q)$-action on the $(p+q-1)$-sphere S^{p+q-1}. This action has codimension-one principal orbits with $\boldsymbol{SO}(p-1) \times \boldsymbol{SO}(q-1)$ as the principal isotropy subgroup. Furthermore, the fixed point set of the restricted $\boldsymbol{SO}(p-1) \times \boldsymbol{SO}(q-1)$-action is diffeomorphic to the circle S^1 for $p \neq 2$ and $q \neq 2$.

In the previous paper [3], we have studied the smooth $\boldsymbol{SO}_0(p, q)$-actions on S^{p+q-1} for $p \geqq 3$ and $q \geqq 3$, each of which is an extension of the above action, and we have shown that such an action is characterized by a pair (ϕ, f) satisfying certain conditions, where ϕ is a smooth one-parameter group on S^1 and $f\colon S^1 \to P_1(\boldsymbol{R})$ is a smooth function.

In this paper, we shall study the smooth $\boldsymbol{SO}_0(p, q)$-actions on S^{p+q-1} in the case $p \geqq 3$ and $q = 1, 2$. In the case $q = 2$, the fixed point set of the restricted $\boldsymbol{SO}(p-1) \times \boldsymbol{SO}(q-1)$-action is diffeomorphic to the 2-sphere S^2, unlike the cases mentioned above. So we shall introduce a triple (S, ϕ, f), instead of the pair, satisfying certain conditions, where S is a circle in S^2, ϕ is a smooth one-parameter group on S, and $f\colon S \to P_1(\boldsymbol{R})$ is a smooth function.

The pair (ϕ, f) was introduced by Asoh [1] to consider smooth $\boldsymbol{SL}(2, \boldsymbol{C})$-actions on the 3-sphere, and was improved by our previous paper [3]. The triple (S, ϕ, f) was introduced by Mukoyama [2] to consider smooth $\boldsymbol{Sp}(2, \boldsymbol{R})$-actions on the 4-sphere. Here, we notice that the Lie groups $\boldsymbol{SL}(2, \boldsymbol{C})$ and $\boldsymbol{Sp}(2, \boldsymbol{R})$ are locally isomorphic to $\boldsymbol{SO}_0(3, 1)$ and $\boldsymbol{SO}_0(3, 2)$, respectively.

The author wishes to express gratitude to the referees and the editors for their helpful advice.

1. Subgroups of $\boldsymbol{SO}(p, q)$. Let $\boldsymbol{SO}(p, q)$ denote the group of matrices in $\boldsymbol{SL}(p+q, \boldsymbol{R})$ which leave invariant the quadratic form

---

1991 *Mathematics Subject Classification*. Primary 57S20.

* Partly supported by the Grants-in-Aid for Scientific Research, The Ministry of Education, Science and Culture, Japan.

$$-x_1^2-\cdots-x_p^2+x_{p+1}^2+\cdots+x_{p+q}^2 .$$

In particular, $\boldsymbol{SO}(p, q)$ contains $\boldsymbol{S}(\boldsymbol{O}(p)\times\boldsymbol{O}(q))$ as a maximal compact subgroup.

Put

$$I_{p,q}=\left[\begin{array}{c|c} -I_p & 0 \\ \hline 0 & I_q \end{array}\right],$$

where I_n denotes the unit matrix of order n. Clearly, a real matrix g of order $p+q$ belongs to $\boldsymbol{SO}(p, q)$ if and only if

$${}^t g I_{p,q} g = I_{p,q} \quad \text{and} \quad \det g = 1 ,$$

where ${}^t g$ denotes the transposed matrix of g.

Let $\mathfrak{so}(p, q)$ denote the Lie algebra of $\boldsymbol{SO}(p, q)$. Then a real matrix X of order $p+q$ belongs to $\mathfrak{so}(p, q)$ if and only if

$$(1.1) \qquad {}^t X I_{p,q} + I_{p,q} X = 0 .$$

Writing X in the form

$$X=\left[\begin{array}{c|c} X_1 & X_2 \\ \hline X_3 & X_4 \end{array}\right],$$

where X_1 is of order p and X_4 is of order q, we see that the condition (1.1) is equivalent to the equality $X_3={}^t X_2$ and the skew-symmetry of X_1, X_4.

Let $\boldsymbol{SO}_0(p, q)$ denote the identity component of $\boldsymbol{SO}(p, q)$. Notice that $\boldsymbol{SO}(p, q)$ has two connected components for $p, q \geqq 1$. We see that

$$\boldsymbol{SO}(p)\times\boldsymbol{SO}(q)=\boldsymbol{SO}_0(p, q)\cap\boldsymbol{SO}(p+q)$$

is a maximal compact subgroup of $\boldsymbol{SO}_0(p, q)$.

Here, we consider the standard representations of $\boldsymbol{SO}(p, 2)$ and $\mathfrak{so}(p, 2)$ on $\boldsymbol{R}^{p+2}$. Let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{p+2}\}$ denote the standard basis of $\boldsymbol{R}^{p+2}$. Let $H(a:b:c)$ (resp. $\mathfrak{h}(a:b:c)$) denote the isotropy subgroup (resp. subalgebra) at $a\boldsymbol{e}_1+b\boldsymbol{e}_{p+1}+c\boldsymbol{e}_{p+2}$ for $(a, b, c)\neq(0, 0, 0)$. Notice that $H(1:0:0)=\boldsymbol{SO}(p-1, 2)$ and $H(0:1:0)=\boldsymbol{SO}(p, 1)$. Put

$$H_0(a:b:c)=\boldsymbol{SO}_0(p, 2)\cap H(a:b:c) .$$

We can show that $H_0(a:b:c)$ is connected for any (a, b, c).

LEMMA 1.2. *Suppose $p\geqq 3$. Let $\mathfrak{g}$ be a proper subalgebra of $\mathfrak{so}(p, 2)$ which contains $\mathfrak{so}(p-1)=\bigcap\mathfrak{h}(a:b:c)$. If*

$$\dim\mathfrak{so}(p, 2)-\dim\mathfrak{g}\leqq p+1 ,$$

then

$$\mathfrak{g}=\mathfrak{h}(a:b:c) \quad \textit{for some} \ (a, b, c)\neq(0, 0, 0)\,,$$

or

$$\mathfrak{g}=\mathfrak{h}(a:b:c)\oplus \boldsymbol{R}^1 \quad \textit{for some} \ (a, b, c)\neq(0, 0, 0)$$

such that $a^2=b^2+c^2$, *where the space* $\boldsymbol{R}^1$ *is generated by the matrix* $b(E_{1,p+1}+E_{p+1,1})+c(E_{1,p+2}+E_{p+2,1})$. *Here* $E_{i,j}$ *denotes the matrix unit.*

PROOF. Let $\boldsymbol{SO}(p-1)$ denote the closed connected subgroup of $\boldsymbol{SO}_0(p, 2)$ corresponding to the subalgebra $\mathfrak{so}(p-1)$. We obtain the desired result, by considering the adjoint representation of $\boldsymbol{SO}(p-1)$ on $\mathfrak{so}(p, 2)$ and the bracket operation on invariant subspaces. We omit the details. q.e.d.

