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GENERAL MULTI-CANONICAL DIVISORS ON TWO-DIMENSIONAL SMOOTHABLE SEMI-LOG-TERMINAL SINGULARITIES

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Abstract. We compute general multi-canonical divisors on two-dimensional smoothable semi-log-terminal singularities. As an application of this result, we give an effective bound for the Gorenstein index of such singularities in terms of the local self-intersection number of their multi-canonical divisor.

1. Introduction. This paper is devoted to certain basic calculations on two-dimensional smoothable semi-log-terminal singularities. If we study minimal or canonical models of one-parameter degenerations of algebraic surfaces, we need to treat singularities that appear in the central fiber. Smoothable semi-log-terminal singularities are those which appear in the central fiber of a minimal model of a degeneration, as well as those which appear in the central fiber of the canonical model of a degeneration which may have large Gorenstein index. Kollár and Shepherd-Barron [5] characterized these singularities, but for the numerical theory of degenerations, we need more detailed information.

In this paper, we calculate general multi-canonical divisors on these singularities. Though the term "general" in algebraic geometry is a relative notion, for a two-dimensional semi-rational singularity (Z, z), we can characterize effective Weil divisors on Z which are sufficiently general (from the viewpoint of the intersection theory) in the associated complete linear system, using the well-known notion of "full sheaf"; and we use this characterization as the definition of the term "general" in this paper (Definition 4.1). We introduce the notions of the λ-expansion and μ-expansion in Section 3, which represent positive integers as certain special sum (Proposition-Definition 3.1); and we show that for a two-dimensional smoothable semi-log-terminal singularity (X, x) (regardless of whether it is normal or non-normal), general members of the complete linear system of the multi-canonical divisor $-nK_X$ in the above sense are described in terms of the λ- and μ-expansion of n (Theorems 4.4 and 4.6). As an application of the above result, we bound their Gorenstein indices in terms of the local self-intersection number of the multi-canonial divisor (Theorem 5.2).

We briefly describe the motivation for such a theorem. Historically, Kollár and Shepherd-Barron [5] asked the following question, as the remaining step to construct the compactification of the moduli of surfaces of general type in characteristic zero.

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Question. Let $\pi: V \to W$ be the canonical model of a one-parameter degeneration of surfaces of general type; i.e.,

(i) V is a threefold with only canonical singularities,

(ii) W is a smooth curve,

(iii) π is a projective morphism whose fibers have only semi-log-canonical singularities and whose general fibers have only canonical singularities,

(iv) the canonical divisor of V is π-ample.

In this situation, does there exist an upper bound for the Gorenstein index of V depending only on deformation invariants of the fiber?

Alexeev [1] showed that such a bound, depending only on the self-intersection number of the canonical divisor of the fiber, exists; thus the construction of the above compactification of the moduli were completed. His boundedness theorem, however, by no means gives an *effective* bound for the Gorenstein index. Seeking this is interesting in its own right.

Theorem 5.2, mentioned above, reduces the problem of effectiveness of the bound in the above question to the problem of effective fixed component freeness of stable surfaces:

Problem. For a stable surface S, find an effective n, depending only on $(K_S)^2$, such that nK_S has no fixed component.

This is still an open problem.

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2. Preliminaries. A two-dimensional semi-log-terminal singularity (for the definition of "semi-log-terminal", see [5, Definition 4.17]) is said to be *smoothable* (cf. Kollár [4, Definition 5.4]) if it admits $\boldsymbol{Q}$-Gorenstein one-parameter deformations to rational double points. Kollár and Shepherd-Barron [5] characterized such singularities. Let us recall their result.

We introduce two types of singularities.

Definition 2.1 ([5, Proposition 3.10]). Let (a, d, m) be a triplet of positive integers such that $m/2 \leq a < m$ and that a is prime to m. We denote by $X_{a,d,m}$ a two-dimensional quotient singularity of the form Soec $\boldsymbol{C}[z_1, z_2]/\langle\rho\rangle$, where $\langle\rho\rangle$ is a cyclic group of order dm^2 acting on Spec $\boldsymbol{C}[z_1, z_2]$ as $(\rho^* z_1, \rho^* z_2) = (\varepsilon^{adm-1} z_1, \varepsilon z_2)$ in which ε is a primitive dm^2th root of unity.

Definition 2.2. Let (a, m) be a pair of positive integers such that $m/2 \leq a < m$ and that a is prime to m. Put $b = m - a$, and let a' (resp. b') be an integer such that $aa' \equiv 1$ (resp. $bb' \equiv 1$) (mod m). Let $\langle\check{\rho}\rangle$ be a cyclic group of order m acting on a normal crossing point $\mathrm{NC}^2 := \mathrm{Spec}\, \boldsymbol{C}[z_1, z_2, z_3]/(z_1 z_2)$ as $(\check{\rho}^* z_1, \check{\rho}^* z_2, \check{\rho}^* z_3) = (\check{\varepsilon}^{a'} z_1, \check{\varepsilon}^{b'} z_2, \check{\varepsilon} z_3)$ where $\check{\varepsilon}$ is a primitive mth root of unity. We denote by $X_{a,m}$ the quotient of NC^2 by this $\langle\check{\rho}\rangle$-action.

Then the result in [5] which motivates this paper is as follows:

THEOREM 2.3 (Kollár and Shepherd-Barron [5, 5.2]). *Let (X, x) be a two-dimensional smoothable semi-log-terminal singularity which is neither a smooth point, a rational double point, a normal crossing point, nor a pinch point. Then (X, x) is analytically isomorphic to $X_{a,d,m}$ or $X_{a,m}$.*

The above $X_{a,d,m}$ and $X_{a,m}$ are the main objects of this paper.

3. The λ-expansion and μ-expansion. In this section we introduce the notions of the λ-expansion and μ-expansion which are needed later to compute general multi-canonical divisors on $X_{a,d,m}$ and $X_{a,m}$. Let m and a be positive integers such that $m/2 \leq a < m$ and that a is prime to m. Put $b = m - a$, and let $a/b = [q_1, q_2, \dots, q_k]$ be the regular continued fraction expansion. Let r_i be the ith remainder of the Euclidean algorithm for a/b and P_i/Q_i the ith convergent for a/b. *We assume that k is even throughout this paper*, because the odd case can be treated in a way similar to the even case, with minor and obvious modifications. For a pair of integers (i, j) such that $1 \leq i \leq k+1$, that $j \geq 1$, and that i is odd (resp. even), we denote $P_{i-2} + Q_{i-2} + (j-1)(P_{i-1} + Q_{i-1})$ by $\lambda_{i,j}$ (resp. $\mu_{i,j}$). We agree that $\lambda_{i,0} = 0$ (resp. $\mu_{i,0} = 0$) for all odd i's (resp. even i's). In this situation, we have the following.

