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LIAPUNOV FUNCTIONALS AND ASYMPTOTIC STABILITY IN INFINITE DELAY SYSTEMS

Dedicated to Professor Junji Kato on his sixtieth birthday

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Abstract. Liapunov's direct method is used to give conditions ensuring the uniform asymptotic stability in a system of infinite delay differential equations. By introducing the concept of a seminorm having a fading memory with respect to the norm on the space of initial functions, we obtain a Liapunov functional with an upper bound larger than those traditionally given. This new approach enables us to unify several well-known theorems in the literature. Examples are provided to illustrate the application of these results.

1. Introduction. In this paper we consider a system of functional differential equations with infinite delay

$$x'(t)=F(t, x_t) \tag{1.1}$$

with $x_t(s)=x(t+s)$ for $-\infty<s\leq 0$ and obtain a Liapunov-type stability theorem in the phase space C_g. Our work provides a unified approach to the stability theory for infinite delay systems.

Let $R=(-\infty, +\infty)$, $R^+=[0, +\infty)$, and $R^-=(-\infty, 0]$, respectively. $|\cdot|$ denotes the Euclidean norm on R^n. $C(A, B)$ denotes the set of all continuous functions $\phi: A\to B$. We define

$$G=\{g\in C(R^-, R^+): g \text{ is nonincreasing and } g(0)=1\}\,. \tag{1.2}$$

For each $g\in G$, we define $(C_g, |\cdot|_g)$ by

$$C_g=\{\phi\in C(R^-, R^n): |\phi|_g<+\infty\} \tag{1.3}$$

where $|\phi|_g=\sup_{s\leq 0}|\phi(s)|/g(s)$. Then $(C_g, |\cdot|_g)$ is a Banach space. We also set for $H>0$

$$C_g(H)=\{\phi\in C_g: |\phi|_g<H\}\,.$$

When $g(s)\equiv 1$, we obtain the classical Banach space $(BC, \|\cdot\|)$,

$$BC=\left\{\phi\in C(R^-, R^n): \|\phi\|=\sup_{s\leq 0}|\phi(s)|<+\infty\right\}$$

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and

$$BC(H)=\{\phi\in BC\colon \|\phi\|<H\}\,.$$

We consider $F\colon R\times C_g(H)\to R^n$ with $F(t,0)=0$ for all $t\in R$ and make the following assumptions on the solutions of (1.1) throughout this paper.

(H_1) For each $(t_0,\phi)\in R\times C_g(H)$, there exists a constant $\sigma>0$ and a continuous function $x\colon(-\infty,t_0+\sigma)\to R^n$ such that $x(t)$ satisfies the equation (1.1) on $[t_0,t_0+\sigma)$ with $x_{t_0}=\phi$. The function x is called a solution of (1.1) and is denoted by $x=x(t_0,\phi)$ or $x(t)=x(t,t_0,\phi)$.

(H_2) For each $(t_0,\phi)\in R\times C_g(H)$, $x(t)=x(t,t_0,\phi)$ is defined on $[t_0,+\infty)$ unless there exists $t_0<\beta<+\infty$ such that $\limsup_{t\to\beta^-}|x(t,t_0,\phi)|=+\infty$.

For discussion on the C_g space, we refer to Burton [3], Corduneanu [6], Haddock [8], Hale and Lunel [9], and Hino, Murakami, and Naito [11].

Let $V\colon R\times C_g(H)\to R^+$ be continuous. The upper right-hand derivative of V along solutions of (1.1) is defined by

$$V'_{(1.1)}(t,\phi)=\limsup_{\delta\to 0^+}\{V(t+\delta,x_{t+\delta}(t,\phi))-V(t,\phi)\}/\delta\,. \tag{1.4}$$

DEFINITION 1.1. The zero solution of (1.1) is g-uniformly stable (g-US) if for each $\varepsilon>0$, there exists a $\delta>0$ such that $[(t_0,\phi)\in R^+\times C_g(H),\ |\phi|_g<\delta,\ t\geq t_0]$ imply $|x(t,t_0,\phi)|<\varepsilon$.

DEFINITION 1.2. The zero solution of (1.1) is g-uniformly asymptotically stable (g-UAS) if it is g-uniformly stable and there exists $\delta>0$ such that for each $\varepsilon>0$ there exists $T>0$ such that $[(t_0,\phi)\in R^+\times C_g(H),\ |\phi|_g<\delta,\ t\geq t_0+T]$ imply $|x(t,t_0,\phi)|<\varepsilon$. If $g\equiv 1$, we write UAS for g-UAS.

DEFINITION 1.3. $W\colon R^+\to R^+$ is called a wedge if W is continuous and strictly increasing with $W(0)=0$. Throughout the paper W and W_j $(j=1,2,\ldots)$ will denote wedges.

DEFINITION 1.4. A continuous function $G\colon R^+\to R^+$ is convex downward if $G[(t+s)/2]\leq[G(t)+G(s)]/2$ for all $t,s\in R^+$.

Jensen's inequality. Let W be convex downward and let $x,p\colon[a,b]\to R^+$ be continuous with $\int_a^b p(s)ds>0$. Then

$$\int_a^b p(s)ds\,W\left[\int_a^b p(s)|x(s)|ds\Big/\int_a^b p(s)ds\right]\leq\int_a^b p(s)W(|x(s)|)ds\,.$$

For reference on Jensen's inequality and its applications, we refer to Becker, Burton, and Zhang [1] and Natanson [16].

