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THE CLASS GROUP OF Z_p-EXTENSIONS OVER TOTALLY REAL NUMBER FIELDS

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Abstract. Let p be an odd prime. We shall give a criterion for p-divisibility of the class number of the n-th layer of a $\boldsymbol{Z}_p$-extension over a certain totally real number field by means of the value of the p-adic zeta function. We shall also discuss the capitulation in such a $\boldsymbol{Z}_p$-extension and a sufficient condition for the Iwasawa λ- and μ-invariants of it to vanish.

1. Introduction. Let p be an odd prime, and k a totally real number field. For any $\boldsymbol{Z}_p$-extension k_∞/k, we denote by k_n the n-th layer of k_∞/k.

In the present paper, we shall prove the following:

THEOREM 1. *Let k and p be as above, and k_∞/k a $\boldsymbol{Z}_p$-extension in which the primes lying above p are totally ramified. We assume that the prime p splits completely in k. Then the following three statements are equivalent*:

(i) $\mathrm{Gal}(L(k_\infty)/k_\infty)\neq 0$,

(ii) *The class number of k_n is divisible by p for all $n\geq 1$,*

(iii) $p\zeta_p(0, k)\equiv 0 \pmod p$,

where $L(k_\infty)/k_\infty$ is the maximal unramified pro-p abelian extension and $\zeta_p(s, k)$ is the p-adic zeta function of k.

One can regard the above theorem as a "totally real" analogue of classical Kummer's criterion for p-divisibility of the class number of the p-th cyclotomic field.

We shall also prove the following two theorems as an application of the argument in the proof of Theorem 1:

THEOREM 2. *Let p and k_∞/k be as in Theorem* 1. *Furthermore, we assume that Leopoldt's conjecture is valid for k and p. Then the following three statements are equivalent*:

(i) *The capitulation of ideals occurs in k_∞/k_1,*

(ii) *The capitulation of ideals occurs in k_∞/k_n for some $n\geq 0$,*

(iii) $M(k_\infty)\neq L(k_\infty)$.

Here $M(k_\infty)$ is the maximal pro-p abelian extension field of k_∞ unramified outside p.

THEOREM 3. *Let p and k_∞/k be as in Theorem* 2. *We assume that the p-Sylow*

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subgroup of the ideal class group of k_n is cyclic for all $n \geq 0$, and that $\lambda^* :=$ $\mathrm{rank}_{Z_p} \mathrm{Gal}(M(k_\infty)/k_\infty) \geq 2$. Then the Iwasawa λ- and μ-invariants of k_∞/k vanish.

We shall prove the above theorems in the next section.

2. Proof of the Theorems. We fix an odd prime p once and for all. We shall use the following notation.

For a field $F \subseteq \bar{\boldsymbol{Q}}$, we write $L(F)$ and $M(F)$ for the maximal unramified pro-p abelian extension field of F and the maximal pro-p abelian extension field of F unramified outside p, respectively. We denote by M^G and M_G the maximal submodule and the maximal quotient module of M on which G acts trivially, respectively, for any group G and a G-module M.

The following proposition is the keystone of the present paper:

PROPOSITION 1. *Let k be a number field and k_∞/k a $\boldsymbol{Z}_p$-extension in which every prime of k above p is ramified. We assume that the prime p is completely decomposed in k. Then $M(k)$ is the maximal subfield of $L(k_\infty)$ which is an abelian extension over k.*

PROOF. We denote by $I_{\mathfrak{p}} \subseteq \mathrm{Gal}(M(k)/k)$ the inertia group for a prime $\mathfrak{p}$ of k above p. It follows from the assumption of the proposition that the pro-p part of the local unit group of $k_{\mathfrak{p}}$ is isomorphic to $\boldsymbol{Z}_p$ where $k_{\mathfrak{p}}$ stands for the completion of k at $\mathfrak{p}$. Hence by class field theory $I_{\mathfrak{p}}$ is isomorphic to a quotient group of $\boldsymbol{Z}_p$. Since $\mathfrak{p}$ is infinitely ramified in $k_\infty \subseteq M(k)$, we see that $I_{\mathfrak{p}} \simeq \boldsymbol{Z}_p$, and that $I_{\mathfrak{p}} \cap \mathrm{Gal}(M(k)/k_\infty) = 0$. This equality implies that the primes of k_∞ above $\mathfrak{p}$ are unramified in $M(k)$. Therefore $M(k)/k_\infty$ is an unramified extension, and $M(k) \subseteq L(k_\infty)$. □

COROLLARY 1. *Let k and k_∞ be as in Proposition 1. Then the following two statements are equivalent*:

(i) $M(k_\infty) \neq k_\infty$,

(ii) $L(k_\infty) \neq k_\infty$.

