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HOLOMORPHIC EXTENSIONS OF MAPS FROM THE BOUNDARY OF KÄHLER MANIFOLDS

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Abstract. In this note, we show that there exists a unique holomorphic extension of a given map which satisfies the tangential Cauchy-Riemann equation on the hyperconvex boundary of a Kähler manifold into a complete Kähler manifold with strongly seminegative curvature, provided there is a plurisubharmonic function on the domain which has at least two positive eigenvalues at some point.

1. **Introduction.** Let Ω be a bounded domain in $\boldsymbol{C}^n$ with smooth connected boundary. The classical theorem on global holomorphic extensions of Bochner [Boc] asserts that if f is a smooth function on $\partial\Omega$ which satisfies the tangential Cauchy-Riemann equation $\bar{\partial}_b f=0$ on $\partial\Omega$, then f can be extended from $\partial\Omega$ to $\bar{\Omega}$ so that f is holomorphic in Ω. When M is a complex manifold, the Bochner type extension problem becomes much harder; even the local extension problem for a hypersurface is not necessarily solvable in general. Some sufficient conditions for local extendability have been found by Lewy [L] and other people (see the reference in [B]). Kohn and Rossi [KR] proved by using the technique of solving the $\bar{\partial}$-Neumann problems the existence of a holomorphic extension of functions, which satisfy the tangential Cauchy-Riemann equation on the boundary of a finite complex manifold with connected boundary whose Levi form has one positive eigenvalue everywhere.

The Bochner type global holomorphic extension problems for maps, namely, extending holomorphically a boundary map satisfying the tangential Cauchy-Riemann equation from the boundary of a domain in a Kähler manifold to another Kähler manifold, have been studied by several authors. In the early 70's, Shiffman [Sh] and Griffiths [Gr] applied the analytic disc technique to study the extension problem and solved a problem posed by Chern in 1970. In the 1980's, Nishikawa and Shiga [NS], [Shi], Siu [S1], [S2] and Wood [W] took a harmonic map approach by using Siu's $\partial\bar{\partial}$-Bochner formula. In fact, the complex-analyticity of certain harmonic maps was studied earlier by Siu and Yau in their resolution of Frankel's conjecture.

Let D be a domain in a Kähler manifold M with smooth compact boundary ∂D. We say that D is *hyperconvex* if D has a smooth defining function with nonnegative trace of the Levi-form on ∂D. The main purpose of this paper is to study the global holomorphic extension problems for maps from the boundary ∂D to another complete

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Kähler manifold N. We shall generalize Bochner's holomorphic extension theorem, for a boundary map $f: \partial D \to N$ satisfying the tangential Cauchy-Riemann equation on ∂D, to Kähler manifolds, provided some geometric conditions on the manifold M and the target manifold N are satisfied. According to Siu [S1], we say that N has strongly seminegative curvature if R denotes the Riemannian curvature tensor, then

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}}(A^\alpha \overline{B^\beta} - C^\alpha \overline{D^\beta})(\overline{A^\delta} B^\gamma - \overline{C^\delta} D^\gamma)$$

is nonnegative for arbitrary complex numbers A^α, B^α, C^α and D^α when $A^\alpha \overline{B^\beta} - C^\alpha \overline{D^\beta} \neq 0$ for at least one pair of indices (α, β). Clearly the strong seminegativity of the curvature tensor implies seminegativity of the sectional curvature. The main theorem of this paper is the following.

THEOREM 1.1. *Let M $(\dim_{\mathbf{C}} M = m > 1)$ be a Kähler manifold and D a hyperconvex domain in M with compact closure and non-empty smooth boundary ∂D. Suppose there is a plurisubharmonic function $\eta \in C^2(\bar{D})$ so that $H(\eta)$ has at least two positive eigenvalues at some point in D. Suppose N is a complete Kähler manifold with strongly seminegative curvature. Let $f: \bar{D} \to N$ be a smooth map satisfying the tangential Cauchy-Riemann equation $\bar{\partial}_b f = 0$ on ∂D. Then there exists a unique holomorphic extension of f.*

In the classical Bochner extension theorem in $\mathbf{C}^m$, ∂D is assumed to be connected. Clearly, some type of convexity on the boundary is needed when domains are general complex manifolds. For instance, take $D = \mathbf{CP}^n \setminus B$ where B is a small geodesic ball in $\mathbf{CP}^n$ biholomorphic to the unit ball in $\mathbf{C}^n$. Then for any non-constant map f from ∂D into N satisfying the tangential Cauchy-Riemann equation, its harmonic extension cannot be holomorphic nor pluriharmonic, since otherwise it must be constant. However, it seems than if we assume there is a pluriharmonic exhaustion function on M (for instance, M is a Stein manifold), and N as before, we may expect to have a holomorphic extension without any convexity conditions on ∂D. In our Theorem 1.1, we assume that ∂D is *hyperconvex*. We would like to point out that hyperconvexity only leads to pluriharmonicity of the harmonic map extension. In order to conclude that the harmonic map extension is holomorphic, one needs either stronger conditions on the curvature of the image spaces and an assumption on the rank of df of the boundary map f or stronger convexity on the boundary of domains (cf. [NS], [S2], [Shi]). Our observation is that the existence of certain plurisubharmonic functions allows us to construct a holomorphic extension under the weaker assumptions as in Theorem 1.1. In particular, we have:

