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WEIERSTRASS REPRESENTATION FOR MINIMAL SURFACES IN HYPERBOLIC SPACE

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Abstract. We give a representation formula for minimal surfaces in hyperbolic space. It is a natural generalization of the Weierstrass-Enneper formula for minimal surfaces in Euclidean space. Furthermore, we define the normal Gauss map and discuss some of its properties.

Introduction. Weierstrass-Enneper formula, which explicitly describes a minimal immersion of a surface into Euclidean space, plays an important role in minimal surface theory.

More generally, Kenmotsu [K] gave a representation formula for surfaces of prescribed mean curvature in Euclidean 3-space and, as a special case, for surfaces of constant mean curvature. By virtue of the formula, if a harmonic map ψ from a Riemann surface Σ into S^2 is given, then one can construct an immersion of a constant mean curvature surface whose Gauss map is ψ. It is remarkable that the harmonic map equation for ψ is the complete integrability condition for a system of partial differential equations of first order which should be satisfied by the corresponding constant mean curvature immersion.

On the other hand, Bryant [B] obtained an explicit representation formula for surfaces of constant mean curvature one (CMC-1 surfaces) in hyperbolic 3-space $H^3(-1)$ of sectional curvature -1: Any CMC-1 surface in $H^3(-1)$ can be constructed from an $\mathfrak{sl}(2, C)$-valued holomorphic 1-form satisfying some conditions (or equivalently a pair of a meromorphic function and a holomorphic 1-form) on a Riemann surface. The study of CMC-1 surfaces in $H^3(-1)$ is making steady progress thanks to Bryant's formula (cf. [U-Y]).

The purpose of this paper is to provide a Weierstrass type representation formula for minimal surfaces in hyperbolic n-space. In Section 2, we consider hyperbolic n-space to be a Lie group equipped with a left invariant metric, which is also obtained by deforming Euclidean space under certain change of the Riemannian metric. We use it as a model of hyperbolic space mainly for the following three reasons: (1) Since it is R^n as a differentiable manifold, an immersion is written in terms of an n-tuple of real-valued functions. (2) We can see that the formula obtained in this paper is a generalization of the Weierstrass formula in the case of R^n. (3) The normal Gauss map,

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which is introduced in Section 5, can be explained in terms of the left translation. In Sections 3 and 4, we get an equation satisfied by a minimal immersion as well as the complete integrability condition for it. We also derive a representation formula from it. In Sections 6 and 7, as a special case, differential geometric invariants of minimal surfaces in hyperbolic 3-space are given explicitly in terms of Weierstrass data.

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2. A model of hyperbolic space. Throughout this paper we use the following convention on the ranges of indices:

$$2 \leq j, k, \cdots \leq n,$$

unless otherwise stated.

On $\boldsymbol{R}^n$, we give a Riemannian metric g_c defined by

$$g_c := (dt)^2 + e^{-2ct}\{(dx^2)^2 + \cdots + (dx^n)^2\},$$

where $(t, x^2, \ldots, x^n)$ is the Cartesian coordinate for $\boldsymbol{R}^n$ and c is a real constant. Then $(\boldsymbol{R}^n, g_c)$ has constant sectional curvature $-c^2$. More precisely, if $c \neq 0$, then it is isometric to hyperbolic space, i.e., a simply connected, complete Riemannian manifold of constant negative sectional curvature. To see this, for instance, the coordinate transformation $y = e^{ct}$, $\tilde{x}^j = cx^j$ $(j = 2, \ldots, n)$ gives a standard upper half-space model of hyperbolic space. If $c = 0$, then it is Euclidean space.

PROPOSITION 2.1. *Let G_c be a Lie group defined by*

$$G_c := \left\{ \begin{bmatrix} 1 & & & & t \\ & e^{ct} & & & x^2 \\ & & \ddots & & \vdots \\ & & & e^{ct} & x^n \\ & & & & 1 \end{bmatrix} ; (t, x^2, \ldots, x^n) \in \boldsymbol{R}^n \right\} \subset GL(n; \boldsymbol{R}).$$

Then $(\boldsymbol{R}^n, g_c)$ is isometric to G_c with a left invariant metric.