PROPOSITION-DEFINITION 3.1. *Let n be an integer such that $1 \leq n \leq m-1$. Then n can be written uniquely as*

$$n = \lambda_{1,j_1} + \sum_{\substack{3 \leq i \leq k-1 \\ i \text{ odd}}} (l_{i-1}\lambda_{i,1} + \lambda_{i,j_i}) + l_k \lambda_{k+1,1} \qquad \left(\text{resp. } n = \sum_{\substack{2 \leq i \leq k \\ i \text{ even}}} (l_{i-1}\mu_{i,1} + \mu_{i,j_i})\right),$$

where $j_1, l_2, j_3, l_4, \dots, j_{k-1}, l_k$ (resp. $l_1, j_2, l_3, j_4, \dots, l_{k-1}, j_k$) are integers that satisfy the following conditions:

(i) (a) $0 \leq l_i \leq q_i + 1$ *for any even (resp. odd) i.*

(b) $0 \leq j_i \leq q_i$ *for any odd (resp. even) i, and $j_i \neq 1$ for $i \geq 2$.*

(ii) *If there exists even (resp. odd) i_0 such that $l_{i_0} = q_{i_0} + 1$, then there exist odd (resp. even) i_1 and i_2 that satisfy the following conditions*:

(a) $0 \leq i_1 < i_0 < i_2 \leq k$.

(b) $l_{i'} = q_{i'}$ *for any even (resp. odd) i' such that $i_1 < i' < i_2$ and that $i' \neq i_0$.*

(c) $j_{i'} = 0$ *for any odd (resp. even) i' such that $i_1 \leq i' \leq i_2$.*

(d) $l_{i_1-1} \leq q_{i_1-1} - 1$ *if $i_1 \geq 2$.*

(e) $l_{i_2+1} \leq q_{i_2+1} - 1$ *if $i_2 \leq k-1$.*

(iii) *If there exists even (resp. odd) i_0 such that $l_{i_0} = q_{i_0}$ and $j_{i_0+1} \geq 2$, then there exists odd (resp. even) i_3 that satisfies the following conditions*:

(a) $0 \leq i_3 < i_0$.

(b) $l_{i'} = q_{i'}$ *for any even (resp. odd) i' such that $i_3 < i' \leq i_0$.*

(c) $j_{i'}=0$ *for any odd* (*resp. even*) i' *such that* $i_3 \leq i' < i_0$.

(d) $l_{i_3-1} \leq q_{i_3-1}-1$ *if* $i_3 \geq 2$.

We call this expression of n *the* λ*-expansion* (*resp.* μ*-expansion*) *of* n *with respect to* a/b.

Let us give an example.

EXAMPLE 3.2. Let $m=9976$ and $a=6961$. Then we have $b=3015$, $a/b=[2, 3, 4, 5, 6, 7]$, $\lambda_{1,1}=1$, $\lambda_{1,2}=2$, $\lambda_{3,1}=3$, $\lambda_{3,2}=13$, $\lambda_{3,3}=23$, $\lambda_{3,4}=33$, $\lambda_{5,1}=43$, $\lambda_{5,2}=268$, $\lambda_{5,3}=493$, $\lambda_{5,4}=718$, $\lambda_{5,5}=943$, $\lambda_{5,6}=1168$ and $\lambda_{7,1}=1393$. Let $n=9503$. We determine l_6, j_5, l_4, j_3, l_2 and j_1 successively in this order by the following manner. In the firste step, we determine l_6, j_5 and n_5'. We determine l_6 by $n=n_5+l_6\lambda_{7,1}$, where n_5 and l_6 are integers such that $0 \leq n_5 < \lambda_{7,1}$. In this case, we have $l_6=6$ and $n_5=1145$. If $n_5 < \lambda_{5,2}$, then we put $j_5=0$; otherwise, we pick up the greatest $\lambda_{5,j'}$ smaller than or equal to n_5 from $\{\lambda_{5,j}\}_{2 \leq j \leq 6}$ and put $j_5=j'$. In this case, we have $j_5=5$. We put $n_5' := n_5 - \lambda_{5,j_5} = 202$. Here we obtain $9503 = n_5' + \lambda_{5,5} + 6\lambda_{7,1}$. In the second step, we repeat the first step for n_5', $\lambda_{5,1}$ and $\{\lambda_{3,j}\}_{2 \leq j \leq 4}$ instead of n, $\lambda_{7,1}$ and $\{\lambda_{5,j}\}_{2 \leq j \leq 6}$ respectively, to determined l_4, j_3 and n_3' instead of l_6, j_5 and n_5' respectively. Namely, we divide 202 by $\lambda_{5,1}$ and obtain $202 = 30 + 4\lambda_{5,1}$, and hence we put $l_4=4$. Since $\lambda_{3,3} \leq 30 < \lambda_{3,4}$, we put $j_3=3$. We put $n_3' = 30 - \lambda_{3,3} = 7$, which we send to the next step. In the last step, we put $l_2=2$ and $j_1=1$ since we divide n_3' by $\lambda_{3,1}$ to obtain the quotient 2 and the remainder $1=\lambda_{1,1}$. As a result, we have $(l_6, j_5, l_4, j_3, l_2, j_1)=(6, 5, 4, 3, 2, 1)$ and $9503 = \lambda_{1,1} + 2\lambda_{3,1} + \lambda_{3,3} + 4\lambda_{5,1} + \lambda_{5,5} + 6\lambda_{7,1}$. This is in fact the λ-expansion of 9503 with respect to 6961/3015. By a similar computation that uses $\mu_{i,j}$'s instead of $\lambda_{i,j}$'s, we obtain $9503 = 0 \cdot \mu_{2,1} + \mu_{2,0} + 2\mu_{4,1} + \mu_{4,0} + 4\mu_{6,1} + \mu_{6,7}$, and this is in fact the μ-expansion of 9503 with respect to 6961/3015.

In fact, we can always compute the λ- and μ-expansion in the above way. Namely, the λ-expansion is characterized as follows:

LEMMA 3.3. *Let* n *be an integer such that* $1 \leq n \leq m-1$. *Let*

$$n = \lambda_{1,j_1} + \sum_{\substack{3 \leq i \leq k-1 \\ i \text{ odd}}} (l_{i-1}\lambda_{i,1} + \lambda_{i,j_i}) + l_k \lambda_{k+1,1}$$

be the λ*-expansion of* n *with respect to* a/b. *Then we have the following*:

(i) *For any odd* h, *we have*

$$l_{h-1} = \max\left\{ l \in \boldsymbol{Z} \,\middle|\, l\lambda_{h,1} \leq \lambda_{1,j_1} + \sum_{\substack{3 \leq i \leq h-2 \\ i \text{ odd}}} (l_{i-1}\lambda_{i,1} + \lambda_{i,j_i}) + l_{h-1}\lambda_{h,1} \right\}.$$

(ii) *For odd* h, *put* $\bar{j}_h := \max\{ j \in \boldsymbol{Z}_{\geq 0} \mid \lambda_{h,j} \leq \lambda_{1,j_1} + \sum_{3 \leq i \leq h,\ i \text{ odd}} (l_{i-1}\lambda_{i,1} + \lambda_{i,j_i}) \}$. *Then we have*

$$j_h = \begin{cases} \bar{j}_h & \text{if } \bar{j}_h \geq 2 \text{ or if } h=1, \\ 0 & \text{otherwise}. \end{cases}$$

Obviously we can characterize the μ-expansion similarly. We leave the elementary proofs of Proposition-Definition 3.1 and Lemma 3.3 to the reader.