LEMMA 1.1. *Let W_1 be a wedge. For any $L>0$, define $W_0(r)=\int_0^r W_1(s)ds/L$ on*

$[0, L]$. *Then* W_0 *is a convex downward wedge such that* $W_0(r) \leq W_1(r)$ *for all* $r \in [0, L]$.

In order to put the problem into its historical context we consider the ordinary differential equation

$$x'(t) = f(t, x(t)) \tag{1.5}$$

where $f: R \times R^n \to R^n$ is continuous. The following result is well-known (see [3, p. 261]).

THEOREM A. *Let* $V: R \times R^n \to R^+$ *be continuous such that*

(i) $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$,

(ii) $V'_{(1.5)}(t, x) \leq -W_3(|x|)$.

Then the zero solution of (1.5) *is uniformly asymptotically stable.*

Extending Theorem A to functional differential equations has been the subject of extensive investigations for many years. For results on equations with finite delay, we refer to Burton [3], Burton and Hatvani [4], Hale and Lunel [9], Kato [13], Krasovskii [15], Wang [18], Yoshizawa [19], and Zhang [20]. Generalizing Theorem A to infinite delay systems is far more difficult. For reference and notation, we state some results of Burton and Zhang [5], Hering [10] for (1.1) with infinite delay. For additional results on stability of functional differential equations with infinite delay we refer to Burton [3], Grippenberg, Londen, and Staffans [7], Hino, Murakami, and Naito [11], Kato [14] and references therein.

THEOREM B (cf. [5]). *Suppose that there exists a continuous functional* $V: R \times BC \to R^+$ *and* $\Phi: R^+ \to R^+$ *with* $\Phi \in L^1[0, +\infty)$ *such that*

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi(0)|) + W_3(\int_{-\infty}^0 \Phi(-s) W_4(|\phi(s)|) ds)$,

(ii) $V'_{(1.1)}(t, \phi) \leq -W_5(|\phi(0)|)$.

Then the zero solution of (1.1) *is* UAS.

A generalization of Theorem B to the space C_g may be found in Zhang [23]. We now state the recent work of Hering [10] in C_g. He define an order in G by $g < g^0$ if and only if $g, g^0 \in G$, $g(s) \leq g^0(s)$ for all $s \leq 0$ and

$$\lim_{N \to +\infty} \left[\sup_{s \leq 0} g(s)/g^0(s-N) \right] = 0$$

THEOREM C (cf. [10]). *Suppose that there exists a continuous functional* $V: R^+ \times C_g \to R^+$, *positive constants* r_1, α, *and* L, *functions* $g^0 \in G$ *with* $g < g^0$ *and* $\eta \in C(R, R^+)$ with $\int_t^{t+L} \eta(s) ds \geq \alpha$ *for all* $t \in R$ *such that*

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi(0)|) + W_3(|\phi|_{g^0})$,

(ii) $V'_{(1.1)}(t, \phi) \leq -\eta(t) W_4(|\phi(0)|)$ *whenever* $|\phi(0)| < r_1$,

(iii) $W_1(r) - W_3(r)$ *is positive, nondecreasing on* $(0, r_1]$.

Then the zero solution of (1.1) *is* g-UAS.

The present paper continues the work of Zhang [21] in which a combination of

Theorem A and Theorem B are obtained in the space BC. By introducing a semi-norm $|\cdot|_B$ on C_g which has a fading memory with respect to $|\cdot|_g$, we will extend Theorem A and Theorem B with $\eta \equiv 1$. The Liapunov functional obtained here has a large upper bound and can be applied to highly perturbed systems.

2. The main result. When (1.1) has an unbounded delay, an example of Seifert [17] shows that if UAS is expected, then (1.1) must have some type of fading memory. It is also believed that in order to prove that the zero solution of (1.1) is UAS or solutions are uniformly ultimately bounded using a Liapunov functional V, the upper bound of V must have a fading memory with respect to the norm on the space of initial functions (see [3], [10], [11]). For each continuous function $\phi: [a, b] \to R^n$, we define

$$\|\phi\|^{[a,b]} = \sup\{|\phi(s)| : a \le s \le b\}.$$

We introduce the following definition which may be found in Zhang [22].

DEFINITION 2.1. A seminorm $|\cdot|_B$ on C_g is said to have a fading memory with respect to $|\cdot|_g$ if $|\phi|_B \le |\phi|_g$ for all $\phi \in C_g$ and if for each $\varepsilon > 0$ and $D > 0$ there exists an $h > 0$ such that

$$|\phi|_B \le \max\{\|\phi\|^{[-\eta,0]}, \varepsilon\}$$

whenever $\eta \ge h$ and $|\phi_{-\eta}|_g \le D$, where

$$|\phi_{-\eta}|_g = \sup_{s \le 0} |\phi(s-\eta)|/g(s) = \sup_{u \le -\eta} |\phi(u)|/g(u+\eta).$$

EXAMPLE 2.1. Let $\alpha: R^- \to R^+$ be continuous and $g \in G$ such that

$$\int_{-\infty}^{0} \alpha(s) g(s) ds \le 1 \quad \text{and} \quad \int_{-\infty}^{0} \alpha(s) ds \le \frac{1}{2}.$$

Define $|\phi|_B = \int_{-\infty}^{0} \alpha(s)|\phi(s)| ds$ for any $\phi \in C_g$. Then $|\cdot|_B$ has a fading memory with respect to $|\cdot|_g$.