Let $N(p, 2)$ denote the subgroup of $\boldsymbol{SO}_0(p, 2)$ consisting of matrices in the form

$$\left[\begin{array}{cc|c} X_1 & 0 & X_2 \\ 0 & I_{p-1} & 0 \\ \hline X_3 & 0 & X_4 \end{array}\right],$$

where X_1 is of order one and X_4 is of order two. Notice that the group $N(p, 2)$ is the identity component of the centralizer of $\boldsymbol{SO}(p-1)$, and $N(p, 2)$ is canonically isomorphic to $\boldsymbol{SO}_0(1, 2)$.

Put

$$m(\theta)=\left[\begin{array}{cc|cc} \cosh\theta & 0 & \sinh\theta & 0 \\ 0 & I_{p-1} & 0 & 0 \\ \hline \sinh\theta & 0 & \cosh\theta & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]; \quad \theta\in\boldsymbol{R}\,.$$

Then we see that $m(\theta)$ is an element of $N(p, 2)$. Let $M(p, 2)$ denote the subgroup of $N(p, 2)$ consisting of matrices $m(\theta)$, $\theta\in\boldsymbol{R}$.

Considering the orbit of $a\boldsymbol{e}_1+b\boldsymbol{e}_{p+1}+c\boldsymbol{e}_{p+2}$, we obtain the following (cf. [3, Proof of Lemma 1.5]):

$$\boldsymbol{SO}_0(p, 2)=(\boldsymbol{SO}(p)\times\boldsymbol{SO}(2))N(p, 2)H_0(a:b:c) \tag{1.3}$$

for each $(a, b, c)\neq(0, 0, 0)$. Moreover, we obtain

$$\boldsymbol{SO}_0(p, 2)=(\boldsymbol{SO}(p)\times\boldsymbol{SO}(2))M(p, 2)H_0(a:b:0) \tag{1.4}$$

for each $(a, b)\neq(0, 0)$.

2. Smooth $SO_0(p, 2)$-actions on S^{p+1}.

Let $\Phi_0: SO_0(p, 2) \times S^{p+1} \to S^{p+1}$ denote the standard action defined by

$$\Phi_0(g, u) = \|gu\|^{-1} gu . \tag{2.1}$$

Its restricted $SO(p) \times SO(2)$-action ψ is by orthogonal transformations and has co-dimension-one principal orbits with $SO(p-1)$ as the principal isotropy subgroup. Put

$$\begin{aligned} G &= SO_0(p, 2), \quad K = SO(p) \times SO(2), \\ H &= SO(p-1), \quad \psi = \Phi_0|_{(K \times S^{p+1})} . \end{aligned} \tag{2.2}$$

Let $F(H)$ denote the fixed point set of the restricted H-action. Then the set $F(H)$ consists of the points

$$xe_1 + ye_{p+1} + ze_{p+2}$$

satisfying $x^2 + y^2 + z^2 = 1$, and is naturally diffeomorphic to the 2-sphere S^2.

Let $\Phi: G \times S^{p+1} \to S^{p+1}$ be a smooth G-action on S^{p+1} such that its restricted K-action coincides with the action ψ, i.e., $\Phi|_{(K \times S^{p+1})} = \psi$.

First we shall show that there exists a smooth function

$$f: F(H) \to P_2(\boldsymbol{R})$$

uniquely determined by the condition: the isotropy subgroup G_Y at $Y \in F(H)$ contains $H_0(a:b:c)$, if $f(Y) = (a:b:c)$. Here, $P_2(\boldsymbol{R})$ denotes the real projective plane while G_Y denotes the isotropy subgroup at Y with respect to the given G-action Φ.

Since G_Y contains $H = SO(p-1)$, G_Y contains a unique subgroup of the form $H_0(a:b:c)$ by Lemma 1.2. It remains only to show the smoothness of f. Let $\mathfrak{g}_Y$ denote the Lie algebra of the subgroup G_Y. Considering the subalgebra $\mathfrak{h}(a:b:c)$, we see that the following elements are contained in $\mathfrak{g}_Y$,

$$\begin{aligned} &a(E_{i,p+1} + E_{p+1,i}) + b(E_{1,i} - E_{i,1}), \\ &a(E_{i,p+2} + E_{p+2,i}) + c(E_{1,i} - E_{i,1}), \\ &b(E_{i,p+2} + E_{p+2,i}) - c(E_{i,p+1} + E_{p+1,i}), \end{aligned}$$

for each $2 \leqq i \leqq p$. Hence we obtain the smoothness of f (cf. [3, §2]).

The subgroup $N(p, 2)$ acts on $F(H)$ via $\Phi|_{(N(p,2) \times F(H))}$, since $N(p, 2)$ is contained in the normalizer of H. On the other hand, the standard representation of $SO_0(1, 2)$ on $\boldsymbol{R}^3$ induces a smooth action of $SO_0(1, 2)$ on the real projective plane $P_2(\boldsymbol{R})$. Via the canonical isomorphism of $N(p, 2)$ onto $SO_0(1, 2)$, we may regard $P_2(\boldsymbol{R})$ as an $N(p, 2)$-manifold.

Put

$$j_1 = \begin{bmatrix} -I_2 & 0 \\ 0 & I_p \end{bmatrix}, \quad j_2 = \begin{bmatrix} I_p & 0 \\ 0 & -I_2 \end{bmatrix}. \tag{2.3}$$

Then these matrices act on $F(H)$ via the orthogonal K-action ψ and act on $P_2(\boldsymbol{R})$ by

$$j_1(a:b:c)=j_2(a:b:c)=(-a:b:c)\,.$$

Put $N(p, 2)^+=N(p, 2)\cup j_1N(p, 2)$. Then, $N(p, 2)^+$ is a subgroup of the normalizer $N(H)$ of H in G, and $N(p, 2)^+$ is naturally isomorphic to $N(H)/H$. Notice that j_2 is contained in $N(p, 2)$, and j_1j_2 commutes with each element of $N(p, 2)$.