In the rest of this section, we introduce the notation relevant to the λ- and μ-expansion needed later. We define a set $I_{a,m}$ of pairs of integers by

$$I_{a,m}=\{(i, j) \mid 1 \leq i \leq k+1;\ 1 \leq j \leq q_i \text{ (for } i \leq k),\ j=1 \text{ (for } i=k+1)\}\ .$$

We denote by $\boldsymbol{L}_{a,m}$ the set of all $\boldsymbol{Z}$-valued functions on $I_{a,m}$. For $v \in \boldsymbol{L}_{a,m}$ and $\iota \in I_{a,m}$, we denote by v_ι the value of v at ι. For $\iota \in I_{a,m}$, we define $\delta^\iota \in \boldsymbol{L}_{a,m}$ by $\delta^\iota_\iota=1$ and $\delta^\iota_\eta=0$ for $\eta \neq \iota$. We agree that $\delta^{(i,0)}=0$ for any integer i.

DEFINITION 3.4. Let n be an integer such that $1 \leq n \leq m-1$. Let

$$n=\lambda_{1,j_1}+\sum_{\substack{3 \leq i \leq k-1 \\ i \text{ odd}}}(l_{i-1}\lambda_{i,1}+\lambda_{i,j_i})+l_k\lambda_{k+1,1}=\sum_{\substack{2 \leq i \leq k \\ i \text{ even}}}(l_{i-1}\mu_{i,1}+\mu_{i,j_i})$$

be the λ- and μ-expansion of n with respect to a/b. We define $v_{a,m}(n) \in \boldsymbol{L}_{a,m}$ as follows:

$$v_{a,m}(n)=\delta^{(1,j_1)}+\sum_{\substack{3 \leq i \leq k-1 \\ i \text{ odd}}}(l_{i-1}\delta^{(i,1)}+\delta^{(i,j_i)})+l_k\delta^{(k+1,1)}+\sum_{\substack{2 \leq i \leq k \\ i \text{ even}}}(l_{i-1}\delta^{(i,1)}+\delta^{(i,j_i)})\ .$$

4. General multi-canonical divisors. In this section, we compute general multi-canonical divisors on two-dimensional smoothable semi-log-terminal singularities in terms of the λ- and μ-expansion introduced in Section 3. Before doing so, we determine what we mean by the term "general" in this paper.

Esnault [3] introduced the notion of "full sheaf" for sheaves on a resolution of a two-dimensional rational singularity. Note that we can generalize this notion for sheaves on a semi-resolution of a two-dimensional semi-rational singularity in an obvious manner and can prove bijective correspondence between the set of isomorphism classes of full sheaves on the semi-resolution and the set of isomorphism classes of reflexive modules on the singularity. (See [3, Lemma and definition (2.2)].) We use this notion of full sheaf to describe a Weil divisor on a two-dimensional semi-rational singularity as a "general member" in the associated complete linear system.

First we fix notation and terminology. For Weil divisors D and D' on a surface singularity Z, we say that D *is linearly equivalent to* D' when there exists a rational function f on Z such that $D-D'=(f)$. For a Weil divisor D on Z, *the complete linear system associated with* D, denoted by $|D|$, is the set of effective divisors that are linearly equivalent to D. When we are given a semi-resolution $p: \tilde{Z} \to Z$ of a two-dimensional semi-rational singularity Z, we denote by $F(\mathscr{M})$ the full sheaf on $\tilde{Z}$ associated with a reflexive module $\mathscr{M}$ on Z.

DEFINITION 4.1. Let (Z, z) be a two-dimensional semi-rational singularity and D a Weil divisor on it. Let $p: \tilde{Z} \to Z$ be the minimal good semi-resolution and $\tilde{D}$ the proper transform of D in $\tilde{Z}$. We call D a *general member of* $|D|$ if $\mathcal{O}_{\tilde{Z}}(\tilde{D}) \simeq F(\mathcal{O}_Z(D))$ and if $\tilde{D}$ intersects the exceptional locus transversely.

Note that general members always exist since the full sheaf is generated by global sections. We give a numerical characterization of "general members" in the next lemma, which justifies Definition 4.1 from the viewpoint of the intersection theory. For a $\boldsymbol{Q}$-Cartier divisor D on a surface singularity (Z, z) and an exceptional curve E of a semi-resolution $p: \tilde{Z} \to Z$, we define a rational number $\alpha_E(D)$ by $p^*D = \tilde{D} + \alpha_E(D)E + E'$, where $\tilde{D}$ is the proper transform of D in $\tilde{Z}$ and E' is a $\boldsymbol{Q}$-linear combination of the exceptional curves other than E.

LEMMA 4.2. *Let (Z, z) be a two-dimensional semi-rational singularity and D a $\boldsymbol{Q}$-Cartier Weil divisor on Z. Let $p: \tilde{Z} \to Z$ be the minimal good semi-resolution. Assume that the proper transform $\tilde{D}$ of D in $\tilde{Z}$ intersects the exceptional locus transversely. Then, D is a general member of $|D|$ if and only if the inequality $\alpha_E(D) \leq \alpha_E(D')$ holds for any $D' \in |D|$ and any exceptional curve E in $\tilde{Z}$.*

PROOF. Suppose that D is a general member of $|D|$ and D' is a member of $|D|$. Let I be the set of exceptional curves in $\tilde{Z}$, and $\boldsymbol{E} = \bigcup_{E \in I} E$. Since $\mathcal{O}_{\tilde{Z}}(\tilde{D})$ is a full sheaf, we have $H^0_{\boldsymbol{E}}(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(\tilde{D})) \simeq H^1_{\boldsymbol{E}}(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(\tilde{D})) \simeq 0$. Hence we obtain

$$(1) \qquad H^0(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(\tilde{D})) \simeq H^0(\tilde{Z} \setminus \boldsymbol{E}, \mathcal{O}_{\tilde{Z}}(\tilde{D})) \simeq H^0(Z, \mathcal{O}_Z(D)),$$

using the exact sequence for local cohomology groups. We regard D' as the zero locus of a section $\mathcal{O}_Z(D)$. Then, (1) means that there exists an effective divisor of the form $\sum_{E \in I} \beta_E E$ such that $\tilde{D}' + \sum_{E \in I} \beta_E E$ is linearly equivalent to $\tilde{D}$, where $\tilde{D}'$ is the proper transform of D' in $\tilde{Z}$. On the other hand, $D' \in |D|$ implies that $\tilde{D}$ and $\tilde{D}' + \sum_{E \in I} (\alpha_E(D') - \alpha_E(D))E$ are linearly equivalent. Therefore we have $\alpha_E(D') - \alpha_E(D) = \beta_E \geq 0$, and hence we have proved the "only if" part.

Next suppose that $\tilde{D}$ and E intersect transversely and that $\alpha_E(D) \leq \alpha_E(D')$ for any $D' \in |D|$. Choose a general member D_0 of $|D|$. We have already shown that $\alpha_E(D_0) \leq \alpha_E(D)$ for any $E \in I$. Thus $\alpha_E(D_0) = \alpha_E(D)$ for any $E \in I$. This means that $\tilde{D}$ and $\tilde{D}_0$ are numerically equivalent. Therefore they are linearly equivalent since Z is a semi-rational singularity. Hence $\mathcal{O}_{\tilde{Z}}(\tilde{D})$ is a full sheaf, and so we have proved the "if" part. ■

COROLLARY 4.3. *Let (Z, z) be a two-dimensional semi-rational singularity and D_1 and D_2 be $\boldsymbol{Q}$-Caritier Weil divisors on (Z, z). Assume that D_1 and D_2 are general members of $|D_1|$ and $|D_2|$ respectively and the proper transforms of D_1 and D_2 in the minimal good semi-resolution have no intersection. Then*

$$D_1 \cdot D_2 = \min\{D_1' \cdot D_2' \mid D_1' \in |D_1|, D_2' \in |D_2|, \dim D_1' \cap D_2' = 0\}.$$

Now we compute general members of the multi-canonical systems of two-dimensional smoothable semi-log-terminal singularities. By Theorem 2.3, it suffices to consider $X_{a,d,m}$ and $X_{a,m}$.

