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CLASSICAL SCHOTTKY GROUPS OF REAL TYPE OF GENUS TWO, III

Dedicated to Professor Fumiyuki Maeda on his sixtieth birthday

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Abstract. There are eight kinds of classical Schottky spaces of real type of genus two. In this paper we consider the spaces of the third, sixth and eighth types. This paper has the following three aims: (1) to represent the shape of the spaces by using multipliers and cross ratios of the fixed points of two generators; (2) to find generators for the Schottky modular groups acting on the above spaces; (3) to determine fundamental regions for the Schottky modular group acting on the spaces.

Introduction. Schottky spaces and their boundaries, and augmented Schottky spaces were studied by Bers [1], Chuckrow [7], Earle [9], Hejhal [13], Sato [21] and others. Furthermore, classical Schottky spaces and classical Schottky groups were studied by Zarrow [30], Jørgensen-Marden-Maskit [14], Marden [17] and Sato [25]. In particular, Schottky spaces and classical Schottky groups of real type were studied by Bobenko [2], Bobenko-Bordag [3] and Sato [24], [26] (see §1 for the definition). In the case of genus two those spaces and groups are classified into eight types (see §1). Purzitsky [20] and Sato [24] obtained fundamental regions for Schottky modular groups acting on the classical Schottky spaces of the first and fourth types, that is, on the space of marked Fuchsian Schottky groups. Furthermore, Sato [26] gave the shape of the classical Schottky spaces of the second, fifth and seventh types and determined fundamental regions for the Schottky modular groups acting on those spaces.

This paper is the final version of the following: the shape of the classical Schottky spaces of real type of genus two and fundamental regions of the Schottky modular groups acting on those spaces. Namely, here we will consider the groups and the spaces of the third, sixth and eighth types as a sequel to our previous papers [24], [26]. This paper has the following three aims: (1) to represent the shape of the spaces of the third, sixth and eighth types by using the coordinates introduced in Sato [22] (Theorem 3); (2) to find generators for the Schottky modular groups acting on the above spaces (Propositions 5.3 and 5.4); (3) to determine fundamental regions for the Schottky modular groups (Theorems 4, 5 and 6).

It is an important problem to decide whether or not a two-generator group

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$G=\langle A_1, A_2\rangle$ is a classical Schottky group. We can solve this problem for the case of two-generator groups of real type by considering the shape of the classical Schottky spaces of real type given in [24], [26] and this paper. For example, (i) the allegedly non-classical Schottky group constructed by Zarrow [30] is a group of the second type. Namely, the group is a classical Schottky group (Sato [25]); (ii) the group due to Jørgensen [14, p. 11] is a boundary group of the classical Schottky space of the sixth type.

The second problem is to find the best lower bound of Jørgensen's numbers for Schottky groups in connection with discreteness of two-generator groups. We solve this problem for classical Schottky groups of real type by using the Schottky modular groups and the fundamental regions for the groups given in [24], [26] and this paper (cf. Gilman [10], [11], [12], Sato [27], [28], [29] for this problem). To be more precise, let $G=\langle A_1, A_2\rangle$ be a classical Schottky group generated by A_1 and A_2. We call

$$J(G):=|\operatorname{tr}^2(A_1)-4|+|\operatorname{tr}(A_1A_2A_1^{-1}A_2^{-1})-2|$$

Jørgensen's number for the marked group $G=\langle A_1, A_2\rangle$ (cf. Jørgensen [14]). Then we have the following:

(i) $J(G)>16$ if G is of the first type (Gilman [12], Sato [27]),
(ii) $J(G)>16$ if G is of the second type (Sato [28]),
(iii) $J(G)>4$ if G is of the third type (Sato [29]),
(iv) $J(G)>4$ if G is of the fourth type (Gilman [12], Sato [27]),
(v) $J(G)>4(1+\sqrt{2})^2$ if G is of the fifth type (Sato [28]),
(vi) $J(G)>16$ if G is of the sixth type (Sato [29]),
(vii) $J(G)>4(1+\sqrt{2})^2$ if G is of the seventh type (Sato [28]),
(viii) $J(G)>16$ if G is of the eighth type (Sato [29]).

Furthermore, it is expected that our results in [24], [26] and this paper are applicable to calculate the Hausdorff dimension of the limit sets of classical Schottky groups of real type (see Beardon [2], [3], Bishop-Jones [4], Doyle [8], Phillips-Sarnak [17] for the Hausdorff dimension of the limit sets of Schottky groups).

In §1 we will state some definitions and consider automorphisms of a free group on two generators. In §2 we will consider relationship among eight kinds of the classical Schottky spaces of real type of genus two (Theorem 1). In §3 we will determine the shape of the classical Schottky spaces of real type of classical generators (Theorem 2) (see §3 for the definition). In §4 we will determine the shape of the classical Schottky spaces of the third, sixth and eighth types (Theorem 3). In §5 we will find generators for the Schottky modular groups acting on those spaces (Propositions 5.3 and 5.4), and determine fundamental regions for the Schottky modular groups (Theorems 4, 5, and 6). In the final section we will collect the main results in [24], [26] and this paper for applications to Jørgensen's numbers and the Hausdorff dimension of the limit sets of classical Schottky groups. Namely, we will represent generators for eight kinds of the Schottky modular groups (Theorem 7) and give eight kinds of the fundamental regions for the Schottky modular groups (Theorem 8).

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1. Preliminaries.

1.1. In this section we will state some definitions and list properties of automorphisms of a free group on two generators. Let $C_1, C_{g+1}; \ldots; C_g, C_{2g}$ be a set of $2g$ $(g \geq 1)$ mutually disjoint Jordan curves on the Riemann sphere which comprise the boundary of a $2g$-ply connected region ω. Suppose there are g Möbius transformations $A_1, \ldots, A_g$ which have the property that A_j maps C_j onto C_{g+j} and $A_j(\omega) \cap \omega = \varnothing$, $1 \leq j \leq g$. Then the g necessarily loxodromic transformations A_g generate a *marked Schottky group* $G = \langle A_1, \ldots, A_g \rangle$ of genus g with ω as a fundamental region. In particular, if all the C_j $(j = 1, 2, \ldots, 2g)$ are circles, then we call $A_1, \ldots, A_g$ a *set of classical generators* of G. A *classical Schottky group* is a Schottky group for which there exists some set of classical generators.

We denote by Möb the group of all Möbius transformations. We say two marked subgroups $G = \langle A_1, \ldots, A_g \rangle$ and $\hat{G} = \langle \hat{A}_1, \ldots, \hat{A}_g \rangle$ of Möb to be *equivalent* if there exists a Möbius transformation T such that $\hat{A}_j = TA_jT^{-1}$ for $j = 1, 2, \ldots, g$. The *Schottky space* (resp. the *classical Schottky space*) of genus g, denoted by $\mathfrak{S}_g$ (resp. $\mathfrak{S}_g^0$), is the set of all equivalence classes of marked Schottky groups (resp. marked classical Schottky groups) of genus $g \geq 1$.

We denote by $\mathfrak{M}_2$ the set of all equivalence classes $[\langle A_1, A_2 \rangle]$ of marked groups $\langle A_1, A_2 \rangle$ generated by loxodromic transformations A_1 and A_2 whose fixed points are all distinct. Let $[\langle A_1, A_2 \rangle] \in \mathfrak{M}_2$. For $j = 1, 2$, let λ_j $(|\lambda_j| > 1)$, p_j and p_{2+j} be the multipliers, the repelling and the attracting fixed points of A_j, respectively. We define t_j by setting $t_j = 1/\lambda_j$. Thus $t_j \in D^* = \{z \,|\, 0 < |z| < 1\}$. We determine a Möbius transformation T by $T(p_1) = 0$, $T(p_3) = \infty$ and $T(p_2) = 1$, and define ρ by $\rho = T(p_4)$. Thus $\rho \in \boldsymbol{C} - \{0, 1\}$. We can define a mapping α of the space $\mathfrak{M}_2$ into $(D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ by setting $\alpha([\langle A_1, A_2 \rangle]) = (t_1, t_2, \rho)$. Then we say $[\langle A_1, A_2 \rangle]$ represents (t_1, t_2, ρ) and (t_1, t_2, ρ) corresponds to $[\langle A_1, A_2 \rangle]$ or $\langle A_1, A_2 \rangle$. We write $t_1 = t_1(G)$, $t_2 = t_2(G)$ and $\rho = \rho(G)$. Conversely, λ_1, λ_2 and p_4 are uniquely determined from a given point $\tau = (t_1, t_2, \rho) \in (D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ under the normalization condition $p_1 = 0$, $p_3 = \infty$ and $p_2 = 1$; we define λ_j $(j = 1, 2)$ and p_4 by setting $\lambda_j = 1/t_j$ and $p_4 = \rho$, respectively. We determine $A_1(z), A_2(z) \in$ Möb from τ as follows: the multiplier, the repelling and the attracting fixed points of $A_j(z)$ are λ_j, p_j and p_{2+j}, respectively. Thus we obtain a mapping β of $(D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ into $\mathfrak{M}_2$ by setting $\beta(\tau) = [\langle A_1(z), A_2(z) \rangle]$. Then we note that $\beta\alpha = \alpha\beta = \mathrm{id}$. Therefore we identify $\mathfrak{M}_2$ with $\alpha(\mathfrak{M}_2)$. Similarly we can define the mapping α^* of $\mathfrak{S}_2$ or $\mathfrak{S}_2^0$ into $(D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ by restricting α to this space, and identify $\mathfrak{S}_2$ (resp. $\mathfrak{S}_2^0$) with $\alpha^*(\mathfrak{S}_2)$ (resp. $\alpha^*(\mathfrak{S}_2^0)$). From now on we denote $\alpha(\mathfrak{M}_2)$, $\alpha^*(\mathfrak{S}_2)$ and $\alpha^*(\mathfrak{S}_2^0)$ by $\mathfrak{M}_2$, $\mathfrak{S}_2$ and $\mathfrak{S}_2^0$, respectively.

We call $G = \langle A_1, A_2 \rangle$ a marked group of *real type* if $(t_1, t_2, \rho) \in \boldsymbol{R}^3 \cap \mathfrak{M}_2$, that is, t_1, t_2 and ρ are all real numbers, where (t_1, t_2, ρ) corresponds to $G = \langle A_1, A_2 \rangle$. Then we can classify marked groups of real type into eight types as follows.

DEFINITION 1.1 (cf. [24]). (1) G is of the first type (Type I) if $t_1>0$, $t_2>0$, $\rho>0$.
(2) G is of the second type (Type II) if $t_1>0$, $t_2<0$, $\rho>0$.
(3) G is of the third type (Type III) if $t_1>0$, $t_2<0$, $\rho<0$.
(4) G is of the fourth type (Type IV) if $t_1>0$, $t_2>0$, $\rho<0$.
(5) G is of the fifth type (Type V) if $t_1<0$, $t_2>0$, $\rho>0$.
(6) G is of the sixth type (Type VI) if $t_1<0$, $t_2<0$, $\rho>0$.
(7) G is of the seventh type (Type VII) if $t_1<0$, $t_2<0$, $\rho<0$.
(8) G is of the eighth type (Type VIII) if $t_1<0$, $t_2>0$, $\rho<0$.

