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POINTWISE CONVERGENCE OF FEJER TYPE MEANS

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Abstract. We study the almost everywhere convergence of polyhedral Fejer type means. We prove positive results for the n-dimensional Euclidean spaces and tori. We then show that these positive results cannot be extended to the whole setting of compact Lie groups.

1. Introduction. A fundamental result of A. N. Kolmogorov (proved in two steps, when he was only nineteen and twenty-two years old!) exhibits a function $f \in L^1(\boldsymbol{T})$ whose Fourier series diverges everywhere (cf. [18, ch. 8]). This means that if $D_N(t)=\sum_{k=-N}^{N} e^{2\pi ikt}$ is the Dirichlet kernel, then the partial sum $S_N f(t)=(f*D_N)(t)=\sum_{k=-N}^{N}\hat{f}(k)e^{2\pi ikt}$ diverges for all t when $N\to+\infty$. To get positive results one can substitute the partial sum with some suitable means. The Fejer kernel

$$K_N(t)=\frac{1}{N+1}\sum_{v=0}^{N} D_v(t)=\sum_{k=-N}^{N}\left(1-\frac{|k|}{N+1}\right)e^{2\pi ikt}=\frac{1}{N+1}\left(\frac{\sin(\pi(N+1)t)}{\sin(\pi t)}\right)^2$$

provides one of the most important examples. In fact, it is well known (cf. [18, ch. 3]) that if $f\in L^1(\boldsymbol{T})$, then $(K_N*f)(t)\to f(t)$ for almost every $t\in\boldsymbol{T}$.

If we now look at the above chain of identities, we see that the Fejer kernel may be seen either as the arithmetic mean of the Dirichlet kernel or, for even N, as $(1/(N+1)$ times) the square of a Dirichlet kernel. This gives rise to two reasonable definitions of the Fejer kernel in several variables. We have to start by defining the Dirichlet kernel on $\boldsymbol{T}^n$.

$$D_N(t)=D_N^B(t)=\sum_{m\in NB} e^{2\pi i m\cdot t}\,, \qquad t\in\boldsymbol{T}^n\,, \quad m\in\boldsymbol{Z}^n$$

where B is a convex body containing the origin in its interior and NB is its dilation. As the first two examples one takes B to be the unit cube or the unit ball. An n-dimensional polyhedron and a convex body whose smooth boundary satisfies good curvature properties are the most familiar generalizations of the cube and the ball respectively. In this paper we are interested in the polyhedral case.

Let P be a compact n-dimensional convex polyhedron in $\boldsymbol{R}^n$ containing the origin in its interior. Following the previous remarks we can define (cf. [16]) the n-dimensional polyhedral Fejer kernel either as

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$$K_{2N}(t)=K_{2N}^{P}(t)=\frac{1}{\operatorname{card}(NP\cap \boldsymbol{Z}^n)}\,|\,D_N^P(t)\,|^2$$

or as

$$H_N(t)=H_N^P(t)=\frac{1}{N+1}\sum_{v=0}^{N} D_v^P(t)\,.$$

We have used different symbols since H_N and K_N coincide only in the 1-dimensional case. We shall call H_N a *Fejer type kernel*. Observe that the graph of $\hat{H}_N(m)$ is a pyramid restricted to the integral points inside NP and that H_N^P may be seen as a polyhedral analogue of the Bochner-Riesz kernel of index one.

Both kernels K_{2N} and H_N have good summability properties. If $f\in L^1(\boldsymbol{T}^n)$, then $\|K_{2N}*f-f\|_1\to 0$ and $\|H_N*f-f\|_1\to 0$ as $N\to+\infty$.

The above result for K_{2N} is obvious since K_{2N} is positive. The result for H_N is contained in [9] (see below for a short proof).

For $n\geq 2$ the study of the pointwise convergence is a different problem. We recall that it is not easy to prove that $(K_{2N}^P*f)(t)\to f(t)$ a.e. for any $f\in L^1(\boldsymbol{T}^n)$, even if P is the unit cube (i.e. even if we can separate the variables). See the last chapter in [18].

The pointwise convergence of polyhedral Fejer type means has not yet been investigated and it is the main object of this paper.

We now introduce the definition of a Fejer type kernel on $\boldsymbol{R}^n$ and relate it to H_N. Let

$$\tilde{H}(t)=\int_{\boldsymbol{R}^n}\int_0^1 \chi_{vP}(\xi)\,dv\,e^{2\pi i t\cdot\xi}d\xi=\int_0^1\int_{vP} e^{2\pi i t\cdot\xi}d\xi\,dv\,,\qquad t\in\boldsymbol{R}^n$$

then

$$H_N(t)=\frac{1}{N+1}\sum_{j=0}^{N} D_j=\sum_{m\in\boldsymbol{Z}^n}\tilde{H}_{1/(N+1)}(t+m)\,,\qquad t\in\boldsymbol{T}^n\,, \tag{1}$$

where χ_{vP} is the characteristic function of the dilated set vP, $\tilde{H}_\varepsilon(t)=\varepsilon^{-n}\tilde{H}(t/\varepsilon)$ and the last identity depends on the Poisson summation formula.