COROLLARY 1.2. *Let M $(\dim_{\mathbf{C}} M = m > 1)$ be a Kähler manifold and D a hyperconvex domain in M with compact closure and non-empty smooth boundary ∂D. Let N be a complete Kähler manifold with strongly seminegative curvature. Suppose there are two holomorphic functions ϕ_1, $\phi_2 \in C^\infty(\bar{D})$ such that $\partial\phi_1$, $\partial\phi_2$ are linearly independent at* *some point in D. If $f:\bar{D}\to N$ is a smooth map satisfying the tangential Cauchy-Riemann equation $\bar{\partial}_b f=0$ on ∂D, then there exists a unique holomorphic extension of f.*

The paper is organized as follows. In Section 2, we introduce basic notation and definitions and prove preliminary results on a smooth map satisfying $\bar{\partial}_b f=0$; and we derive some technical lemmas for use in later section. Finally, in Section 3, we prove Theorem 1.1 and some corollaries.

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2. Some lemmas for $\bar{\partial}$ and $\bar{\partial}_b$ operators. Throughout this paper, we always assume N to be a Kähler manifold of complex dimension n with a Kähler metric tensor

$$g=\sum_{\alpha,\beta} g_{\alpha\bar{\beta}}dw^\alpha d\bar{w}^\beta\,; \tag{2.1}$$

and let M be a Kähler manifold of complex dimension $m>1$ with a Kähler metric tensor

$$h=\sum_{i,j} h_{i\bar{j}}dz^i d\bar{z}^j\,. \tag{2.2}$$

Let T_N denote the real tangent bundle of N when N is regarded as a real manifold. The complex structure of N gives a decomposition of $T_N\otimes \boldsymbol{C}$ into the tangent vectors of type (1, 0) and type (0, 1), i.e.,

$$T_N\otimes \boldsymbol{C}=T_N^{1,0}\oplus T_N^{0,1}\,.$$

If M is a complex manifold and $f:M\to N$ is a smooth map, then $df:T_M\to T_N$ gives rise to a map

$$df\otimes \boldsymbol{C}\colon T_M\otimes \boldsymbol{C}\to T_N\otimes \boldsymbol{C}\,.$$

Composing this with the projection map

$$\Pi_{1,0}\colon T_M\otimes \boldsymbol{C}\to T_M^{1,0}\,,$$

we have

$$\Pi_{1,0}\circ(df\otimes \boldsymbol{C})\colon T_M\otimes \boldsymbol{C}\to T_N^{1,0}\,;$$

and this is equivalent to a bundle map

$$T_M\otimes \boldsymbol{C}\to f^*T_N^{1,0}$$

of $\boldsymbol{C}$-vector bundles over M. Thus the $\bar{\partial}$-operator can be defined as follows: $\bar{\partial}f$ is the bundle map $T_M^{0,1}\to f^*T_N^{1,0}$ if we compose the above with the inclusion map: $T_M^{0,1}\to f^*T_N^{1,0}$. Therefore, $\bar{\partial}f$ is a smooth section of $\boldsymbol{C}$-vector bundle

$$\mathrm{Hom}_{\boldsymbol{C}}(T_M^{0,1}, f^*T_N^{1,0}) = (T_M^{1,0})^* \otimes f^*T_N^{1,0}$$

over M. In other words $\bar{\partial}f$ is an $f^*T_N^{1,0}$-valued (0, 1)-form on M. Let $\{z^1, \ldots, z^m\}$ be a local holomorphic coordinate chart on M and (w^α) a holomorphic coordinate chart on N. Then $\bar{\partial}f$ can be simply expressed as $\partial_{\bar{i}} f^\alpha = \bar{\partial}_i f^\alpha$, where $\partial_{\bar{i}} = \bar{\partial}_i = \partial/\partial\bar{z}^i$. We also set $\partial_i = \partial/\partial z^i$, $\partial_\alpha = \partial/\partial w^\alpha$, and $\partial_{\bar{\alpha}} = \partial/\partial\bar{w}^\alpha$. Similarly, we may define ∂f, $\partial\bar{f}$, and $\overline{\partial f}$, where ∂f is an $f^*T^{1,0}$-valued $(1,0)$-form on M represented by $(\partial_i f^\alpha)$. $\partial\bar{f}$ is an $f^*T^{0,1}$-valued $(1,0)$-form on M represented by $(\partial_i \bar{f}^\alpha)$. Finally $\overline{\partial f}$ is an $f^*T^{0,1}$-valued (0, 1)-form on M represented by $(\partial_{\bar{i}} \bar{f}^\alpha)$. It is clear that $\partial\bar{f}$ is the complex conjugate of $\bar{\partial}f$ and $\overline{\partial f}$ is the complex conjugate of ∂f.

In order to define the $\bar{\partial}_b$-operator, we recall that one can introduce the tangential Cauchy-Riemann operator extrinsically or intrinsically. The intrinsic approach leads to the definition of abstract CR-manifolds and the problem of embedding abstract CR-manifolds into $\boldsymbol{C}^n$. This direction has been studied extensively by many people. To study the Bochner type global extension problem, it is natural to require that the almost complex structure used to define the tangential Cauchy-Riemann operator at the boundary ∂M is consistent with the complex structure of M. This extrinsic approach and the first formal definition of $\bar{\partial}_b$ on the boundary of a finite complex manifold is due to Kohn and Rossi in [KR]. Let us recall a definition [KR]. A *finite complex manifold* is a pair $\{M, M'\}$ where M' is a complex manifold and M is an open submanifold of M' such that (a) the closure $\bar{M}$ of M is compact; (b) the boundary ∂M is a C^∞-submanifold of M' and (c) for any point $P \in \partial M$, there exists a coordinate neighborhood U of P with coordinates $t^2, \ldots, t^{2m}$, ρ such that $\rho(x) < 0$ if $x \in U \cap M$, and $\rho(x) > 0$ if $x \in U \cap (M' \setminus \bar{M})$. Our treatment on a domain D in M with compact closure and smoothly connected boundary is equivalent to (D, M) being a finite manifold.