The components of the coordinates (t_1, t_2, ρ) have the following meaning. If ρ is positive (resp. negative), then the axes of A_1 and A_2 are disjoint (resp. intersect). If $t_j>0$ (resp. $t_j<0$) for $j=1, 2$, then A_j leaves the upper half plane invariant (resp. A_j interchanges the upper and the lower half planes). Concequently, $G=\langle A_1, A_2\rangle$ is a Schottky group of Type I or Type IV, that is, a Fuchsian Schottky group if and only if both t_1 and t_2 are positive. For geometrical meaning of t_j and ρ, see Sato [21], [22], [23].

For each $k=\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VIII}$, we call the set of all equivalence classes of marked groups (resp. marked Schottky groups and marked classical Schottky groups) of Type k the *real space* (resp. the *real Schottky space* and the *real classical Schottky space*) of Type k, and denote it by $R_k\mathfrak{M}_2$ (resp. $R_k\mathfrak{S}_2$ and $R_k\mathfrak{S}_2^0$).

1.2. Let $G=\langle A_1, A_2\rangle$ be a marked free group on two generators.

THEOREM A (Neumann [18]). *The group Φ_2 of automorphisms of $G=\langle A_1, A_2\rangle$ has the following presentation*:

$$\Phi_2=\langle N_1, N_2, N_3 \mid (N_2N_1N_2N_3)^2=1, \\ N_3^{-1}N_2N_3N_2N_1N_3N_1N_2N_1=1, N_1N_3N_1N_3=N_3N_1N_3N_1\rangle,$$

where N_1 takes (A_1, A_2) to (A_1, A_2^{-1}), N_2 takes (A_1, A_2) to (A_2, A_1) and N_3 takes (A_1, A_2) to (A_1, A_1A_2).

We call the mappings N_1, N_2 and N_3 the *Nielsen transformations*.

DEFINITION 1.2. Let ϕ_1, ϕ_2 be elements of Φ_2. We say ϕ_1 and ϕ_2 are *equivalent* if $\phi_1(G)$ is equivalent to $\phi_2(G)$ for a Schottky group G, and expressed as $\phi_1\sim\phi_2$.

REMARKS. (1) We can regard the Nielsen transformations N_j $(j=1, 2, 3)$ and hence $\phi\in\Phi_2$ as automorphisms of the space of all equivalence classes of marked free groups on two generators (cf. [24]).

(2) From the above (1) and Definition 1.2, we have the following: If $\langle A_1, A_2\rangle\sim\langle\hat{A}_1, \hat{A}_2\rangle$ and $\phi_1\sim\phi_2$ $(\phi_1, \phi_2\in\Phi_2)$, then $\phi_1(\langle A_1, A_2\rangle)\sim\phi_2(\langle\hat{A}_1, \hat{A}_2\rangle)$.

DEFINITION 1.3. Let ϕ be in Φ_2 and let m_j $(j=1, 2)$ be the numbers of the Nielsen transformations N_j contained in ϕ. If m_1+m_2 is even, we say that ϕ is an orientation preserving automorphism. The *Schottky modular group* of genus two, which is denoted by $\mathrm{Mod}(\mathfrak{S}_2)$, is the set of all equivalence classes of orientation preserving automorphisms of $\mathfrak{S}_2$. We denote by $[\Phi_2(\mathfrak{S}_2)]$ the set of all equivalence classes of automorphisms of $\mathfrak{S}_2$ and call it the *extended Schottky modular group* of genus two.

1.3. Let (t_1, t_2, ρ) be the point in $\mathfrak{S}_2$ corresponding to a marked Schottky group $G=\langle A_1, A_2\rangle$. Let $(t_1(j), t_2(j), \rho(j))$ be the images of (t_1, t_2, ρ) under the Nielsen transformations N_j $(j=1, 2, 3)$. We set $X=\rho-t_2-\rho t_1t_2+t_1$ and $Y=\rho-t_2+\rho t_1t_2-t_1$. Then by straightforward calculations, we have the following.

LEMMA 1.1 (Sato [24, Lemma 2.1]). (1) *$t_1(1)=t_1$, $t_2(1)=t_2$ and $\rho(1)=1/\rho$.*
(2) *$t_1(2)=t_2$, $t_2(2)=t_1$ and $\rho(2)=\rho$.*
(3) *$t_1(3)=t_1$, $t_2(3)+1/t_2(3)=Y^2/t_1t_2(\rho-1)^2-2$, and $\rho(3)+1/\rho(3)=X^2/t_1\rho(1-t_2)^2-2$.*

2. Relationship among the real Schottky spaces.

2.1. In this section we will consider relationship among the real schottky spaces $R_k\mathfrak{S}_2$ $(k=\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}, \mathrm{VII}, \mathrm{VIII})$. Throughout this section, let N_j $(j=1, 2, 3)$ be the Nielsen transformations defined in §1.

PROPOSITION 2.1. *Let $R_k\mathfrak{S}_2$ $(k=\mathrm{I}, \mathrm{II}, \dots, \mathrm{VIII})$ be the Schottky spaces of type k, and let N_j $(j=1, 2, 3)$ be the Nielsen transformations defined in* §1, *Then*

(i) *Let $\tau=(t_1, t_2, \rho)\in R_{\mathrm{I}}\mathfrak{S}_2$. Then $N_1(\tau)$ is contained in $R_{\mathrm{I}}\mathfrak{S}_2$, $N_2(\tau)$ is contained in $R_{\mathrm{I}}\mathfrak{S}_2$ and $N_3^\delta(\tau)$ is contained in $R_{\mathrm{I}}\mathfrak{S}_2$, where $\delta=+1$ or -1.*

(ii) *Let $\tau=(t_1, t_2, \rho)\in R_{\mathrm{II}}\mathfrak{S}_2$. Then $N_1(\tau)$ is contained in $R_{\mathrm{II}}\mathfrak{S}_2$, $N_2(\tau)$ is contained in $R_{\mathrm{V}}\mathfrak{S}_2$ and $N_3^\delta(\tau)$ is contained in $R_{\mathrm{II}}\mathfrak{S}_2$, where $\delta=+1$ or -1.*

(iii) *Let $\tau=(t_1, t_2, \rho)\in R_{\mathrm{III}}\mathfrak{S}_2$. Then $N_1(\tau)$ is contained in $R_{\mathrm{III}}\mathfrak{S}_2$, $N_2(\tau)$ is contained in $R_{\mathrm{VIII}}\mathfrak{S}_2$ and $N_3^\delta(\tau)$ is contained in $R_{\mathrm{III}}\mathfrak{S}_2$, where $\delta=+1$ or -1.*

(iv) *Let $\tau=(t_1, t_2, \rho)\in R_{\mathrm{IV}}\mathfrak{S}_2$. Then $N_1(\tau)$ is contained in $R_{\mathrm{IV}}\mathfrak{S}_2$, $N_2(\tau)$ is contained in $R_{\mathrm{IV}}\mathfrak{S}_2$ and $N_3^\delta(\tau)$ is contained in $R_{\mathrm{IV}}\mathfrak{S}_2$, where $\delta=+1$ or -1.*

(v) *Let $\tau=(t_1, t_2, \rho)\in R_{\mathrm{V}}\mathfrak{S}_2$. Then $N_1(\tau)$ is contained in $R_{\mathrm{V}}\mathfrak{S}_2$, $N_2(\tau)$ is contained in $R_{\mathrm{II}}\mathfrak{S}_2$ and $N_3^\delta(\tau)$ is contained in $R_{\mathrm{VII}}\mathfrak{S}_2$, where $\delta=+1$ or -1.*

(vi) *Let $\tau=(t_1, t_2, \rho)\in R_{\mathrm{VI}}\mathfrak{S}_2$. Then $N_1(\tau)$ is contained in $R_{\mathrm{VI}}\mathfrak{S}_2$, $N_2(\tau)$ is contained in $R_{\mathrm{VI}}\mathfrak{S}_2$ and $N_3^\delta(\tau)$ is contained in $R_{\mathrm{VIII}}\mathfrak{S}_2$, where $\delta=+1$ or -1.*

(vii) *Let $\tau=(t_1, t_2, \rho)\in R_{\mathrm{VII}}\mathfrak{S}_2$. Then $N_1(\tau)$ is contained in $R_{\mathrm{VII}}\mathfrak{S}_2$, $N_2(\tau)$ is contained in $R_{\mathrm{VII}}\mathfrak{S}_2$ and $N_3^\delta(\tau)$ is contained in $R_{\mathrm{V}}\mathfrak{S}_2$, where $\delta=+1$ or -1.*

(viii) *Let $\tau=(t_1, t_2, \rho)\in R_{\mathrm{VIII}}\mathfrak{S}_2$. Then $N_1(\tau)$ is contained in $R_{\mathrm{VIII}}\mathfrak{S}_2$, $N_2(\tau)$ is contained in $R_{\mathrm{III}}\mathfrak{S}_2$ and $N_3^\delta(\tau)$ is contained in $R_{\mathrm{VI}}\mathfrak{S}_2$, where $\delta=+1$ or -1.*

PROOF. (i) Our assertion in the cases (ii), (v) and (vii) are proved in Sato [26]. Here we only prove the case of (iii). Let $\tau=(t_1, t_2, \rho)\in R_{\mathrm{III}}\mathfrak{S}_2$. Then we easily see $N_1(\tau)$ is contained in $R_{\mathrm{III}}\mathfrak{S}_2$ and $N_2(\tau)$ is contained in $R_{\mathrm{VIII}}\mathfrak{S}_2$ by Lemma 1.1 and the definitions of $R_{\mathrm{III}}\mathfrak{S}_2$ and $R_{\mathrm{VIII}}\mathfrak{S}_2$. We have only to prove $N_3^\delta(\tau)$ is contained in $R_{\mathrm{III}}\mathfrak{S}_2$. Set

$$A_1 = \frac{1}{t_1^{1/2}} \begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix}$$

and

$$A_2 = \frac{1}{t_2^{1/2}(\rho-1)} \begin{pmatrix} \rho - t_2 & \rho(t_2-1) \\ 1-t_2 & t_2\rho - 1 \end{pmatrix}.$$

Then $\langle A_1, A_2 \rangle$ represents (t_1, t_2, ρ). We set $N_3(\tau) = (t_1^*, t_2^*, \rho^*)$. Let p and q be the two solutions of the equation

$$t_1(1-t_2)z^2 - (\rho - t_2 - \rho t_1 t_2 + t_1)z + \rho(1-t_2) = 0 .$$

Then p and q are the fixed points of A_1A_2. We may assume that p and q are the repelling and the attracting fixed points of A_1A_2, respectively. Since $pq = \rho/t_1 < 0$ and $\rho^* = q/p$, we have $\rho^* < 0$. Furthermore, since

$$t_2^* + 1/t_2^* + 2 = (\rho - t_2 + t_1 t_2 \rho - t_1)^2 / t_1 t_2 (\rho - 1)^2 < 0 ,$$

we have $t_2^* < 0$. Noting that $t_1^* = t_1$, we have $N_3(\tau) = R_{\mathrm{III}}\mathfrak{S}_2$. By the same method as above, we see that $N_3^{-1}(\tau) \in R_{\mathrm{III}}\mathfrak{S}_2$.