In the following section we shall prove that $(\tilde{H}_\varepsilon*f)(t)\to f(t)$ a.e. for any $f\in L^1(\boldsymbol{R}^n)$, and that $(H_N*f)(t)\to f(t)$ a.e. for any $f\in L^1(\boldsymbol{T}^n)$. Our argument relies on pointwise estimates for $\tilde{H}$. It turns out that the decay of $\tilde{H}$ at infinity depends on the direction chosen, so that $\tilde{H}$ does not admit suitable radial bounds and the maximal operator $\sup_{\varepsilon>0}|\,\tilde{H}_\varepsilon*f(t)\,|$ cannot be controlled via the Hardy-Littlewood maximal function. This difficulty can be overcomed by appealing to an argument of H. S. Shapiro (see Theorem 3 below).

In the last part of this paper we shall prove that these results cannot be extended to all compact Lie groups. We shall consider a compact simple simply connected Lie group G and we shall define polyhedral Dirichlet kernels D_N and the associated Fejer type kernels H_N. When the rank of G is greater than one, we shall prove that there exists a central function $f \in L^1(G)$ such that $(H_N * f)(x)$ diverges a.e. As a corollary of the proof we shall produce, for any $p < n/m$ (here n is the dimension of G and m is the number of positive roots), a central function $g \in L^p(G)$ such that $(D_N * g)(x)$ diverges a.e. This almost complements the known fact that $(D_N * g)(x) \to g(x)$ a.e. whenever g is a central function in $L^p(G)$, for $p > n/m$ (see [12]). The proof of our result is essentially a combination of arguments in [8], [4], [15] and [16] and we shall only sketch it. We refer the interested reader to [12] and [6] for the related problem concerning the L^p convergence.

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2. Fejer type kernels on $\boldsymbol{R}^n$ and $\boldsymbol{T}^n$. Our main results are the following.

THEOREM 1. *Let $\tilde{H}_\varepsilon(t) = \varepsilon^{-n}\tilde{H}(t/\varepsilon)$. Then for any $f \in L^1(\boldsymbol{R}^n)$, $(\tilde{H}_\varepsilon * f)(t) \to f(t)$ a.e. as $\varepsilon \to 0$.*

THEOREM 2. *Let $\boldsymbol{T}^n$ be the n-dimensional torus, let P be a polyhedron with vertices in $\boldsymbol{Z}^n$ and let*

$$H_N(t) = \frac{1}{N+1} \sum_{j=0}^{N} D_j$$

be the Fejer type kernel defined on $\boldsymbol{T}^n$. Then, for any $f \in L^1(\boldsymbol{T}^n)$, $H_N * f(t) \to f(t)$ a.e. on $\boldsymbol{T}^n$ as $N \to \infty$.

We need the following result from [10] (see also [14] and [11] for related problems).

THEOREM 3 (Shapiro). *Let k be a measurable function satisfying the following conditions: there exists a sequence $\{A_j\}$ of centrally symmetric convex bodies about $t=0$ such that, for any $t \in \boldsymbol{R}^n$,*

$$|k(t)| \le \sum_{j=1}^{\infty} \alpha_j \frac{1}{|A_j|} \chi_{A_j}(t)$$

with

$$\alpha_j > 0\,, \quad \alpha_j \to 0\,, \quad \sum_{j=1}^{\infty} \alpha_j \log\left(\frac{1}{\alpha_j}\right) \le c$$

($|A_j|$ is the Lebesgue measured of A_j). Then, for any $f \in L^1(\boldsymbol{R}^n)$, $(k_\varepsilon * f)(t) \to \hat{k}(0) f(t)$ a.e.

PROOF OF THEOREM 1. Let t be a regular point in ∂vP with outward unit normal $\omega(t)$. Then the divergence theorem implies

$$\tilde{H}(\xi) = \int_0^1 dv \int_{vP} e^{2\pi i \xi \cdot t} dt = \frac{1}{2\pi i |\xi|^2} \int_0^1 dv \int_{\partial vP} \xi \cdot \omega(t) e^{2\pi i \xi \cdot t} dt'$$

where dt' denotes the $(n-1)$-dimensional measure. Let P' be a face of P. We suppose it is contained in $(1, t_2, \dots, t_n)$ and therefore, since $\omega(t)$ is constant on P', it is enough to estimate

$$(2)\quad \min\left(\frac{|\xi_1|}{|\xi|^2}\left|\int_0^1 e^{2\pi i v\xi_1}dv\int_{vP'} e^{2\pi i(t_2\xi_2+\cdots+t_n\xi_n)}dt_2\dots dt_n\right|, 1\right)\leq \min\left(\frac{1}{|\xi|}|\hat{\chi}_T(\xi)|, 1\right)$$

where T is the pyramid having P' as a face and the origin as the opposite vertex. We split T into simplices. Let S be one of them with vertices $a_0, \dots, a_n$. The identity

$$\hat{\chi}_S(\xi)=\frac{n!|S|}{(2\pi i)^n}\sum_{j=0}^{n}\frac{e^{-2\pi i\xi\cdot a_j}}{\prod_{k\neq j}\xi\cdot(a_k-a_j)}$$

has been pointed out in [9]. We treat one of the above terms (say $j=0$) and we observe that the applicability of Shapiro's theorem is not affected by a linear change of variables. Therefore we can assume $a_k-a_j=e_k$ and we are reduced to bounding

$$G(\xi)=\min\left(\frac{1}{|\xi||\xi_1\cdot\xi_2\cdot\cdots\cdot\xi_n|}, 1\right).$$