4.1. $X_{a,d,m}$. The symbols b, k, q_i, r_i, P_i and Q_i, which represent numbers derived from m and a, are the same as in Section 3. Let $p_{a,d,m}\colon \tilde{X}_{a,d,m} \to X_{a,d,m}$ be the minimal resolution. We denote its exceptional locus, which is a chain of $\boldsymbol{P}^1$'s (see, e.g., Brieskorn [2]), by $\bigcup_{(i,j)\in I(X_{a,d,m})} E_{i,j}$, where the index set $I(X_{a,d,m})$ is defined by

$$I(X_{a,d,m}) = \left\{ (i,j) \in \boldsymbol{Z}\times\boldsymbol{Z} \;\middle|\; \begin{array}{l} 1 \le i \le k+1\,; \\ 1\le j \le q_i \text{ (for } i<k),\ 1\le j\le q_k-1 \text{ (for } i=k)\,, \\ 1 \le j \le d \text{ (for } i=k+1)\,. \end{array} \right\},$$

and the indices are assigned to the exceptional $\boldsymbol{P}^1$'s by the following rule:

(i) $(E_{1,1})^2 = -2$.

(ii) $E_{i,j}\cdot E_{i',j'} = 1$ if either (a) $i=i'$ and $|j-j'|=1$, (b) $\{(i,j),(i',j')\} = \{(i,q_i),(i+2,1)\}$, or (c) $\{(i,j),(i',j')\} = \{(k,q_k-1),(k+1,d)\}$.

Note that

$$(E_{i,j})^2 = \begin{cases} -2-q_{i-1} & \text{if } \ j=1 \text{ and } (i,j)\ne(k+1,d)\,, \\ -2 & \text{if } \ j\ge 2 \text{ and } (i,j)\ne(k+1,d)\,, \\ -3 & \text{if } \ (i,j)=(k+1,d) \text{ and } d\ge 2\,, \\ -3-q_k & \text{if } \ (i,j)=(k+1,d) \text{ and } d=1\,. \end{cases}$$

(cf. Kollár and Shepherd-Barron [5, Proposition 3.11].)

Let us recall the $\lambda_{i,j}$'s and $\mu_{i,j}$'s introduced in the preceding section. For $(i,j)\in I(X_{a,d,m})$ and $\zeta = [\gamma_1:\gamma_2] \in \boldsymbol{P}^1 \setminus \{[1:0],[0:1]\}$, we set

$$C^{i,j}(\zeta) := \begin{cases} (\gamma_1 z_1^{\lambda_{i,j}} + \gamma_2 z_2^{-\lambda_{i,j}+dm\{r_{i-1}-(j-1)r_i\}})/\langle\rho\rangle & \text{if } \ i \text{ is odd}\,, \\ (\gamma_1 z_1^{-\mu_{i,j}+dm\{r_{i-1}-(j-1)r_i\}} + \gamma_2 z_2^{\mu_{i,j}})/\langle\rho\rangle & \text{if } \ i \text{ is even}\,. \end{cases}$$

Then $C^{i,j}(\zeta)$ is an irreducible curve on $X_{a,d,m}$ whose proper transform in $\tilde{X}_{a,d,m}$ intersects $E_{i,j}$ transversely and does not intersect the other exceptional curves.

In this situation, we have the following theorem.

THEOREM 4.4 (the $X_{a,d,m}$ case). *Let n be an integer such that $1\le n\le m-1$. Let*

$$n = \lambda_{1,j_1} + \sum_{\substack{3\le i\le k-1 \\ i \text{ odd}}} (l_{i-1}\lambda_{i,1} + \lambda_{i,j_i}) + l_k\lambda_{k+1,1} = \sum_{\substack{2\le i\le k \\ i \text{ even}}} (l_{i-1}\mu_{i,1} + \mu_{i,j_i})$$

be the λ- and μ-expansion of n with respect to a/b. Then

$$C^{1,j_1}(\bar{\zeta}^{(1)}) + \sum_{\substack{3\le i\le k-1 \\ i \text{ odd}}} \left(\sum_{1\le h_{i-1}\le l_{i-1}} C^{i,1}(\zeta^{(i)}_{h_{i-1}}) + C^{i,j_i}(\bar{\zeta}^{(i)}) \right) + \sum_{1\le h_k\le l_k} C^{k+1,1}(\zeta^{(k+1)}_{h_k})$$
$$+ \sum_{\substack{2\le i\le k \\ i \text{ even}}} \left(\sum_{1\le h_{i-1}\le l_{i-1}} C^{i,1}(\zeta^{(i)}_{h_{i-1}}) + C^{i,j_i}(\bar{\zeta}^{(i)}) \right)$$

is a general member of the $(-n)$-canonical system $|-nK_{X_{a,d,m}}|$, where $\zeta^{(i)}_h$, $\bar{\zeta}^{(i)} \in$

$P^1\setminus\{[1:0],[0:1]\}$ such that $\zeta_h^{(i)}\neq\zeta_{h'}^{(i)}$ if $h\neq h'$, and $C^{i,0}(\zeta)$'s are regarded as empty sets.

For the proof of Theorem 4.4, we need the following proposition whose proof is elementary.

PROPOSITION 4.5. *Put $\boldsymbol{T}$, v and $\boldsymbol{T}_{\min}$ as follows:*

$$\boldsymbol{T}=\{(s,t)\in \boldsymbol{Z}_{\geq 0}\times \boldsymbol{Z}_{\geq 0}\,|\, s+(dmb-1)t\equiv dmbn \pmod{dm^2}\}\,,$$

$$v=\min\{s+t\,|\,(s,t)\in \boldsymbol{T}\}\,,\quad \boldsymbol{T}_{\min}=\{(s,t)\in \boldsymbol{T}\,|\, s+t=v\}\,.$$

Then we have the following:

(i) *If $d=1$ and $m-(P_{k-1}+Q_{k-1})\leq n<m$, then $v=2n-m+2(P_{k-1}+Q_{k-1})$ and $\boldsymbol{T}_{\min}=\{(n+P_{k-1}+Q_{k-1},\, n-m+P_{k-1}+Q_{k-1})\}$.*

(ii) *Otherwise, $v=2n$ and $\boldsymbol{T}_{\min}=\{(n,n)\}$.*

PROOF OF THEOREM 4.4. For simplicity, we assume that $d\geq 2$. Modifications needed to treat the case $d=1$ are left to the reader.