LEMMA 2.4. *The function f: $F(H)\to P_2(\boldsymbol{R})$ is $N(p, 2)^+$-equivariant.*

PROOF. Let $n\in N(p, 2)^+$ and $(a:b:c)\in P_2(\boldsymbol{R})$. We can write

$$n(a\boldsymbol{e}_1+b\boldsymbol{e}_{p+1}+c\boldsymbol{e}_{p+2})$$

in the form $a'\boldsymbol{e}_1+b'\boldsymbol{e}_{p+1}+c'\boldsymbol{e}_{p+2}$ in the standard representation space $\boldsymbol{R}^{p+2}$ of $\boldsymbol{SO}_0(p, 2)$. In this case, we obtain

$$nH_0(a:b:c)n^{-1}=H_0(a':b':c')$$

in $\boldsymbol{SO}_0(p, 2)$, and

$$n(a:b:c)=(a':b':c')$$

in $P_2(\boldsymbol{R})$.

Suppose $f(Y)=(a:b:c)$. We see that

$$H_0(a:b:c)\subset G_Y\,,$$

by the definition of the function f. Hence we obtain

$$H_0(a':b':c')=nH_0(a:b:c)n^{-1}\subset nG_Yn^{-1}=G_{\Phi(n,Y)}\,.$$

Therefore,

$$f(\Phi(n, Y))=(a':b':c')=n(a:b:c)=nf(Y)\,.$$

The equation means that f is $N(p, 2)^+$-equivariant. q.e.d.

By the definition of the function f, we see that

$$N(p, 2)_Y^+\supset N(p, 2)^+\cap H_0(a:b:c) \tag{2.5}$$

for $f(Y)=(a:b:c)$.

3. Certain symmetric matrices. Let a smooth action ϕ of $N(p, 2)^+$ on $F(H)$ and a smooth function f: $F(H)\to P_2(\boldsymbol{R})$ be given. Suppose that f is $N(p, 2)^+$-equivariant. Let $P(Y)$ denote the symmetric matrix of order $p+2$ defined by

$$P(Y)=(a^2+b^2+c^2)^{-1}V(a, b, c)\,{}^tV(a, b, c)$$

for $f(Y)=(a:b:c)$, where $V(a, b, c)=a\boldsymbol{e}_1+b\boldsymbol{e}_{p+1}+c\boldsymbol{e}_{p+2}$, and define

$$U(Y)=\{g\in G\,|\,gP(Y)^tg=P(Y)\}\,.$$

Then, clearly the identity component $U_0(Y)$ of $U(Y)$ coincides with $H_0(a:b:c)$, and there is a positive real number $\lambda(n, Y)$ such that

$$nP(Y)^tn=\lambda(n, Y)P(\phi(n, Y)) \tag{3.1}$$

for each $Y\in F(H)$ and $n\in N(p, 2)^+$.

LEMMA 3.2. *Suppose* $kP(Y)^tk=P(Y')$ *for* $Y, Y'\in F(H)$ *and* $k\in \boldsymbol{SO}(p)\times I_2$. *Then, there are the following possibilities*: (1) $k\in H$ *and* $f(Y)=f(Y')$, (2) $j_1k\in H$ *and* $j_1f(Y)=f(Y')$, *or* (3) $f(Y)=f(Y')=(0:b:c)$.

PROOF. Put $f(Y)=(a:b:c)$. We have

$$k(a\boldsymbol{e}_1+b\boldsymbol{e}_{p+1}+c\boldsymbol{e}_{p+2})=a'\boldsymbol{e}_1+b\boldsymbol{e}_{p+1}+c\boldsymbol{e}_{p+2}$$

in the standard representation space $\boldsymbol{R}^{p+2}$ of $\boldsymbol{SO}_0(p, 2)$ and $f(Y')=(a':b:c)$ by the assumption. Moreover, $a'=\pm a$.

If $a'=a\neq 0$, then $k\in H$, which is the case (1). If $a'=-a\neq 0$, then $j_1k\in H$, which is the case (2). If $a'=a=0$, then it is the case (3). q.e.d.

4. Construction of $\boldsymbol{SO}_0(p, 2)$-actions. Let a smooth action ϕ of $N(p, 2)^+$ on $F(H)$ and a smooth function f: $F(H)\to P_2(\boldsymbol{R})$ be given.

Suppose that the restriction of ϕ on $K\cap N(p, 2)^+$ coincides with the restriction of the orthogonal K-action ψ, while f is $N(p, 2)^+$-equivariant and satisfies the condition (2.5). The condition (2.5) can be restated as

$$N(p, 2)_Y^+\supset N(p, 2)^+\cap U_0(Y) \tag{4.1}$$

for each $Y\in F(H)$.

We shall show how to construct a smooth $G=\boldsymbol{SO}_0(p, 2)$-action on the $(p+1)$-sphere S^{p+1} from the pair (ϕ, f). We use the notation in (2.2) and (2.3). Put $K'=\boldsymbol{SO}(p)\times I_2$.

By (1.3) and the facts that $U_0(Y)=H_0(a:b:c)$ for $f(Y)=(a:b:c)$, and that $N(p, 2)$ contains $I_p\times \boldsymbol{SO}(2)$, we obtain the decomposition

$$G=K'N(p, 2)U_0(Y) \tag{4.2}$$

for each $Y\in F(H)$.

Take $(g, X)\in G\times S^{p+1}$. Let us choose

$$k\in K\,,\ \ Y\in F(H)\,;\ \ \psi(k, Y)=X\,,$$

$$k_0\in K'\,,\ \ n\in N(p, 2)\,,\ \ h\in U_0(Y)\,;\ \ gk=k_0nh\,,$$

and put

$$\Phi(g, X)=\psi(k_0, \phi(n, Y))\,. \tag{4.3}$$

We shall show that Φ is well-defined and is a smooth G-action on S^{p+1}. For the proof, we need the following:

LEMMA 4.4. *Let* $Y \in F(H)$. *Suppose* $knh = k'n'h'$ *for* $k, k' \in K'$, $n, n' \in N(p, 2)$, *and* $h, h' \in U_0(Y)$. *Then*

$$\psi(k, \phi(n, Y)) = \psi(k', \phi(n', Y)) .$$

PROOF. We obtain

$$knP(Y)\,{}^t n\,{}^t k = k'n'P(Y)\,{}^t n'\,{}^t k' .$$

Then, by (3.1)

$$\lambda k P(\phi(n, Y))\,{}^t k = \lambda' k' P(\phi(n', Y))\,{}^t k'$$

for certain positive real numbers λ, λ'. Comparing the traces of both sides, we obtain $\lambda = \lambda'$ and

$$kP(\phi(n, Y))\,{}^t k = k'P(\phi(n', Y))\,{}^t k' .$$

Then we have the following possibilities, by Lemma 3.2,

(1) $k'^{-1}k \in H$,

(2) $j_1 k'^{-1} k \in H$,

(3) $f(\phi(n, Y)) = f(\phi(n', Y)) = (0 : b : c)$.