Let

$$E_h=E_h^{(n)}=\{\xi\in \boldsymbol{R}^n : |\xi||\xi_1\cdot\xi_2\cdot\cdots\cdot\xi_n|\leq 2^h\}.$$

Then

$$G(\xi)\leq\sum_{h=1}^{\infty}G(\xi)\cdot\chi_{E_h\setminus E_{h-1}}(\xi)+\chi_{E_0}(\xi)\leq 2\sum_{h=0}^{\infty}2^{-h}\chi_{E_h}(\xi).$$

We claim that there exist a family $\{Q_j\}$ of centrally symmetric n-dimensional intervals about $\xi=0$ and a sequence $\{\alpha_j\}$ satisfying

$$(3)\qquad \chi_{E_0}(\xi)\leq\sum_{j=0}^{\infty}\alpha_j\frac{1}{|Q_j|}\chi_{Q_j}(\xi)$$

$$\alpha_j>0, \quad \alpha_j\to 0, \quad \sum_{j=1}^{\infty}\alpha_j\log\left(\frac{1}{\alpha_j}\right)\leq c.$$

Assuming this to be true we observe that $E_h=2^{h/(n+1)}E_0$ so that

$$\chi_{E_h}(\xi)\leq\sum_{j=0}^{\infty}\alpha_j\frac{1}{|Q_j|}\chi_{R_{j,h}}(\xi)$$

where $R_{j,h}=2^{h/(n+1)}Q_j$. Since $|R_{j,h}|=2^{nh/(n+1)}|Q_j|$ we obtain

$$G(\xi)\leq 2\sum_{h=0}^{\infty}2^{-h}\sum_{j=0}^{\infty}\alpha_j\frac{1}{|Q_j|}\chi_{R_{j,h}}(\xi)=2\sum_{j,h}\alpha_j 2^{-h/(n+1)}\frac{1}{|R_{j,h}|}\chi_{R_{j,h}}(\xi)$$

with

$$\sum_{j,h} \alpha_j 2^{-h/(n+1)} \log\left(2^{h/(n+1)} \frac{1}{\alpha_j}\right) < +\infty$$

as a consequence of the assumptions on $\{\alpha_j\}$. We shall prove (3) in the following proposition. □

REMARK. Observe that (2), the inequality

$$\int_{\Sigma_{n-1}} |\hat{\chi}_P(\rho\sigma)| \, d\sigma \le c(2+\rho)^{-n} \log^{n-1}(2+\rho)$$

(proved in [2]) and an integration in polar coordinates yield a simple proof of the bound $\|\tilde{H}\|_{L^1} < \infty$ already proved in [9].

PROPOSITION 4. *For every dimension n there exist a family $\{Q_j^{(n)}\}$ of centrally symmetric n-dimensional intervals about $\xi = 0$ and a sequence $\{\alpha_j^{(n)}\}$ satisfying*

$$\chi_{E_0^{(n)}}(\xi) \le \sum_{j=0}^{\infty} \alpha_j^{(n)} \frac{1}{|Q_j^{(n)}|} \chi_{Q_j^{(n)}}(\xi)$$

$$\alpha_j^{(n)} > 0\,, \quad \alpha_j^{(n)} \to 0\,, \quad \sum_{j=1}^{\infty} \alpha_j^{(n)} \log\left(\frac{1}{\alpha_j^{(n)}}\right) < +\infty\,.$$

PROOF. The proof is by induction on the dimension n. If $n=1$ then $E_0^{(1)}$ is a symmetric interval and the above is trivial. We assume the case $n-1$ and we shall prove the case n. By symmetry we can study

$$E_+^{(n)} = \{\xi \in E_0^{(n)} : 0 < \xi_1 \le \xi_k, \, k > 1\}$$

in place of $E_0^{(n)}$. Let $\xi \in E_+^{(n)}$. Then $1 \ge |\xi| \xi_1 \xi_2 \cdot \cdots \cdot \xi_n \ge \xi_1^{n+1}$ and $\xi_1 \le 1$. Assuming $\xi_1 \in [2^{-k}, 2^{-k+1}]$, we have

$$\sqrt{\xi_2^2 + \cdots + \xi_n^2}(\xi_2 \cdot \cdots \cdot \xi_n) \le \frac{|\xi|}{\xi_1}(\xi_1 \xi_2 \cdot \cdots \cdot \xi_n) \le 2^k$$