We denote by $\boldsymbol{L}(X_{a,d,m})$ the set of all $\boldsymbol{Z}$-valued functions on $I(X_{a,d,m})$. We regard $I_{a,m}$ as a subset of $I(X_{a,d,m})$ by an injective map that sends $(i,j)\in I_{a,m}$ to $(i,j)\in I(X_{a,d,m})$ if $(i,j)\neq(k,q_k)$ and to $(k+1,d)\in I(X_{a,d,m})$ if $(i,j)=(k,q_k)$. We extend $v_{a,m}(n)$ to a function on $I(X_{a,d,m})$ by $v_{a,m}(n)_\iota=0$ for $\iota\notin I_{a,m}$ and regard it as an element of $\boldsymbol{L}(X_{a,d,m})$. We define $v'(n)\in \boldsymbol{L}(X_{a,d,m})$ by $v'(n)_\iota=\deg_{E_\iota}F(\omega_{X_{a,d,m}}^{[-n]})$ for $\iota\in I(X_{a,d,m})$, where $\omega_{X_{a,d,m}}^{[-n]}$ is the triple dual of the nth tensor power of the dualizing sheaf of $X_{a,d,m}$. Then the theorem is restated as follows:

$$v_{a,m}(n)=v'(n)\,. \tag{2}$$

We shall give an arithmetical characterization of $v'(n)$ and show that it forces the above equality.

For a Weil divisor D on $X_{a,d,m}$, we define $\mathrm{ev}(D)\in \boldsymbol{L}(X_{a,d,m})$ by $\mathrm{ev}(D)_\iota=\tilde{D}\cdot E_\iota$ for $\iota\in I(X_{a,d,m})$, where $\tilde{D}$ is the proper transform of D in $\tilde{X}_{a,d,m}$. We extend $\lambda_{i,j}$ (resp. $\mu_{i,j}$) to a function on $I(X_{a,d,m})$ by $\lambda_{i,j}:=-\mu_{i,j}+dm\{r_{i-1}-(j-1)r_i\}$ for even i (resp. $\mu_{i,j}:=-\lambda_{i,j}+dm\{r_{i-1}-(j-1)r_i\}$ for odd i). For $v\in \boldsymbol{L}(X_{a,d,m})$, we define $\alpha(v)\in \boldsymbol{L}(X_{a,d,m})$ by $\alpha(v)_\iota=\sum_{\eta\in I(X_{a,d,m})}v_\eta\alpha_{E_\iota}(C^\eta(\zeta^\eta))$ for $\iota\in I(X_{a,d,m})$, where

$$\alpha_{E_{i,j}}(C^{i',j'}(\zeta^{i',j'}))=\begin{cases}(dm^2)^{-1}\lambda_{i,j}\mu_{i',j'} & \text{if } i \text{ and } i' \text{ are odd and } (i,j)\preceq(i',j')\,,\\ & \text{if } i \text{ and } i' \text{ are even and } (i,j)\succ(i',j')\,,\\ & \text{or if } i \text{ is odd and } i' \text{ is even}\,,\\ (dm^2)^{-1}\mu_{i,j}\lambda_{i',j'} & \text{otherwise}\,.\end{cases}$$

("$\preceq$" means the lexicographic order.) Note that $\alpha(v)_\iota=\alpha_{E_\iota}(\mathrm{ev}(D))$ for a Weil divisor D on $X_{a,d,m}$ and for $\iota\in I(X_{a,d,m})$. Let $\boldsymbol{L}_n:=\{\mathrm{ev}(D)\,|\,D\in|-nK_{X_{a,d,m}}|\}$. Then by Lemma 4.2, $v'(n)$ is the element of $\boldsymbol{L}_n$ that is characterized by $\alpha(v'(n))_\eta\leq\alpha(v)_\eta$ for any $v\in \boldsymbol{L}_n$ and any $\eta\in I(X_{a,d,m})$.

$\boldsymbol{L}_n$ is characterized as follows. Put $d'=\lceil 2^{-1}d\rceil$. Put I' and I'' as follows:

$$I'\{(i,j)\in I(X_{a,d,m})\mid i \text{ is odd, and } j\le d' \text{ if } i=k+1.\}\,,\quad I''=I(X_{a,d,m})\setminus I'\,.$$

For $v\in\boldsymbol{L}(X_{a,d,m})$, define $s(v)$ and $t(v)$ by $s(v)=\sum_{\iota\in I'}v_\iota\lambda_\iota$ and $t(v)=\sum_{\iota\in I''}v_\iota\mu_\iota$ respectively. Then, since $\rho^*\phi=\varepsilon^{(adm-1)s(\mathrm{ev}((\phi)/\langle\rho\rangle))+t(\mathrm{ev}((\phi)/\langle\rho\rangle))}\phi$ for a $\langle\rho\rangle$-semi-invariant $\phi\in\boldsymbol{C}[z_1,z_2]$ and $\rho^*(dz_1\wedge dz_2)^{\otimes(-n)}=\varepsilon^{-dman}(dz_1\wedge dz_2)^{\otimes(-n)}$, we know that $\boldsymbol{L}_n$ consists of the elements of $\boldsymbol{L}(X_{a,d,m})$ satisfying the following conditions:

(i) $v_\iota\ge 0$ for any $\iota\in I(X_{a,d,m})$. (ii) $s(v)+(dmb-1)t(v)\equiv dmbn \pmod{dm^2}$.
Since $s(v_{a,m}(n))=t(v_{a,m}(n))=n$, we have $v_{a,m}(n)\in\boldsymbol{L}_n$. Therefore, we have

$$\alpha(v'(n))_\eta\le\alpha(v_{a,m}(n))_\eta \tag{3}$$

for any $\eta\in I(X_{a,d,m})$.

To prove (2), we first show

$$s(v'(n))=s(v_{a,m}(n))\,,\quad t(v'(n))=t(v_{a,m}(n))\,. \tag{4}$$

For this purpose, we look at $\alpha(v)_{(k+1,d')}$ and $\alpha(v)_{(k+1,d'+1)}$ for $v\in\boldsymbol{L}(X_{a,d,m})$. We have

$$dm^2\alpha(v)_{(k+1,d')}=\beta_1(s(v)+t(v))-\beta_2t(v)\,,$$

where $\beta_1=(d-d'+1)m-(P_{k-1}+Q_{k-1})>0$ and $\beta_2=(d-2d'+2)m-2(P_{k-1}+Q_{k-1})>0$, and

$$dm^2\alpha(v)_{(k+1,d'+1)}=\beta_3(s(v)+t(v))+\beta_4t(v)\,,$$

where $\beta_3=(d-d')m-(P_{k-1}+Q_{k-1})>0$ and $\beta_4=(2d'-d)m+2(P_{k-1}+Q_{k-1})>0$. If $(s(v'(n)),t(v'(n)))\ne(s(v_{a,m}(n)),t(v_{a,m}(n)))=(n,n)$, then we have $s(v'(n))+t(v'(n))>s(v_{a,m}(n))+t(v_{a,m}(n))$ by Proposition 4.5. Thus we have $\alpha(v'(n))_{(k+1,d')}>\alpha(v_{a,m}(n))_{(k+1,d')}$ or $\alpha(v'(n))_{(k+1,d'+1)}>\alpha(v_{a,m}(n))_{(k+1,d'+1)}$ by the above equalities. This contradicts (3). Thus we obtain (4).