PROOF. For an arbitrary $\tilde{a} = (a, a^j) \in G_c$, we denote by $L_{\tilde{a}}$ the left translation by $\tilde{a}$. Then $L_{\tilde{a}}(t, x^j) = (t + a, e^{ca}x^j + a^j)$ and

$$\begin{aligned} L_{\tilde{a}}^* g_c &= \{d(t+a)\}^2 + e^{-2c(t+a)} \sum \{d(e^{ca}x^j + a^j)\}^2 \\ &= dt^2 + e^{-2ct} \sum (dx^j)^2 = g_c . \end{aligned}$$

$\square$

Let $\mathfrak{g}$ denote the Lie algebra of G_c and $\langle\ ,\ \rangle$ an inner product on $\mathfrak{g}$ induced by g_c under the identification $\mathfrak{g} \cong T_eG$. We choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for $\mathfrak{g}$ as follows:

$$
e_1 = \begin{bmatrix} 0 & & & 1 \\ & c & & \\ & & \ddots & \\ & & & c \\ & & & 0 \end{bmatrix}, \quad e_j = E_{jn} \quad (j=2, \dots, n). \tag{2.1}
$$

Here E_{jn} is an $n \times n$-matrix whose (j, n)-entry is 1 and the other entries are 0. We compute the Lie bracket as

$$
[e_1, e_j] = ce_j, \quad [e_j, e_k] = 0 \quad (j, k = 2, \dots, n). \tag{2.2}
$$

For $X \in \mathfrak{g}$ we denote by $\mathrm{ad}(X)^*$ the *adjoint* operator of $\mathrm{ad}(X)$, that is, it is an element in $\mathfrak{gl}(\mathfrak{g})$ defined by the equation

$$
\langle [X, Y], Z \rangle = \langle Y, \mathrm{ad}(X)^*(Z) \rangle
$$

for any $Y, Z \in \mathfrak{g}$. Let U be the symmetric bilinear operator on $\mathfrak{g}$ defined by

$$
U(X, Y) = \frac{1}{2} \{ \mathrm{ad}(X)^*(Y) + \mathrm{ad}(Y)^*(X) \} .
$$

LEMMA 2.2. *For the Lie algebra* $\mathfrak{g}$ *of* G_c, *we have*

$$
U(e_1, e_1) = 0, \quad U(e_1, e_j) = U(e_j, e_1) = \frac{c}{2} e_j, \quad U(e_j, e_k) = -c\delta_{jk} e_1 .
$$

Here $\{e_1, e_j\}$ *is an orthonormal basis given in* (2.1) *and* δ_{jk} *is Kronecker's delta.*

3. Harmonic map equation for $\varphi : \Omega \to G$. Let Ω be a domain in $\boldsymbol{R}^2$ and (u, v) the usual coordinate for Ω. Let G be a Lie group endowed with a left invariant Riemannian metric ds_G^2 and $\langle \ , \ \rangle$ an inner product on the Lie algebra $\mathfrak{g}$ of G induced by ds_G^2 via the identification $T_e G \cong \mathfrak{g}$. Recall that a smooth map $\varphi : \Omega \to G$ is said to be *harmonic* if it is a critical point of the energy functional $E(\varphi) = (1/2) \int_\Omega |d\varphi|^2$ under every compactly supported variation of φ.

For simplicity we may assume that G is a matrix group.

LEMMA 3.1. *A smooth map* $\varphi : \Omega \to (G, ds_G^2)$ *is harmonic if and only if*

$$
\begin{aligned}
&\frac{\partial}{\partial u}\left(\varphi^{-1}\frac{\partial \varphi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\varphi^{-1}\frac{\partial \varphi}{\partial v}\right) \\
&\qquad - \mathrm{ad}\left(\varphi^{-1}\frac{\partial \varphi}{\partial u}\right)^*\left(\varphi^{-1}\frac{\partial \varphi}{\partial u}\right) - \mathrm{ad}\left(\varphi^{-1}\frac{\partial \varphi}{\partial v}\right)^*\left(\varphi^{-1}\frac{\partial \varphi}{\partial v}\right) = 0
\end{aligned} \tag{3.1}
$$

holds.