PROOF. For any $\phi \in C_g$, we have

$$|\phi|_B = \int_{-\infty}^{0} \alpha(s)|\phi(s)| ds \le |\phi|_g \int_{-\infty}^{0} \alpha(s) g(s) ds \le |\phi|_g.$$

Let $\varepsilon > 0$ and $D > 0$ be given. Then there exists an $h > 0$ such that $2D\int_{-\infty}^{-h} \alpha(s) g(s) ds < \varepsilon$. If $\eta \ge h$ and $|\phi_{-\eta}|_g \le D$, then

$$\begin{aligned} |\phi|_B = \int_{-\infty}^{0} \alpha(s)|\phi(s)| ds &= \int_{-\eta}^{0} \alpha(s)|\phi(s)| ds + \int_{-\infty}^{-\eta} \alpha(s)|\phi(s)| ds \\ &\le \|\phi\|^{[-\eta,0]} \int_{-\eta}^{0} \alpha(s) ds + \sup_{u \le -\eta} \frac{|\phi(u)|}{g(u+\eta)} \int_{-\infty}^{-\eta} \alpha(s) g(s+\eta) ds \end{aligned}$$

$$\leq \|\phi\|^{[-\eta,0]}\int_{-\infty}^{0}\alpha(s)ds+D\int_{-\infty}^{-\eta}\alpha(s)g(s)ds$$

$$\leq\frac{1}{2}\|\phi\|^{[-\eta,0]}+\frac{\varepsilon}{2}\leq\max\{\|\phi\|^{[-\eta,0]},\varepsilon\}.$$

Thus, $|\cdot|_B$ has a fading memory with respect to $|\cdot|_g$.

EXAMPLE 2.2. Let $g^*\in G$ and $g<g^*$, that is, $g(s)\leq g^*(s)$ for all $s\in R^-$ and

$$\lim_{N\to+\infty}\left[\sup_{s\leq 0}\frac{g(s)}{g^*(s-N)}\right]=0.$$

Then $|\cdot|_{g^*}$ has a fading memory with respect to $|\cdot|_g$.

PROOF. For any $\varepsilon>0$ and $D>0$, there exists $h>0$ such that $\sup_{s\leq 0} g(s)/g^*(s-N)\leq \varepsilon/D$ for $N\geq h$. If $\eta\geq h$ and $|\phi_{-\eta}|_g<D$, then

$$|\phi|_{g^*}=\max\left\{\sup_{-\eta\leq s\leq 0}|\phi(s)|/g^*(s),\ \sup_{s\leq-\eta}|\phi(s)|/g^*(s)\right\}$$

$$\leq\max\left\{\|\phi\|^{[-\eta,0]},\ \sup_{u\leq 0}|\phi(u-\eta)|/g^*(u-\eta)\right\}$$

$$\leq\max\left\{\|\phi\|^{[-\eta,0]},\ |\phi_{-\eta}|_g\sup_{u\leq 0}g(u)/g^*(u-\eta)\right\}$$

$$\leq\max\{\|\phi\|^{[-\eta,0]},\varepsilon\}.$$

Also, $|\phi|_{g^*}\leq|\phi|_g$ for all $\phi\in C_g$. Therefore, $|\cdot|_{g^*}$ has a fading memory with respect to $|\cdot|_g$.

REMARK 2.1. It follows from Example 2.2 that $|\cdot|_g$ has a fading memory with respect to $\|\cdot\|$ if $g(s)\to+\infty$ as $s\to-\infty$.

EXAMPLE 2.3. Let $\rho: R^-\to R^+$ be continuous and $g\in G$ such that

$$\int_{-\infty}^{0}\rho(s)g(s)ds\leq 1\quad\text{and}\quad\int_{-\infty}^{0}\rho(s)ds\leq\frac{1}{2}.$$

Define $|\phi|_B=\int_{-\infty}^{0}\rho(s)\|\phi\|^{[s,0]}ds$ for any $\phi\in C_g$. It was shown in Zhang [22] that $|\cdot|_B$ has a fading memory with respect to $|\cdot|_g$. For more properties of $|\cdot|_B$ defined here with $|\cdot|_g=\|\cdot\|$ we refer to Huang and Wang [12].

DEFINITION 2.2. Let $\alpha: R^+\to R^+$ be continuous and $\int_{-\infty}^{0}\alpha(-s)g(s)ds\leq 1$ for some $g\in G$. We adopt the notation in Burton [2] and define

$$|||\phi|||=\int_{-\infty}^{0}\alpha(-s)|\phi(s)|ds$$

for any $\phi \in C_g$ throughout of the rest of this paper.

THEOREM 2.1. *Suppose that there exists a continuous functional* $V: R \times C_g(H) \to R^+$, *a seminorm* $|\cdot|_B$ *on* C_g *having a fading memory with respect to* $|\cdot|_g$, *and positive constants* γ, $K \in (0, H)$ *such that the following conditions hold for* $(t, \phi) \in R \times C_g(K)$.