Next we show $v_{a,m}(n)_\eta=v'(n)_\eta$ for $\eta\in I'$ by induction on the lexicographic order (which is denoted by "$\preceq$") in I'. Let η be an element of I' which is not $(1,1)$. Assume $v_{a,m}(n)_\iota=v'(n)_\iota$ for any $\iota\in I'$ such that $\iota\succ\eta$. We show $v_{a,m}(n)_\eta=v'(n)_\eta$ under this induction hypothesis. For arbitrary $v\in\boldsymbol{L}(X_{a,d,m})$, we have

$$dm^2\alpha(v)_{\eta^l}=-dm^2v_\eta+\left(s(v)-\sum_{\iota\in I',\iota\succ\eta}v_\iota\lambda_\iota\right)\mu_{\eta^l}+\left(t(v)+\sum_{\iota\in I',\iota\succ\eta}v_\iota\mu_\iota\right)\lambda_{\eta^l}\,, \tag{5}$$

where $\eta^l:=\max\{\iota\in I'\mid\iota\prec\eta\}$. The induction hypothesis and (4) imply

$$s(v_{a,m}(n))-\sum_{\iota\in I',\iota\succ\eta}v_{a,m}(n)_\iota\lambda_\iota=s(v'(n))-\sum_{\iota\in I',\iota\succ\eta}v'(n)_\iota\lambda_\iota \tag{6}$$

and $t(v_{a,m}(n))+\sum_{\iota\in I',\iota\succ\eta}v_{a,m}(n)_\iota\mu_\iota=t(v'(n))+\sum_{\iota\in I',\iota\succ\eta}v'(n)_\iota\mu_\iota$. Thus from (3) and (5) we obtain

$$v'(n)_\eta\ge v_{a,m}(n)_\eta\,. \tag{7}$$

Note that (6) is written as

$$\sum_{\iota\in I',\iota\preceq\eta} v_{a,m}(n)_\iota\lambda_\iota=\sum_{\iota\in I',\iota\preceq\eta} v'(n)_\iota\lambda_\iota\,. \tag{8}$$

By Lemma 3.3, we know that (7) and (8) imply $v'(n)_\eta=v_{a,m}(n)_\eta$.

The same argument shows $v_{a,m}(n)_\eta=v'(n)_\eta$ for $\eta\in I''$. Hence we obtain the equality (2). ■

4.2. $X_{a,m}$. The symbols b, q_i, k, r_i, P_i, Q_i, $\lambda_{i,j}$ and $\mu_{i,j}$ are the same as in Section 3. Let $p_{a,m}\colon \tilde{X}_{a,m}\to X_{a,m}$ be the minimal good semi-resolution. We denote its exceptional locus by $\bigcup_{(i,j)\in I_{a,m}} E_{i,j}$, where the indices are assigned to the exceptional $\boldsymbol{P}^1$'s by the following rule:

(i) $(E_{1,1})^2=-2$.

(ii) $E_{i,j}\cdot E_{i',j'}=1$ if either (a) $i=i'$ and $|j-j'|=1$, or (b) $\{(i,j),(i',j')\}=\{(i,q_i),(i+2,1)\}$.

(iii) $E_{k+1,1}$ and E_{k,q_k} intersect the double curve of $\tilde{X}_{a,m}$.

Note that

$$(E_{i,j})^2=\begin{cases} -2-q_{i-1} & \text{if } j=1 \text{ and } (i,j)\neq(k+1,1)\,,\\ -2 & \text{if } j\geq 2\,,\\ -1-q_k & \text{if } (i,j)=(k+1,1)\,. \end{cases}$$

For $(i,j)\in I_{a,m}$ and $\zeta=[\gamma_1:\gamma_2]\in\boldsymbol{P}^1\setminus\{[1:0],[0:1]\}$, we set

$$\check{C}^{i,j}(\zeta):=\begin{cases} (z_2,\gamma_1 z_3^{\lambda_{i,j}}+\gamma_2 z_1^{r_{i-1}-(j-1)r_i})/\langle\check{\rho}\rangle & \text{if } i \text{ is odd}\,,\\ (z_1,\gamma_1 z_2^{r_{i-1}-(j-1)r_i}+\gamma_2 z_3^{\mu_{i,j}})/\langle\check{\rho}\rangle & \text{if } i \text{ is even}\,. \end{cases}$$

Then $\check{C}^{i,j}(\zeta)$ is an irreducible curve on $X_{a,m}$ whose proper transform in $\tilde{X}_{a,m}$ intersects $E_{i,j}$ transversely and does not intersect the other exceptional curves.

In this situation, we have the following theorem, whose proof is left to the reader since it is almost parallel to that of Theorem 4.4.

THEOREM 4.6 (the $X_{a,m}$ case). *Let n be an integer such that $1\leq n\leq m-1$. Let*

$$n=\lambda_{1,j_1}+\sum_{\substack{3\leq i\leq k-1\\ i\text{ odd}}}(l_{i-1}\lambda_{i,1}+\lambda_{i,j_i})+l_k\lambda_{k+1,1}=\sum_{\substack{2\leq i\leq k\\ i\text{ even}}}(l_{i-1}\mu_{i,1}+\mu_{i,j_i})$$

be the λ- and μ-expansion of n with respect to a/b. Then

$$\check{C}^{1,j_1}(\bar{\zeta}^{(1)})+\sum_{\substack{3\leq i\leq k-1\\ i\text{ odd}}}\left(\sum_{1\leq h_{i-1}\leq l_{i-1}}\check{C}^{i,1}(\zeta^{(i)}_{h_{i-1}})+\check{C}^{i,j_i}(\bar{\zeta}^{(i)})\right)+\sum_{1\leq h_k\leq l_k}\check{C}^{k+1,1}(\zeta^{(k+1)}_{h_k})$$
$$+\sum_{\substack{2\leq i\leq k\\ i\text{ even}}}\left(\sum_{1\leq h_{i-1}\leq l_{i-1}}\check{C}^{i,1}(\zeta^{(i)}_{h_{i-1}})+\check{C}^{i,j_i}(\bar{\zeta}^{(i)})\right)$$

is a general member of the $(-n)$-canonical system $|-nK_{X_{a,m}}|$, where $\zeta^{(i)}_h$, $\bar{\zeta}^{(i)}\in\boldsymbol{P}^1\setminus\{[1:0],[0:1]\}$ such that $\zeta^{(i)}_h\neq\zeta^{(i)}_{h'}$ if $h\neq h'$, and $\check{C}^{i,0}(\zeta)$'s are regarded as empty sets.

5. Local intersection numbers. As an application of the result in Section 4, we shall state and prove Theorem 5.2 in this section. To state the theorem we first define a function $B(M, N)$.

DEFINITION 5.1. For a sequence of positive integers $L=(L_1, L_2, \dots, L_{J(L)})$ and a positive integer N, we define a sequence

$$A(L, N)=(A(L, N)_{-1},\ A(L, N)_0,\ A(L, N)_1, \dots, A(L, N)_{J(L)})$$

by the formulas $A(L, N)_{-1}=A(L, N)_0=N$ and $A(L, N)_j=L_jA(L, N)_{j-1}+A(L, N)_{j-2}$ $(1\leq j\leq J(L))$. For a pair of positive integers (M, N), we define $B(M, N)$ by

$$B(M, N)=\max_L A(L, N)_{J(L)}\,,$$

where $L=(L_1, L_2, \dots, L_{J(L)})$ runs through all sequences of positive integers such that $\sum_{1\leq j\leq J(L)}L_j=M$.