PROOF. Let φ_t, $-\varepsilon < t < \varepsilon$, be a smooth variation of $\varphi = \varphi_0$ such that $\varphi_t|_{\partial\Omega} = \varphi|_{\partial\Omega}$, where $\partial\Omega$ is the boundary of Ω, and put

$$\Lambda=\frac{d}{dt}(\varphi^{-1}\varphi_t)\big|_{t=0}:\Omega\to\mathfrak{g}\,.$$

It is easy to see that the energy density $e(\varphi)$ of φ is

$$e(\varphi)=\frac{1}{2}\left(\left|\varphi^{-1}\frac{\partial\varphi}{\partial u}\right|^2+\left|\varphi^{-1}\frac{\partial\varphi}{\partial v}\right|^2\right).$$

Thus

$$\begin{aligned}\frac{d}{dt}E(\varphi_t)\bigg|_{t=0}=\int_\Omega\bigg\{&\left\langle\frac{d}{dt}\left(\varphi_t^{-1}\frac{\partial\varphi_t}{\partial u}\right)\bigg|_{t=0},\varphi^{-1}\frac{\partial\varphi}{\partial u}\right\rangle\\&+\left\langle\frac{d}{dt}\left(\varphi_t^{-1}\frac{\partial\varphi_t}{\partial v}\right)\bigg|_{t=0},\varphi^{-1}\frac{\partial\varphi}{\partial v}\right\rangle\bigg\}dudv\,.\end{aligned}$$

We have

$$\frac{d}{dt}\left(\varphi_t^{-1}\frac{\partial\varphi_t}{\partial u}\right)\bigg|_{t=0}=\left[\varphi^{-1}\frac{\partial\varphi}{\partial u},\Lambda\right]+\frac{\partial\Lambda}{\partial u},\quad\frac{d}{dt}\left(\varphi_t^{-1}\frac{\partial\varphi_t}{\partial v}\right)\bigg|_{t=0}=\left[\varphi^{-1}\frac{\partial\varphi}{\partial v},\Lambda\right]+\frac{\partial\Lambda}{\partial v}\,.$$

Therefore

$$\begin{aligned}&\frac{d}{dt}E(\varphi_t)\big|_{t=0}\\&=\int_\Omega\left\{\left\langle\left[\varphi^{-1}\frac{\partial\varphi}{\partial u},\Lambda\right]+\frac{\partial\Lambda}{\partial u},\varphi^{-1}\frac{\partial\varphi}{\partial u}\right\rangle+\left\langle\left[\varphi^{-1}\frac{\partial\varphi}{\partial v},\Lambda\right]+\frac{\partial\Lambda}{\partial v},\varphi^{-1}\frac{\partial\varphi}{\partial v}\right\rangle\right\}dudv\\&=\int_\Omega\left\langle\Lambda,\mathrm{ad}\left(\varphi^{-1}\frac{\partial\varphi}{\partial u}\right)^*\left(\varphi^{-1}\frac{\partial\varphi}{\partial u}\right)\right\rangle dudv-\int_\Omega\left\langle\Lambda,\frac{\partial}{\partial u}\left(\varphi^{-1}\frac{\partial\varphi}{\partial u}\right)\right\rangle dudv\\&\quad+\int_\Omega\left\langle\Lambda,\mathrm{ad}\left(\varphi^{-1}\frac{\partial\varphi}{\partial v}\right)^*\left(\varphi^{-1}\frac{\partial\varphi}{\partial v}\right)\right\rangle dudv-\int_\Omega\left\langle\Lambda,\frac{\partial}{\partial v}\left(\varphi^{-1}\frac{\partial\varphi}{\partial v}\right)\right\rangle dudv\,.\end{aligned}$$

□

REMARK. The formula in Lemma 3.1 was used by [U] when G is equipped with a bi-invariant metric.

Let $z=u+iv$ and let $\mathfrak{g}^{\boldsymbol{C}}$ be the complexification of $\mathfrak{g}$. We extend U complex linearly to a bilinear form on $\mathfrak{g}^{\boldsymbol{C}}$. Using the complex coordinate z, the equation (3.1) can be written as

$$\frac{\partial}{\partial\bar z}\left(\varphi^{-1}\frac{\partial\varphi}{\partial z}\right)+\frac{\partial}{\partial z}\left(\varphi^{-1}\frac{\partial\varphi}{\partial\bar z}\right)-2U\left(\varphi^{-1}\frac{\partial\varphi}{\partial z},\varphi^{-1}\frac{\partial\varphi}{\partial\bar z}\right)=0\,.\tag{3.2}$$

Let $\theta=\varphi^{-1}d\varphi=Adz+\bar Ad\bar z$ denote the pull-back of the Maurer-Cartan form of G

by φ. Then the equation (3.2) can be written as

$$A_{\bar{z}}+\bar{A}_z=2U(A,\bar{A})\,. \tag{3.3}$$

By the Maurer-Cartan equation, we have

$$A_{\bar{z}}-\bar{A}_z=[A,\bar{A}]\,. \tag{3.4}$$

The equations (3.3) and (3.4) above are reduced to the following single equation:

$$A_{\bar{z}}=U(A,\bar{A})+\frac{1}{2}[A,\bar{A}]\,. \tag{3.5}$$

The equation (3.4) is the complete integrability condition for the differential equation $\varphi^{-1}d\varphi=Adz+\bar{A}d\bar{z}$. Hence if, conversely, a map A from a simply connected Riemann surface Σ into $\mathfrak{g}^{\boldsymbol{C}}$ satisfying (3.5) is given, then a solution to $\varphi^{-1}d\varphi=Adz+\bar{A}d\bar{z}$ exists and defines a harmonic map from Σ into G.