In the case (1), we see that

$$n^{-1}n' = (n^{-1}k'^{-1}kn)(hh'^{-1})$$

is contained in $N(p, 2)_Y^+$ by (4.1). Hence

$$\phi(n, Y) = \phi(n', Y) ,$$

and we obtain the desired equation by $k'^{-1}k \in H$.

In the case (2), we see that

$$n^{-1} j_1 n' = (n^{-1} j_1 k'^{-1} kn)(hh'^{-1})$$

is contained in $N(p, 2)_Y^+$ by (4.1). Hence

$$\phi(n, Y) = \phi(j_1 n', Y) ,$$

and we obtain the desired equation by $j_1 k'^{-1} k \in H$.

In the case (3), we see that

$$\psi(j_1, \phi(n, Y)) = \phi(n, Y) ,$$

by (2.5). Hence

$$\phi(n, Y) = y e_{p+1} + z e_{p+2} ,$$

and we see that

$$\psi(k, \phi(n, Y)) = \psi(k', \phi(n, Y))$$

for any $k, k' \in K'$. On the other hand,

$$n'n^{-1} = (k'^{-1}k)(nhh'^{-1}n^{-1})$$

is contained in $N(p, 2)^{+}_{\phi(n,Y)}$ by (4.1). Hence

$$\phi(n, Y) = \phi(n', Y),$$

and we obtain the desired equation. q.e.d.

PROPOSITION 4.5. *Φ in* (4.3) *is well-defined and is an abstract G-action such that* $\Phi|_{(K \times S^{p+1})} = \psi$.

PROOF. For $(g, X) \in G \times S^{p+1}$, let us choose

$$X = \psi(k_1, Y_1) = \psi(k_2, Y_2)$$

as in (4.3), where $k_i \in K$, $Y_i \in F(H)$, and

$$gk_1 = k_0 nh,$$

where $k_0 \in K'$, $n \in N(p, 2)$, $h \in U_0(Y_1)$.

We have $k_1^{-1}k_2 = k'k''$, where $k' \in K'$ and $k'' \in K \cap N(p, 2)$. Then, we have the following possibilities,

(1) $k' \in H$, (2) $j_1 k' \in H$, and (3) $\psi(j_1, Y_1) = Y_1$.

In the case (1), we see that $Y_1 = \psi(k'', Y_2)$, and

$$gk_2 = k_0(nk'')(k''^{-1}hk'k''),$$

where $nk'' \in N(p, 2)$ and $k''^{-1}hk'k'' \in U_0(Y_2)$. Hence we obtain

$$\psi(k_0, \phi(nk'', Y_2)) = \psi(k_0, \phi(n, Y_1)).$$

In the case (2), we see that $\psi(j_1, Y_1) = \psi(k'', Y_2)$, and

$$gk_2 = (k_0 j_1)(j_1 n j_1 k'')(k''^{-1} j_1 hk'k''),$$

where $k_0 j_1 \in K'$, $j_1 n j_1 k'' \in N(p, 2)$, and $k''^{-1} j_1 hk'k'' \in U_0(Y_2)$. Hence we obtain

$$\psi(k_0 j_1, \phi(j_1 n j_1 k'', Y_2)) = \psi(k_0, \phi(n, Y_1)).$$

In the case (3), we see that $Y_1 = \psi(k'', Y_2)$, $k' \in U_0(Y_1)$ and

$$gk_2 = k_0(nk'')(k''^{-1}hk'k''),$$

where $nk'' \in N(p, 2)$ and $k''^{-1}hk'k'' \in U_0(Y_2)$. Hence we obtain

$$\psi(k_0, \phi(nk'', Y_2)) = \psi(k_0, \phi(n, Y_1)).$$

Combining these results and Lemma 4.4, we see that Φ in (4.3) is well-defined.

Take $g, g' \in G$ and $X \in S^{p+1}$. As in (4.3), let us choose $X = \psi(k, Y)$, where $k \in K$, $Y \in F(H)$, and

$$gk = k_0 nh\,, \quad g'k_0 = k_1 n_1 h_1\,,$$

where $k_0, k_1 \in K'$, $n, n_1 \in N(p, 2)$, $h \in U_0(Y)$ and $h_1 \in U_0(\phi(n, Y))$. Then,

$$\begin{aligned}\Phi(g', \Phi(g, X)) &= \Phi(g', \psi(k_0, \phi(n, Y))) = \psi(k_1, \phi(n_1, \phi(n, Y))) \\ &= \psi(k_1, \phi(n_1 n, Y)) = \Phi(g'g, X)\,,\end{aligned}$$

because

$$g'gk = k_1(n_1 n)(n^{-1} h_1 nh)\,,$$

$n_1 n \in N(p, 2)$ and $n^{-1} h_1 nh \in U_0(Y)$.

This shows that Φ in (4.3) is an abstract G-action.

Finally, take $(k, X) \in K \times S^{p+1}$ and put $X = \psi(k_1, Y)$, where $k_1 \in K$, $Y \in F(H)$. We have $kk_1 = k'k''$, where $k' \in K'$ and $k'' \in K \cap N(p, 2)$. Then,

$$\Phi(k, X) = \psi(k', \phi(k'', Y)) = \psi(k'k'', Y) = \psi(k, X)\,.$$

This shows $\Phi|_{(K \times S^{p+1})} = \psi$. q.e.d.

Notice that the continuity of Φ is unknown at this stage. In the remainder of this section, we shall show the smoothness of the G-action Φ.

Define $S = f^{-1}(P_1(\boldsymbol{R}))$. This means that the set S consists of the points $Y \in F(H)$ such that $f(Y) = (* : * : 0)$. Then the points $\pm \boldsymbol{e}_1$ are contained in S.

Considering the orbits of $I_p \times \boldsymbol{SO}(2)$, we see that the function f is transversal to $P_1(\boldsymbol{R})$ at each point of $S - \{\pm \boldsymbol{e}_1\}$. Clearly S is invariant under the restricted $M(p, 2)$-action and the actions $\psi(j_\varepsilon, -)$ for $\varepsilon = 1, 2$.

Consequently, we see that S is a one-dimensional closed submanifold of $F(H)$.

The subset of S consisting of the points Y with $f(Y) = (a : b : 0)$ such that $ab > 0$ has two connected components. Denote by S_+ the component contained in the upper hemisphere. Then there is a smooth positive-valued function β on S_+ such that $f(Y) = (1 : \beta(Y) : 0)$.