The proof in the cases (i), (iv), (vi) and (viii) are done similarly to the above, and so we omit them. q.e.d.

REMARK. For $R_k\mathfrak{S}_2^0$ $(k = \mathrm{I}, \mathrm{II}, \dots, \mathrm{VIII})$, the same results as above hold.

We have the following theorem by Proposition 2.1 and Corollary to Lemma 2.1.

THEOREM 1. *Let N_j $(j = 1, 2, 3)$ be the Nielsen transformations defined in* §1. *Let X be the classical Schottky space $R_k\mathfrak{S}_2^0$ or the Schottky space $R_k\mathfrak{S}_2$ of type k $(k = \mathrm{I}, \mathrm{II}, \dots, \mathrm{VIII})$. Then*

(i) $N_1(R_kX) = R_kX$ *for each* $k = \mathrm{I}, \mathrm{II}, \dots, \mathrm{VIII}$.

(ii) (1) $N_2(R_kX) = R_kX$ *for* $k = \mathrm{I}, \mathrm{IV}, \mathrm{VI}, \mathrm{VII}$.

(2) $N_2(R_{\mathrm{II}}X) = R_{\mathrm{V}}X$ *and* $N_2(R_{\mathrm{V}}X) = R_{\mathrm{II}}X$.

(3) $N_2(R_{\mathrm{III}}X) = R_{\mathrm{VIII}}X$ *and* $N_2(R_{\mathrm{VIII}}X) = R_{\mathrm{III}}X$.

(iii) (1) $N_3(R_kX) = R_kX$ *for* $k = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$.

(2) $N_3(R_{\mathrm{V}}X) = R_{\mathrm{VII}}X$ *and* $N_3(R_{\mathrm{VII}}X) = R_{\mathrm{V}}X$.

(3) $N_3(R_{\mathrm{VI}}X) = R_{\mathrm{VIII}}X$ *and* $N_3(R_{\mathrm{VIII}}X) = R_{\mathrm{VI}}X$.

3. Shape of $R_k\mathfrak{S}_2^{00}$.

3.1 We denote by $\mathfrak{S}_g^{00}$ the space of all equivalence classes of the following marked classical Schottky groups $G = \langle A_1, \dots, A_g \rangle$ of genus g: $A_1, \dots, A_g$ is a set of all classical generators of G (see §1 for the definition). We set $R_k\mathfrak{S}_2^{00} := \mathfrak{S}_2^{00} \cap R_k\mathfrak{S}_2^0$ $(k = \mathrm{I}, \mathrm{II}, \dots, \mathrm{VIII})$. We call the space $R_k\mathfrak{S}_2^{00}$ the *classical Schottky space of real type of classical generators*. In this section we will determine the shape of the spaces

$R_k\mathfrak{S}_2^{00} := \mathfrak{S}_2^{00} \cap R_k\mathfrak{S}_2^0$ ($k=$II, VI, VIII).

Let $\tau=(t_1, t_2, \rho)\in(D^*)^2\times(\boldsymbol{C}-\{0, 1\})$. Throughout this section we let

$$A_1(z) := z/t_1$$

and

$$A_2(z) := \{(\rho-t_2)z+\rho(t_2-1)\}/\{(1-t_2)z+(\rho t_2-1)\}\,.$$

Then we note that $\langle A_1(z), A_2(z)\rangle$ represents $\tau=(t_1, t_2, \rho)$.

We set

$$\begin{aligned}M_{\text{III}}(0)=\{(t_1, t_2, \rho)\in\boldsymbol{R}^3 \,|\, &(1+t_1)((-\rho)^{1/2}+1/(-\rho)^{1/2})\\ &<(1-t_1)((-t_2)^{1/2}+1/(-t_2)^{1/2}),\ \rho<0,\ 0<t_1<1\}\,,\\ M_{\text{VI}}(1)=\{(t_1, t_2, \rho)\in\boldsymbol{R}^3 \,|\, &-(1+t_1\rho^{1/2})/(\rho^{1/2}+t_1)<t_2<\rho,\\ &-1/t_1^2<\rho<-1,\ -1<t_1<0\}\,,\\ M_{\text{VI}}(-1)=\{(t_1, t_2, \rho)\in\boldsymbol{R}^3 \,|\, &-(\rho^{1/2}+t_1)/(1+t_1\rho^{1/2})<t_2<0,\\ &-1\le\rho<-t_1^2,\ -1<t_1<0\}\,,\end{aligned}$$

and

$$M_{\text{VIII}}(0)=\left\{(t_1, t_2, \rho)\in\boldsymbol{R}^3 \,\middle|\, 0<t_2<\frac{((-\rho)^{1/2}-(-t_1)^{1/2})(1-(-t_1)^{1/2}(-\rho)^{1/2})}{((-\rho)^{1/2}+(-t_1)^{1/2})(1+(-t_1)^{1/2}(-\rho)^{1/2})},\right.$$
$$\left. 1/t_1<\rho<t_1,\ -1<t_1<0\right\}.$$

3.2. THEOREM 2. *Let $R_k\mathfrak{S}_2^{00}$ ($k=$III, VI, VIII) be the classical Schottky spaces of classical generators, and let $M_{\text{III}}(0)$, $M_{\text{VI}}(1)$, $M_{\text{VI}}(-1)$ and $M_{\text{VIII}}(0)$ be the spaces defined above. Then*

(i) $R_{\text{VI}}\mathfrak{S}_2^{00}=M_{\text{VI}}(1)\cup M_{\text{VI}}(-1)$,

(ii) $R_{\text{III}}\mathfrak{S}_2^{00}=M_{\text{III}}(0)$,

(iii) $R_{\text{VIII}}\mathfrak{S}_2^{00}=M_{\text{VIII}}(0)$.

PROOF. (i) 1) First we will show that $M_{\text{VI}}(1)\subseteq R_{\text{VI}}\mathfrak{S}_2^{00}$. Let $\tau=(t_1, t_2, \rho)\in M_{\text{VI}}(1)$ and let $\langle A_1, A_2\rangle$ represent τ. If we can choose four circles C_j ($j=1, 2, 3, 4$) satisfying the following two conditions, then we easily see that we have $\tau\in R_{\text{VI}}\mathfrak{S}_2^{00}$:

(1) C_j ($j=1, 2, 3, 4$) are the circles perpendicular to the real axis such that $A_1(C_1)=C_3$ and $A_2(C_2)=C_4$.

(2) For $j=1, 2, 3, 4$, the points a_j and b_j satisfy the inequality

$$a_3<a_1<0<b_1<a_2<1<b_2<a_4<\rho<b_4<b_3\,,$$

where a_j and b_j ($a_j<b_j$) are the intersection points of the circles C_j with the real axis.

We take a_j and b_j ($j=1, 2, 3, 4$) as follows: $a_1=-\rho^{1/2}$, $b_1=-t_1\rho^{1/2}-t_1\varepsilon$;

$a_2=-t_1\rho^{1/2}+\varepsilon$, $b_2=A_2^{-1}(-\rho^{1/2}/t_1-\varepsilon)$; $a_3=-\rho^{1/2}-\varepsilon$, $b_3=-\rho^{1/2}/t_1$; $a_4=A_2(a_2)=A_2(-t_1\rho^{1/2}+\varepsilon)=\{(\rho-t_2)(-t_1\rho^{1/2}+\varepsilon)+\rho(t_2-1)\}/\{(1-t_2)(-t_1\rho^{1/2}+\varepsilon)+(\rho t_2-1)\}$, $b_4=-\rho^{1/2}/t_1-\varepsilon$, where $\varepsilon>0$ is chosen sufficiently small. Then we easily see the following by noting $1<\rho<1/t_1^2$:

$$a_3<a_1<0<b_1<a_2<1<b_2\,,\quad a_4<\rho<b_4<b_3\,.$$

Since $1<\rho^{1/2}<-(1+t_1t_2)/(t_1+t_2)$, we can show by straightforward calculations that $b_2<a_4$ for sufficiently small $\varepsilon>0$. Furthermore, we easily see that $a_3=A_1(b_1)$, $b_3=A_1(a_1)$, $a_4=A_2(a_2)$ and $b_4=A_2(b_2)$, that is, $A_1(C_1)=C_3$ and $A_2(C_2)=C_4$.

Similarly, we can prove $M_{\mathrm{VI}}(-1)\subseteq R_{\mathrm{VI}}\mathfrak{S}_2^{00}$. Hence we have $M_{\mathrm{VI}}(1)\cup M_{\mathrm{VI}}(-1)\subseteq R_{\mathrm{VI}}\mathfrak{S}_2^{00}$.

2) Next we show that $M_{\mathrm{VI}}(1)\cup M_{\mathrm{VI}}(-1)\supseteq R_{\mathrm{VI}}\mathfrak{S}_2^{00}$. Let $\tau=(t_1,t_2,\rho)\in R_{\mathrm{VI}}\mathfrak{S}_2^{00}$. It is easily seen that if $\tau\in R_{\mathrm{VI}}\mathfrak{S}_2^{00}$, then $1<\rho<1/t_1^2$ and $1<\rho<1/t_2^2$ for $\rho>1$, and $t_1^2<\rho<1$ and $t_2^2<\rho<1$ for $0<\rho<1$. We will show that if $\tau\notin M_{\mathrm{VI}}(1)\cup M_{\mathrm{VI}}(-1)$, then $\tau\notin R_{\mathrm{VI}}\mathfrak{S}_2^{00}$. We only consider the case where $\rho>1$, since we can similarly treat the case where $0<\rho<1$.