Let D be a domain in M with non-empty smooth boundary ∂D. If $P \in \partial D$, we let $H_P(\partial D)$ denote the complex subspace of $T_P(M) \otimes \boldsymbol{C}$ consisting of those vectors which are tangent to ∂D. Thus $H_P(\partial D) \cap T_P^{0,1}$ is a complex $(m-1)$-dimensional subspace of $H_P(\partial D)$. If f is a function on ∂D, we say that f satisfies the tangential Cauchy-Riemann equation $\bar{\partial}_b f = 0$ on ∂D if $\xi(f) = 0$ for every $\xi \in H_P(\partial D) \cap T_M^{0,1}$. If $f : \partial D \to N$ is a map, we say that f satisfies the tangential Cauchy-Riemann equation $\bar{\partial}_b f = 0$ on ∂D if for any $z \in \partial D$, there is an open neighborhood $U_z \subset M$ and a local holomorphic coordinate chart (V, ϕ) on N around $f(z)$ such that

$$\bar{\partial}_b(\phi^\alpha \circ f) = 0 \qquad \text{on} \quad U_z \cap \partial D\,, \quad \alpha = 1, \ldots, n\,.$$

It is easy to show that the above definition is independent of the choice of the coordinate charts (V, ϕ). For completeness, we give a proof of this fact.

PROPOSITION 2.1. *Suppose $P \in \partial D$ is an arbitrary point. If (V, ϕ) and (V', ψ) are any two complex coordinate charts for N around $f(P)$, then there is a neighborhood U_p of P in M such that*

$$\bar{\partial}_b(\phi^\alpha \circ f)=0 \quad on \quad U_p \cap \partial D \Longleftrightarrow \bar{\partial}_b(\psi^\beta \circ f)=0 \quad on \quad U_p \cap \partial D\,.$$

PROOF. Since (V, ϕ) and (V', ψ) are two complex charts around $f(p)$, there is an $n \times n$ matrix $A=(A_{\alpha\beta})$ with $A_{\alpha\beta}$ are holomorphic functions on $\phi(V \cap V')$ so that $\psi = A\phi$. Thus, let U_P be a neighborhood of P in M such that $f(U_P \cap \bar{D}) \subset V \cap V'$. For any $x \in U_p \cap \partial D$ and $\xi \in H_x(\partial D) \cap T_x^{0,1}(M)$, we have $\xi(\phi^\beta \circ f)(x)=0$ by assumption. Now we have

$$\begin{aligned}\xi(\psi^\alpha \circ f)(x) &= \xi(A_{\alpha\beta}\phi^\beta \circ f)(x) = \xi(A_{\alpha\beta}(\phi \circ f)\phi^\beta \circ f)(x)\\ &= \xi(A_{\alpha\beta}(\phi \circ f))(x)\phi^\beta(f(x)) + A_{\alpha\beta}\xi(\phi^\beta \circ f)(x)\\ &= \langle \xi, d(A_{\alpha\beta}(\phi \circ f))\rangle_x \phi^\beta(f(x)) = \langle \xi, dA_{\alpha\beta} \circ d(\phi \circ f)\rangle_x \phi^\beta(f(x)) = 0\end{aligned}$$

since $A_{\alpha\beta}$ is holomorphic. ■

Let h (in (2.2)) be a Kähler metric for M and ω the Kähler form

$$\omega = \sqrt{-1}\sum_{i,j} h_{i\bar{j}}\, dz^i \wedge d\bar{z}^j\,. \tag{2.3}$$

Let D be a *hyperconvex* domain in M with smooth boundary. Then there is a defining function $\rho \in C^\infty(\bar{D})$ so that for each $P \in \partial D$, we may choose local holomorphic coordinates $\{z^1, \dots, z^m\}$ around P such that $z^2, \dots, z^m$ are complex tangential to ∂D at P, with $|\partial\rho| = 1$ at P, and the trace of the Levi form on $H_P(\partial D)$ with respect to the Kähler metric $h_{i\bar{j}}$ on M at P is

$$\sum_{i,j=2}^{m} h^{i\bar{j}}(P)\frac{\partial^2\rho}{\partial z^i \partial \bar{z}^j}(P) \geq 0\,.$$

The following lemma is necessary in proving the pluriharmonicity of harmonic extensions.

LEMMA 2.2. *Let M be a Kähler manifold and D a hyperconvex domain in M with compact closure and non-empty boundary ∂D. Let N be a complete Kähler manifold. Let $f: \bar{D} \to N^n$ be a smooth map such that $\bar{\partial}_b f = 0$ on ∂D. Then*

$$\int_D \partial\bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{m-2} \leq 0\,, \tag{2.4}$$

where g is a Kähler metric on N.