4. Minimal surface in hyperbolic space. In this section we investigate harmonic maps and minimal immersions $\varphi:\Sigma\to(\boldsymbol{R}^n, g_c)$ using the results obtained in the previous section.

For a smooth map $\varphi:\Sigma\to(\boldsymbol{R}^n, g_c)$, we write $\varphi(z)=(t(z), x^j(z))$. The following is obtained by straightforward calculation.

LEMMA 4.1.

$$A=t_z e_1+\sum_{j=2}^{n} e^{-ct}x_z^j e_j\,.$$

LEMMA 4.2. *φ is harmonic if and only if the following equations hold*:

$$\begin{cases} t_{z\bar{z}}+ce^{-2ct}\sum_{j=2}^{n}x_z^j x_{\bar{z}}^j=0 \\ x_{z\bar{z}}^j-c(t_{\bar{z}}x_z^j+t_z x_{\bar{z}}^j)=0 \qquad (j=2,\dots,n)\,. \end{cases}$$

PROOF. We have only to check that the above equations is equivalent to (3.3). □

By Lemma 4.2 and the maximum principle for subharmonic functions, if $\varphi:\Sigma\to(\boldsymbol{R}^n, g_c)$ is a harmonic map from a *compact* Riemann surface then φ must be a constant map. Hence we assume that Σ is *noncompact* from now on.

Put $\xi=t_z dz$ and $\omega^j=e^{-ct}x_z^j dz$. By Lemma 4.2, these $(1,0)$-forms satisfy the equations

$$\begin{cases} \bar{\partial}\xi=c\sum_{j=2}^{n}\omega^j\wedge\bar{\omega}^j \\ \bar{\partial}\omega^j=c\bar{\omega}^j\wedge\xi\,. \end{cases} \tag{4.1}$$

PROPOSITION 4.3. *Let Σ be a simply connected Riemann surface. If an n-tuple of $(1,0)$-forms (ξ,ω^j) on Σ satisfy* (4.1), *then*

$$(t(z), x^j(z)) = \left(2\int_{z_0}^{z} \mathrm{Re}\,\xi,\ 2\int_{z_0}^{z} e^{ct(z)}\,\mathrm{Re}\,\omega^j\right)$$

gives a harmonic map from Σ into $(\boldsymbol{R}^n, g_c)$. Furthermore if the conformality condition

$$\begin{cases} \xi\cdot\xi+\sum_{j=2}^{n}\omega^j\cdot\omega^j=0\,, \\ \xi\cdot\bar{\xi}+\sum_{j=2}^{n}\omega^j\cdot\bar{\omega}^j\neq 0 \end{cases} \tag{4.2}$$

is satisfied, then the above map defines a minimal surface in $(\boldsymbol{R}^n, g_c)$. Conversely, any harmonic map from Σ into $(\boldsymbol{R}^n, g_c)$ or any minimal surface in $(\boldsymbol{R}^n, g_c)$ can be represented in this way.

PROOF. We have only to show that the equation (4.1) is equivalent to (3.5) and recall that minimal immersions are equivalent to conformal harmonic maps. □

REMARK. In Proposition 4.3, if we assume only the weak-conformality condition instead of (4.2), i.e., $\xi\cdot\bar{\xi}+\sum_{j=2}^{n}\omega^j\cdot\bar{\omega}^j$ can admit zero points, then the map obtained defines a branched minimal immersion in the sense of Gulliver-Osserman-Roydon (cf. Proposition 2.4 and Examples 2.5 of [G-O-R]).