We easily rewrite the above $B(M, N)$ in an explicit form as

$$B(M, N)=\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{M+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{M+2}\right\}N\,.$$

Now we state the main theorem of this section:

THEOREM 5.2. *Let (X, x) be a two-dimensional smoothable semi-log-terminal singularity, and n a positive integer. Let D and D' be members in $|nK_X|$ without common components. Then, the Gorenstein index of X is bounded by an effectively computable function of n and the local intersection number of D and D' as follows*:

$$\text{index}(X, x)\leq B(D\cdot D'+1, n)\,. \tag{9}$$

The rest of this paper is mainly devoted to the proof of the above theorem. First, if X is Gorenstein, the above theorem is trivial. Thus, it suffices to consider the cases $X=X_{a,d,m}$ and $X=X_{a,m}$ (see Theorem 2.3). Secondly, it can be easily seen that $B(M, N)$ is strictly increasing function with respect to M if we fix N. Thus it suffices to prove the inequality (9) for D and D' in $|nK_X|$ such that $D\cdot D'$ attains the minimal value, that is, by Corollary 4.3, D and D' which are general members in $|nK_X|$ whose proper transforms in the minimal good semi-resolution have no intersection. Thirdly, we easily obtain $\text{index}(X_{a,d,m}, x)=\text{index}(X_{a,m}, x)=m$. Finally, it suffices to consider the case $n<\text{index}(X, x)$ because $B(M, N)\geq N$ holds for all pairs of positive integers M, N. Hence, summing up all the above, we know that Theorem 5.2 follows from the next claim.

CLAIM I. *Let $X=X_{a,d,m}$ or $X=X_{a,m}$. Let n be an integer such that $1\leq n\leq m-1$. Let D and D' be general members of $|nK_X|$ whose proper transforms in the minimal good semi-resolution of X have no intersection. Then*

$$m \leq B(D \cdot D' + 1, n). \tag{10}$$

We know that general members of $|nK_{X_{a,d,m}}|$ and $|nK_{X_{a.m}}|$ are expressed in terms of the λ- and μ-expansion of $m-n$ with respect to a/b (Theorems 4.4 and 4.6). Therefore we can write $D \cdot D'$ in (10) in terms of such expansions. For this purpose, we introduce the following notation. For a pair of positive integers (i, j) such that $i \leq k+1$ and that i is odd (resp. even), we denote $P_{i-2}+(j-1)P_{i-1}$ by $\bar{\lambda}_{i,j}$ (resp. $\bar{\mu}_{i,j}$). We define subsets $I^o_{a,m}$ and $I^e_{a.m}$ of $I_{a,m}$ by $I^o_{a,m}=\{(i, j) \in I_{a,m} \mid i \text{ is odd.}\}$ and $I^e_{a,m}=I_{a,m} \setminus I^o_{a,m}$ respectively. We denote by $\preceq$ the lexicographic order in $I_{a,m}$. We define a $\boldsymbol{Z}$-valued symmetric bilinear form "$\circ$" on $\boldsymbol{L}_{a,m}$ by

$$\delta^\iota \circ \delta^\eta = \begin{cases} -\lambda_\iota \bar{\lambda}_\eta & (\iota, \eta \in I^o_{a,m}, \iota \preceq \eta) \\ -\bar{\lambda}_\iota \lambda_\eta & (\iota, \eta \in I^o_{a,m}, \iota \succ \eta) \\ \mu_\iota \bar{\mu}_\eta & (\iota, \eta \in I^e_{a,m}, \iota \preceq \eta) \\ \bar{\mu}_\iota \mu_\eta & (\iota, \eta \in I^e_{a,m}, \iota \succ \eta) \\ 0 & (\text{otherwise}). \end{cases}$$

With $\nu \in \boldsymbol{L}_{a,m}$, we associate an integer $\sigma(\nu)$ (resp. $\tau(\nu)$) by $\sigma(\nu)=\sum_{\iota \in I^o_{a,m}} \nu_\iota \lambda_\iota$ (resp. $\tau(\nu)=\sum_{\iota \in I^e_{a,m}} \nu_\iota \mu_\iota$). Then we have the following lemma.

LEMMA 5.3. *Let D and D' be as in Claim* I. *Then we have*

$$D \cdot D' = \nu_{a,m}(m-n) \circ \nu_{a,m}(m-n).$$

PROOF. In the case $X=X_{a,d,m}$, the lemma follows from the following intersection formula. For simplicity, we assume that $d \geq 2$. Let ν and $\hat{\nu}$ be elements in $\boldsymbol{L}_{a,m}$. We regard ν and $\hat{\nu}$ as elements in $\boldsymbol{L}(X_{a,d,m})$ in the same way as we regard $\nu_{a,m}(n)$ as an element in $\boldsymbol{L}(X_{a,d,m})$ in the proof of Theorem 4.4. Let C and $\hat{C}$ be divisors on $X_{a,d,m}$ of the forms $\sum_{\iota \in I(X_{a,d,m})} \sum_{1 \leq h_\iota \leq \nu_\iota} C^\iota(\zeta^\iota_{h_\iota})$ and $\sum_{\iota \in I(X_{a,d,m})} \sum_{1 \leq \hat{h}_\iota \leq \hat{\nu}_\iota} C^\iota(\hat{\zeta}^\iota_{\hat{h}_\iota})$ respectively. Assume that the proper transforms of C and $\hat{C}$ in $\tilde{X}_{a,d,m}$ have no intersection. Then we have

$$C \cdot \hat{C} = \nu \circ \hat{\nu} + (dm^2)^{-1}\{(dma-1)\sigma(\nu)\sigma(\hat{\nu}) + \sigma(\nu)\tau(\hat{\nu}) + \tau(\nu)\sigma(\hat{\nu}) - (dma+1)\tau(\nu)\tau(\hat{\nu})\}.$$

We leave the case $X=X_{a,m}$ to the reader. ∎

Therefore we can rewrite Claim I as follows.

CLAIM II. *For an integer n such that $1 \leq n \leq m-1$,*

$$m \leq B(\nu_{a,m}(m-n) \circ \nu_{a,m}(m-n) + 1, n).$$

We can easily deduce Claim II from the following two propositions.

PROPOSITION 5.4. *Let n be an integer such that $0<n<m-(P_{k-1}+Q_{k-1})$. Put $i(n):=\max\{i \mid 0 \leq i \leq k-1, P_i+Q_i \leq n\}$. Then*

$$\nu_{a,m}(n) \circ \nu_{a,m}(n) \geq \frac{n}{P_{i(n)}+Q_{i(n)}}. \tag{11}$$

PROPOSITION 5.5. *Let n be an integer such that $m-(P_{k-1}+Q_{k-1})\leq n\leq m-1$. Put $i'(n)$ and $j(n)$ as follows*:

$$i'(n):=\min\{i\,|\,0\leq i\leq k-1,\, m-n\leq P_i+Q_i\}\,,\quad j(n):=\left\lceil \frac{m-n}{P_{i(n)-1}+Q_{i(n)-1}}\right\rceil-1\,.$$

Then

$$v_{a,m}(n)\circ v_{a,m}(n)\geq \sum_{i'(n)\leq h\leq k} q_h-j(n)\,. \tag{12}$$

To complete the proof of Theorem 5.2, we shall prove Proposition 5.4 and 5.5 in the rest of this paper. We start with the following easy lemma. With $v\in \boldsymbol{L}_{a,m}$, we associate an integer $\bar{\sigma}(v)$ (resp. $\bar{\tau}(v)$) by $\bar{\sigma}(v)=\sum_{\iota\in I^o_{a,m}} v_\iota\bar{\lambda}_\iota$ (resp. $\bar{\tau}(v)=\sum_{\iota\in I^e_{a,m}} v_\iota\bar{\mu}_\iota$).