(i) $W_1(|\phi(0)|) \le V(t, \phi) \le W_2(|\phi(0)| + |||\phi|||) + W_3(|\phi|_B)$,

(ii) $V'_{(1.1)}(t, \phi) \le -W_4(|\phi(0)|)$,

(iii) $W_1(r) - W_3(r) > 0$ for $r \in (0, \gamma]$.

Then the zero solution of (1.1) *is* g-UAS.

PROOF. For any $(t, \phi) \in R \times C_g(K)$, we have

$$\begin{aligned} V(t, \phi) &\le W_2(|\phi(0)| + |||\phi|||) + W_3(|\phi|_B) \\ &\le W_2(|\phi(0)| + |\phi|_g) + W_3(|\phi|_g) \le W_2^*(|\phi|_g) \end{aligned}$$

where $W_2^*(r) = W_2(2r) + W_3(r)$. For any $\varepsilon > 0$ ($\varepsilon < K$), there exists a $\delta > 0$ ($\delta < K - \varepsilon$) such that $W_2^*(\delta) < W_1(\varepsilon)$. Let $x(t) = x(t, t_0, \phi)$ be a solution of (1.1) with $|\phi|_g < \delta$. We claim that $|x(t)| < \varepsilon$ for all $t \ge t_0$. Notice that $|x(t_0)| < \varepsilon$. Suppose that there exists a $t_1 > t_0$ such that $|x(t_1)| = \varepsilon$ and $|x(t)| < \varepsilon$ for $t \in [t_0, t_1)$. Then

$$\begin{aligned} |x_t|_g &= \sup_{s \le 0} \frac{|x(t+s)|}{g(s)} \le \sup_{-(t-t_0) \le s \le 0} |x(t+s)| + \sup_{s \le -(t-t_0)} \frac{|x(t+s)|}{g(s)} \\ &\le \varepsilon + \sup_{u \le 0} \frac{|x(u+t_0)|}{g(u-(t-t_0))} \le \varepsilon + |\phi|_g < \varepsilon + \delta < K \end{aligned} \tag{2.1}$$

for all $t \in [t_0, t_1]$. Since $V'_{(1.1)}(t, x_t) \le -W_4(|x(t)|)$, it follows that

$$W_1(|x(t)|) \le V(t, x_t) \le V(t_0, \phi) \le W_2^*(|\phi|_g) < W_2^*(\delta) < W_1(\varepsilon)\,.$$

This implies that $|x(t)| < \varepsilon$ for all $t \in [t_0, t_1]$. In particular, $|x(t_1)| < \varepsilon$. We have a contradiction. Thus, $|x(t)| < \varepsilon$ for all $t \ge t_0$ and the zero solution of (1.1) is g-US.

By the above argument, we may choose $\delta > 0$ for $\varepsilon_0 = \min\{1, \gamma, K/2\}$, where $\gamma > 0$ is given in (iii), such that $[|\phi|_g < \delta, t \ge t_0]$ implies $|x(t)| < \varepsilon_0$ and $|x_t|_g < K$. To complete the proof, we must show that for each $\varepsilon > 0$ there exists $T > 0$ such that $[(t_0, \phi) \in R \times C_g(H), |\phi|_g \le \delta, t \ge t_0 + T]$ implies $|x(t, t_0, \phi)| < \varepsilon$. Let $\varepsilon > 0$ be given and find a constant M with $0 < M < \gamma$ such that

$$W_2(3M) + W_3(M) < W_1(\varepsilon)\,. \tag{2.2}$$

By the condition (iii), there exists a $\sigma > 0$ such that $0 < \sigma < M$ and

$$W_1(r) - W_3(r) \ge \sigma + W_2(3\sigma) \tag{2.3}$$

for $r \in [M, \gamma]$. Since W_1 is uniformly continuous on $[\sigma, \gamma]$, there is a constant m such that $0 < m < M - \sigma$ and

(2.4) $$W_1(r)-W_1(r-m)<\sigma$$

for all $r\in[M,\gamma]$. This yields

(2.5) $$W_1(r-m)-W_3(r)>W_1(r)-\sigma-W_3(r)\geq\sigma+W_2(3\sigma)-\sigma=W_2(3\sigma)$$

for $r\in[M,\gamma]$. Since $|\cdot|_B$ has a fading memory with respect to $|\cdot|_g$, for $D=K$ and $\sigma>0$ defined above there exists an $h>0$ such that

(2.6) $$|\psi|_B\leq\max\{\|\psi\|^{[-h,0]},\sigma\}$$

whenever $\psi\in C_g$ and $|\psi_{-h}|_g\leq D$. We also choose $h>0$ so large that $D\int_{-\infty}^{-h}\alpha(-s)g(s)ds<\sigma$. Let $x(t)=x(t,t_0,\phi)$ be a solution of (1.1) with $|\phi|_g<\delta$. Since $|x_t|_g<D$ for all $t\geq t_0$, using (2.6) we obtain

(2.7) $$|x_t|_B\leq\max\{\|x\|^{[t-h,t]},\sigma\}$$

for all $t\geq t_0+h$. Moreover, for any $\tau\geq t_0$ and $t\geq\tau$ we have

$$V(t,x_t)\leq V(\tau,x_\tau)-\int_\tau^t W_4(|x(s)|)ds\leq W_2^*(D)-\int_\tau^t W_4(|x(s)|)ds\,.$$