PROOF. (ii) $\Rightarrow$ (i) is obvious. We assume that $M(k_\infty) \neq k_\infty$. It is known that $\mathrm{Gal}(M(k_\infty)/k_\infty)$ is a finitely generated $\boldsymbol{Z}_p[[\Gamma]]$-module, where $\Gamma = \mathrm{Gal}(k_\infty/k)$ (cf. [4, Theorem 4]). Since $\mathrm{Gal}(M(k)/k_\infty) \simeq \mathrm{Gal}(M(k_\infty)/k_\infty)_\Gamma = \mathrm{Gal}(M(k_\infty)/k_\infty)/(\gamma - 1)\mathrm{Gal}(M(k_\infty)/k_\infty)$, where γ is a topological generator of Γ, we have $M(k) \neq k_\infty$ by Nakayama's lemma. Since $M(k) \subseteq L(k_\infty)$ from Proposition 1, we have $L(k_\infty) \neq k_\infty$. □

Now we shall give a proof of Theorem 1.

PROOF OF THEOREM 1. Let $X = \mathrm{Gal}(L(k_\infty)/k_\infty)$, $Y = \mathrm{Gal}(L(k_\infty)/k_\infty L(k)) \subseteq X$ and $\nu_n = (\gamma^{p^n} - 1)/(\gamma - 1) \in \boldsymbol{Z}_p[[\Gamma]]$ for $n \geq 0$, where $\Gamma = \mathrm{Gal}(k_\infty/k)$ and γ is a fixed topological generator of Γ. It is known that X is a finitely generated $\boldsymbol{Z}_p[[\Gamma]]$-module and $\mathrm{Gal}(L(k_n)/k_n) \simeq X/\nu_n Y$ for all $n \geq 0$ (cf. [4, Theorems 5 and 6]). So (ii)$\Rightarrow$(i) is obvious. By Nakayama's lemma, (i) implies $X/\nu_n X \neq 0$ for all $n \geq 1$. Hence we see that (i) $\Rightarrow$ (ii).

To show (i) $\Leftrightarrow$ (iii), we recall the following. We denote by k_∞^c/k the cyclotomic

Z_p-extension, and we fix a topological generator γ_0 of $\mathrm{Gal}(k_\infty^c/k)$. Let $\kappa \in 1+pZ_p$ be the number such that $\zeta^{\tilde{\gamma}_0}=\zeta^\kappa$ for any p-power-th root of unity ζ, where $\tilde{\gamma}_0$ is the image of γ_0 under the natural isomorphism $\mathrm{Gal}(k_\infty^c/k)\simeq \mathrm{Gal}(k_\infty^c(\zeta_p)/k(\zeta_p))$, ζ_p being a primitive p-th root of unity. Since p is unramified in $k/\boldsymbol{Q}$, we have $v_p(\kappa-1)=v_p(p)$, where v_p stands for the p-adic valuation. It is known that there exists a power series $F(T)\in Z_p[\![T]\!]$ such that $\zeta_p(s,k)=F(\kappa^s-1)/(\kappa^s-\kappa)$ for $s\in Z_p$ (cf. [1]). Iwasawa's main conjecture proved by Wiles [8] asserts that $F(\kappa(1+T)^{-1}-1)\in Z_p[\![T]\!]$ is a generator of the characteristic ideal of the finitely generated torsion $Z_p[\![T]\!]$-module $\mathrm{Gal}(M(k_\infty^c)/k_\infty^c)$, where we identify $Z_p[\![\mathrm{Gal}(k_\infty^c/k)]\!]$ with $Z_p[\![T]\!]$ by sending γ_0-1 to T as usual (cf. [7, Theorem 7.1]).