PROPOSITION 4.4. *Assume that $c\neq 0$. If $\varphi:\Sigma\to(\boldsymbol{R}^n, g_c)$ is a conformal harmonic map, i.e., a minimal surface, then there exist no points where both ξ and $\bar{\partial}\xi$ vanish.*

PROOF. Assume that such a point p exists. By the equation $\bar{\partial}\xi=c\sum_{j=2}^{n}\omega^j\wedge\bar{\omega}^j$, we have $\sum_{j=2}^{n}\omega^j\wedge\bar{\omega}^j\big|_p=0$. Hence $\omega^j\big|_p=0$ holds for any $j=2,\dots,n$. Therefore $(\xi\cdot\bar{\xi}+\sum\omega^j\cdot\bar{\omega}^j)\big|_p=0$, a contradiction to the conformality. □

LEMMA 4.5. *If $c\neq 0$, then the equations* (4.1) *and* (4.2) *are equivalent to*

$$\begin{cases} \xi\cdot\xi+\sum_{j=2}^{n}\omega^j\cdot\omega^j=0\,, \\ \text{There does not exist any point } p\in\Sigma \text{ such that } \xi\big|_p=0 \text{ and } \bar{\partial}\xi\big|_p=0, \\ \bar{\partial}\omega^j=c\bar{\omega}^j\wedge\xi\ (j=2,\dots,n). \end{cases} \tag{4.3}$$

PROOF. Put $\xi=fdz$ and $\omega^j=h^jdz$ for some smooth functions f, h^j. From the first condition of (4.3), we have $f^2+\sum_{j=2}^{n}(h^j)^2=0$. Differentiating this equations with respect to $\bar{z}$, we have $ff_{\bar{z}}+\sum_{j=2}^{n}h^jh^j_{\bar{z}}=0$. Using this and the third condition of (4.3), we have

$$\begin{aligned} f\bar{\partial}\xi=ff_{\bar{z}}d\bar{z}\wedge dz &= -\sum_{j=2}^{n}h^jh^j_{\bar{z}}d\bar{z}\wedge dz=-\sum_{j=2}^{n}h^j\bar{\partial}\omega^j \\ &= -\sum_{j=2}^{n}h^jc\bar{\omega}^j\wedge\xi=c\sum_{j=2}^{n}h^jfdz\wedge\bar{\omega}^j=cf\sum_{j=2}^{n}\omega^j\wedge\bar{\omega}^j\,. \end{aligned}$$

Thus

$$\bar{\partial}\xi=c\sum_{j=2}^{n}\omega^j\wedge\bar{\omega}^j \tag{4.4}$$

holds outside the zeros of f. Since the set of zeros of f has no interior points because of the second condition, (4.4) holds on Σ. Finally, we have only to show that $\xi\cdot\bar{\xi}+\sum_{j=2}^{n}\omega^j\cdot\bar{\omega}^j\neq 0$. If $\xi|_p\neq 0$, then clearly $(\xi\cdot\bar{\xi}+\sum_{j=2}^{n}\omega^j\cdot\bar{\omega}^j)|_p\neq 0$. Assume that $\xi|_p=0$. Then $\bar{\partial}\xi\neq 0$ because of the second condition of (4.3), so $(\sum_{j=2}^{n}\omega^j\wedge\bar{\omega}^j)|_p\neq 0$ by (4.4). Hence at least one of the $\omega^j|_p$'s is non-zero. Therefore $(\xi\cdot\bar{\xi}+\sum_{j=2}^{n}\omega^j\cdot\bar{\omega}^j)|_p\neq 0$ holds. □

Thus we have the following:

THEOREM 4.6. *Assume that $c\neq 0$. Let Σ be a simply connected Riemann surface. If an n-tuple of* $(1,0)$*-forms* (ξ,ω^j) *on Σ satisfies* (4.3), *then*

$$(t(z),x^j(z))=\left(2\int_{z_0}^{z}\operatorname{Re}\xi,\,2\int_{z_0}^{z}e^{ct(z)}\operatorname{Re}\omega^j\right)$$

gives a minimal surface in $(\boldsymbol{R}^n,g_c)$. *Conversely, any minimal surface in* $(\boldsymbol{R}^n,g_c)$ *can be represented in this way.*

5. Gaussian curvature and normal Gauss map. Let $\varphi:\Sigma\to(\boldsymbol{R}^n,g_c)$ be a minimal surface with data (ξ,ω^j). The induced metric ds_Σ^2 is $2(\xi\cdot\bar{\xi}+\sum_{j=2}^{n}\omega^j\cdot\bar{\omega}^j)$. Using a local coordinate z, we write $\xi=fdz$ and $\omega^j=h^jdz$. Then $ds_\Sigma^2=2(|f|^2+\sum_{j=2}^{n}|h^j|^2)dzd\bar{z}$ and the conditions (4.1) and (4.2) are

$$f_{\bar{z}}=-c\left(\sum_{j=2}^{n}|h^j|^2\right),\quad h_{\bar{z}}^j=c\bar{h}^jf,\quad (f)^2+\sum_{j=2}^{n}(h^j)^2=0\,. \tag{5.1}$$