Suppose that $\tau\notin M_{\mathrm{VI}}(1)$, $1<\rho<1/t_1^2$ and $1<\rho<t_2^2$. Then we have $t_2\leq-(1+\rho^{1/2}t_1)/(\rho^{1/2}+t_1)$. If $t_2=-(1+\rho^{1/2}t_1)/(\rho^{1/2}+t_1)$, then we see by straightforward calculations that $A_1^{-2}A_2^2$ is parabolic, and hence τ is not contained in $R_{\mathrm{VI}}\mathfrak{S}_2^{00}$. If $-(1-\rho^{1/2}t_1)/(\rho^{1/2}-t_1)<t_2<-(1+\rho^{1/2}t_1)/(\rho^{1/2}+t_1)$, then $A_1^{-2}A_2^2$ is elliptic and hence τ is not contained in $R_{\mathrm{VI}}\mathfrak{S}_2^{00}$. Furthermore, if $t_2<-1/\rho^{1/2}$, then τ is not a point of $R_{\mathrm{VI}}\mathfrak{S}_2^{00}$, since $1<\rho<1/t_2^2$. Noting that $-(1-\rho^{1/2}t_1)/(\rho^{1/2}-t_1)<-1/\rho^{1/2}$, we have that if $\tau\notin M_{\mathrm{VI}}(1)$ and $\rho>1$, then $\tau\notin R_{\mathrm{VI}}\mathfrak{S}_2^{00}$. A similar argument shows that if $\tau\notin M_{\mathrm{VI}}(-1)$ and $\rho<1$, then $\tau\notin R_{\mathrm{VI}}\mathfrak{S}_2^{00}$. Hence we have $M_{\mathrm{VI}}(1)\cup M_{\mathrm{VI}}(-1)\supseteq R_{\mathrm{VI}}\mathfrak{S}_2^{00}$. By combining 1) with 2) we have the desired result $R_{\mathrm{VI}}\mathfrak{S}_2^{00}=M_{\mathrm{VI}}(1)\cup M_{\mathrm{VI}}(-1)$.

(ii) 1) First we will show that $M_{\mathrm{III}}(0)\subseteq R_{\mathrm{III}}\mathfrak{S}_2^{00}$. Let $\tau=(t_1,t_2,\rho)\in M_{\mathrm{III}}(0)$ and let $\langle A_1,A_2\rangle$ represent τ. If we can choose four circles C_j $(j=1,2,3,4)$ satisfying the following two conditions, then we have $\tau\in R_{\mathrm{III}}\mathfrak{S}_2^{00}$:

(1) C_j $(j=1,2,3,4)$ are circles perpendicular to the real axis such that $A_1(C_1)=C_3$ and $A_2(C_2)=C_4$.

(2) For $j=1,2,3,4$, the points a_j and b_j satisfy the inequality

$$a_3<a_4<\rho<b_4<a_1<0<b_1<a_2<1<b_2<b_3$$

where a_j and b_j $(a_j<b_j)$ are the intersection points of the circles C_j with the real axis.

We take a_j and b_j $(j=1,2,3,4)$ as follows: $a_1=A_2(q)+\varepsilon$, $b_1=t_1q-\varepsilon$; $a_2=t_1q$, $b_2=q$; $a_3=A_2(q)/t_1+\varepsilon/t_1$, $b_3=q-\varepsilon/t_1$; $a_4=A_2(t_1q)$, $b_4=A_2(q)$, where $q=\{(1+t_1)(1-\rho t_2)\}/2t_1(1-t_2)$ and $\varepsilon>0$ is a constant chosen to be sufficiently small. Then we easily see the following:

$$a_4<\rho<b_4<a_1<0<b_1<a_2<1<b_2<b_3\,.$$

Since $(1+t_1)((-\rho)^{1/2}+1/(-\rho)^{1/2})<(1-t_1)((-t_2)^{1/2}+1/(-t_2)^{1/2})$, we can show by straightforward calculations that $a_3<a_4$ for sufficiently small $\varepsilon>0$. Furthermore, we easily see that $a_3=A_1(a_1)$, $b_3=A_1(b_1)$, $a_4=A_2(a_2)$ and $b_4=A_2(b_2)$, that is, $C_3=A_1(C_1)$ and $C_4=A_2(C_2)$. Hence we have $M_{\mathrm{III}}(0)\subseteq R_{\mathrm{III}}\mathfrak{S}_2^{00}$.

2) We can similarly prove $M_{\mathrm{III}}(0)\supseteq R_{\mathrm{III}}\mathfrak{S}_2^{00}$ to the above (i) 2), and so omit the proof. By combining 1) with 2), we have the desired result $R_{\mathrm{III}}\mathfrak{S}_2^{00}=M_{\mathrm{III}}(0)$.

(iii) 1) First we will show that $M_{\mathrm{VIII}}(0)\subseteq R_{\mathrm{VIII}}\mathfrak{S}_2^{00}$. Let $\tau=(t_1, t_2, \rho)\in M_{\mathrm{VIII}}(0)$ and let $\langle A_1, A_2\rangle$ represent τ. If we can choose four circles C_j $(j=1, 2, 3, 4)$ satisfying the following two conditions, then we easily see $\tau\in R_{\mathrm{VIII}}\mathfrak{S}_2^{00}$:

(1) C_j $(j=1, 2, 3, 4)$ are the circles perpendicular to the real axis such that $A_1(C_1)=C_3$ and $A_2(C_2)=C_4$.

(2) For $j=1, 2, 3, 4$, the points a_j and b_j satisfy the inequality

$$a_3<a_4<\rho<b_4<a_1<0<b_1<a_2<1<b_2<b_3$$

where a_j and b_j $(a_j<b_j)$ are the intersection points of the circles C_j with the real axis.

We take a_j and b_j $(j=1, 2, 3, 4)$ as follows: $a_1=-(-t_1)^{1/2}(-\rho)^{1/2}$, $b_1=(-t_1)^{1/2}(-\rho)^{1/2}$; $a_2=(-t_1)^{1/2}(-\rho)^{1/2}+\varepsilon$, $b_2=(-\rho)^{1/2}/(-t_1)^{1/2}-\varepsilon$; $a_3=-(-\rho)^{1/2}/(-t_1)^{1/2}$, $b_3=(-\rho)^{1/2}/(-t_1)^{1/2}$; $a_4=A_2((-\rho)^{1/2}/(-t_1)^{1/2}-\varepsilon)$, $b_4=A_2((-t_1)^{1/2}(-\rho)^{1/2}+\varepsilon)$, where $\varepsilon>0$ is chosen to be sufficiently small. Then we easily see the following:

$$a_4<\rho<b_4\,,\quad a_1<0<b_1<a_2<1<b_2<b_3\,.$$

Since

$$0<t_2<\frac{((-\rho)^{1/2}-(-t_1)^{1/2})(1-(-t_1)^{1/2}(-\rho)^{1/2})}{((-\rho)^{1/2}+(-t_1)^{1/2})(1+(-t_1)^{1/2}(-\rho)^{1/2})}\,,$$

we can show by straightforward calculations that $a_3<a_4$ and $b_4<a_1$ for a sufficiently small $\varepsilon>0$. Furthermore, we easily see that $a_3=A_1(b_1)$, $b_3=A_1(a_1)$, $a_4=A_2(b_2)$, and $b_4=A_2(a_2)$, that is, $C_3=A_1(C_1)$ and $C_4=A_2(C_2)$. Hence we have $M_{\mathrm{VIII}}(0)\subseteq R_{\mathrm{VIII}}\mathfrak{S}_2^{00}$.

2) We can similarly prove $M_{\mathrm{VIII}}(0)\supseteq R_{\mathrm{VIII}}\mathfrak{S}_2^{00}$ to the above (i) 2), and so omit the proof. By combining 1) with 2), we have the desired result $R_{\mathrm{VIII}}\mathfrak{S}_2^{00}=M_{\mathrm{VIII}}(0)$. q.e.d.

4. The domains of existence.

4.1. In this section we will determine the shape of the real classical Schottky spaces $R_{\mathrm{III}}\mathfrak{S}_2^0$, $R_{\mathrm{VI}}\mathfrak{S}_2^0$ and $R_{\mathrm{VIII}}\mathfrak{S}_2^0$ in $\boldsymbol{R}^3$. We set

$$R_{\mathrm{III}}^3=\{(t_1, t_2, \rho)\in\boldsymbol{R}^3\,|\,0<t_1<1,\ -1<t_2<0,\ \rho<0\}$$
$$R_{\mathrm{VI}}^3=\{(t_1, t_2, \rho)\in\boldsymbol{R}^3\,|\,-1<t_1<0,\ -1<t_2<0,\ \rho>0\}$$
$$R_{\mathrm{VIII}}^3=\{(t_1, t_2, \rho)\in\boldsymbol{R}^3\,|\,-1<t_1<0,\ 0<t_2<1,\ \rho<0\}\,.$$

Refer to the previous section for the definitions of $M_{\mathrm{III}}(0)$, $M_{\mathrm{VI}}(1)$, $M_{\mathrm{VI}}(-1)$ and $M_{\mathrm{VIII}}(0)$. Throughout this section let N_j $(j=1, 2, 3)$ be the Nielsen transformations defined in §1. By straightforward calculations we have the following.

PROPOSITION 4.1. *Let $M_{\mathrm{III}}(0)$, $M_{\mathrm{VI}}(1)$, $M_{\mathrm{VI}}(-1)$ and $M_{\mathrm{VIII}}(0)$ be the domains defined in* §3, *and let N_j ($j=1, 2, 3$) be the Nielsen transformations defined in* §1. *Then*

(1) $N_1 M_{\mathrm{III}}(0)=M_{\mathrm{III}}(0)$, $N_3 M_{\mathrm{III}}(0)=M_{\mathrm{III}}(0)$, $N_2 M_{\mathrm{III}}(0)=M_{\mathrm{VIII}}(0)$.

(2) $N_1 M_{\mathrm{VI}}(1)=M_{\mathrm{VI}}(-1)$, $N_1 M_{\mathrm{VI}}(-1)=M_{\mathrm{VI}}(1)$, $N_3^2 M_{\mathrm{VI}}(-1)=M_{\mathrm{VI}}(1)$, $N_2 M_{\mathrm{VI}}(1)=M_{\mathrm{VI}}(1)$, $N_2 M_{\mathrm{VI}}(-1)=M_{\mathrm{VI}}(-1)$, $N_3 M_{\mathrm{VI}}(-1)=M_{\mathrm{VIII}}(0)$.

(3) $N_1 M_{\mathrm{VIII}}(0)=M_{\mathrm{VIII}}(0)$, $N_2 M_{\mathrm{VIII}}(0)=M_{\mathrm{III}}(0)$, $N_3 M_{\mathrm{VIII}}(0)=M_{\mathrm{VI}}(1)$.

4.2. Inductively we now define the following domains. Let δ denote the number $+1$ or -1, and let $-\delta$ denote -1 or $+1$ according as δ is $+1$ or -1.

We define $M_{\mathrm{VI}}(\delta(2k+1)):=N_3^{\delta 2k} M_{\mathrm{VI}}(\delta 1)$ and $M_{\mathrm{VIII}}(\delta 2k)=N_3^{\delta 2k} M_{\mathrm{VIII}}(0)$ for $k=1, 2, 3, \ldots$, where $M_{\mathrm{VIII}}(-0)=M_{\mathrm{VIII}}(0)$. Then we easily see the following.