Now we shall prove (i) $\Leftrightarrow$ (iii). From Corollary 1 we obtain (i) $\Leftrightarrow \mathrm{Gal}(M(k_\infty)/k_\infty)\neq 0$. First we assume that Leopoldt's conjecture is valid for k and p. Then k_∞/k must be the cyclotomic Z_p-extension k_∞^c/k. Since $\mathrm{Gal}(M(k_\infty^c)/k_\infty^c)$ has no non-trivial finite $Z_p[\![\Gamma]\!]$-submodule (cf. [3]), we find from Iwasawa's main conjecture that $\mathrm{Gal}(M(k_\infty^c)/k_\infty^c)=0$ is equivalent to $F(\kappa(1+T)^{-1}-1)\in Z_p[\![T]\!]^\times$. This in turn is equivalent to $F(0)=(1-\kappa)\zeta_p(0,k)\in Z_p^\times$. Thus we have proved (i) $\Leftrightarrow$ (iii) under the validity of Leopoldt's conjecture for k and p. If Leopoldt's conjecture is not valid for k and p, $\mathrm{Gal}(M(k)/k_\infty)$ is infinite, hence especially $\mathrm{Gal}(M(k_\infty)/k_\infty)\neq 0$. Thus statement (i) holds in this case by Corollary 1. On the other hand, since $\mathrm{Gal}(M(k_\infty^c)/k_\infty^c)$ is also infinite, $F(\kappa(1+T)^{-1}-1)$ is not a unit in $Z_p[\![T]\!]$ by Iwasawa's main conjecture. Hence $(1-\kappa)\zeta_p(0,k)\equiv 0 \pmod p$ as in the above argument, namely, statement (iii) holds. This completes the proof of Theorem 1.

REMARK 1. In the case of $k=\boldsymbol{Q}(\zeta_q+\zeta_q^{-1})$ where q is an odd prime satisfying $p\equiv 1 \pmod q$, Kim [5, Theorem 2] proved "(iii) $\Rightarrow$ (ii)" part of the above Theorem 1. (Note that $p\zeta_p(0,k)\equiv \pm\prod_{1\neq\chi\in \mathrm{Gal}(k/\boldsymbol{Q})^\wedge} B_{1,\chi\omega^{-1}} \pmod p$ in this case, where ω is the Teichmüller character for p.) His method of proof is different from ours and based on the theory of cyclotomic units.

To prove Theorems 2 and 3, we need the following proposition which may be of interest by itself:

PROPOSITION 2. *Let k and k_∞/k be as in Theorem 1. We assume that Leopoldt's conjecture is valid for k and p. Then*

$$\mathrm{Gal}(M(k_\infty)/L(k_\infty))_\Gamma \simeq \mathrm{Gal}(L(k_\infty)/k_\infty)^\Gamma=(\mathrm{Gal}(L(k_\infty)/k_\infty)_{\mathrm{finite}})^\Gamma,$$

where $\Gamma=\mathrm{Gal}(k_\infty/k)$ and $\mathrm{Gal}(L(k_\infty)/k_\infty)_{\mathrm{finite}}$ is the maximal finite $Z_p[\![\Gamma]\!]$-submodule of $\mathrm{Gal}(L(k_\infty)/k_\infty)$.

PROOF. From the exact sequence

$$0\longrightarrow \mathrm{Gal}(M(k_\infty)/L(k_\infty))\longrightarrow \mathrm{Gal}(M(k_\infty)/k_\infty)\longrightarrow \mathrm{Gal}(L(k_\infty)/k_\infty)\longrightarrow 0,$$

we get the exact sequence

$$\begin{aligned}\mathrm{Gal}(M(k_\infty)/k_\infty)^\Gamma \longrightarrow \mathrm{Gal}(L(k_\infty)/k_\infty)^\Gamma \longrightarrow \mathrm{Gal}(M(k_\infty)/L(k_\infty))_\Gamma \\ \longrightarrow \mathrm{Gal}(M(k_\infty)/k_\infty)_\Gamma \xrightarrow{f} \mathrm{Gal}(L(k_\infty)/k_\infty)_\Gamma \longrightarrow 0\,.\end{aligned}$$

Let $L(k_\infty)^{\mathrm{ab}}$ be the maximal abelian extension field over k which is contained in $L(k_\infty)$. Then $\mathrm{Gal}(L(k_\infty)^{\mathrm{ab}}/k_\infty)\simeq \mathrm{Gal}(L(k_\infty)/k_\infty)_\Gamma$. By Proposition 1, we have $L(k_\infty)^{\mathrm{ab}}=M(k)$. Since $\mathrm{Gal}(M(k)/k_\infty)\simeq \mathrm{Gal}(M(k_\infty)/k_\infty)_\Gamma$, the homomorphism f in the above exact sequence is an isomorphism. On the other hand, it follows from the validity of Leopoldt's conjecture for k and p that $\mathrm{Gal}(M(k)/k_\infty)\simeq \mathrm{Gal}(M(k_\infty)/k_\infty)_\Gamma$ is finite, and hence a generator of the characteristic ideal of the $\boldsymbol{Z}_p[\![\Gamma]\!]$-module $\mathrm{Gal}(M(k_\infty)/k_\infty)$ is prime to $\gamma-1$. Therefore $\mathrm{Gal}(M(k_\infty)/k_\infty)^\Gamma=0$ since $\mathrm{Gal}(M(k_\infty)/k_\infty)$ has no non-trivial finite $\boldsymbol{Z}_p[\![\Gamma]\!]$-submodule (cf. [3]). Thus we have