LEMMA 4.6. *For $(\theta, Y) \in \boldsymbol{R} \times S_+$, we have $\phi(m(\theta), Y) \in S_+$ if and only if*

$$(1 + \beta(Y) \tanh \theta)(\beta(Y) + \tanh \theta) > 0\,. \tag{4.7}$$

PROOF. Since f is $N(p, 2)^+$-equivariant, we obtain

$$f(\phi(m(\theta), Y)) = (1 + \beta(Y) \tanh \theta : \beta(Y) + \tanh \theta : 0)\,.$$

Then the "only if" part is clear. Suppose (4.7) holds. Then,

$$\phi(m(\theta), Y) \in S_+ \cup j_1 j_2(S_+)$$

and we see that $\phi(m(\theta), Y)$ is not contained in $j_1 j_2(S_+)$ by considering orbits of the

$M(p, 2)$-action. q.e.d.

Define

$$D_+ = \{(\theta, Y) \in \boldsymbol{R} \times S_+ \mid \phi(m(\theta), Y) \in S_+\},$$

$$W_+ = \{(g, Y) \in G \times S_+ \mid \pm \operatorname{trace}(gP(Y)\,{}^t g) \neq (1-\beta(Y)^2)(1+\beta(Y)^2)^{-1}\}.$$

Clearly D_+ is an open set of $\boldsymbol{R} \times S_+$ and W_+ is an open set of $G \times S_+$. Notice that

$$\operatorname{trace}(gP(Y)\,{}^t g) = \cosh 2\theta + 2\beta(Y)(1+\beta(Y)^2)^{-1} \sinh 2\theta$$

for the decomposition $g = km(\theta)u$, where $k \in K$, $\theta \in \boldsymbol{R}$, $u \in U_0(Y)$ and $Y \in S_+$.

Now, we have the following results, whose proof is quite similar to that of Lemma 4.7 in [3].

LEMMA 4.8. *For $(g, Y) \in G \times S_+$, we have $(g, Y) \in W_+$ if and only if there is a decomposition $g = km(\theta)u$, where $k \in K$, $\theta \in \boldsymbol{R}$, and $u \in U_0(Y)$ such that $(\theta, Y) \in D_+$.*

LEMMA 4.9. *There is a smooth mapping $\Delta : W_+ \to (K/H) \times D_+$ defined by $\Delta(g, Y) = (kH, (\theta, Y))$, where $g = km(\theta)u$; $k \in K$, $\theta \in \boldsymbol{R}$, and $u \in U_0(Y)$.*

By the definition of S_+, there exists only one point $\boldsymbol{w}_0$ of S such that $f(\boldsymbol{w}_0) = (0:1:0)$ and $\boldsymbol{w}_0$ is contained in the closure of S_+. Define

$$S_1(\Phi) = \{\Phi(g, \boldsymbol{e}_1) \mid g \in G\},$$

$$S_1(\Phi_0) = \{\Phi_0(g, \boldsymbol{e}_1) \mid g \in G\},$$

$$S_2(\Phi) = \{\Phi(g, \boldsymbol{w}_0) \mid g \in G\},$$

$$S_2(\Phi_0) = \{\Phi_0(g, \boldsymbol{e}_{p+1}) \mid g \in G\}.$$

Here, Φ is the G-action defined by (4.3) and Φ_0 is the standard G-action. By (4.3) and the conditions on ϕ and f, we see that there are positive real numbers $r_1, r_2 < 1$ such that

$$S_1(\Phi) = \{\boldsymbol{v} \oplus \boldsymbol{w} \in S(\boldsymbol{R}^p \oplus \boldsymbol{R}^2) \mid \|\boldsymbol{w}\| < r_1\},$$

$$S_2(\Phi) = \{\boldsymbol{v} \oplus \boldsymbol{w} \in S(\boldsymbol{R}^p \oplus \boldsymbol{R}^2) \mid \|\boldsymbol{v}\| < r_2\}.$$

On the other hand, it is clear that

$$S_1(\Phi_0) = \{\boldsymbol{v} \oplus \boldsymbol{w} \in S(\boldsymbol{R}^p \oplus \boldsymbol{R}^2) \mid \|\boldsymbol{v}\| > \|\boldsymbol{w}\|\},$$

$$S_2(\Phi_0) = \{\boldsymbol{v} \oplus \boldsymbol{w} \in S(\boldsymbol{R}^p \oplus \boldsymbol{R}^2) \mid \|\boldsymbol{v}\| < \|\boldsymbol{w}\|\}.$$

Define two G-maps $F_\varepsilon : S_\varepsilon(\Phi) \to S_\varepsilon(\Phi_0)$; $\varepsilon = 1, 2$ by

$$F_1(\Phi(g, \boldsymbol{e}_1)) = \Phi_0(g, \boldsymbol{e}_1),$$

$$F_2(\Phi(g, \boldsymbol{w}_0)) = \Phi_0(g, \boldsymbol{e}_{p+1}).$$

Notice that the smoothness of the G-action Φ is unknown at this stage.

LEMMA 4.10. *F_1 and F_2 are diffeomorphisms.*

PROOF. We can write

$$\phi(m(\theta), e_1) = x(\theta)e_1 + y(\theta)e_{p+1} + z(\theta)e_{p+2} .$$

Considering the action $\psi(j_2, -)$, we see that $x(\theta)$ is an even function while $y(\theta)$ and $z(\theta)$ are odd functions. Hence there exist smooth even functions $u(\theta)$, $v(\theta)$ such that

$$y(\theta) = \theta \cdot u(\theta) , \quad z(\theta) = \theta \cdot v(\theta) .$$

Considering the function f, we see that $(u(\theta), v(\theta)) \neq (0, 0)$ for each θ.

Define $a: \boldsymbol{R} \to \boldsymbol{SO}(2)$ and $\tau: \boldsymbol{R} \to (-r_1, r_1)$ by

$$a(\theta) = (u(\theta)^2 + v(\theta)^2)^{-1/2} \begin{bmatrix} u(\theta) & v(\theta) \\ -v(\theta) & u(\theta) \end{bmatrix},$$

$$\tau(\theta) = \theta(u(\theta)^2 + v(\theta)^2)^{1/2} .$$

Then

$$a(\theta) \begin{bmatrix} y(\theta) \\ z(\theta) \end{bmatrix} = \begin{bmatrix} \tau(\theta) \\ 0 \end{bmatrix}.$$

Since the curve $\phi(m(\theta), e_1)$ is transverse to each latitude, we see that τ is a diffeomorphism. Then, we obtain a K-equivariant diffeomorphism $h_0: S_1(\Phi) \to S_1(\Phi)$ defined by

$$h_0(v \oplus w) = v \oplus a\tau^{-1}(\|w\|)w .$$

Next, there is an odd function $s_1: (-r_1, r_1) \to \boldsymbol{R}$ determined by

$$s_1(\tau(\theta)) = (\cosh 2\theta)^{-1/2} \sinh \theta ,$$

and hence there is a smooth even function $\sigma_1: (-r_1, r_1) \to \boldsymbol{R}$ such that $s_1(\tau(\theta)) = \tau(\theta)\sigma_1(\tau(\theta))$. Then, we obtain a K-equivariant diffeomorphism $h_1: S_1(\Phi) \to S_1(\Phi_0)$ defined by

$$h_1(v \oplus w) = c_1 v \oplus \sigma_1(\|w\|)w ,$$

where c_1 is a positive scalar.