PROPOSITION 4.2. *Let N_1 be the Nielsen transformation defined in* §1. *Then*

(1) $N_1 M_{\mathrm{VI}}(\delta(2k+1))=M_{\mathrm{VI}}(-\delta(2k+1))$.

(2) $N_1 M_{\mathrm{VIII}}(\delta 2k)=M_{\mathrm{VIII}}(-\delta 2k)$.

Next we define some domains $M_l(n_1, n_2)$ ($l=$III, VI, VIII) of length two.

DEFINITION 4.1. Domains $M_l(1, n_0)$, and $M_l(-1, -n_0)$ ($l=$III, VI, VIII) are defined as follows. (1) $M_{\mathrm{III}}(1, n_0):=N_2 M_{\mathrm{VIII}}(n_0)$, $M_{\mathrm{III}}(-1, -n_0):=N_2 M_{\mathrm{VIII}}(-n_0)$, where $n_0=2k$ ($k=0, 1, 2, \ldots$).

(2) $M_{\mathrm{VI}}(1, n_0):=N_2 M_{\mathrm{VI}}(n_0)$, $M_{\mathrm{VI}}(-1, -n_0):=N_2 M_{\mathrm{VI}}(-n_0)$, where $n_0=2k-1$ ($k=1, 2, 3, \ldots$).

(3) $M_{\mathrm{VIII}}(1, n_0):=N_2 M_{\mathrm{III}}(n_0)=N_2 M_{\mathrm{III}}(0)=M_{\mathrm{VIII}}(0)$, $M_{\mathrm{VIII}}(-1, -n_0):=M_{\mathrm{VIII}}(-0)$ for $n_0=2k$ ($k=0, 1, 2, \ldots$).

REMARKS. By Proposition 4.1 we have $M_{\mathrm{III}}(1, 0)=M_{\mathrm{III}}(0)$, $M_{\mathrm{VI}}(1, 1)=M_{\mathrm{VI}}(1)$ and $M_{\mathrm{VI}}(-1, -1)=M_{\mathrm{VI}}(-1)$.

DEFINITION 4.2. Domains $M_{\mathrm{III}}(k+1, n_0)$, $M_{\mathrm{III}}(-(k+1), -n_0)$, $M_{\mathrm{VI}}(2k+1, n_0)$, $M_{\mathrm{VI}}(-(2k+1), -n_0)$, $M_{\mathrm{VIII}}(2k, n_0)$, $M_{\mathrm{VIII}}(-2k, -n_0)$ are defined as follows. (1) $M_{\mathrm{III}}(k+1, n_0):=N_3^k M_{\mathrm{III}}(1, n_0)$, $M_{\mathrm{III}}(-(k+1), -n_0):=N_3^{-k} M_{\mathrm{III}}(-1, -n_0)$, ($k=0, 1, 2, \ldots$), where $n_0=2m$ ($m=0, 1, 2, \ldots$).

(2) $M_{\mathrm{VI}}(2k+1, n_0):=N_3^{2k} M_{\mathrm{VI}}(1, n_0)$, $M_{\mathrm{VI}}(-(2k+1), -n_0):=N_3^{-2k} M_{\mathrm{VI}}(-1, -n_0)$, ($k=1, 2, 3, \ldots$), where $n_0=2k-1$ ($k=1, 2, 3, \ldots$).

(3) $M_{\mathrm{VIII}}(2k, n_0):=N_3 M_{\mathrm{VI}}(2k-1, n_0)$, $M_{\mathrm{VIII}}(-2k, -n_0):=N_3^{-1} M_{\mathrm{VI}}(-(2k+1), -n_0)$, ($k=1, 2, 3, \ldots$), where $n_0=2m-1$ ($m=1, 2, 3, \ldots$).

We easily see that $N_1 M_l(n_1, n_0)=M_l(-n_1, -n_0)$ ($l=$III, VI, VIII).

DEFINITION 4.3. Domains $M_l(0, n_0)$, and $M_l(-0, -n_0)$ ($l=$III, VI, VIII) are defined as follows. (1) $M_{\mathrm{VIII}}(0, n_0):=N_3^{-1} M_{\mathrm{VI}}(1, n_0)$, $M_{\mathrm{III}}(-0, -n_0)=N_3 M_{\mathrm{VI}}(-1, -n_0)$, $n_0=2m-1$ ($m=1, 2, 3, \ldots$).

(2) $M_{\mathrm{III}}(0, n_0):=N_3^{-1} M_{\mathrm{III}}(1, n_0)$, $M_{\mathrm{III}}(-0, -n_0)=N_3 M_{\mathrm{III}}(-1, -n_0)$, $n_0=2m$ ($m=0, 1, 2, \ldots$).

(3) $M_{\mathrm{VI}}(0, n_0) := N_3^{-1} M_{\mathrm{VIII}}(1, n_0)$, $M_{\mathrm{VI}}(-0, -n_0) = N_3 M_{\mathrm{VIII}}(-1, -n_0)$, $n_0 = 2m$ ($m = 0, 1, 2, \ldots$).

REMARK. $M_{\mathrm{VI}}(0, n_0) = N_3^{-1} M_{\mathrm{VIII}}(1, n_0) = N_3^{-1} N_2 M_{\mathrm{III}}(0) = N_3^{-1} M_{\mathrm{VIII}}(0) = M_{\mathrm{VI}}(-1)$.

4.3. We will define some domains $M_l(n_k, \ldots, n_1, n_0)$ ($l=$III, VI, VIII) of length $k+1$ ($k \geq 2$). Let n_0 and n_1 be the integers as in §4.2 for each case III, VI, VIII. For simplicity, we write

$$\delta(n_k, \ldots, n_0) = \begin{cases} (n_k, \ldots, n_0) & \text{if} \quad \delta = +1 \\ (-n_k, \ldots, -n_0) & \text{if} \quad \delta = -1 \,. \end{cases}$$

DEFINITION 4.4. Let $k \geq 2$ be integers. Domains $M_l(\delta(1, n_k, \ldots, n_1, n_0))$ ($l=$III, VI, VIII) are defined as follows.

(1) $M_{\mathrm{III}}(\delta(1, n_k, \ldots, n_1, n_0)) := N_2 M_{\mathrm{VIII}}(\delta(n_k, \ldots, n_1, n_0))$.

(2) $M_{\mathrm{VI}}(\delta(1, n_k, \ldots, n_1, n_0)) := N_2 M_{\mathrm{VI}}(\delta(n_k, \ldots, n_1, n_0))$.

(3) $M_{\mathrm{VIII}}(\delta(1, n_k, \ldots, n_1, n_0)) := N_2 M_{\mathrm{III}}(\delta(n_k, \ldots, n_1, n_0))$.

DEFINITION 4.5. Domains $M_l(\delta(0, n_k, \ldots, n_1, n_0))$ ($l=$III, VI, VIII) are defined as follows.

(1) $M_{\mathrm{VIII}}(\delta(0, n_k, \ldots, n_1, n_0)) := N_3^{-\delta} M_{\mathrm{VI}}(\delta(1, n_k, \ldots, n_1, n_0))$.

(2) $M_{\mathrm{VI}}(\delta(0, n_k, \ldots, n_1, n_0)) := N_3^{-\delta} M_{\mathrm{VIII}}(\delta(1, n_k, \ldots, n_1, n_0))$.

(3) $M_{\mathrm{III}}(\delta(0, n_k, \ldots, n_1, n_0)) := N_3^{-\delta} M_{\mathrm{III}}(\delta(1, n_k, \ldots, n_1, n_0))$.

DEFINITION 4.6. Let $k \geq 2$ be integers. Domains $M_l(\delta(m+1, n_k, \ldots, n_1, n_0))$ ($l=$ III, VI, VIII) are defined as follows. For $m = 1, 2, 3, \ldots$,

(1) $M_{\mathrm{III}}(\delta(m+1, n_k, \ldots, n_1, n_0)) := N_3^{\delta m} M_{\mathrm{III}}(\delta(1, n_k, \ldots, n_1, n_0))$.

(2) $M_{\mathrm{VI}}(\delta(2m+1, n_k, \ldots, n_1, n_0)) := N_3^{\delta 2m} M_{\mathrm{VI}}(\delta(1, n_k, \ldots, n_1, n_0))$.

(3) $M_{\mathrm{VI}}(\delta(2m, n_k, \ldots, n_1, n_0)) := N_3^{\delta 2m} M_{\mathrm{VI}}(\delta(0, n_k, \ldots, n_1, n_0))$.

(4) $M_{\mathrm{VIII}}(\delta(2m+1, n_k, \ldots, n_1, n_0)) := N_3^{\delta 2m} M_{\mathrm{VIII}}(\delta(2m-1, n_k, \ldots, n_1, n_0))$.

(5) $M_{\mathrm{VIII}}(\delta(2m, n_k, \ldots, n_1, n_0)) := N_3^{\delta 2m} M_{\mathrm{VIII}}(\delta(0, n_k, \ldots, n_1, n_0))$.

4.4. Next we will consider relationship among the domains defined in the above. By replacing Types II, V, and VII in the previous paper [26] with Types III, VIII and VI, respectively and replacing the surfaces $F_{\mathrm{II}}^{+}(n_k, \ldots, n_1, n_0)$, $F_{\mathrm{V}}^{+}(n_k, \ldots, n_1, n_0)$ and $F_{\mathrm{VII}}^{+}(n_k, \ldots, n_1, n_0)$ with the domains $M_{\mathrm{III}}(n_k, \ldots, n_1, n_0)$, $M_{\mathrm{VIII}}(n_k, \ldots, n_1, n_0)$ and $M_{\mathrm{VI}}(n_k, \ldots, n_1, n_0)$, respectively, we have the same relationship among the domains $M_l(n_k, \ldots, n_1, n_0)$ as in §5 in the paper [26]. We will omit the detail here.

Noting that $R_{\mathrm{III}}\mathfrak{S}_2^0 = \bigcup \{\phi(R_{\mathrm{III}}\mathfrak{S}_2^{00}) \mid \phi \in \mathrm{Mod}\,\mathfrak{S}_2\} \cap R_{\mathrm{III}}^3$, $R_{\mathrm{VI}}\mathfrak{S}_2^0 = \bigcup \{\phi(R_{\mathrm{VI}}\mathfrak{S}_2^{00}) \mid \phi \in \mathrm{Mod}\,\mathfrak{S}_2\} \cap R_{\mathrm{VI}}^3$ and $R_{\mathrm{VIII}}\mathfrak{S}_2^0 = \bigcup \{\phi(R_{\mathrm{VIII}}\mathfrak{S}_2^{00}) \mid \phi \in \mathrm{Mod}\,\mathfrak{S}_2\} \cap R_{\mathrm{VIII}}^3$, we have the following theorem by Theorem 2.