PROOF. Since M is Kähler, $d\omega = 0$ and $d = \partial + \bar{\partial}$. Also, we have $\bar{\partial}^2 = 0$ hence $\partial\bar{\partial} = d\bar{\partial}$. Thus

$$\int_D \partial\bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle_N \wedge \omega^{m-2} = \int_D d\bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{m-2} = \int_D d(\bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{m-2})$$

$$= \int_{\partial D} \bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{m-2}$$

by the Stokes theorem. We will prove that the last integral vanishes by using the assumption $\bar{\partial}_b f = 0$ on ∂D. Indeed, given any $P \in \partial D$, we may choose local coordinates without loss of generality such that $z^2, \ldots, z^m$ are the complex tangential directions near P, and z^1 is the complex normal direction to ∂D near P. Moreover, we let $\rho = \rho(z^1, \ldots, z^m, \bar{z}^1, \ldots, \bar{z}^m)$ be a real-valued smooth function defined in a neighborhood U of P such that $D \cap U = \{x \in U: \rho(x) < 0\}$ are $d\rho(x) \neq 0$ on $\partial D \cap U$; and $(\partial\rho/\partial z^j)(0) = (\partial\rho/\partial \bar{z}^j)(0) = 0$ for all $j = 2, \ldots, m$ and $(\partial\rho/\partial z^1)(0) = 1$. To handle the boundary term, we need to use the tangential Cauchy-Riemann equation (see [NS]). Since $\bar{\partial}_b f = 0$ on ∂D, we may rewrite $\bar{\partial}_b f = 0$ in $U \cap \partial D$ in terms of $\rho = \rho(x)$ as

$$\bar{\partial}_b f^\alpha = \sum_{i=2}^{m} (\partial_{\bar{i}} f^\alpha - \partial_{\bar{1}} f^\alpha (\partial_{\bar{i}}\rho / \partial_{\bar{1}}\rho)) d\bar{z}^i, \qquad 1 \leq \alpha \leq n.$$

Thus

$$\partial_{\bar{i}} f^\alpha = \partial_{\bar{1}} f^\alpha (\partial_{\bar{i}}\rho / \partial_{\bar{1}}\rho), \qquad 1 \leq \alpha \leq n.$$

Without loss of generality, we may assume that $g^{\alpha\beta}(f(P)) = \delta_{\alpha\beta}$ and $\partial g^{\alpha\beta}(f(P)) = \bar{\partial} g^{\alpha\beta}(f(P)) = 0$. Using such facts and direct computation, at P, we have

$$\begin{aligned} &\bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{m-2} \\ &= -(m-2)!(\sqrt{-1})^{m-2} \sum_{1 \leq \alpha \leq n, 2 \leq i \leq m} \partial_{\bar{1}} f^\alpha \partial_{\bar{i}} \partial_i \rho \, d\bar{z}^n \wedge d\bar{z}^i \wedge dz^i \wedge_{2 \leq l \leq m, l \neq i} (dz^l \wedge d\bar{z}^l) \\ &= -\left(\sum_{i=2}^{m} \frac{\partial^2 \rho}{\partial z^i \partial \bar{z}^i}\right) \sum_\alpha \left|\frac{\partial f^\alpha}{\partial \bar{z}^n}\right|^2 W, \end{aligned}$$

where W is a positive $(2m-1)$-form on ∂D. Since the trace of the Levi form on the boundary ∂D are nonnegative, we see that $\sum_{i=2}^m (\partial^2\rho/\partial\bar{z}^i \partial z^i)$ coincides with the trace of the Levi form of ρ at $P \in \partial D$ which is nonnegative. Since P is an arbitrary point of ∂D and the sign of trace of the Levi form on ∂D is invariant under biholomorphic changes of variables, we have

$$\int_D \partial\bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle_N \wedge \omega^{m-2} = \int_{\partial D} \bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{m-2} \leq 0.$$

■

Recall that a map $f: M \to N$ is *harmonic* if

$$\Delta_M f^\alpha + \sum_{\beta,\gamma,i,j} \Gamma^\alpha_{\beta\gamma} \partial_i f^\beta \bar{\partial}_j f^\gamma h^{i\bar{j}} = 0, \qquad \alpha = 1, \ldots, n,$$

where $\Delta_M = 2h^{i\bar{j}} \partial_i \partial_{\bar{j}}$ and $(h^{i\bar{j}})$ is the inverse matrix of the matrix $(h_{i\bar{j}})$ of the Kähler metric of M, and $\Gamma^{\alpha}_{\beta\gamma}$ is the Christoffel symbol of N. A C^1-map $f: M \to N$ is *holomorphic* if $\bar{\partial} f = 0$ on M. We say that a smooth map $f: M \to N$ is *pluriharmonic* if $D\bar{\partial} f = 0$ on M where $D\bar{\partial} f$ is the ∂ exterior derivative of the $f^*T^{1,0}$-valued (0, 1)-form $\bar{\partial} f$ on M. In local coordinates, we may write

$$D\bar{\partial} f^{\alpha} = \partial\bar{\partial} f^{\alpha} + \sum_{\beta,\gamma} \Gamma^{\alpha}_{\beta\gamma} \partial f^{\beta} \bar{\partial} f^{\gamma}, \qquad \alpha = 1, \ldots, n.$$

The following are well known: (a) any pluriharmonic map is a harmonic map; (b) any holomorphic map is a pluriharmonic map when both M and N are Kähler.