This implies that there exists a constant $L>0$ depending on D such that for each $\tau\geq t_0$, there is a number $t^*\in[\tau,\tau+L]$ with $|x(t^*)|<\sigma$. Consequently, we can find a sequence $\{t_n\}$ such that

(2.8) $$t_{n-1}+h\leq t_n\leq t_{n-1}+h+L\quad\text{and}\quad|x(t_n)|<\sigma$$

for $n=1,2,\ldots$. For any $t\geq t_0+h$, we have

$$\begin{aligned}V(t,x_t)&\leq W_2\left[|x(t)|+\int_{t-h}^t\alpha(t-s)|x(s)|ds+\int_{-\infty}^{t-h}\alpha(t-s)|x(s)|ds\right]\\&\quad+W_3[\max\{\|x\|^{[t-h,t]},\sigma\}]\\&\leq W_2\left[|x(t)|+\int_{t-h}^t\alpha(t-s)|x(s)|ds+D\int_{-\infty}^{-h}\alpha(-s)g(s)ds\right]\\&\quad+\max[W_3(\|x\|^{[t-h,t]}),W_3(\sigma)]\,.\end{aligned}$$

Thus, for $n\geq1$ we have

$$V(t_n,x_{t_n})\leq W_2\left[2\sigma+\int_{t_n-h}^{t_n}\alpha(t_n-s)|x(s)|ds\right]+\max[W_3(\|x\|^{[t_n-h,t_n]}),W_3(\sigma)]\,.$$

Notice that $|x(t)|<\varepsilon_0\leq\gamma$ for all $t\geq t_0$. By Lemma 1.1, there exists a convex downward wedge W_4^* such that $W_4^*(r)\leq W_4(r)$ for $0\leq r\leq\gamma$. Without loss of generality, we assume that W_4 is convex downward. Thus, for $t\geq t_0$, we have

(2.9) $$V'_{(1.1)}(t,x_t)\leq-W_4(|x(t)|)\,.$$

Let $Q=1+\sup\{\alpha(s): 0\le s\le h\}$ and J be the integer such that

$$W_2^*(D)-(J-1)hW_4(\sigma/hQ)<0\,.$$

For any integer n and $t\ge t_{n+J}$, integrate (2.9) from t_n to t and use Jensen's inequality to obtain

$$\begin{aligned} V(t,x_t)&\le V(t_n,x_{t_n})-\int_{t_n}^{t}W_4(|x(s)|)ds\le W_2^*(D)-\sum_{j=n+1}^{n+J}\int_{t_j-h}^{t_j}W_4(|x(s)|)ds\\ &\le W_2^*(D)-\sum_{j=n+1}^{n+J}hW_4\left(\frac{1}{h}\int_{t_j-h}^{t_j}|x(s)|ds\right). \end{aligned}$$

We now claim that there is an integer k, $n+1\le k\le n+J$, such that

$$Q\int_{t_k-h}^{t_k}|x(s)|ds<\sigma\,. \tag{2.10}$$

Indeed, suppose $Q\int_{t_j-h}^{t_j}|x(s)|ds\ge\sigma$ for all j with $n+1\le j\le n+J$. Then

$$V(t,x_t)\le W_2^*(D)-\sum_{j=n+1}^{n+J}hW_4(\sigma/hQ)=W_2^*(D)-(J-1)hW_4(\sigma/hQ)<0\,,$$

a contradiction. Thus, (2.10) holds. By (2.8) and (2.10), there is a subsequence $\{s_k\}$ of $\{t_n\}$ such that

$$Q\int_{s_k-h}^{s_k}|x(s)|ds<\sigma$$

and $s_{k-1}+h\le s_k\le s_{k-1}+J(h+L)$ for $n=1,2,\dots$ with $s_0=t_0$. Thus,

$$V(s_k,x_{s_k})\le W_2(3\sigma)+\max[W_3(\|x\|^{[s_k-h,s_k]}),\,W_3(\sigma)]\,.$$

Let $I_j=[s_j-h,\,s_j]$. On each I_j we have either

(A) $\|x\|^{[s_j-h,s_j]}\le M$ or

(B) $|x(\tau_j)|>M$ for some $\tau_j\in I_j$.

If (A) holds, then for $t\ge s_j$ we have

$$W_1(|x(t)|)\le V(t,x_t)\le V(s_j,x_{s_j})\le W_2(3\sigma)+W_3(M)\le W_2(3M)+W_3(M)<W_1(\varepsilon)\,.$$