THEOREM 3. *Let $R_l\mathfrak{S}_2^0$ ($l=$III, VI, VIII) be the classical Schottky spaces of type l and let $M_l(n_k, \ldots, n_1, n_0)$ be the domains defined in §§4.2 and 4.3. Then*

$$R_{\mathrm{III}}\mathfrak{S}_2^0 = \bigcup \{M_{\mathrm{III}}(n_k, \ldots, n_1, n_0) \mid n_0 = \pm 2m \ (m = 0, 1, 2, \ldots),$$

$$n_j=0,\ \pm 1,\ \pm 2, \dots (j=1, 2, \dots, k)\,;\ k=0, 1, 2, \dots\}\,;$$

$$R_{\mathrm{VI}}\mathfrak{S}_2^0=\bigcup\{M_{\mathrm{VI}}(n_k, \dots, n_1, n_0)\,|\,n_0=\pm(2m-1)\ (m=1, 2, 3, \dots),$$
$$n_j=0,\ \pm 1,\ \pm 2, \dots (j=1, 2, \dots, k)\,;\ k=0, 1, 2, \dots\}\,;$$

$$R_{\mathrm{VIII}}\mathfrak{S}_2^0=\bigcup\{M_{\mathrm{VIII}}(n_k, \dots, n_1, n_0)\,|\,n_0=\pm 2m\ (m=0, 1, 2, \dots),$$
$$n_j=0,\ \pm 1,\ \pm 2, \dots (j=1, 2, \dots, k)\,;\ k=0, 1, 2, \dots\}\,.$$

5. Fundamental regions.

5.1. In this section we will determine fundamental regions for $[\Phi_2]$ and $\mathrm{Mod}(\mathfrak{S}_2^0)$ acting on $R_{\mathrm{III}}\mathfrak{S}_2^0$, $R_{\mathrm{VI}}\mathfrak{S}_2^0$ and $R_{\mathrm{VIII}}\mathfrak{S}_2^0$, respectively. We denote by $\mathrm{Mod}(R_l\mathfrak{S}_2^0)$ (resp. $[R_l\Phi_2]$) the restriction of $\mathrm{Mod}(\mathfrak{S}_2^0)$ (resp. $[\Phi_2]$) to $R_l\mathfrak{S}_2^0$, that is, the set of all equivalence classes of orientation preserving automorphisms (resp. the set of all equivalence classes of automorphisms) in $R_l\mathfrak{S}_2^0$ for $l=$ III, VI, VIII. We call $\mathrm{Mod}(R_l\mathfrak{S}_2^0)$ and $[R_l\Phi_2]$ the *Schottky modular group of type l* and the *extended Schottky modular group of type l*, respectively.

Throughout this section, let N_j $(j=1, 2, 3)$ be the Nielsen transformations defined in §1. We denote by $[\phi]$ the equivalence class of $\phi\in\Phi_2$. We use the symbol ϕ for an element $[\phi]$ in $[\Phi_2]$ or $\mathrm{Mod}(\mathfrak{S}_2^0)$ when there is no fear of confusion. We denote by $W(\phi_1, \phi_2, \dots, \phi_n)$ a word on $\phi_1, \phi_2, \dots, \phi_n$. We denote by $SW(\phi_1, \phi_2, \dots, \phi_n)$ (resp. $S[W(\phi_1, \phi_2, \dots, \phi_n)]$ the set of all words on $\phi_1, \phi_2, \dots, \phi_n$ (resp. the set of all equivalence classes of words on $\phi_1, \phi_2, \dots, \phi_n$). We easily see the following two lemmas by Theorem 1.

LEMMA 5.1. *Let* $\mathrm{Mod}(R_l\mathfrak{S}_2^0)$ *and* $[R_l\Phi_2]$ $(l=\mathrm{III, VI, VIII})$ *be the Schottky modular group and the extended Schottky modular group of type l, respectively, and let N_2 and N_3 be the Nielsen transformations defined in* §1. *If* $\phi\in\mathrm{Mod}(R_{\mathrm{VIII}}\mathfrak{S}_2^0)$ *(resp.* $\phi\in[R_{\mathrm{VIII}}\Phi_2]$), *then* $N_2\phi N_2\in\mathrm{Mod}(R_{\mathrm{III}}\mathfrak{S}_2^0)$ *(resp.* $N_2\phi N_2\in[R_{\mathrm{III}}\Phi_2]$) *and* $N_3^{-1}\phi N_3\in\mathrm{Mod}(R_{\mathrm{VI}}\mathfrak{S}_2^0)$ *(resp.* $N_3^{-1}\phi N_3\in[R_{\mathrm{VI}}\Phi_2]$).

LEMMA 5.2. *Let* $\mathrm{Mod}(R_l\mathfrak{S}_2^0)$, $[R_l\Phi_2]$ $(l=\mathrm{III, VI, VIII})$, N_2 *and* N_3 *be as in Lemma* 5.1. *Then*

(1) *If* $\psi\in\mathrm{Mod}(R_{\mathrm{III}}\mathfrak{S}_2^0)$ *(resp.* $\psi\in[R_{\mathrm{III}}\Phi_2]$), *then there exists* $\phi\in\mathrm{Mod}(R_{\mathrm{VIII}}\mathfrak{S}_2^0)$ *(resp.* $\phi\in[R_{\mathrm{VIII}}\Phi_2]$) *with* $\psi=N_2\phi N_2^{-1}$.

(2) *If* $\psi\in\mathrm{Mod}(R_{\mathrm{VI}}\mathfrak{S}_2^0)$ *(resp.* $\psi\in[R_{\mathrm{VI}}\Phi_2]$), *then there exists* $\phi\in\mathrm{Mod}(R_{\mathrm{VIII}}\mathfrak{S}_2^0)$ *(resp.* $\phi\in[R_{\mathrm{VIII}}\Phi_2]$) *with* $\psi=N_3^{-1}\phi N_3$.

PROPOSITION 5.1. *Let* $\mathrm{Mod}(R_l\mathfrak{S}_2^0)$, $[R_l\Phi_2]$ $(l=\mathrm{III, VI, VIII})$, N_2 *and* N_3 *be as in Lemma* 5.1. *Then*

(1) $\mathrm{Mod}(R_{\mathrm{III}}\mathfrak{S}_2^0)=N_2(\mathrm{Mod}(R_{\mathrm{VIII}}\mathfrak{S}_2^0))N_2$ *and* $[R_{\mathrm{III}}\Phi_2]=N_2[R_{\mathrm{VIII}}\Phi_2]N_2$.

(2) $\mathrm{Mod}(R_{\mathrm{VI}}\mathfrak{S}_2^0)=N_3^{-1}(\mathrm{Mod}(R_{\mathrm{VIII}}\mathfrak{S}_2^0))N_3$ *and* $[R_{\mathrm{VI}}\Phi_2]=N_3^{-1}[R_{\mathrm{VIII}}\Phi_2]N_3$.

PROOF. This follows from Lemmas 5.1 and 5.2. q.e.d.

5.2. By straightforward calculations, we have the following lemma and proposition.

LEMMA 5.3. *Let* N_j $(j=1, 2, 3)$ *be the Nielsen transformations defined in* §1. *Then*
(1) $N_1(R_{\mathrm{VIII}}\mathfrak{S}_2^0)=R_{\mathrm{VIII}}\mathfrak{S}_2^0$.
(2) $(N_2W(N_1, N_3)N_2)(R_{\mathrm{VIII}}\mathfrak{S}_2^0)=R_{\mathrm{VIII}}\mathfrak{S}_2^0$.
(3) $(N_3^{\pm 1}W(N_1, N_2)N_3^{\pm 1})(R_{\mathrm{VIII}}\mathfrak{S}_2^0)=R_{\mathrm{VIII}}\mathfrak{S}_2^0$.

PROPOSITION 5.2. *Let* $[R_{\mathrm{VIII}}\Phi_2]$ *be the extended Schottky modular group of type* VIII. *Then the set* $[R_{\mathrm{VIII}}\Phi_2]$ *consists of all equivalence classes of words on* N_1, $N_2W_\alpha N_2$, $N_3^{\pm 1}W_\beta N_3^{\pm 1}$ *with* $W_\alpha\in SW[N_1, N_3]$, $W_\beta\in SW[N_1, N_2]$, *where* $SW[N_1, N_2]$ *(resp.* $SW[N_1, N_3]$*) is the set of all equivalence classes of words on* N_1 *and* N_2 *(resp.* N_1 *and* N_3*).*

By the same method as in Lemmas 7.4 and 7.5 in Sato [26] we have the following lemmas.

LEMMA 5.4. *Let* N_j $(j=1, 2, 3)$ *be the Nielsen transformations defined in* §1. *Then the group* $\{[N_2W_\alpha N_2] \mid W_\alpha\in SW(N_1, N_3)\}$ *is generated by* $[N_1]$ *and* $[N_2N_3N_2]$, *where* $SW(N_1, N_3)$ *is the set of all equivalence classes of words on* N_1 *and* N_3.

LEMMA 5.5. *Let* N_j $(j=1, 2, 3)$ *be the Nielsen transformations defined in* §1. *Then*

(1) *The group* $\{[N_3W_\alpha N_3] \mid W_\alpha\in SW(N_1, N_2)\}$ *is generated by* $[N_1N_3^{-2}N_1](=[N_3^{-2}])$, $[N_3N_1N_3](=[N_1])$ *and* $[N_3N_2N_3]$, *where* $SW(N_1, N_2)$ *is the set of all equivalence classes of words on* N_1 *and* N_3.

(2) *The group* $\{[N_3^{-1}W_\alpha N_3] \mid W_\alpha\in SW(N_1, N_2)\}$ *is generated by* $[N_3^{-1}N_1N_3]$ *and* $[N_3^{-1}N_2N_3]$, *where* $SW(N_1, N_3)$ *is the set of all equivalence classes of words on* N_1 *and* N_3.

By Proposition 5.2., Lemmas 5.4 and 5.5, we have the following two propositions. The proofs are omitted, since the propositions are proved similarly to Propositions 7.3 and 7.4 in [26] by noting that $N_3^{-1}N_1N_3\sim N_1N_3^2$.

PROPOSITION 5.3. *Let* $\mathrm{Mod}(R_{\mathrm{VIII}}\mathfrak{S}_2^0)$ *and* $[R_{\mathrm{VIII}}\Phi_2]$ *be the Schottky modular group and the extended Schottky modular group of type* VIII, *and let* N_j $(j=1, 2, 3)$ *be the Nielsen transformations defined in* §1. *Then*
(1) $[R_{\mathrm{VIII}}\Phi_2]$ *is generated by* $[N_1]$, $[N_3^2]$, *and* $[N_2N_3N_2]$.
(2) $\mathrm{Mod}(R_{\mathrm{VIII}}\mathfrak{S}_2^0)$ *is generated by* $[N_3^2]$ *and* $[N_2N_3N_2]$.