Next we prove a pointwise identity if $f: M \to N$ is a harmonic map satisfying the tangential Cauchy-Riemann equation, and N has strongly seminegative curvature. The lemma below is known (cf. [S1], [NS], [Shi]), although stated and proved in slightly different forms. We will sketch a proof for the readers' convenience.

LEMMA 2.3. *Let M be a Kähler manifold and D a hyperconvex domain in M with compact closure and non-empty smooth boundary. Let N be a complete Kähler manifold with strongly seminegative curvature. Let $f: D \to N$ be a harmonic map such that $f \in C^2(\bar{D}, N)$ and $\bar{\partial}_b f = 0$ on ∂D. Then*

$$\partial\bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{m-2} = 0 \qquad \text{on} \quad D \tag{2.5}$$

and

$$\bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{m-2} = \partial\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{m-2} = 0 \qquad \text{on} \quad \partial D. \tag{2.6}$$

In particular, f is pluriharmonic.

PROOF. Since N has nonpositive sectional curvature, the regularity theory for harmonic maps asserts that $f \in C^{\infty}(D) \cap C^2(\bar{D})$. Next, let us recall the $\partial\bar{\partial}$-Bochner formula for harmonic map of Siu [S1]:

$$\begin{aligned}\partial\bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{m-2} &= \langle R^N, \bar{\partial} f \wedge \partial \bar{f} \wedge \partial f \wedge \overline{\partial f}\rangle \wedge \omega^{m-2} - \langle g, D\bar{\partial} f \wedge \bar{D}\partial \bar{f}\rangle \wedge \omega^{m-2} \\ &= \sigma\omega^m - \mathscr{X}\omega^m\end{aligned}$$

where σ and $-\mathscr{X}$ are some nonnegative functions as shown in [S1]. Now we integrate the above identity over D and apply Lemma 2.2. We have

$$0 \geq \int_D \partial\bar{\partial}\langle g, \bar{\partial} f \wedge \partial \bar{f}\rangle \wedge \omega^{m-2} = \int_D (\sigma - \mathscr{X})\omega^m \geq 0.$$

Therefore

$$\int_D (\sigma - \mathscr{X})\omega^m = 0.$$

This immediately implies that

$$\sigma=0 \quad \text{and} \quad \mathscr{X}=0\,. \tag{2.7}$$

Therefore, (2.5) holds. Moreover, by Lemma 2.2, we have

$$0=\int_{\partial D}\bar{\partial}\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}$$

and

$$\bar{\partial}\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}$$

is a nonpositive $(2m-1)$ differential form on ∂D. Therefore, we have

$$\bar{\partial}\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}=0$$

on ∂M. By taking complex conjugate, we have

$$\partial\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}=0\,.$$

Therefore (2.6) follows. In particular, $\mathscr{X}=0$ implies that $D\bar{\partial}f^{\alpha}=0$ for all $\alpha=1,\ldots,n$. This means that f is pluriharmonic. ■

3. Proof of Theorem 1.1. Before we start Theorem 1.1, we need the following lemma.

LEMMA 3.1. *Let M ($\dim_{C}M=m>1$) be a Kähler manifold and D a hyperconvex domain in M with compact closure and non-empty smooth boundary ∂D. Let N be a complete Kähler manifold with strongly seminegative curvature. Let $f: D\to N$ be a harmonic map such that $f\in C^{2}(\bar{D},N)$ and $\bar{\partial}_{b}f=0$ on ∂D. Then*

$$\int_{D}\bar{\partial}\partial\eta\wedge\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}=0 \tag{3.1}$$

for all $\eta\in C^{2}(\bar{D})$.

PROOF. By (2.5) and integration by parts, we have

$$\begin{aligned}0&=\int_{D}\eta\partial\bar{\partial}\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}=\int_{D}\eta d\bar{\partial}\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}\\&=\int_{\partial D}\eta\bar{\partial}\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}-\int_{D}\partial\eta\wedge\bar{\partial}\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}\,.\end{aligned}$$

Now applying integration by parts again, we have

$$\begin{aligned}\int_{D}\partial\eta\wedge\bar{\partial}\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}&=\int_{\partial D}\partial\eta\wedge\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}\\&-\int_{D}\bar{\partial}\partial\eta\wedge\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\omega^{m-2}+\int_{D}\partial\eta\wedge\langle g,\bar{\partial}f\wedge\partial\bar{f}\rangle\wedge\bar{\partial}\omega^{m-2}\,.\end{aligned}$$

Since M is Kähler, we have $\bar{\partial}\omega=0$ and $\partial\omega=0$. Therefore by combining the above calculations, we have

$$\int_D \bar{\partial}\partial\eta \wedge \langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge \omega^{m-2}$$
$$= -\int_{\partial D} \eta\bar{\partial}\langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge \omega^{m-2} + \int_{\partial D} \partial\eta \wedge \langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge \omega^{m-2}.$$

Therefore, the lemma follows if we prove

$$\int_{\partial D} \eta\bar{\partial}\langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge w^{m-2} = \int_{\partial D} \partial\eta\langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge w^{m-2} = 0. \tag{3.2}$$

By Lemma 2.3, we have

$$\int_{\partial D} \eta\bar{\partial}\langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge w^{m-2} = 0.$$

Now we prove

$$\int_{\partial D} \partial\eta \wedge \langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge w^{m-2} = 0. \tag{3.3}$$