If we put $\lambda=2(|f|^2+\sum_{j=2}^{n}|h^j|^2)$, then the Gaussian curvature K is given by

$$K=-\frac{1}{2\lambda}\Delta\log\lambda=-\frac{2}{\lambda}\left\{\frac{\lambda\lambda_{z\bar{z}}-|\lambda_z|^2}{\lambda^2}\right\}.$$

In terms of f and h^j, K is computed to be

$$K=-c^2-\frac{(|f|^2+\sum_{j=2}^{n}|h^j|^2)(|f_z|^2+\sum_{j=2}^{n}|h_z^j|^2)-|\bar{f}f_z+\sum_{j=2}^{n}\bar{h}^jh_z^j|^2}{(|f|^2+\sum_{j=2}^{n}|h^j|^2)^3}\,.$$

Let $\boldsymbol{P}(\mathfrak{g}^{\boldsymbol{C}})$ be a projective space of complex lines in $\mathfrak{g}^{\boldsymbol{C}}$. $Z=[Z_1;\dots;Z_n]$ denotes a complex line spanned by $\sum_{\alpha=1}^{n}Z_\alpha e_\alpha$, where $\{e_k\}$ is the orthonormal basis chosen in (2.1). Define a map $\Phi:\Sigma\to\boldsymbol{P}(\mathfrak{g}^{\boldsymbol{C}})$ by

$$\Phi(p)=[A|_p]=[\xi|_p;\omega^2|_p;\dots;\omega^n|_p]=[f(p);h^2(p);\dots;h^n(p)]\,.$$

We call Φ the *normal Gauss map* of φ. In the case $c=0$, Φ is exactly the Gauss map. In the following, we see that Φ in case $c\neq 0$ is a geometrically natural generalization of the Gauss map.

In fact, Φ has values in the hyperquadric $\boldsymbol{Q}_{n-2}=\{Z\in\boldsymbol{P}(\mathfrak{g}^{\boldsymbol{C}});\sum_{\alpha=1}^{n}(Z_\alpha)^2=0\}$. It is known that the hyperquadric $\boldsymbol{Q}_{n-2}$ is diffeomorphic to the Grassmannian manifold

$\mathrm{Gr}_2(\mathfrak{g})$ of oriented 2-planes in $\mathfrak{g}$. Under this identification $\boldsymbol{Q}_{n-2} \cong \mathrm{Gr}_2(\mathfrak{g})$, $[\varphi(p)]$ is identified with $\varphi^{-1}(\varphi_*(\partial/\partial v|_p)) \wedge \varphi^{-1}(\varphi_*(\partial/\partial u|_p)) \in \mathrm{Gr}_2(\mathfrak{g})$. So Φ is essentially equal to the map $\tilde{\Phi}: \Sigma \to \mathrm{Gr}_2(\mathfrak{g})$ so defined that $\tilde{\Phi}$ assigns a point $p \in \Sigma$ to an oriented 2-plane $L_{\varphi(p)^{-1}*}(\varphi_* T_p \Sigma)$ in $\mathfrak{g}$.

It should be remarked that the normal Gauss map is different from the Gauss map due to Obata [O] or the hyperbolic Gauss map [E].

6. The 3-dimensional case. We consider the case $n=3$, i.e., minimal surfaces in $(\boldsymbol{R}^3, g_c)$. We introduce functions F, G and derive formulas similar to those in the case $\boldsymbol{R}^3$:

$$F = h^2 - ih^3, \quad G = \frac{f}{h^2 - ih^3}.$$

Strictly speaking, G is not a function but a smooth map into the Riemann sphere. In fact, G is the composite of Φ, the diffeomorphism $\boldsymbol{Q}_1 \to \boldsymbol{S}^2$ and the stereographic projection $\boldsymbol{S}^2 \to \boldsymbol{C} \cup \{\infty\}$. So we also call G the normal Gauss map.