and therefore $(\xi_2, \ldots, \xi_n) \in 2^{k/n} E_0^{(n-1)}$. Hence we have

$$\begin{aligned} \chi_{E_+^{(n)}}(\xi) &\le \sum_{k=1}^{+\infty} \chi_{[2^{-k}, 2^{-k+1}]}(\xi_1) \chi_{2^{k/n} E_0^{(n-1)}}(\xi_2, \ldots, \xi_n) \\ &\le \sum_{k=1}^{+\infty} \chi_{[-2^{-k+1}, 2^{-k+1}]}(\xi_1) \chi_{2^{k/n} E_0^{(n-1)}}(\xi_2, \ldots, \xi_n) \end{aligned}$$

and using the induction hypothesis

$$\chi_{E_+^{(n)}}(\xi) \le \sum_{k=1}^{+\infty} \chi_{[-2^{-k+1}, 2^{-k+1}]}(\xi_1) \sum_{j=1}^{\infty} \alpha_j^{(n-1)} \frac{1}{|Q_j^{(n-1)}|} \chi_{2^{k/n} Q_j^{(n-1)}}(\xi_2, \ldots, \xi_n)\,.$$

We consider the centrally symmetric sets $Q_{j,k}^{(n)}=[-2^{-k+1}, 2^{-k+1}]\times(2^{k/n}Q_j^{(n-1)})$. Then $|Q_{j,k}^{(n)}|=2^{2-k/n}|Q_j^{(n-1)}|$ and

$$\chi_{E_+^{(n)}}(\xi)\leq\sum_{k,j}\alpha_j^{(n-1)}2^{2-k/n}\frac{1}{|Q_{j,k}^{(n)}|}\chi_{Q_{j,k}^{(n)}}(\xi)\,.$$

Since

$$\sum_{k,j}\alpha_j^{(n-1)}2^{-k/n+2}\log\left(2^{k/n+2}\frac{1}{\alpha_j^{(n-1)}}\right)<+\infty$$

the proposition and the theorem are proved. □

Theorem 2 follows from Theorem 1 and the general result in [1, p. 276]. However we like to propose the following direct proof.

Proof of Theorem 2. Since the convolution commutes with translations it will be enough to prove the almost everywhere convergence of $H_N*f(t)$ for a ball of t, say $|t|<1/4$. Let $Q=[-1/2, 1/2]^n$. By (1),

$$\begin{aligned}(H_N*f)(t)&=\int_Q f(s)\sum_{m\in\mathbf{Z}^n}\tilde{H}_{1/(N+1)}((t-s)+m)ds\\&=\sum_{m\in\mathbf{Z}^n}\int_Q f(s)\tilde{H}_{1/(N+1)}((t-s)+m)ds\\&=\int_Q f(s)\tilde{H}_{1/(N+1)}(t-s)ds+\sum_{m\neq0}\int_Q f(s)\tilde{H}_{1/(N+1)}((t-s)+m)ds\\&=A(t)+B(t)\,.\end{aligned}$$

Observe that for almost every t

$$A(t)=\int_{\mathbf{R}^n}(f\chi_Q)(s)\tilde{H}_{1/(N+1)}(t-s)ds\to(f\chi_Q)(t)$$

by the previous theorem. Now we have to prove that $B(t)\to0$ a.e. Arguing as in the proof of the previous theorem, we can control $\tilde{H}(\xi)$ with a sum of terms of the form

$$\min\left(|\xi|^{-1}\prod_{k=1}^n|\xi\cdot(a_k-a_0)|^{-1}, 1\right)$$

where $a_0,\ldots,a_n$ denote the vertices of a simplex. Let $p_k=a_k-a_0$. We are reduced to showing that, for almost every t,

$$\sum_{m\neq0}\int_Q|f(s)|N^n\min\left(N^{-n-1}|t-s+m|^{-1}\prod_{k=1}^n|(t-s+m)\cdot p_k|^{-1}, 1\right)ds\to0 \tag{4}$$

as $N\to+\infty$.

Let

$$R = \sup_{\substack{t,s \in Q \\ k=1,\dots,n}} |(t-s)\cdot p_k|\,.$$

For every $\Lambda \subseteq \{1,\dots,n\}$ we denote by A_Λ the set of all $m \in \boldsymbol{Z}^n$, $m \neq 0$, such that $|m\cdot p_k| > 2R$ when $k \in \Lambda$ and $|m \cdot p_k| \leq 2R$ when $k \notin \Lambda$. Since $\{A_\Lambda\}$ is a partition of $Z^n \diagdown \{0\}$ we can split the sum over m in (4) and are reduced to considering

$$\text{(5)} \qquad \sum_{A_\Lambda} \int_Q |f(s)|\, N^n \min\left(N^{-n-1}|t-s+m|^{-1} \prod_{k=1}^n |(t-s+m)\cdot p_k|^{-1},\, 1\right) ds$$

for a given Λ. We consider three different cases.