Since $\bar{\partial}_b f=0$ on ∂D, we compute the integrand $\partial\eta \wedge \langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge w^{m-2}$ as follows. Let $P \in \partial D$ be any point. Take local complex coordinates $z^1, \ldots, z^m$ in M around P such that $\partial/\partial z^2, \ldots, \partial/\partial z^m$ are the complex tangential vector fields to ∂D at P. Since $\bar{\partial}_b f=0$, we have at P

$$\langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge \omega^{m-2} = \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\bar{\partial}_i f^\alpha \partial_j \bar{f}^\beta d\bar{z}^i \wedge dz^j \wedge \omega^{m-2}$$
$$= \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\bar{\partial}_1 f^\alpha \partial_1 \bar{f}^\beta d\bar{z}^1 \wedge dz^1 \wedge \omega^{m-2}.$$

We observe that only the tangential part of $\partial\eta$ contributes to the above form. Therefore

$$\partial\eta \wedge \langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge \omega^{m-2} = \sum_{k=2} \partial_k\eta \wedge \langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge \omega^{m-2}.$$

Since ∂D is a compact manifold, we let ϕ_j, $j=1, \ldots, K$, be a partition of unity on ∂D such that ϕ_j has support in U_j and each U_j is contained in a single complex coordinate chart. On each U_j, we can carry out integration by parts and obtain

$$\int_{U_j} \phi_j\partial\eta \wedge \langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge w^{m-2}$$
$$= \int_{U_j} \phi_j\partial(\eta \wedge \langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge w^{m-2}) - \int_{U_j} \phi_j\eta\partial(\langle g, \bar{\partial}f \wedge \partial\bar{f}\rangle \wedge w^{m-2})$$

$$= \int_{U_j} \phi_j \partial(\eta \wedge \langle g, \bar{\partial} f \wedge \partial \bar{f} \rangle \wedge w^{m-2}) = -\int_{U_j} \eta (\partial \phi_j) \wedge \langle g, \bar{\partial} f \wedge \partial \bar{f} \rangle \wedge w^{m-2},$$

where we have used (2.6) in Lemma 2.3 were ω is the Kähler form in the second equality. Thus

$$\int_{\partial D} \partial\eta \wedge \langle g, \bar{\partial} f \wedge \partial \bar{f} \rangle \wedge w^{m-2} = \sum_{j=1}^{K} \int_{U_j} \phi_j \partial \eta \wedge \langle g, \bar{\partial} f \wedge \partial \bar{f} \rangle \wedge w^{m-2}$$
$$= \sum_{j=1}^{K} \int_{U_j} \eta (\partial \phi_j) \wedge \langle g, \bar{\partial} f \wedge \partial \bar{f} \rangle \wedge w^{m-2} = \int_D \eta \sum_{j=1}^{K} (\partial \phi_j) \wedge \langle g, \bar{\partial} f \wedge \partial \bar{f} \rangle \wedge w^{m-2}$$
$$= \int_D \eta \partial \left(\sum_{j=1}^{K} \phi_j \right) \langle g, \bar{\partial} f \wedge \partial \bar{f} \rangle \wedge w^{m-2} = 0,$$

and (3.3) follows. Therefore, we have completed the proof of (3.2) and (3.3). Hence (3.1) holds. ■

Let η be a real-valued function on D. We let $H(\eta)$ denote the complex Hessian of η. We say that η is plurisubharmonic on D if and only if $H(\eta)$ is positive semi-definite. Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Since the sectional curvature of N is nonpositive, the existence theorem of Hamilton [H] and Schoen [Sc] asserts that there is a unique harmonic map u smooth up to boundary which solves the Dirichlet problem $u = f$ on ∂D. For convenience, we still use f to denote the harmonic extension u on $\bar{D}$. By our assumption, there is a real-valued function $\eta \in C^2(\bar{D})$ such that its complex Hessian $H(\eta)$ is positive semidefinite on D and $H(\eta)$ has at least two positive eigenvalues at some point $P_0 \in D$. We shall prove that $\bar{\partial}\partial\eta \wedge \langle g, \bar{\partial} f \wedge \partial \bar{f} \rangle \wedge \omega^{m-2}$ is a seminegative (m, m)-form. Since holomorphic changes of coordinates preserve the orientation, the sign of any (m, m)-form is unchanged under different holomorphic coordinates. For any $P \in \partial D$, without loss of generality, we choose a normal coordinate at P, i.e., at P the Kähler form has the expression

$$\omega = \sqrt{-1} \sum_k dz^k \wedge d\bar{z}^k.$$

In particular, for any $i \neq j$, we have

$$dz^i \wedge d\bar{z}^i \wedge dz^j \wedge d\bar{z}^j \wedge \omega^{m-2} = (\sqrt{-1})^{m-2} dz^i \wedge d\bar{z}^i \wedge dz^j \wedge d\bar{z}^j \wedge \left(\sum_k dz^k \wedge d\bar{z}^k \right)^{m-2}$$
$$= (\sqrt{-1})^{m-2} (m-2)! dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m$$
$$= \frac{(m-2)!}{m!} (\sqrt{-1})^{m-2} \left(\sum_k dz^k \wedge d\bar{z}^k \right)^m = -\frac{(m-2)!}{m!} \omega^m.$$