By definition, we see that

$$h_1 h_0(\phi(m(\theta), e_1)) = \Phi_0(m(\theta), e_1)$$

for each θ, and hence

$$h_1 h_0(\Phi(g, e_1)) = \Phi_0(g, e_1)$$

for each $g \in G$. Therefore, $F_1 = h_1 h_0$ is a diffeomorphism.

Similarly, we can write

$$\phi(m(\theta), \boldsymbol{w}_0) = x(\theta)\boldsymbol{e}_1 + y(\theta)\boldsymbol{e}_{p+1} + z(\theta)\boldsymbol{e}_{p+2} .$$

Considering the action $\psi(j_1, -)$, we see that $x(\theta)$ is an odd function while $y(\theta)$ and $z(\theta)$ are even functions. By an argument similar to that above, we see that $x(\theta)$ is a diffeomorphism of $\boldsymbol{R}$ onto the interval $(-r_2, r_2)$ and there is a smooth mapping $b: \boldsymbol{R} \to \boldsymbol{SO}(2)$ such that

$$b(\theta)\begin{bmatrix} y(\theta) \\ z(\theta) \end{bmatrix} = \begin{bmatrix} (1-x(\theta)^2)^{1/2} \\ 0 \end{bmatrix}.$$

Next, there is an odd function $s_2: (-r_2, r_2) \to \boldsymbol{R}$ determined by

$$s_2(x(\theta)) = (\cosh 2\theta)^{-1/2} \sinh \theta ,$$

and hence there is a smooth even function $\sigma_2: (-r_2, r_2) \to \boldsymbol{R}$ such that $s_2(x(\theta)) = x(\theta)\sigma_2(x(\theta))$.

Then, we obtain K-equivariant diffeomorphisms $h_2: S_2(\Phi) \to S_2(\Phi)$ and $h_3: S_2(\Phi) \to S_2(\Phi_0)$ defined by

$$h_2(\boldsymbol{v} \oplus \boldsymbol{w}) = \boldsymbol{v} \oplus bx^{-1}(\|\boldsymbol{v}\|)\boldsymbol{w} ,$$

$$h_3(\boldsymbol{v} \oplus \boldsymbol{w}) = \sigma_2(\|\boldsymbol{v}\|)\boldsymbol{v} \oplus c_2\boldsymbol{w} ,$$

where c_2 is a positive scalar. Then, we see that

$$h_3h_2(\phi(m(\theta), \boldsymbol{w}_0)) = \Phi_0(m(\theta), \boldsymbol{e}_{p+1})$$

for each θ, and hence

$$h_3h_2(\Phi(g, \boldsymbol{w}_0)) = \Phi_0(g, \boldsymbol{e}_{p+1})$$

for each $g \in G$. Therefore, $F_2 = h_3h_2$ is a diffeomorphism.

This completes the proof of Lemma 4.10.

PROPOSITION 4.11. *Φ in (4.3) is a smooth G-action.*

PROOF. By Lemma 4.10 we see that the restrictions of Φ to $G \times S_1(\Phi)$ and $G \times S_2(\Phi)$ are smooth. Define

$$W(\phi) = \{(g, \phi(k, Y)) \mid g \in G, k \in K \text{ and } (gk, Y) \in W_+\} .$$

Then, we see that $W(\Phi)$ is an open set of $G \times S^{p+1}$, since W_+ is an open set of $G \times S_+$. Furthermore, we see that the restriction of Φ to $W(\Phi)$ is smooth, since Δ is smooth by Lemma 4.9. Consequently, we obtain the smoothness of Φ, because three open sets $W(\Phi)$, $G \times S_1(\Phi)$ and $G \times S_2(\Phi)$ cover $G \times S^{p+1}$. q.e.d.

Thus we have proved the following.

THEOREM 4.12. *Suppose $p \geqq 3$. Then, there is a one-to-one correspondence between*

the set of smooth $\boldsymbol{SO}_0(p, 2)$-actions on S^{p+1} whose restricted $K=\boldsymbol{SO}(p)\times\boldsymbol{SO}(2)$-action is the standard orthogonal action and the set of pairs (ϕ, f), where ϕ is a smooth $N(p, 2)^+$-action on $S^2=F(H)$ whose restriction on $K\cap N(p, 2)^+$ coincides with the standard K-action and $f: S^2\to P_2(\boldsymbol{R})$ is a smooth $N(p, 2)^+$-equivariant function satisfying the condition (4.1).

5. Construction of (ϕ, f). In the previous section, we saw how to construct a smooth $\boldsymbol{SO}_0(p, 2)$-action on S^{p+1} from a pair (ϕ, f), where ϕ is a smooth $N(p, 2)^+$-action on $S^2=F(H)$ and $f: S^2\to P_2(\boldsymbol{R})$ is a smooth $N(p, 2)^+$-equivariant function, satisfying certain conditions.

We now consider how to construct such a pair (ϕ, f).

Put $J_\varepsilon=\psi(j_\varepsilon, -)$ for $\varepsilon=1, 2$ and $J=J_1J_2$ on $S^2=F(H)$. Then J_1, J_2 are involutions on S^2, and J is the antipodal involution. Let S be a one-dimensional closed submanifold of S^2, which transversely intersects each latitude and is invariant under the involutions J_1 and J_2. In particular, $\pm\boldsymbol{e}_1$ are contained in S.

Put $K''=I_p\times\boldsymbol{SO}(2)$. K'' acts orthogonally on S^2 via the restricted action of ψ, and the set of K''-orbits coincides with the set of latitudes.

Suppose a smooth one-parameter group ϕ_0 on S and a smooth map $f_0: S\to P_1(\boldsymbol{R})$ satisfy the following condition:

$$
(5.1)\qquad
\begin{aligned}
&\text{(a)}\quad J_\varepsilon\phi_0(\theta, Y)=\phi_0(-\theta, J_\varepsilon(Y)), \qquad \varepsilon=1, 2,\\
&\text{(b)}\quad f_0J_\varepsilon(Y)=j_\varepsilon f_0(Y), \qquad \varepsilon=1, 2,\\
&\text{(c)}\quad f_0\phi_0(\theta, Y)=m(\theta)f_0(Y),\\
&\text{(d)}\quad f_0(Y)=(1:0:0)\Leftrightarrow j_2(Y)=Y,\\
&\text{(e)}\quad f_0(Y)=(0:1:0)\Leftrightarrow j_1(Y)=Y.
\end{aligned}
$$

Here, $P_1(\boldsymbol{R})$ is the subspace of $P_2(\boldsymbol{R})$ consisting of the points $(*:*:0)$. $P_1(\boldsymbol{R})$ is invariant under the actions j_ε $(\varepsilon=1, 2)$ and $m(\theta)$.