(i) $\Lambda = \{1,\dots,n\}$. This means $|m\cdot p_k| > 2R$ for $k=1,\dots,n$ and therefore, since $m \neq 0$ and $|t| \leq 1/4$,

$$|t-s+m|^{-1} \prod_{k=1}^n |(t-s+m)\cdot p_k|^{-1} \leq \text{const}\, |m|^{-1} \prod_{k=1}^n |m\cdot p_k|^{-1}\,.$$

Since

$$\text{(6)} \qquad \sum_{\substack{|m\cdot p_k| > 2R \\ k=1,\dots,n}} |m|^{-1} \prod_{k=1}^n |m\cdot p_k|^{-1}$$

converges it follows that (5) is bounded up to a constant by

$$\frac{1}{N}\int_Q |f(s)|\,ds$$

which tends to zero as $N \to +\infty$. To see that (6) converges observe that

$$|m| \geq \text{const} \prod_{k=1}^n |m\cdot p_k|^{1/n}$$

and therefore (6) is bounded by

$$\sum_{\substack{|m\cdot p_k| > 2R \\ k=1,\dots,n}} \prod_{k=1}^n |m\cdot p_k|^{-1-1/n}\,.$$

However it is not difficult to show that one can substitute the above series by an integral and, after a change of variable, one is reduced to bounding

$$\int_{|\xi_j| > R} \frac{d\xi}{|\xi_1 \xi_2 \cdots \xi_n|^{1+1/n}}$$

which converges.

(ii) Λ is a proper non-empty subset of $\{1,\dots,n\}$. We can suppose $\Lambda = \{1,\dots,r\}$

with $r<n$. Then

$$|(t-s+m)\cdot p_k|\geq|m\cdot p_k|-R\geq\frac{1}{2}|m\cdot p_k|$$

for $k=1,\dots,r$. Hence (5) is bounded by

$$\sum_{A_\Lambda}\int_Q|f(s)|\min\left(N^{-1}|m|^{-1}\prod_{k=1}^{r}|m\cdot p_k|^{-1}\prod_{k=r+1}^{n}|(t-s+m)\cdot p_k|^{-1},N^n\right)ds\,.$$

Since $m, p_k\in Z^n$ and $|m\cdot p_k|\leq 2R$ for $k=r+1,\dots,n$ it follows that in this case $m\cdot p_k$ takes only a finite number of values. We fix one of these values, say z_k, and we are reduced to showing that

$$\sum_{A_\Lambda}\int_Q|f(s)|\min\left(N^{-1}|m|^{-1}\prod_{k=1}^{r}|m\cdot p_k|^{-1}\prod_{k=r+1}^{n}|(t-s)\cdot p_k+z_k|^{-1},N^n\right)ds\to 0$$

for almost every t. To see this we apply the dominated convergence theorem on the measure space $A_\Lambda\times Q$. First of all we observe that for every fixed t

$$|f(s)|\min\left(N^{-1}|m|^{-1}\prod_{k=1}^{r}|m\cdot p_k|^{-1}\prod_{k=r+1}^{n}|(t-s)\cdot p_k+z_k|^{-1},N^n\right)\to 0 \tag{7}$$

as $N\to+\infty$ for almost every $(m,s)\in A_\Lambda\times Q$. Moreover using the inequality

$$\min(N^{-1}a,N^n)\leq a^{n/(n+1)} \tag{8}$$

the term in (7) is bounded, up to a constant, by $F(m)G(s,t)$, where

$$F(m)=\left(|m|^{-1}\prod_{k=1}^{r}|m\cdot p_k|^{-1}\right)^{n/(n+1)},\quad G(s,t)=f(s)\prod_{k=r+1}^{n}|(t-s)\cdot p_k+z_k|^{-n/(n+1)}\,.$$

Observe first that $\int_Q G(s,t)ds$ is the convolution of two functions of $L^1(Q)$ and therefore the integral is finite for almost every t. Moreover

$$\sum_{A_\Lambda}F(m)<+\infty\,.$$

To see this observe that

$$|m|>\text{const}\prod_{k=1}^{r}|m\cdot p_k|^{1/r}$$

so that

$$\sum_{A_\Lambda}F(m)\leq\text{const}\sum_{A_\Lambda}\left(\prod_{k=1}^{r}|m\cdot p_k|^{-1-1/r}\right)^{n/(n+1)}=\text{const}\sum_{A_\Lambda}\prod_{k=1}^{r}|m\cdot p_k|^{-1-(n-r)/(n+1)r}\,.$$

Since $r<n$ the convergence follows similarly to the one of (6).

From the above considerations it follows that

$$\sum_{A_\Lambda} \int_Q F(m)G(s,t)ds < +\infty$$

for almost every t and therefore the convergence in (7) is dominated.