This implies that $|x(t)|<\varepsilon$ for $t\ge s_j$. Now suppose (B) holds. Let $M_j=\|x\|^{[s_j-h,s_j]}\le\gamma$. We will show that $|x(t)|<M_j-m$ for all $t\ge s_j$, where m is given in (2.4). Indeed, if there exists a $t^*\ge s_j$ such that $|x(t^*)|=M_j-m$, then

$$\begin{aligned} W_1(M_j-m)&=W_1(|x(t^*)|)\le V(t^*,x_{t^*})\le V(s_j,x_{s_j})\\ &\le W_2(3\sigma)+\max[W_3(\|x\|^{[s_j-h,s_j]}),\,W_3(\sigma)]\le W_2(3\sigma)+W_3(M_j)\,, \end{aligned}$$

which contradicts (2.5). Thus, $|x(t)|<M_j-m$ for all $t\ge s_j$. Now choose the first positive integer N such that $1-Nm\le M$. If (B) holds on I_j for $j=1,2,\dots N$, then for $t\ge s_N$ we

have

$$|x(t, t_0, \phi)| < M_N - m < M_{N-1} - 2m < \cdots < 1 - Nm \leq M.$$

Thus, (A) must occur on some I_j with $j \leq N+1$, that is, $|x(t, t_0, \phi)| < \varepsilon$ for $t \geq s_{N+1} \geq s_j$. Notice that

$$s_{N+1} \leq t_0 + (N+1)J(h+L) =: t_0 + T.$$

Therefore, $|x(t, t_0, \phi)| < \varepsilon$ for $t \geq t_0 + T$ and the proof is complete.

COROLLARY 2.1. *Suppose that there exists a continuous functional* $V\colon R \times BC(H) \to R^+$, $g \in G$ *with* $g(s) \to +\infty$ *as* $s \to -\infty$, *and positive constants* γ, $K \in (0, H)$ *such that the following conditions hold for* $(t, \phi) \in R \times BC(K)$.

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi(0)| + |||\phi|||) + W_3(|\phi|_g)$,

(ii) $V'(t, \phi) \leq -W_4(|\phi(0)|)$

(iii) $W_1(r) - W_3(r) > 0$ *for* $r \in (0, \gamma)$.

Then the zero solution of (1.1) *is* UAS.

REMARK 2.2. We can replace the seminorm $|||\phi|||$ in the condition (i) of Theorem 2.1 by $\int_{-\infty}^{0} \alpha(-s) W(|\phi(s)|) ds$ for any $\phi \in C_g(K)$ with $\int_{-\infty}^{0} \alpha(-s) W(\gamma g(s)) ds < +\infty$ for all $\gamma \geq 0$ and get the same result.

In part (i) of Theorem 2.1 we can replace the norm $|||\phi|||$ by

$$|\phi|_{\alpha(t)} = \int_{-\infty}^{0} \alpha(t, t+s) |\phi(s)| ds$$

with $\alpha(t, s) \geq 0$ satisfying the following conditions:

(H_3) For each $h > 0$ there exists a positive constant L depending on h such that $\sup\{\alpha(t, t+s)\colon -h \leq s \leq 0, t \in R\} \leq L$.

(H_4) For each $\varepsilon > 0$, there exists $J > 0$ such that

$$\int_{-\infty}^{-J} \alpha(t, t+s) ds < \varepsilon$$

for all $t \in R$.

By Burton [3, p. 282, Theorem 4.3.1], there exists a function $g \in G$ such that $\int_{-\infty}^{0} \alpha(t, t+s) g(s) ds \leq 1$ for all $t \in R$. Moreover, for each $\varepsilon > 0$, there exists a constant $J > 0$ such that

$$\int_{-\infty}^{-J} \alpha(t, t+s) g(s) ds < \varepsilon$$

for all $t \in R$.

THEOREM 2.2. *Suppose that there exists a continuous functional* $V\colon R \times C_g(H) \to R^+$, *a seminorm* $|\cdot|_B$ *on* C_g *having a fading memory with respect to* $|\cdot|_g$, *and positive* *constants* γ, $K \in (0, H)$ *such that the following conditions hold for* $(t, \phi) \in R \times C_g(K)$.

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi(0)| + |\phi|_{\alpha(t)}) + W_3(|\phi|_B)$,

(ii) $V'_{(1.1)}(t, \phi) \leq -W_4(|\phi(0)|)$,

(iii) $W_1(r) - W_3(r) > 0$ *for* $r \in (0, \gamma)$.

Then the zero solution of (1.1) *is* g-UAS.

Since the proof is so similar to that of Theorem 2.1 it will not be given here.

3. Examples. In this section we present two examples to show that the combination of the norms $|||\phi|||$, $|\phi|_{\alpha(t)}$, and $|\phi|_B$ yields very interesting results in perturbation.

EXAMPLE 3.1. Consider the equation

$$x'(t) = A(t)x(t) + \int_{-\infty}^{t} C(t, s, x(s))ds + f(t, x_t) \tag{3.1}$$

where

$$A(t) = \begin{pmatrix} a(t) & 0 \\ 0 & a(t) \end{pmatrix},$$

$a: R \to R^+$ and $C: R^4 \to R^2$ are continuous. There exists a positive constant $Q > 0$ and a continuous function $E: R^+ \to R^+$ with $E \in L^1[0, +\infty)$ such that

$$|C(t, s, x)| \leq QE(t-s)|x|$$

for all $x \in R^2$. We define $\alpha(t) = \int_t^{+\infty} E(u)du$ and assume $\alpha \in L^1[0, +\infty)$. By Burton [3, p. 282] there exists a function $g \in G$ such that $\int_{-\infty}^{0} E(-s)g(s)ds < +\infty$ and $\int_{-\infty}^{0} \alpha(-s)g(s)ds < +\infty$. Without loss of generality, we assume that $\int_{-\infty}^{0} \alpha(-s)g(s)ds \leq 1$. For this fixed $g \in G$, we consider the initial function space C_g and assume $f: R \times C_g \to R^2$ is well-defined. Then the right-hand side of (3.1) can be written as

$$D(t, \phi) = A(t)\phi(0) + \int_{-\infty}^{0} C(t, t+s, \phi(s))ds + f(t, \phi)$$

as a function on $R \times C_g$. We assume that for each $(t_0, \phi) \in R \times C_g$, there exists a unique solution $x(t) = x(t, t_0, \phi)$ of (3.1) defined on $[t_0, +\infty)$. Suppose the following conditions hold.