Thus, at the point P, we have

$$\sum_{\alpha,\beta} g_{\alpha\bar\beta}\bar\partial\partial\eta\wedge\bar\partial f^{\alpha}\wedge\partial\bar f^{\beta}\wedge\omega^{m-2}=\sum_{\alpha\beta}\sum_{i,j,s,t} g_{\alpha\bar\beta}\partial^2_{i\bar j}\eta\bar\partial_s f^{\alpha}\partial_t\bar f^{\beta}d\bar z^j\wedge dz^i\wedge d\bar z^s\wedge dz^t\wedge\omega^{m-2}.$$

Since ω^{m-2} is an $(m-2, m-2)$-form, either $s=i$, $t=j$ or $i=j$, $s=t$. Hence

$$\begin{aligned}
&\sum_{\alpha,\beta}\sum_{i,j,s,t} g_{\alpha\bar\beta}\partial^2_{i\bar j}\eta\bar\partial_s f^{\alpha}\partial_t\bar f^{\beta}d\bar z^j\wedge dz^i\wedge d\bar z^s\wedge dz^t\wedge\omega^{m-2}\\
&=\sum_{\alpha,\beta,i\neq j} g_{\alpha\bar\beta}\frac{\partial^2\eta}{\partial z^i\partial\bar z^j}\bar\partial_i f^{\alpha}\partial_j\bar f^{\beta}d\bar z^j\wedge dz^i\wedge d\bar z^i\wedge dz^j\wedge\omega^{m-2}\\
&\quad+\sum_{\alpha,\beta,i\neq j} g_{\alpha\bar\beta}\frac{\partial^2\eta}{\partial z^i\partial\bar z^i}\bar\partial_j f^{\alpha}\partial_j\bar f^{\beta}d\bar z^i\wedge dz^i\wedge d\bar z^j\wedge dz^j\wedge\omega^{m-2}\\
&=\sum_{\alpha,\beta,i,j} g_{\alpha\bar\beta}\left(-\frac{\partial^2\eta}{\partial z^i\partial\bar z^j}\bar\partial_i f^{\alpha}\partial_j\bar f^{\beta}+\frac{\partial^2\eta}{\partial z^i\partial\bar z^i}\bar\partial_j f^{\alpha}\partial_j\bar f^{\beta}\right)dz^i\wedge d\bar z^i\wedge dz^j\wedge d\bar z^j\wedge\omega^{m-2}\\
&=-\frac{(m-2)!}{m!}\sum_{\alpha,\beta,i,j} g_{\alpha\bar\beta}\left(-\frac{\partial^2\eta}{\partial z^i\partial\bar z^j}\bar\partial_i f^{\alpha}\partial_j\bar f^{\beta}+\frac{\partial^2\eta}{\partial z^i\partial\bar z^i}\bar\partial_j f^{\alpha}\partial_j\bar f^{\beta}\right)\omega^m.
\end{aligned}$$

Therefore at P, we have

$$\begin{aligned}
&\bar\partial\partial\eta\wedge\langle g,\bar\partial f\wedge\partial\bar f\rangle\wedge\omega^{m-2}\\
&\quad=-\frac{(m-2)!}{m!}\sum_{\alpha,\beta,i,j} g_{\alpha\bar\beta}\left[(\bar\partial f^{\beta})^*\left(\mathrm{tr}\left(\frac{\partial^2\eta}{\partial z^i\partial\bar z^j}\right)I_m-\left(\frac{\partial^2\eta}{\partial z^i\partial\bar z^j}\right)\right)\bar\partial f^{\alpha}\right]\omega^m
\end{aligned}$$

on D where I_m is the $m\times m$ identity matrix and $\bar\partial f^{\alpha}$ is viewed as a column vector with components $\bar\partial_j f^{\alpha}$, and $(\bar\partial f^{\alpha})^*=(\partial\bar f^{\alpha})^t$ as a row vector which is the adjoint of $\bar\partial f$. Since the matrix

$$\mathrm{tr}\left(\frac{\partial^2\eta}{\partial z^i\partial\bar z^j}\right)I_m-\left(\frac{\partial^2\eta}{\partial z^i\partial\bar z^j}\right)$$

is positive semidefinite at any point $P\in D$, we may write it as T^*T where T is an $m\times m$ matrix for each $P\in D$. Thus

$$(\bar\partial f^{\beta})^*\left(\mathrm{tr}\left(\frac{\partial^2\eta}{\partial z^i\partial\bar z^j}\right)I_m-\left(\frac{\partial^2\eta}{\partial z^i\partial\bar z^j}\right)\right)\bar\partial f^{\alpha}=(T\bar\partial f^{\beta})^*(T\bar\partial f^{\alpha}).$$

Since N is Kähler, $(g_{\alpha\bar\beta})$ is positive definite. Let $\lambda(P)$ be the smallest eigenvalue of $(g_{\alpha\bar\beta})$ at $f(P)$. Then the matrix $(g_{\alpha\bar\beta})-\lambda(P)I_n$ is positive semidefinite. Therefore, we have

$$\begin{aligned}
-\bar\partial\partial\eta\wedge\langle g,\bar\partial f\wedge\partial\bar f\rangle\wedge\omega^{m-2}&=\frac{(m-2)!}{m!}g_{\alpha\bar\beta}(T\bar\partial f^{\beta})^*(T\bar\partial f^{\alpha})\omega^m\\
&\geq\frac{\lambda(p)(m-2)!}{m!}\sum_{\alpha}(T\bar\partial f^{\alpha})^*(T\bar\partial f^{\alpha})\omega^m\geq 0
\end{aligned}$$

on D. Thus, combining this and Lemma 3.1, we conclude

$$\bar{\partial}\partial\eta \wedge \langle g, \bar{\partial}f \wedge \partial\bar{f} \rangle \wedge \omega^{m-2} = 0 .$$