(iii) $\Lambda = \varnothing$. In this case $A_\Lambda = \{m \in Z^n : m \neq 0, |m \cdot p_k| \leq 2R, k = 1, \dots, n\}$ is finite since the p_k's span $\boldsymbol{R}^n$, so that it is enough to show that for every $m \in A_\Lambda$

$$\int_Q |f(s)| N^n \min\left(N^{-n-1} |t-s+m|^{-1} \prod_{k=1}^n |(t-s+m) \cdot p_k|^{-1}, 1 \right) ds \tag{9}$$

tends to zero as $N \to +\infty$ for almost every t. Observe now that since $|t| \leq 1/4$ and $m \neq 0$ we have $|t-s+m|$ bounded away from zero. Hence (9) is controlled, up to a constant, by

$$\int_Q |f(s)| N^n \min\left(N^{-n-1} \prod_{k=1}^n |(t-s+m) \cdot p_k|^{-1}, 1 \right) ds .$$

Now we argue as in the previous case. Indeed,

$$|f(s)| N^n \min\left(N^{-n-1} \prod_{k=1}^n |(t-s+m) \cdot p_k|^{-1}, 1 \right) \to 0$$

as $N \to +\infty$ for almost every s. Moreover by (8) we have

$$N^n \min\left(N^{-n-1} \prod_{k=1}^n |(t-s+m) \cdot p_k|^{-1}, 1 \right) \leq \prod_{k=1}^n |(t-s+m) \cdot p_k|^{-n/(n+1)} .$$

Since the latter function is in $L^1(Q)$ we conclude that the convergence is dominated for almost every t and therefore (9) tends to zero for almost every t. □

3. Fejer type kernels on compact Lie groups. In this section we prove that the positive results on the pointwise convergence of Fejer type means on $\boldsymbol{T}^n$ cannot be extended to the whole setting of compact Lie groups.

We first need to set the notation.

Let G be an n-dimensional compact simple simply connected Lie group. Every integrable function f on G has a Fourier series

$$f \sim \sum_\lambda d_\lambda \chi_\lambda * f$$

where d_λ and χ_λ are the dimension and the character of the irreducible unitary representation λ, respectively. Let T be a maximal torus of G, and let $\mathfrak{t}$ and $\mathfrak{g}$ be the Lie algebras of T and G. We choose a positive system Φ^+ in the set of roots of G and let $\{\alpha_1, \dots, \alpha_l\}$ be the associated system of simple roots. We shall write $m = \operatorname{card}(\Phi^+)$. We denote by W the Weyl group generated by the reflections σ_j in the hyperplanes

$\alpha_j(H)=0$ ($j=1,\dots,l$), and we consider W acting both on $\mathfrak{t}$ and on the dual $\mathfrak{t}^*$. The Killing form B defines a positive definite inner product $(\cdot,\cdot)=-B(\cdot,\cdot)$ in $\mathfrak{t}$. For every $\lambda\in i\mathfrak{t}^*$ there exists a unique $H_\lambda\in\mathfrak{t}$ such that $\lambda(H)=i(H_\lambda, H)$ for every $H\in\mathfrak{t}$. The vectors $H_j=4\pi i H_{\alpha_j}/\alpha_j(H_{\alpha_j})$ generate the lattice Ker(exp). The elements $\lambda\in i\mathfrak{t}^*$ satisfying $\lambda(H)\in 2\pi i\boldsymbol{Z}$ for all $H\in$ Ker(exp) give the set Λ of the weights of G, and the fundamental weights are defined by the relations $\lambda_j(H_j)=2\pi i\delta_{jk}$, $j,k=1,\dots,l$. If μ and λ are weights we write $\mu\preceq\lambda$ if $\lambda-\mu$ is a sum of positive roots.

The set $\Sigma=\{\lambda\in\Lambda:\lambda=\sum_{j=1}^{l}m_j\lambda_j,\ m_j\in\boldsymbol{N}\}$ of the dominant weights can be naturally identified with the set of the equivalence classes of unitary irreducible representations of G. A dominant weight λ is non-singular if $m_j>0$ for every $j=1,\dots,l$.

If ξ is a character of T, there exists a unique $\lambda\in i\mathfrak{t}^*$ such that

$$\xi\circ\exp H=e^{\lambda(H)}=e^{i(H_\lambda,H)}$$

for $H\in T$. The character χ_λ of a representation λ splits (on T) as

$$\chi_\lambda=\sum_{\mu\preceq\lambda}m_\lambda(\mu)\xi_\mu$$

where $m_\lambda(\mu)$ is the multiplicity of the weight μ in the representation corresponding to the dominant weight λ.

For $\lambda\in\Sigma$ and $t=\exp(H)$ in T we define the alternating sum and the symmetric sum

$$A(\lambda)(t)=\sum_{\sigma\in W}\det(\sigma)e^{\sigma(\lambda)(H)}\,,\qquad S(\lambda)(t)=\sum e^{\sigma(\lambda)(H)}$$

where the last sum is over the orbit of λ under the action of the Weyl group.