THEOREM 6.1. *Let φ be a minimal immersion from a Riemann surface Σ into $(\boldsymbol{R}^3, g_c)$. If the normal Gauss map G of φ is constant, then $|G|=1$ and φ is totally geodesic.*

PROOF. If $G=0$, then $f=0$. Differentiating f with respect to $\bar{z}$, we have $f_{\bar{z}}=0$. Thus $h^2=h^3=0$ by (5.1), a contradiction to the assumption that φ is an immersion. Hence G cannot be identically 0. If $G=\infty$, then $h^2-ih^3=0$. So we have $f=0$ by conformality. By the argument similar to the above, G cannot be identically ∞.

Assume that $G=k$ were k is constant ($\neq 0, \infty$), that is,

$$f = k(h^2 - ih^3). \tag{6.1}$$

Differentiating f with respect to $\bar{z}$, we have $f_{\bar{z}} = k(h^2_{\bar{z}} - ih^3_{\bar{z}})$. By (5.1),

$$-(|h^2|^2 + |h^3|^2) = k(\bar{h}^2 - i\bar{h}^3)f. \tag{6.2}$$

(6.1), (6.2) and the third equation of (5.1) yield

$$(h^2 - ih^3)(|h^2|^2 + |h^3|^2 - |h^2 + ih^3|^2) = 0. \tag{6.3}$$

From the former part of this proof, $h^2 - ih^3$ cannot be identically zero on any open subset. Hence, by continuity and (6.3), $(|h^2|^2 + |h^3|^2 - |h^2 + ih^3|^2) = 0$, i.e.,

$$h^2\bar{h}^3 = \bar{h}^2 h^3. \tag{6.4}$$

Conformality and (6.4) yield $|f|^2 = |h^2|^2 + |h^3|^2$. On the other hand, $|f|^2 = |k|^2(|h^2|^2 + |h^3|^2)$ holds by (6.1). Therefore $|k|=1$.

We have only to show that φ is totally geodesic. Differentiating the equation (6.4) and using (5.1), we have

$$h^2_z \bar{h}^3 = \bar{h}^2 h^3_z. \tag{6.5}$$

It follows from (6.4) and (6.5) that $(h^2, h^3) = \mu(h_z^2, h_z^3)$ for some function μ. We also have $f = \mu f_z$ by (6.1), so $(f, h^2, h^3) = \mu(f_z, h_z^2, h_z^3)$. This means that the Gaussian curvature K is identically equal to $-c^2$ by the formula in Section 5. Therefore φ is totally geodesic. □

Assume that G is not constant. By the conformality $(f)^2 + (h^2)^2 + (h^3)^2 = 0$, we have

$$f = FG, \quad h^2 = \frac{1}{2}F(1 - G^2), \quad h^3 = \frac{i}{2}F(1 + G^2).$$

The condition (4.1) is

$$F_{\bar z} = -c|F|^2|G|^2\bar G, \quad G_{\bar z} = \frac{c\bar F}{2}(|G|^4 - 1). \tag{6.6}$$

PROPOSITION 6.2. *The normal Gauss map G of a minimal immersion $\varphi: \Sigma \to (\boldsymbol{R}^3, g_c)$ satisfies*

$$G_{z\bar z} + \frac{2|G|^2\bar G}{1 - |G|^4}G_z G_{\bar z} = 0, \tag{6.7}$$

$$\frac{\bar G^i G_{\bar z}}{1 - |G|^4} \in C^\infty(\Sigma) \qquad (i = 0, 2). \tag{6.8}$$

PROOF. By straightforward computation, we see that if (F, G) is a solution to (6.6), then G satisfies (6.7). As for (6.8),

$$\frac{G_{\bar z}}{1 - |G|^4} = -\frac{c\bar F}{2}, \quad \frac{\bar G^2 G_{\bar z}}{1 - |G|^4} = -\frac{c\bar F\bar G^2}{2} = c(\bar h^2 - \bar F),$$

hence they are smooth. □

LEMMA 6.3. *The equation* (6.7) *is the harmonic map equation for a map*

$$G: \Sigma \to (\boldsymbol{C} \cup \{\infty\}, |1 - |w|^4|^{-1} dw d\bar w).$$

Strictly speaking, $|1 - |w|^4|^{-1} dw d\bar w$ is not a Riemannian metric on $\boldsymbol{C} \cup \{\infty\}$, because it diverges on $|w| = 1$, i.e., the equator of the Riemann sphere.