If (ϕ, f) is given, then $S=f^{-1}(P_1(\boldsymbol{R}))$, f_0 is a restriction of f, and $\phi_0(\theta, Y)=\phi(m(\theta), Y)$.

Now, we shall show how to construct (ϕ, f) from the triple (S, ϕ_0, f_0) satisfying the condition (5.1).

First, we show that f_0 can be extended uniquely to a K''-equivariant map $f: S^2\to P_2(\boldsymbol{R})$. Suppose

$$\psi(k_1, Y_1)=\psi(k_2, Y_2)$$

for $k_i\in K''$, $Y_i\in S$. Then, we obtain

$$k_1f_0(Y_1)=k_2f_0(Y_2),$$

because, by assumption we have the following two possibilities:

(1) $Y_1=Y_2=\pm\boldsymbol{e}_1$,

(2) $Y_1 = J_2^\varepsilon Y_2$ and $k_1^{-1}k_2 = j_2^\varepsilon$ $(\varepsilon = 0, 1)$.
So, we can define

$$f(\psi(k, Y)) = kf_0(Y) \qquad \text{for} \quad k \in K'' \text{ and } Y_i \in S. \tag{5.2}$$

Next, we construct an $N(p, 2)$-action ϕ on S^2. We obtain the decomposition

$$N(p, 2) = K''M(p, 2)(H_0(a:b:0) \cap N(p, 2))$$

for each $(a, b) \neq (0, 0)$ by (1.4). Since $U_0(Y) = H_0(a:b:0)$ for $f_0(Y) = (a:b:0)$, we obtain

$$N(p, 2) = K''M(p, 2)(U_0(Y) \cap N(p, 2))$$

for each $Y \in S$.

Take $(g, X) \in N(p, 2) \times S^2$. Let us choose

$$k \in K'', \ Y \in S; \ \psi(k, Y) = X,$$

$$k_0 \in K'', \ h \in U_0(Y) \cap N(p, 2); \ gk = k_0 m(\theta) h,$$

and put

$$\phi(g, X) = \psi(k_0, \phi_0(\theta, Y)). \tag{5.3}$$

We shall show that ϕ is well-defined and is a smooth $N(p, 2)$-action on S^2. We need the following lemma.

LEMMA 5.4. *Suppose* $km(\theta)h = k'm(\theta')h'$ *for* $k, k' \in K''$, $h, h' \in U_0(Y) \cap N(p, 2)$ *for* $Y \in S$. *Then*

$$\psi(k, \phi_0(\theta, Y)) = \psi(k', \phi_0(\theta', Y)).$$

PROOF. We obtain

$$kP(\phi_0(\theta, Y))\,{}^t k = k'P(\phi_0(\theta', Y))\,{}^t k'$$

from (5.1, c) in a way similar to that in the case (3.1). Then we have the following possibilities by direct calculation:

(1) $f_0\phi_0(\theta, Y) = f_0\phi_0(\theta', Y) = (1:0:0)$,

(2) $k = k'$,

(3) $kj_2 = k'$.

In the case (1), we see that $\phi_0(\theta, Y) = \phi_0(\theta', Y) = \pm \boldsymbol{e}_1$ by considering the orbits of ϕ_0. Hence we obtain

$$\psi(k, \phi_0(\theta, Y)) = \psi(k', \phi_0(\theta', Y)) = \pm \boldsymbol{e}_1$$

for any $k, k' \in K''$.

In the case (2), we see that $m(\theta)h = m(\theta')h'$. Then we obtain

$$m(\theta - \theta') = h'h^{-1} \in U_0(Y).$$

Then $\theta=\theta'$ by the definition of $U_0(Y)=H_0(a:b:0)$. Hence we obtain

$$\psi(k, \phi_0(\theta, Y))=\psi(k', \phi_0(\theta', Y)) .$$

In the case (3), we see that $m(\theta)h=j_2m(\theta')h'$. Put $2\tau=\theta+\theta'$. Then we obtain

$$I_{p+2}=m(-\theta)j_2m(\theta')h'h^{-1}=m(-\tau)j_2m(\tau)h'h^{-1} .$$

Hence we obtain

$$j_2=m(\tau)h'h^{-1}m(-\tau)\in H_0(c:d:0) ,$$

for $f_0\phi_0(\tau, Y)=(c:d:0)$. Then $d=0$, and hence $\phi_0(\tau, Y)$ is J_2-invariant. Therefore,

$$\phi_0(\theta', Y)=J_2\phi_0(\theta, Y) .$$

Hence we obtain

$$\psi(k, \phi_0(\theta, Y))=\psi(k', \phi_0(\theta', Y)) .$$

q.e.d.

PROPOSITION 5.5. *ϕ in (5.3) is well-defined and is a smooth $N(p, 2)$-action.*

PROOF. For $(g, X)\in N(p, 2)\times S^2$, let us choose

(i) $$X=\psi(k_1, Y_1)=\psi(k_2, Y_2) ,$$

where $k_1, k_2\in K''$, $Y_1, Y_2\in S$, and

(ii) $$gk_1=k_0m(\theta)h ,$$

where $k_0\in K''$, $h\in U_0(Y_1)\cap N(p, 2)$.

By the condition (i), we have the following possibilities:

(1) $Y_1=Y_2=\pm e_1$,

(2) $k_1^{-1}k_2=j_2^\varepsilon$, $Y_1=J_2^\varepsilon(Y_2)$ for $\varepsilon=0, 1$.

In the case (1), we obtain

$$k_1^{-1}k_2\in U_0(Y_1)=U_0(Y_2) ,$$

and

$$gk_2=k_0m(\theta)hk_1^{-1}k_2 , \quad hk_1^{-1}k_2\in U_0(Y_2) .$$

Therefore, ϕ in (5.3) is well-defined in this case by Lemma 5.4.

In the case (2), we obtain

$$gk_2=k_0j_2^\varepsilon m((-1)^\varepsilon\theta)j_2^\varepsilon hj_2^\varepsilon ,$$

where $k_0j_2^\varepsilon\in K''$, $j_2^\varepsilon hj_2^\varepsilon\in U_0(Y_2)$. Hence we can write

$$\psi(k_0j_2^\varepsilon, \phi_0((-1)^\varepsilon\theta, Y_2))=\psi(k_0, \phi_0(\theta, J_2^\varepsilon(Y_2))=\psi(k_0, \phi_0(\theta, Y_1)) .$$

Therefore, ϕ in (5.3) is well-defined in this case by Lemma 5.4. Consequently, ϕ in (5.3) is well-defined.