For the character χ_λ and the dimension d_λ of the representation corresponding to the dominant weight λ we have the Weyl formulas:

$$\chi_\lambda(t)=\Delta^{-1}(t)A(\lambda+\beta)(t)\,,\qquad d_\lambda=\prod_{\alpha\in\Phi^+}\frac{(\lambda+\beta,\alpha)}{(\beta,\alpha)}$$

where $\beta=(\sum_{\alpha\in\Phi^+}\alpha)/2$ and

$$\Delta(t)=A(\beta)(t)=(-2i)^m\prod_{\alpha\in\Phi^+}\sin(i\alpha(H)/2)\,.$$

A function f on G is said to be central if $f(xyx^{-1})=f(y)$ for any $x,y\in G$. A reference for the theory is [17].

Let ω be a dominant nonsingular weight and let $P^+(\omega)$ be the set of all dominant λ's such that $(\lambda_j,\lambda)\le(\lambda_j,\omega)$ for every $j=1,\dots,l$. The polyhedron $P(\omega)$ is defined to be the union of the satured hull of the dominant weights $\lambda\in P^+(\omega)$: i.e. $P(\omega)=\bigcup_{\sigma\in W}\sigma(P^+(\omega))$. We now fix a nonsingular large ω and for any nonnegative integer N we write P_N and P_N^+ in place of $P(N\omega)$ and $P^+(N\omega)$ respectively. We denote by D_N the polyhedral Dirichlet kernel

$$D_N = \sum_{\lambda \in P_N^+} d_\lambda \chi_\lambda . \tag{10}$$

Then the Fejer type kernel H_N is defined as

$$H_N = \frac{1}{N+1} \sum_{j=0}^{N} D_j .$$

We need the following result from [16] (see also [15] and [8]).

LEMMA 5. *Let P be a polyhedron in $\boldsymbol{R}^n$ and let $V = V_P \subset \boldsymbol{Z}^n$ be the set of vertices. Let $[a_1, b_1], \ldots, [a_s, b_s]$ be a maximal set of pairwise nonparallel edges of P. For $h = 1, \ldots, s$ let $m_h \in \boldsymbol{Z}^n$ such that $[0, m_h]$ is a segment of minimal length parallel to $[a_h, b_h]$. Let $E(t) = \prod_{h=1}^{s} (e^{2\pi i v_h \cdot t} - 1)$. Then, for any large natural number N,*

$$\sum_{m \in NP} e^{2\pi i m \cdot t} = E^{-1}(t) \cdot \sum_{a \in V} e^{2\pi i N a \cdot t} \cdot F_a(t)$$

where the F_a's are trigonometric polynomials with integral coefficients, independent of N.

We can now prove the following:

THEOREM 6. *Let G be a compact simple simply connected Lie group of rank greater than one. Then there exists a central function $f \in L^1(G)$ such that*

$$\limsup_{N \to +\infty} |(f * H_N)(x)| = +\infty$$

for almost every $x \in G$.

PROOF. The function f can be constructed (cf. e.g. [13, p. 272]) after we prove that

$$\limsup_{N \to +\infty} |H_N(t)| = +\infty \tag{11}$$

for almost every $t \in T$.

In order to prove (11) we argue as in [16]. For any positive root α let $\mathcal{D}_\alpha$ denote the partial derivative with respect to the tangent vector H_α. Then, by the previous lemma and by [5], there exists a polynomial F independent of N such that

$$\begin{aligned} H(t) &= \frac{1}{N+1} \sum_{j=0}^{N} D_j(t) = \frac{1}{N+1} \sum_{j=0}^{N} \sum_{\lambda \in P_j^+} d_\lambda \Delta^{-1}(t) A(\lambda + \beta)(t) \\ &= \frac{1}{N+1} \Delta^{-1}(t) \prod_{\alpha \in \Phi^+} \mathcal{D}_\alpha \left(\sum_{j=0}^{N} \sum_{\lambda \in P^+(j\omega + \beta)} S(\lambda)(t) \right) \\ &= \frac{1}{N+1} \Delta^{-1}(t) \prod_{\alpha \in \Phi^+} \mathcal{D}_\alpha \left(\sum_{j=0}^{N} \Delta^{-1}(t) \sum_{\sigma \in W} \xi_{\sigma(j\omega)}(t) \det(\sigma) F(\sigma(t)) \right) \end{aligned}$$

$$=\frac{1}{N+1}\Delta^{-1}(t)\prod_{\alpha\in\Phi^+}\mathscr{D}_\alpha\left(\Delta^{-1}(t)\sum_{\sigma\in W}\det(\sigma)F(\sigma(t))\frac{\xi_{\sigma((N+1)\omega)}(t)-1}{\xi_{\sigma(\omega)}(t)-1}\right)$$

$$=A_N(t)+\frac{1}{N+1}\Delta^{-2}(t)\prod_{\alpha\in\Phi^+}\mathscr{D}_\alpha\left(\sum_{\sigma\in W}\det(\sigma)F(\sigma(t))\frac{\xi_{\sigma((N+1)\omega)}(t)-1}{\xi_{\sigma(\omega)}(t)-1}\right)$$

$$=A_N(t)+B_N(t)+\frac{1}{N+1}\Delta^{-2}(t)\sum_{\sigma\in W}\det(\sigma)\frac{\prod_{\alpha\in\Phi^+}\mathscr{D}_\alpha(F(\sigma(t))(\xi_{\sigma((N+1)\omega)}(t)-1))}{\xi_{\sigma(\omega)}(t)-1}$$

$$=A_N(t)+B_N(t)+N^{m-1}\cdot C_N(t)$$

where the C_N's are polynomials with a bounded number of terms, and with coefficients uniformly bounded and uniformly away from zero. Then (cf., e.g., [4, p. 381]) we have

$$\limsup_{N\to\infty}|C_N(t)|>0$$

for almost every $t\in T$.