(i) There are positive constants p and h with $ph < 1$ such that $|f(t, \phi)| \leq p\|\phi\|^{[-h,0]}$ for all $\phi \in C_g$ with $|\phi|_g \leq 1$. Assume that $g(s) = 1$ for all $s \in [-h, 0]$.

(ii) $a(t) + (1 + qh)Q\int_0^{+\infty} E(s)ds + q < -\delta$, for some $\delta > 0$, where $q = p/(1-ph)$. Then the zero solution of (3.1) is g-UAS.

PROOF. For $(t, \phi) \in R \times C_g$, we define

$$V(t,\phi)=|\phi(0)|+Q(1+qh)\int_{-\infty}^{0}\alpha(-s)|\phi(s)|ds+q\int_{-h}^{0}\|\phi\|^{[s,0]}ds\,.$$

Notice that $h|\phi(0)|\le\int_{-h}^{0}\|\phi\|^{[s,0]}ds\le h\|\phi\|^{[-h,0]}$. Define $W_1(r)=(1+qh)r$, $W_2(r)=(1+Q+qhQ)r$, $W_3(r)=qhr$, and $|\phi|_B=\|\phi\|^{[-h,0]}$. It is clear that $|\cdot|_B$ has a fading memory with respect to $|\cdot|_g$. Thus,

$$W_1(|\phi(0)|)\le V(t,\phi)\le W_2(|\phi(0)|+|||\phi|||)+W_3(|\phi|_B)\,.$$

Moreover, $W_1(r)-W_3(r)>0$ for $r\in R^+$. Let $x(t)=x(t,t_0,\phi)$ be a solution of (3.1). Then

$$V(t,x_t)=|x(t)|+Q(1+qh)\int_{-\infty}^{t}\alpha(t-s)|x(s)|ds+q\int_{t-h}^{t}\|x\|^{[s,t]}ds\,.$$

If $x(t)\neq 0$, then

$$\begin{aligned}\frac{d}{dt}|x(t)|&=\frac{1}{|x(t)|}[x_1(t)x_1'(t)+x_2(t)x_2'(t)]\\&\le a(t)|x(t)|+\int_{-\infty}^{t}QE(t-s)|x(s)|ds+|f(t,x_t)|\,.\end{aligned}$$

For $x(t)=0$, we have

$$|x'(t)|\le a(t)|x(t)|+\int_{-\infty}^{t}QE(t-s)|x(s)|ds+|f(t,x_t)|\,.$$

Thus,

$$\begin{aligned}V'_{(3.1)}(t,x_t)\le{}&a(t)|x(t)|+\int_{-\infty}^{t}QE(t-s)|x(s)|ds+|f(t,x_t)|\\&+Q(1+qh)\int_{0}^{+\infty}E(s)ds|x(t)|-Q(1+qh)\int_{-\infty}^{t}E(t-s)|x(s)|ds\\&+q|x(t)|-q\|x\|^{[t-h,t]}+q\int_{t-h}^{t}\frac{d}{dt}\|x\|^{[s,t]}ds\end{aligned}\tag{3.2}$$

For each fixed s, if $\|x\|^{[s,t]}=|x(\theta)|$ with $s\le\theta<t$ and $|x(\theta)|>|x(\tau)|$ for all $\theta<\tau\le t$, then $(d/dt)\|x\|^{[s,t]}=0$. If $\|x\|^{[s,t]}=|x(t)|$, then

$$\frac{d}{dt}\|x\|^{[s,t]}\le Q\int_{-\infty}^{t}E(t-s)|x(s)|ds+|f(t,x_t)|\,.\tag{3.3}$$

Substitute (3.3) into (3.2) to obtain

$$V'_{(3.1)}(t,x_t)\le a(t)|x(t)|+|f(t,x_t)|+Q(1+qh)\int_{0}^{+\infty}E(s)ds|x(t)|$$

$$
\begin{aligned}
&+q|x(t)|-q\|x\|^{[t-h,t]}+qh|f(t,x_t)| \\
&\le (a(t)+Q(1+qh)\alpha(0)+q)|x(t)|+(1+qh)|f(t,x_t)|-q\|x\|^{[t-h,t]} \\
&\le -\delta|x(t)|+(1+qh)p\|x\|^{[t-h,t]}-q\|x\|^{[t-h,t]}=-\delta|x(t)|\,.
\end{aligned}
$$

All conditions of Theorem 2.1 are satisfied. Thus, the zero solution of (3.1) is g-UAS.

EXAMPLE 3.2. Consider the scalar equation

$$
x'(t)=a(t)x^3(t)+\int_{-\infty}^{t} b(t,s)x^3(s)ds+\sum_{k=1}^{+\infty} a_k(t)x^3(t-h_k) \tag{3.4}
$$

where $\{h_k\}$ is a sequence of real numbers with $0<h_1<h_2<\cdots$ and the functions $a(t)$, $a_k(t)$, and $b(t,s)$ are continuous. Suppose there are positive constants δ and M such that the following conditions are satisfied:

(i) $a(t)+\int_t^{+\infty}|b(u,t)|du+\sum_{k=1}^{\infty}|a_k(t+h_k)|\le-\delta$.