Hence, at any point $P \in D$, we have

$$T\bar{\partial}f^\alpha = 0, \qquad \alpha = 1, \ldots, n .$$

Since at $P_0 \in D$, the complex Hessian of η has at least two positive eigenvalues, by continuity, the complex Hessian of η has at least two positive eigenvalues in a small neighborhood U_{P_0} of P_0. Thus, in U_{P_0}, the matrix

$$\operatorname{tr}\left(\frac{\partial^2\eta}{\partial z^i \partial \bar{z}^j}\right) I_m - \left(\frac{\partial^2\eta}{\partial z^i \partial \bar{z}^j}\right)$$

is positive definite, and so T is non-singular. Therefore, $T\bar{\partial}f^\alpha = 0$ implies $\bar{\partial}f^\alpha = 0$ for all $\alpha = 1, \ldots, n$, i.e., we have proved

$$\bar{\partial}_j f^\alpha \equiv 0 \qquad \text{for all} \quad \alpha \text{ and } j$$

in the small neighborhood U_{P_0} of P_0. Therefore f is holomorphic in a small neighborhood of P_0. Since f is a harmonic map from D to N, the analytic continuation theorem of Siu (see [S1, Proposition 4]) implies that f is holomorphic on the whole D. ■

COROLLARY 3.2. *Let Ω be a bounded smooth hyperconvex domain in C^m with $m > 1$. Let N be a complete Kähler manifold with strongly seminegative curvature. Let $f: \Omega \to N$ be a smooth map satisfying $\bar{\partial}_b f = 0$ on $\partial\Omega$. Then there is a unique holomorphic extension F of $f: \partial\Omega \to N$ such that $F: \bar{\Omega} \to N$ is holomorphic in Ω.*

PROOF. Since Ω is a bounded hyperconvex domain in C^m with smooth boundary, the function $\eta(z) = |z|^2$ is smooth on $\bar{\Omega}$. Note that the complex Hessian of η is the identity matrix, and hence is positive definite everywhere on Ω. Then, as a direct consequence of Theorem 1.1, we have obtain Corollary 3.2. ■

By Hartogs' theorem and Corollary 3.2, we have:

COROLLARY 3.3. *Let Ω be a smoothly bounded hyperconvex domain in C^m with $m > 1$. Then $\partial\Omega$ is connected.*

PROOF. Suppose $\partial\Omega$ is not connected. Since Ω is connected, there is a connected component of the boundary $\partial\Omega$ which bounds a domain Ω_1 in C^m so that $\Omega_2 = \Omega \cup \bar{\Omega}_1$ is a bounded domain in C^n. It is clear that $\partial\Omega = \partial\Omega_2 \cup \partial\Omega_1$. Obviously both Ω_1 and Ω_2 are hyperconvex by assumption and $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$. Now we consider a function on $\partial\Omega$ such that $f = 0$ on $\partial\Omega_1$ and $f = 1$ on $\partial\Omega_2$. f trivially satisfies the tangential Cauchy-Riemann equation on the boundary $\partial\Omega$. By Corollary 3.2, we have a holomorphic function f on Ω which has nontangential limit 1 on $\partial\Omega_2$ and 0 on $\partial\Omega_1$. Hartogs' theorem asserts that f can be extended to a holomorphic function on Ω_2, which equals 1 on $\partial\Omega_2$. This implies $f \equiv 1$ on Ω_2, but $f \equiv 0$ on $\partial\Omega_1 \subset \Omega_2$, a contradiction. ■

References

[Boc] S. Bochner, Analytic and meromorphic continuation by means of Green's formula, Ann. of Math. 44 (1943), 652–673.

[B] A. Boggess, CR manifolds and the tangential Cauchy-Riemann complex, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL (1991).

[H] R. Hamilton, Harmonic maps of manifolds with boundary, Lecture Notes in Math. 471, Springer-Verlag, Berlin-New York (1975).

[Gr] P. Griffiths, Two theorems on extensions of holomorphic mapping, Inven. Math. 14 (1971), 27–62.

[KR] J. J. Kohn and H. Rossi, On the extension of holomorphic functions from the boundary of a complex manifold, Ann. of Math. 81 (1965), 451–472.

[L] H. Lewy, On the local character of solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables, Ann. of Math. 64 (1956), 514–522.

[NS] S. Nishikawa and K. Shiga, On the holomorphic equivalence of bounded domains in complete Kähler manifolds of nonpositive curvature, J. Math. Soc. Japan 35 (1983), 273–278.

[Sh] B. Shiffman, Extension of holomorphic map into Hermitian manifolds, Math. Ann. 194 (1971), 249–264.

[Shi] K. Shiga, An application of harmonic mapping to complex analytic geometry, Research Report of Gifu University, No. 17 (1981), 57–60.

[S1] Y. T. Siu, The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds, Ann. of Math. 112 (1980), 73–111.

[S2] Y. T. Siu, Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems, J. Differential Geometry 17 (1982), 55–138.

[Sc] R. Schoen, Existence and regularity for some geometric variational problems, Thesis, Stanford University (1977).

[W] J. C. Wood, An extension theorem for holomorphic mappings, Math. Proc. Camb. Phil. Soc. 88 (1980), 125–127.

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