We omit the proof of Lemma 6.3 for it is straightforward. From Lemma 6.3, we get:

PROPOSITION 6.4. *The normal Gauss map of a minimal surface in $(\boldsymbol{R}^3, g_c)$ is a harmonic map into the space $(\boldsymbol{C} \cup \{\infty\}, |1 - |w|^4|^{-1} dw d\bar w)$.*

The induced metric ds_Σ^2 and the $(2, 0)$-part of the complexification of the second fundamental form $(\mathrm{II}^{\boldsymbol{C}})^{2,0}$, which is called the Hopf differential, are computed to be

$$ds_\Sigma^2 = |F|^2(1 + |G|^2)^2 dz d\bar z. \qquad (\mathrm{II}^{\boldsymbol{C}})^{2,0} = FG_z dz dz.$$

In particular, G cannot be holomorphic. It is verified that the Hopf differential is holomorphic. The Gaussian curvature K is given by

$$K=-c^2-4\frac{|G_z|^2}{|F|^2(1+|G|^2)^4}.$$

In particular, φ is totally geodesic if G is anti-holomorphic.

Conversely, the following lemma holds.

LEMMA 6.5. *Let G be a solution to* (6.7) *satisfying the condition* (6.8), *which is not a holomorphic map. Then there exists a function F which satisfies* (6.6) *together with G.*

PROOF. Put $F=2c^{-1}\bar{G}_z/(|G|^4-1)$. Then

$$F_{\bar{z}}=\frac{2}{c}\frac{\bar{G}_{z\bar{z}}(|G|^4-1)-\bar{G}_z(2GG_{\bar{z}}\bar{G}^2+2G^2\bar{G}\bar{G}_{\bar{z}})}{(|G|^4-1)^2}.$$

Making use of (6.2), we have $F_{\bar{z}}=-c|F|^2|G|^2\bar{G}$. □

Hence we have:

THEOREM 6.6. *Let Σ be a simply-connected Riemann surface and $G: \Sigma \to \boldsymbol{C}\cup\{\infty\}$ a solution to* (6.7) *satisfying the condition* (6.8), *which is not a holomorphic map. Then there exists a branched minimal immersion from Σ into hyperbolic* 3*-space whose normal Gauss map is G.*

7. Examples.

For simplicity, we deal only with the case $c=1$.

EXAMPLE 7.1. Let $\Sigma=\{z=u+iv\in\boldsymbol{C}; u>0\}$ and

$$\xi=\frac{dz}{z+\bar{z}}, \quad \omega^2=\frac{idz}{z+\bar{z}}, \quad \omega^3=0.$$

Then $(\xi, \omega^2, \omega^3)$ satisfies (4.3) and the surface obtained is totally geodesic:

$$t(z)=\int_1^z\frac{dz+d\bar{z}}{z+\bar{z}}=\log(z+\bar{z})=\log(2u),$$

$$x^2(z)=\int_1^z e^{\log 2u}\left(\frac{idz-id\bar{z}}{z+\bar{z}}\right)=\int_1^z 2u\frac{-2dv}{2u}=-2v,$$

$$x^3(z)=\text{constant}.$$

EXAMPLE 7.2. Let $\rho(u)$ be a solution to the ordinary differential equation

$$\frac{d\rho}{du}=(e^{-2\rho}-a^2e^{2\rho})^{1/2},$$

where a is a real constant.

On an appropriate domain Σ, if we put $\rho=\rho(u)=\rho((z+\bar{z})/2)$ and

$$G=\left(\frac{e^{-\rho}+ae^{\rho}}{e^{-\rho}-ae^{\rho}}\right)^{1/2},$$

then G satisfies (6.2). By Lemma 6.1, F can be computed as $F=(e^{-\rho}-ae^{\rho})/2$.

Hence,

$$\xi=\frac{1}{2}(e^{-2\rho}-a^2e^{2\rho})^{1/2}dz\,,\quad \omega^2=\frac{1}{2}ae^{\rho}dz\,,\quad \omega^3=-\frac{i}{2}e^{-\rho}dz$$

satisfy (4.3) and the minimal immersion obtained is given by

$$t(z)=\rho(u)\,,\quad x^2(z)=a\int e^{2\rho(u)}du\,,\quad x^3(z)=v\,.$$

This minimal surface is known to be the rotational minimal surface of parabolic type given by do Carmo and Dajczer [D-D].

The induced metric, the Hopf differential and the Gaussian curvature are computed to be

$$ds_{\Sigma}^2=e^{-2\rho}dzd\bar{z}\,,\quad (\mathrm{II}^{\mathbf{C}})^{2,0}=\frac{a}{2}dzdz\,,\quad K=-(1+a^2e^{2\rho})\,.$$

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