PROPOSITION 5.4. *Let* $\mathrm{Mod}(R_l\mathfrak{S}_2^0)$ *and* $[R_l\Phi_2]$ $(l=\mathrm{III}, \mathrm{VI})$ *be the Schottky modular group and the extended Schottky modular group of type* l, *and let* N_j $(j=1, 2, 3)$ *be the Nielsen transformations defined in* §1. *Then*
(1) (i) $[R_{\mathrm{III}}\Phi_2]$ *is generated by* $[N_1]$, $[N_2N_3^2N_2]$ *and* $[N_3]$.
(ii) $\mathrm{Mod}(R_{\mathrm{III}}\mathfrak{S}_2^0)$ *is generated by* $[N_3]$ *and* $[N_2N_3^2N_2]$.
(2) (i) $[R_{\mathrm{VI}}\Phi_2]$ *is generated by* $[N_1]$, $[N_1N_2]$ *and* $[N_3^2](=[N_1N_3^{-2}N_1])$.
(ii) $\mathrm{Mod}(R_{\mathrm{VI}}\mathfrak{S}_2^0)$ *is generated by* $[N_3^2]$ *and* $[N_1N_2]$.

5.3. We will introduce some surfaces in $\boldsymbol{R}^3_{\mathrm{VI}}$:

$$B_1 := \{(t_1, t_2, \rho) \in M_{\mathrm{VI}}(-1) \mid t_2 = t_1, \{(t_1 + 1/t_1)/2\}^{-2} < \rho < 1, -1 < t_1 < 0\} ;$$

$$B_2 := \{(t_1, t_2, \rho) \in M_{\mathrm{VI}}(1) \mid t_2 = t_1, 1 < \rho < \{(t_1 + 1/t_1)/2\}^2, -1 < t_1 < 0\} ;$$

$$B_3 := \{(t_1, t_2, \rho) \in M_{\mathrm{VI}}(-1) \mid \rho = (t_2 - t_1)/(1 - t_1 t_2), t_{20} < t_2 < 0, -1 < t_1 < 0\} ,$$

where

$$t_{20} = [1 - 4t_1 + t_1^2 - \{(1 - 4t_1 + t_1^2)^2 - 4t_1^2\}^{1/2}]/2t_1 .$$

We note that $t_1 < t_{20} < 0$ in this case.

$$B_4 := \{(t_1, t_2, \rho) \in M_{\mathrm{VI}}(1) \mid \rho = (1 - t_1 t_2)/(t_2 - t_1), t_{20} < t_2 < 0, -1 < t_1 < 0\} ;$$

$$B_5 := \{(t_1, t_2, \rho) \in \partial M_{\mathrm{VI}}(-1) \mid t_2 = 0, -t_1 < \rho < 1, -1 < t_1 < 0\} ;$$

$$B_6 := \{(t_1, t_2, \rho) \in \partial M_{\mathrm{VI}}(1) \mid t_2 = 0, 1 < \rho < -1/t_1, -1 < t_1 < 0\} ;$$

$$B_7 := \{(t_1, t_2, \rho) \in \partial M_{\mathrm{VI}}(-1) \mid \rho^{1/2} = -(t_1 + t_2)/(1 + t_1 t_2), t_1 < t_2 < t_{20}, -1 < t_1 < 0\} ;$$

$$B_8 := \{(t_1, t_2, \rho) \in \partial M_{\mathrm{VI}}(1) \mid \rho^{1/2} = -(1 + t_1 t_2)/(t_1 + t_2), t_1 < t_2 < t_{20}, -1 < t_1 < 0\} ;$$

$$B_9 := \{(t_1, t_2, \rho) \in \partial M_{\mathrm{VI}}(-1) \cap \partial M_{\mathrm{VI}}(1) \mid \rho = 1, t_1 < t_2 < t_{20}, -1 < t_1 < 0\} .$$

By Lemma 1.1 we have

LEMMA 5.6. *Let B_1, B_2, B_3 and B_4 be the surfaces defined in the above, and let N_1 and N_2 be the Nielsen transformations defined in* §1. *Then $N_1 N_2(B_1) = B_2$ and $N_3^2(B_3) = B_4$.*

We denote by $F^-_{\mathrm{VI}}([\Phi_2])$ (resp. $F^+_{\mathrm{VI}}([\Phi_2])$) the subregions in $M_{\mathrm{VI}}(-1)$ (resp. $M_{\mathrm{VI}}(1)$) bounded by the surfaces B_1, B_3, B_5, B_7 and B_9 (resp. B_2, B_4, B_6, B_8 and B_9). Then $F^+_{\mathrm{VI}}([\Phi_2]) = N_1(F^-_{\mathrm{VI}}[\Phi_2])$. We set $F_{\mathrm{VI}}([\Phi_2]) = F^-_{\mathrm{VI}}([\Phi_2])$ and $F_{\mathrm{VI}}(\mathfrak{S}_2^0) = F^+_{\mathrm{VI}}([\Phi_2]) \cup F^-_{\mathrm{VI}}([\Phi_2])$. Then we have the following by Proposition 5.4(2) and Lemma 5.6.

THEOREM 4. *Let* $\mathrm{Mod}(R_{\mathrm{VI}}\mathfrak{S}_2^0)$ *and* $[R_{\mathrm{VI}}\Phi_2]$ *be the Schottky modular group and the extended Schottky modular group of type* VI, *and let* $F_{\mathrm{VI}}(\mathfrak{S}_2^0)$ *and* $F_{\mathrm{VI}}([\Phi_2])$ *be the domains defined in the above. Then*

(1) $F_{\mathrm{VI}}(\mathfrak{S}_2^0)$ *is a fundamental region in* $R_{\mathrm{VI}}\mathfrak{S}_2^0$ *for* $\mathrm{Mod}(R_{\mathrm{VI}}\mathfrak{S}_2^0)$.

(2) $F_{\mathrm{VI}}([\Phi_2])$ *is a fundamental region in* $R_{\mathrm{VI}}\mathfrak{S}_2^0$ *for* $[R_{\mathrm{VI}}\Phi_2]$.

5.4. Next we will describe fundamental regions in $R_{\mathrm{III}}\mathfrak{S}_2^0$ for $\mathrm{Mod}(R_{\mathrm{III}}\mathfrak{S}_2^0)$ and $[R_{\mathrm{III}}\Phi_2]$. We define some surfaces:

$$B_1 := \{(t_1, t_2, \rho) \in R_{\mathrm{III}}\mathfrak{S}_2^0 \mid \rho = (t_2 - t_1^{1/2})/(1 - t_1^{1/2} t_2), t_{21} < t_2 < 0, 0 < t_1 < 1\} ,$$

where $t_{21} = t_{21}(t_1)$ is the t_2-coordinate of the intersection point of two curves

$$\rho = (t_2 - t_1^{1/2})/(1 - t_1^{1/2} t_2)$$

and

$$(1+t_1)((-\rho)^{1/2}+1/(-\rho)^{1/2})=(1-t_1)((-t_2)^{1/2}+1/(-t_2)^{1/2})$$

for $0<t_1<1$;

$$B_2:=\{(t_1,t_2,\rho)\in R_{\mathrm{III}}\mathfrak{S}_2^0 \mid \rho=(1-t_1^{1/2}t_2)/(t_2-t_1^{1/2}),\ t_{21}<t_2<0,\ 0<t_1<1\}\ ;$$

$$B_3:=\{(t_1,t_2,\rho)\in R_{\mathrm{III}}\mathfrak{S}_2^0 \mid (1+t_1)((-\rho)^{1/2}+1/(-\rho)^{1/2})$$
$$=(1-t_1)((-t_2)^{1/2}+1/(-t_2)^{1/2}),\ \rho<0,\ 0<t_1<1\}\ ;$$

$$B_4:=\{(t_1,t_2,\rho)\in \partial R_{\mathrm{III}}\mathfrak{S}_2^0 \mid t_2=0,\ -1<\rho<-t_1^{1/2},\ 0<t_1<1\}\ ;$$

$$B_5:=\{(t_1,t_2,\rho)\in \partial R_{\mathrm{III}}\mathfrak{S}_2^0 \mid t_2=0,\ -1/t_1^{1/2}<\rho\le -1,\ 0<t_1<1\}\ ;$$

$$B_6:=\{(t_1,t_2,\rho)\in R_{\mathrm{III}}\mathfrak{S}_2^0 \mid \rho=-1,\ -((1-t_1^{1/2})/(1+t_1^{1/2}))^2<t_2<0,\ 0<t_1<1\}\ .$$

Let $F_{\mathrm{III}}(\mathfrak{S}_2^0)$ (resp. $F_{\mathrm{III}}([\Phi_2])$ be the subregions of $M_{\mathrm{III}}(0)$ bounded by B_1, B_2, B_3, B_4 and B_5 (resp. B_1, B_3, B_4 and B_6).

We can show the following theorem by a similar argument to the proof of Theorem 4. Namely, noting that $B_2=N_3(B_1)$, we have the theorem by Proposition 5.4(1).

THEOREM 5. *Let* $\mathrm{Mod}(R_{\mathrm{III}}\mathfrak{S}_2^0)$ *and* $[R_{\mathrm{III}}\Phi_2]$ *be the Schottky modular group and the extended Schottky modular group of type* III, *and let* $F_{\mathrm{III}}(\mathfrak{S}_2^0)$ *and* $F_{\mathrm{III}}([\Phi_2])$ *be the domains defined in the above. Then*

(1) $F_{\mathrm{III}}(\mathfrak{S}_2^0)$ *is a fundamental region in* $R_{\mathrm{III}}\mathfrak{S}_2^0$ *for* $\mathrm{Mod}(R_{\mathrm{III}}\mathfrak{S}_2^0)$.

(2) $F_{\mathrm{III}}([\Phi_2])$ *is a fundamental region in* $R_{\mathrm{III}}\mathfrak{S}_2^0$ *for* $[R_{\mathrm{III}}\Phi_2]$.