(ii) $\int_{-\infty}^{t}\int_t^{+\infty}|b(u,s)|duds\le M$. For each $h>0$ there exists a positive constant L depending on h such that $\sup\{\int_t^{+\infty}|b(u,s)|du\colon -h\le s\le 0,\ t\in R\}\le L$ and for each $\varepsilon>0$, there exists $J>0$ such that

$$
\int_{-\infty}^{-J}\int_t^{+\infty}|b(u,t+s)|duds<\varepsilon
$$

for all $t\in R$.

(iii) $\sum_{k=1}^{\infty}a_k^*h_k<+\infty$, $a_k^*=\sup_{t\in R}|a_k(t)|$.

Then the zero solution of (3.4) is UAS.

PROOF. For each $\phi\in BC(R^-,R)$, define

$$
V(t,\phi)=|\phi(0)|+\int_{-\infty}^{0}\int_t^{+\infty}|b(u,s+t)||\phi(s)|^3duds+\sum_{k=1}^{\infty}\int_{-h_k}^{0}|a_k(t+s+h_k)||\phi(s)|^3ds\,.
$$

Let $x(t)=x(t,t_0,\phi)$ be a solution of (3.4). Then

$$
V(t,x_t)=|x(t)|+\int_{-\infty}^{t}\int_t^{+\infty}|b(u,s)||x(s)|^3duds+\sum_{k=1}^{\infty}\int_{t-h_k}^{t}|a_k(s+h_k)||x(s)|^3ds
$$

and

$$
\begin{aligned}
V'_{(3.4)}(t,x_t)\le{}& a(t)|x(t)|^3+\int_{-\infty}^{t}|b(t,s)||x(s)|^3ds+\sum_{k=1}^{\infty}|a_k(t)||x^3(t-h_k)| \\
&+\int_t^{+\infty}|b(u,t)|du|x^3(t)|-\int_{-\infty}^{t}|b(t,s)||x(s)|^3ds \\
&+\sum_{k=1}^{\infty}|a_k(t+h_k)||x^3(t)|-\sum_{k=1}^{\infty}|a_k(t)||x^3(t-h_k)|
\end{aligned}
$$

$$\leq\left(a(t)+\int_t^{+\infty}|b(u,t)|du+\sum_{k=1}^{\infty}|a_k(t+h_k)|\right)|x(t)|^3\leq-\delta|x(t)|^3 .$$

Next, define $\alpha(t,s)=(1/M)\int_t^{+\infty}|b(u,s)|du$. Then $\int_{-\infty}^t\alpha(t,s)ds\leq 1$ and $\alpha(t,s)$ satisfies (H_3) and (H_4). Choose a constant K such that $0<K<1$ and $K\sum_{k=1}^{\infty}a_k^*h_k<1/2$. Then define $|\phi|_B=K\sum_{k=1}^{\infty}a_k^*h_k\|\phi\|^{[-h_k,0]}$. Thus, for $\phi\in BC(R^-,R)$ with $\|\phi\|\leq K$ we have

$$|\phi(0)|\leq V(t,\phi)\leq(M+1)(|\phi(0)|+|\phi|_{\alpha(t)})+K|\phi|_B .$$

Define $W_1(r)=r$, $W_2(r)=(M+1)r$, and $W_3(r)=Kr$. Then $W_1(r)-W_3(r)>0$ for $r>0$. It remains to show that $|\cdot|_B$ has a fading memory with respect to $\|\cdot\|$.

It is clear that $|\phi|_B\leq\|\phi\|K\sum_{k=1}^{\infty}a_k^*h_k\leq\|\phi\|$. For any $\varepsilon>0$ and $D>0$, there exists $h>0$ such that

$$DK\sum_{h_k\geq h}a_k^*h_k<\frac{\varepsilon}{2} .$$

If $[\mu\geq h,\ \|\phi\|^{(-\infty,-\mu]}\leq D]$, we have

$$\begin{aligned}|\phi|_B&=K\sum_{h_k<\mu}a_k^*h_k\|\phi\|^{[-h_k,0]}+K\sum_{h_k\geq\mu}a_k^*h_k\|\phi\|^{[-h_k,0]}\\&\leq\|\phi\|^{[-\mu,0]}K\sum_{h_k<\mu}a_k^*h_k+K\sum_{h_k\geq\mu}a_k^*h_k\|\phi\|^{[-h_k,-\mu]}+K\sum_{h_k\geq\mu}a_k^*h_k\|\phi\|^{[-\mu,0]}\\&=\|\phi\|^{[-\mu,0]}K\sum_{k=1}^{\infty}a_k^*h_k+KD\sum_{h_k\geq\mu}a_k^*h_k\leq\frac{1}{2}\|\phi\|^{[-\mu,0]}+\frac{\varepsilon}{2}\leq\max\{\|\phi\|^{[-\mu,0]},\varepsilon\} .\end{aligned}$$

We conclude that all conditions of Theorem 2.2 are satisfied and the zero solution of (3.4) is UAS. This completes the proof.

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