$$\mathrm{Gal}(M(k_\infty)/L(k_\infty))_\Gamma \simeq \mathrm{Gal}(L(k_\infty)/k_\infty)^\Gamma\,.$$

Since a generator of the characteristic ideal of the $\boldsymbol{Z}_p[\![\Gamma]\!]$-module $\mathrm{Gal}(L(k_\infty)/k_\infty)$ is prime to $\gamma-1$ for the same reason for $\mathrm{Gal}(M(k_\infty)/k_\infty)$, we obtain

$$(\mathrm{Gal}(L(k_\infty)/k_\infty)/\mathrm{Gal}(L(k_\infty)/k_\infty)_{\mathrm{finite}})^\Gamma=0\,.$$

Hence $\mathrm{Gal}(L(k_\infty)/k_\infty)^\Gamma=(\mathrm{Gal}(L(k_\infty)/k_\infty)_{\mathrm{finite}})^\Gamma$. □

PROOF OF THEOREM 2. Let X and Y be as in the proof of Theorem 1. We first note that every ideal class of k_n which capitulates in k_∞ is contained in the p-Sylow subgroup of the ideal class group of k_n. From [6, Corollary], (ii) is equivalent to $X_{\mathrm{finite}}\neq 0$ which in turn is equivalent to $(X_{\mathrm{finite}})^\Gamma\neq 0$. Hence we have (ii)$\Leftrightarrow$(iii) by Proposition 2 and Nakayama's lemma. Furthermore, the subgroup of the ideal class group of k_1 consisting of all ideal classes which capitulate in k_∞ is isomorphic to $\mathrm{Im}(X_{\mathrm{finite}}\to X/\nu_1 Y)$ by [6, Proposition]. As in the proof of [6, Proposition], the multiplication-by-ν_1 map $X/X_{\mathrm{finite}}\xrightarrow{\nu_1} X/X_{\mathrm{finite}}$ is injective. Hence we see that the natural map $X_{\mathrm{finite}}/\nu_1 X_{\mathrm{finite}}\to X/\nu_1 X$ is injective. Therefore if no ideals in k_1 capitulate in k_∞, namely $X_{\mathrm{finite}}\subseteq \nu_1 Y\subseteq \nu_1 X$, then $X_{\mathrm{finite}}/\nu_1 X_{\mathrm{finite}}=0$, which is equivalent to $X_{\mathrm{finite}}=0$ by Nakayama's lemma. Thus we have (ii)$\Rightarrow$(i). Since (i)$\Rightarrow$(ii) is obvious, this conclude the proof of Theorem 2.

PROOF OF THEOREM 3. Let A_n be the p-Sylow subgroup of the ideal class group of k_n. Then $X=\mathrm{Gal}(L(k_\infty)/k_\infty)\simeq \mathrm{proj\,lim}\,A_n$, where the projective limit is taken with respect to the norm maps. Since X is cyclic over $\boldsymbol{Z}_p$, we have $\mathrm{Gal}(M(k_\infty)/L(k_\infty))\neq 0$ by the assumption $\lambda^*\geq 2$. Hence $X_{\mathrm{finite}}\neq 0$ by Proposition 2. If X is infinite, then $X\simeq \boldsymbol{Z}_p$, a contradiction to $X_{\mathrm{finite}}\neq 0$. Therefore X is finite, namely, the Iwasawa λ- and μ-invariants of k_∞/k vanish. □

REMARK 2. The cyclicity of A_2 guarantees the cyclicity of all A_n (cf. [2, Theorem 1(2)]). If k is a real abelian number field, one knows λ^* by computing the Iwasawa power series attached to the Kubota-Leopoldt p-adic L-function. Therefore one can effectively verify whether the assumptions of Theorem 3 hold, at least, for real abelian number fields.

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