5.5. Finally we will describe fundamental regions in $R_{\mathrm{VIII}}\mathfrak{S}_2^0$ for $\mathrm{Mod}(R_{\mathrm{VIII}}\mathfrak{S}_2^0)$ and $R_{\mathrm{VIII}}[\Phi_2]$. We set

$$T_1(t_1,\rho)=\frac{((-\rho)^{1/2}-(-t_1)^{1/2})(1-(-t_1)^{1/2}(-\rho)^{1/2})}{((-\rho)^{1/2}+(-t_1)^{1/2})(1+(-t_1)^{1/2}(-\rho)^{1/2})}$$

and

$$T_2(t_1,\rho)=(t_1-\rho)^2/(1-t_1\rho)^2\ .$$

We define some surfaces:

$$B_1:=\{(t_1,t_2,\rho)\in M_{\mathrm{VIII}}(0) \mid t_2=T_2(t_1,\rho),\ \rho_1<\rho<t_1,\ -1<t_1<0\}\ ,$$

where $\rho_1=\rho(t_1)$ is the ρ-coordinate of the intersection point of two curves $t_2=T_1(t_1,\rho)$ and $t_2=T_2(t_1,\rho)$ for $-1<t_1<0$;

$$B_2:=\{(t_1,t_2,\rho)\in M_{\mathrm{VIII}}(0) \mid t_2=T_2(t_1,\rho)^{-1},\ 1/t_1<\rho<1/\rho_1,\ -1<t_1<0\}\ ;$$

$$B_3:=\{(t_1,t_2,\rho)\in \partial M_{\mathrm{VIII}}(0) \mid t_2=T_1(t_1,\rho),\ -1<\rho<\rho_1,\ -1<t_1<0\}\ ;$$

$$B_4:=\{(t_1,t_2,\rho)\in \partial M_{\mathrm{VIII}}(0) \mid t_2=T_1(t_1,\rho),\ 1/\rho_1<\rho<-1,\ -1<t_1<0\}\ ;$$

$B_5 := \{(t_1, t_2, \rho) \in \partial M_{\mathrm{VIII}}(0) \mid t_2 = 0, -1 < \rho < t_1, -1 < t_1 < 0\}$;

$B_6 := \{(t_1, t_2, \rho) \in \partial M_{\mathrm{VIII}}(0) \mid t_2 = 0, 1/t_1 < \rho \leq -1, -1 < t_1 < 0\}$;

$B_7 := \{(t_1, t_2, \rho) \in M_{\mathrm{VIII}}(0) \mid \rho = -1, 0 < t_2 < \{(1-(-t_1)^{1/2})/(1+(-t_1)^{1/2})\}^2, -1 < t_1 < 0\}$.

Let $F_{\mathrm{VIII}}(\mathfrak{S}_2^0)$ (resp. $F_{\mathrm{VIII}}([\Phi_2])$) be the subregions in $R_{\mathrm{VIII}}\mathfrak{S}_2^0$ bounded by B_1, B_2, B_3, B_4, B_5 and B_6 (resp. B_1, B_3, B_5 and B_7).

We can show the following theorem by an entirely similar argument to the proof of Theorem 4. Namely, noting that $B_2 = N_2 N_3 N_2(B_1)$, we can prove the theorem by Proposition 5.3.

THEOREM 6. *Let* $\mathrm{Mod}(R_{\mathrm{VIII}}\mathfrak{S}_2^0)$ *and* $[R_{\mathrm{VIII}}\Phi_2]$ *be the Schottky modular group and the extended Schottky modular group of type* VIII, *and let* $F_{\mathrm{VIII}}(\mathfrak{S}_2^0)$ *and* $F_{\mathrm{VIII}}([\Phi_2])$ *be the domains defined in the above. Then*

(1) $F_{\mathrm{VIII}}(\mathfrak{S}_2^0)$ *is a fundamental region in* $R_{\mathrm{VIII}}\mathfrak{S}_2^0$ *for* $\mathrm{Mod}(R_{\mathrm{VIII}}\mathfrak{S}_2^0)$.

(2) $F_{\mathrm{VIII}}([\Phi_2])$ *is a fundamental region in* $R_{\mathrm{VIII}}\mathfrak{S}_2^0$ *for* $[R_{\mathrm{VIII}}\Phi_2]$.

6. Conclusion.

6.1. In this section we will collect together the main results in [24], [26] and this paper and apply them to calculate Jørgensen's numbers as in [27], [28], [29] and the Hausdorff dimensions of the limit sets of Schottky groups. Namely, we will list the fundamental regions $F_l(\mathrm{Mod}(\mathfrak{S}_2^0))$ ($l = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VIII}$) for classical Schottky groups of real type of genus two and list generators of the Schottky modular groups $\mathrm{Mod}(R_l\mathfrak{S}_2^0)$.

THEOREM 7. *Let* $F_l(\mathrm{Mod}(\mathfrak{S}_2^0))$ ($l = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VIII}$) *be the fundamental regions for classical Schottky groups* $\mathrm{Mod}(R_l\mathfrak{S}_2^0)$ *of type l. Then*

(1) $F_{\mathrm{I}}(\mathrm{Mod}(\mathfrak{S}_2^0)) = \{(t_1, t_2, \rho) \in R_{\mathrm{I}}\mathfrak{S}_2^0 \mid \rho(t_1, t_2)^{-1} < \rho < \rho(t_1, t_2), \rho \neq 1, 0 < t_2 < 1, 0 < t_1 < 1\}$,

where $\rho(t_1, t_2) = (1 + t_1^{1/2} t_2)/(t_1^{1/2} + t_2)$.

(2) $F_{\mathrm{II}}(\mathrm{Mod}(\mathfrak{S}_2^0)) = \{(t_1, t_2, \rho) \in R_{\mathrm{II}}\mathfrak{S}_2^0 \mid (1 + t_1^{1/2} t_2)/(t_1^{1/2} + t_2) < \rho < \{(1 - t_1^{1/2} t_2)/(t_1^{1/2} - t_2)\}^2, -1 < t_2 < 0, 0 < t_1 < 1\}$.

(3) $F_{\mathrm{III}}(\mathrm{Mod}(\mathfrak{S}_2^0)) = \{(t_1, t_2, \rho) \in R_{\mathrm{III}}\mathfrak{S}_2^0 \mid \rho^*(T_1, T_2) < \rho < -1, t_2^*(t_1, \rho) < t_2 < 0, 0 < t_1 < 1\}$,

where $\rho^*(T_1, T_2) = \{4 - T_1 T_2 + (4 - T_1^2)^{1/2}(4 - T_2^2)^{1/2}\}/2(T_2 - T_1)$, $T_1 = t_1 + 1/t_1$, $T_2 = t_2 + 1/t_2$, *and* $t_2^*(t_1, t_2)$ *is* t_2 *satisfying the equation* $(1 + t_1)\{(-\rho)^{1/2} + 1/(-\rho)^{1/2}\} = (1 - t_1)\{(-t_2)^{1/2} + 1/(-t_2)^{1/2}\}$.

(4) $F_{\mathrm{IV}}(\mathrm{Mod}(\mathfrak{S}_2^0)) = \{(t_1, t_2, \rho) \in R_{\mathrm{IV}}\mathfrak{S}_2^0 \mid \rho^*(t_1, t_2) < \rho < 1/\rho^*(t_1, t_2), t_2 < t_1, 0 < t_2 < t_2^*(t_1, \rho), 0 < t_1 < 1\}$,

where $\rho^*(t_1, t_2) = (1 - t_1^{1/2} t_2)/(t_2 - t_1^{1/2})$ *and* $t_2^*(t_1, \rho)$ *is* t_2 *satisfying the equation* $2 t_1^{1/2} t_2^{1/2}(1 - \rho) = (-\rho)^{1/2}(1 - t_1)(1 - t_2)$.

(5) $F_{\mathrm{V}}(\mathrm{Mod}(\mathfrak{S}_2^0)) = \{(t_1, t_2, \rho) \in R_{\mathrm{V}}\mathfrak{S}_2^0 \mid (1 - t_1 t_2)/(t_2 - t_1) < \rho$

$<\{(1-t_2^{1/2}t_1)/(t_2^{1/2}-t_1)\}^2, 0<t_2<1, -1<t_1<0\}.$

(6) $F_{\mathrm{VI}}(\mathrm{Mod}(\mathfrak{S}_2^0))=\{(t_1, t_2, \rho)\in R_{\mathrm{VI}}\mathfrak{S}_2^0 | (t_2-t_1)/(1-t_1t_2)<\rho<(1-t_1t_2)/(t_2-t_1),$
$\{-(t_1+t_2)/(1+t_1t_2)\}^2<\rho<\{-(1+t_1t_2)/(t_1+t_2)\}^2,$
$\rho\neq 1, t_1<t_2<0, -1<t_1<0\}.$

(7) $F_{\mathrm{VII}}(\mathrm{Mod}(\mathfrak{S}_2^0))=\{(t_1, t_2, \rho)\in R_{\mathrm{VII}}\mathfrak{S}_2^0 |$
$\{(-t_1)^{1/2}+(-t_2)^{1/2}\}/(1-(-t_1)^{1/2}(-t_2)^{1/2})$
$<(-\rho)^{1/2}<(1-(-t_1)^{1/2}(-t_2)^{1/2})/((-t_1)^{1/2}+(-t_2)^{1/2})),$
$t_2<t_1, -1<t_1<0\}.$

(8) $F_{\mathrm{VIII}}(\mathrm{Mod}(\mathfrak{S}_2^0))=\{(t_1, t_2, \rho)\in R_{\mathrm{VIII}}\mathfrak{S}_2^0 | 0<t_2$
$<t_2(t_1, \rho), 1/t_1<\rho<-1, -1<t_1<0\},$

where $t_2(t_1, \rho)=\{(-\rho)^{1/2}-(-t_1)^{1/2}\}\{1-(-t_1)^{1/2}(-\rho)^{1/2}\}/\{(-\rho)^{1/2}+(-t_1)^{1/2})(1+(-t_1)^{1/2}(-\rho)^{1/2}\}.$

6.2. THEOREM 8. *Let* N_j $(j=1, 2, 3)$ *be the Nielsen transformations defined in* §1, *that is,* $N_1: (A_1, A_2)\mapsto(A_1^{-1}, A_2)$, $N_2: (A_1, A_2)\mapsto(A_2, A_1)$ *and* $N_3: (A_1, A_2)\mapsto(A_1, A_1A_2)$. *Let* $\mathrm{Mod}(R_l\mathfrak{S}_2^0)$ $(l=\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VIII})$ *be the Schottky modular group of type* l. *Then*

(1) $\mathrm{Mod}(R_{\mathrm{I}}\mathfrak{S}_2^0)=\langle N_1N_3N_1, N_1N_2\rangle.$
(2) $\mathrm{Mod}(R_{\mathrm{II}}\mathfrak{S}_2^0)=\langle N_3, N_2N_3^2N_2\rangle.$
(3) $\mathrm{Mod}(R_{\mathrm{III}}\mathfrak{S}_2^0)=\langle N_3, N_2N_3^2N_2\rangle.$
(4) $\mathrm{Mod}(R_{\mathrm{IV}}\mathfrak{S}_2^0)=\langle N_1N_3N_1, N_1N_2\rangle.$
(5) $\mathrm{Mod}(R_{\mathrm{V}}\mathfrak{S}_2^0)=\langle N_3^2, N_2N_3N_2\rangle.$
(6) $\mathrm{Mod}(R_{\mathrm{VI}}\mathfrak{S}_2^0)=\langle N_3^2, N_1N_2\rangle.$
(7) $\mathrm{Mod}(R_{\mathrm{VII}}\mathfrak{S}_2^0)=\langle N_3^2, N_1N_2\rangle.$
(8) $\mathrm{Mod}(R_{\mathrm{VIII}}\mathfrak{S}_2^0)=\langle N_3^2, N_2N_3N_2\rangle.$

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