Then one checks through similar computations that

$$|A_N(t)+B_N(t)|\le\gamma(t)\cdot o(N^{m-1}),\qquad \text{a.e.}$$

with $\gamma(t)$ independent of N. This implies (11) since $m>1$ whenever $l>1$. □

The previous argument suggests the new part of the following theorem. Here D_N is the Dirichlet kernel defined in (10).

THEOREM 7. *Let G be a compact simple simply connected Lie group.*

(i) *Let $p>n/m$. Then any central function $f\in L^p(G)$ satisfies $(D_N*f)(x)\to f(x)$ for almost every $x\in G$.*

(ii) *Let $p<n/m$. Then there exists a central function $f\in L^p(G)$ such that $(D_N*f)(x)$ diverges for almost every $x\in G$.*

PROOF. (i) has been proved in [12] for a larger class of polyhedral Dirichlet kernels.

Here we sketch the proof of (ii). The rank one case has already been implicitly proved in [7, p. 127], we therefore assume $l>1$. Our argument is simply a refinement of the one in [4].

It is enough to prove that $((D_{2N}-D_N)*f)(x)$ diverges a.e. By [3] we have a sequence $\{\Phi_N\}$ of central trigonometric polynomials $\Phi_N=\sum_\lambda a_\lambda^{(N)}d_\lambda\chi_\lambda$ satisfying

$$\begin{aligned} a_\lambda^{(N)}&=1 && \text{if } \lambda\in P_{2N}^+\setminus P_N^+\\ a_\lambda^{(N)}&=0 && \text{if } \lambda\notin P_{4N}^+\\ 0\le a_\lambda^{(N)}&\le 1 && \text{for any } \lambda\\ \|\Phi_N\|_1&\le\text{const}. \end{aligned}$$

The above assumptions on the a_λ's imply $\|\Phi_N\|_2\le\text{const}\,N^{n/2}$. Then, for any $1<p<2$

we have, by the Hölder inequality,

$$\|\Phi_N\|_p=\left\{\int_G |\Phi_N|^{2-p}|\Phi_N|^{2p-2}\right\}^{1/p}\leq\left\{\int_G |\Phi_N|\right\}^{(2-p)/p}\left\{\int_G |\Phi_N|^2\right\}^{(p-1)/p}\leq \mathrm{const}_p N^{n/p'}$$

where p and p' are conjugate exponents. An argument similar to the one we used in the proof of the previous theorem yields

$$\begin{aligned}
&N^{-n/p'}\Phi_N*(D_{2N}-D_N)(t)N^{-n/p'}\sum_{\lambda\in P^+_{2N}\setminus P^+_N} d_\lambda\chi_\lambda(t)\\
&=N^{-n/p'}\Delta^{-1}(t)\sum_{\lambda\in P^+_{2N}\setminus P^+_N} d_\lambda A(\lambda+\beta)(t)\\
&=N^{-n/p'}\Delta^{-1}(t)\prod_{\alpha\in\Phi^+}\mathscr{D}_\alpha\left(\sum_{\lambda\in P^+(2N\omega+\beta)\setminus P^+(N\omega+\beta)} S(\lambda)(t)\right)\\
&=N^{-n/p'}\Delta^{-1}(t)\prod_{\alpha\in\Phi^+}\mathscr{D}_\alpha\left(\Delta^{-1}(t)\sum_{\sigma\in W}\det(\sigma)(\xi_{\sigma(2N\omega)}(t)F_2(\sigma(t))-\xi_{\sigma(N\omega)}(t)F_1(\sigma(t)))\right)\\
&=\tilde{A}_N(t)+N^{-n/p'}\Delta^{-2}(t)\prod_{\alpha\in\Phi^+}\mathscr{D}_\alpha\left(\sum_{\sigma\in W}\det(\sigma)(\xi_{\sigma(2N\omega)}(t)F_2(\sigma(t))-\xi_{\sigma(N\omega)}(t)F_1(\sigma(t)))\right)\\
&=\tilde{A}_N(t)+N^{m-n/p'}\Delta^{-2}(t)\tilde{C}_N(t)\,,
\end{aligned}$$

where F_1 and F_2 are polynomials independent of N.

Now we observe that $\tilde{C}_N(t)$ behaves like $C_N(t)$ in the previous theorem. We also point out that $\tilde{A}_N(t)=o(N^{m-n/p'})$. The explicit construction of the function f in the statement of the theorem is now easy as long as we recall that $m>n/p'$. □

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