Finally, as in the proof of Proposition 4.5, we see easily that ϕ in (5.3) is an abstract action. Notice that the smoothness of ϕ can be proved in a way similar to that of Φ defined by (4.3). q.e.d.

By definition, ϕ is compatible with the antipodal involution $J=\psi(j_1 j_2, -)$. Therefore, we can extend ϕ to an action of $N(p, 2)^+$ on S^2.

PROPOSITION 5.6. *The map $f: S^2 \to P_2(\boldsymbol{R})$ defined by* (5.2) *is $N(p, 2)^+$-equivariant and smooth.*

PROOF. Take $(g, X) \in N(p, 2) \times S^2$. Let us choose

$$k \in K'', \ Y \in S\,; \ \psi(k, Y) = X\,,$$

$$k_0 \in K'', \ h \in U_0(Y) \cap N(p, 2)\,; \ gk = k_0 m(\theta) h\,.$$

Then, by definition

$$\phi(g, X) = \psi(k_0, \phi_0(\theta, Y))\,.$$

Thus we obtain

$$\begin{aligned} f(\phi(g, X)) &= k_0 f_0(\phi_0(\theta, Y)) = k_0 m(\theta) f_0(Y) = k_0 m(\theta) h f_0(Y) \\ &= gk f_0(Y) = g f(\psi(k, Y)) = g f(X)\,. \end{aligned}$$

Thus f is $N(p, 2)$-equivariant. By definition

$$f_0(-X) = f_0(J_1 J_2 X) = j_1 j_2 f_0(X) = f_0(X) \qquad \text{for} \quad X \in S\,.$$

Thus f is $N(p, 2)^+$-equivariant.

Finally, we show the smoothness of f. The smoothness of f on $S^2 - \{\pm \boldsymbol{e}_1\}$ is obvious by the definition (5.2). The smoothness of f around $\pm \boldsymbol{e}_1$ follows from the fact that the orbit of $\pm \boldsymbol{e}_1$ with respect to the $N(p, 2)$-action is open. q.e.d.

Now, it remains only to show that the pair (ϕ, f) satisfies the condition (4.1).

LEMMA 5.7. *The following condition holds for each $Y \in F(H)$.*

$$N(p, 2)_Y^+ \supset N(p, 2)^+ \cap U_0(Y)\,.$$

PROOF. By the definition of ϕ, we see that

$$N(p, 2)_Y \supset N(p, 2) \cap U_0(Y)\,.$$

It remains to show that

$$(1) \qquad j_1 j_2 k m(\theta) \in U_0(Y) \qquad \text{for} \quad k \in K''$$

implies

$$j_1 j_2 km(\theta) \in N(p, 2)_Y^+ .$$

Put $f_0(Y) = (a : b : 0)$. Denote by

$$\begin{bmatrix} s & t \\ -t & s \end{bmatrix}, \qquad s^2 + t^2 = 1$$

the $SO(2)$-factor of k. Then (1) implies

$$\begin{aligned} &① \qquad a\cosh\theta + b\sinh\theta = -a \\ &② \qquad a\sinh\theta + b\cosh\theta = -bs \\ &③ \qquad 0 = -bt . \end{aligned}$$

We obtain $b \neq 0$ by ①, hence $t = 0$ and $s = \pm 1$ by ③. Calculating ① $\times \sinh\theta -$ ② $\times (1 + \cosh\theta)$, we obtain $s = -1$, and hence $k = j_2$. Then $(a+b)e^\theta = b - a$ by ① + ②. Thus we see that

$$|a| < |b| \quad \text{and} \quad \theta = \log\frac{b-a}{a+b}.$$

Hence, there exists θ_0 such that

$$(a : b) = (-\sinh\theta_0 : \cosh\theta_0) .$$

Then $f_0(\phi_0(\theta_0, Y)) = (0 : 1 : 0)$ and $\theta = 2\theta_0$. Thus

$$J_2\phi_0(\theta_0, Y) = -\phi_0(\theta_0, Y)$$

by (5.1, e), and we have

$$\begin{aligned} \phi(j_1 j_2 km(\theta), Y) &= -\psi(k, \phi_0(\theta, Y)) = -\psi(j_2, \phi_0(2\theta_0, Y)) \\ &= -\phi_0(-\theta_0, \psi(j_2, \phi_0(\theta_0, Y))) = -\phi_0(-\theta_0, -\phi_0(\theta_0, Y)) \\ &= -\phi_0(-\theta_0, \phi_0(\theta_0, -Y)) = -(-Y) = Y . \end{aligned}$$

Consequently, $j_1 j_2 km(\theta) \in N(p, 2)_Y^+$. q.e.d.

Thus we have proved the following.

THEOREM 5.8. *There is a one-to-one correspondence between the set of pairs* (ϕ, f) *given in Theorem* 4.12 *and the set of triples* (S, ϕ_0, f_0) *satisfying the condition* (5.1).

REMARK. By Asoh [1, §9–§11], we can show that there exist infinitely many smooth $SO_0(p, 2)$-actions on S^{p+1} which are topologically mutually distinct. In fact, we can construct a smooth $SO_0(p, 2)$-action on S^{p+1} which has just $2m$ open orbits for each positive integer m.

6. Concluding remark. We can prove the following result by an argument similar

to that in [3].

THEOREM. *Suppose* $p \geqq 3$. *Then, there is a one-to-one correspondence between the set of smooth* $\boldsymbol{SO}_0(p, 1)$*-actions on* S^p *whose restricted* $\boldsymbol{SO}(p)$*-action is the standard orthogonal action and the set of pairs* (ϕ, f) *satisfying the conditions* (i) *to* (iv) *in* §3 *of* [3], *where* ϕ *is a smooth one-parameter group on* S^1 *and* $f: S^1 \to P_1(\boldsymbol{R})$ *is a smooth function.*

REFERENCES

[1] T. ASOH, On smooth $SL(2, C)$ actions on 3-manifolds, Osaka J. Math. 24 (1987), 271–298.
[2] K. MUKŌYAMA, Smooth $Sp(2, \boldsymbol{R})$-actions on the 4-sphere, Tôhoku Math. J. 48 (1996), 543–560.
[3] F. UCHIDA, On smooth $\boldsymbol{SO}_0(p, q)$-actions on S^{p+q-1}, Osaka J. Math. 26 (1989), 775–787.

DEPARTMENT OF MATHEMATICAL SCIENCES
FACULTY OF SCIENCE
YAMAGATA UNIVERSITY
KOSHIRAKAWA, YAMAGATA 990
JAPAN