LEMMA 5.6. *Let v, $\hat{v}\in \boldsymbol{L}_{a,m}$. Assume*:

(i) *If $\iota,\eta\in I^o_{a,m}$, $v_\iota\neq 0$ and $\hat{v}_\eta\neq 0$, then $\iota\preceq\eta$. If $\iota,\eta\in I^e_{a,m}$, $v_\iota\neq 0$ and $\hat{v}_\eta\neq 0$, then $\iota\preceq\eta$.*

(ii) *$\sigma(v)=\tau(v)$, $\sigma(\hat{v})=\tau(\hat{v})$, and $\bar{\sigma}(\hat{v})=\bar{\tau}(\hat{v})$.*

Then, $v\circ\hat{v}=0$.

The proof is easy.

For the proof of Proposition 5.4, we introduce two types of special elements of $\boldsymbol{L}_{a,m}$, namely, $\varphi(\iota)$ and $\psi(\iota,\eta)$.

DEFINITION-LEMMA 5.7. (i) *For $\iota=(i,j)\in I_{a,m}$ such that $i\neq k+1$, we define $\varphi(\iota)\in \boldsymbol{L}_{a,m}$ by $\varphi(\iota)=-\delta^{(i,1)}+\delta^{(i,j)}+(j-1)\delta^{(i+1,1)}$.*

(ii) *We have the following formulas*:

$$\sigma(\varphi(\iota))=\tau(\varphi(\iota))=(j-1)(P_{i-1}+Q_{i-1})\,,$$
$$\bar{\sigma}(\varphi(\iota))=\bar{\tau}(\varphi(\iota))=(j-1)P_{i-1}\,,$$
$$\varphi(\iota)\circ\varphi(\iota)=j-1\,.$$

DEFINITION-LEMMA 5.8. (i) *Let $\iota=(i_1,j_1)$ and $\eta=(i_2,j_2)$ be elements in $I_{a,m}$. We say that ψ* is defined for *(ι,η) if it satisfies the following conditions*: (a) $i_2\neq k+1$, (b) *i_1+i_2 is even*, (c) *$\iota\preceq(i_2,1)$. For a pair $(\iota,\eta)=((i_1,j_1),(i_2,j_2))$ for which ψ is defined, we define $\psi(\iota,\eta)\in \boldsymbol{L}_{a,m}$ by*

$$\psi(\iota,\eta)=\begin{cases} -\delta^{\iota^l}+\delta^{\iota}+\displaystyle\sum_{\substack{\iota\preceq(i,1)\preceq\eta\\ i\ \text{odd},\, i\neq 1}} q_{i-1}\delta^{(i,1)}-\delta^{(i_2,1)}+\delta^{\eta}+j_2\delta^{(i_2+1,1)} & (\iota\in I^o_{a,m})\\ -\delta^{\iota^r}+\delta^{\iota}+\displaystyle\sum_{\substack{\iota\preceq(i,1)\preceq\eta\\ i\ \text{even}}} q_{i-1}\delta^{(i,1)}-\delta^{(i_2,1)}+\delta^{\eta}+j_2\delta^{(i_2+1,1)} & (\iota\in I^e_{a,m})\,,\end{cases}$$

where $\iota^l:=\max\{\iota'\in I^o_{a,m}\,|\,\iota'\prec\iota\}$ and $\iota^r:=\max\{\iota'\in I^e_{a,m}\,|\,\iota'\prec\iota\}$.

(ii) *Let $(\iota,\eta)=((i_1,j_1),(i_2,j_2))$ be a pair for which ψ is defined, and (i',i'') a pair of integers. We say that (ι,η)* is of type (i',i'') *if it satisfies the following*:

$$i_2=i''\,,\quad i_1=\begin{cases} i' & ((i_1, j_1)=(1, 1) \text{ or } j_1\geq 2) \\ i'+2 & (\textit{otherwise})\,. \end{cases}$$

(iii) *Let $(\imath, \eta)$ be a pair for which ψ is defined. Assume that $\eta=(i'', j'')$ and that $(\imath, \eta)$ is of type (i', i''). Then we have the following formulas*:

$$\sigma(\psi(\imath, \eta))=\tau(\psi(\imath, \eta))=j''(P_{i''-1}+Q_{i''-1})\,,$$

$$\bar{\sigma}(\psi(\imath, \eta))=j''P_{i''-1}\,,$$

$$\bar{\tau}(\psi(\imath, \eta))=\begin{cases} j''P_{i''-1} & (\imath\neq(2, 1)) \\ j''P_{i''-1}+1 & (\imath=(2, 1))\,, \end{cases}$$

$$\psi(\imath, \eta)\circ\psi(\imath, \eta)=\begin{cases} j''+\displaystyle\sum_{\substack{i'+1\leq i\leq i''-1 \\ i\text{ even}}} q_i & (\imath\in I^o_{a,m}) \\ j''+\displaystyle\sum_{\substack{i'+1\leq i\leq i''-1 \\ i\text{ odd}}} q_i & (\imath\in I^e_{a,m})\,. \end{cases}$$

We can check the above formulas by direct computations.

PROOF OF PROPOSITION 5.4. We use induction on $i(n)$. If $i(n)=0$, then the λ- and μ-expansions of n are $n=\lambda_{1,n}=n\mu_{2,1}$, i.e., $v_{a,m}(n)=\delta^{(1,n)}+n\delta^{(2,1)}$. Hence, a direct computation yields $v_{a,m}(n)\circ v_{a,m}(n)=n$. Therefore we are done in this case.

Let i be an integer such that $1\leq i\leq k-1$. Assume that the inequality (11) holds for all n's such that $i(n)<i$. We shall show that the inequality (11) holds for n such that $i(n)=i$ under this induction hypothesis. We only treat the case where i is odd since the proof for even i is similar.

Let n be an integer such that $i(n)=i$. Write $n=j(P_i+Q_i)+n'$ such that $0\leq n'<P_i+Q_i$. Note that $1\leq j\leq q_{i+1}$ (resp. $1\leq j\leq q_k-1$) if $i\leq k-3$ (resp. $i=k-1$). We divide the proof into three cases: (I) $n'=0$, (II) $1\leq n'<P_{i-1}+Q_{i-1}$, and (III) $P_{i-1}+Q_{i-1}\leq n'<P_i+Q_i$.

Case (I). In this case,the λ-expansion of n is $n=j\lambda_{i+2,1}$ and the μ-expansion of n is

$$n=\begin{cases} (q_1+1)\mu_{2,1}+\displaystyle\sum_{\substack{4\leq h\leq i-1 \\ h\text{ even}}} q_{h-1}\mu_{h,1}+(q_i-1)\mu_{i+1,1}+\mu_{i+1,j} & (i\geq 3, j\geq 2) \\ (q_1+1)\mu_{2,1}+\displaystyle\sum_{\substack{4\leq h\leq i-1 \\ h\text{ even}}} q_{h-1}\mu_{h,1}+q_i\mu_{i+1,1} & (i\geq 3, j=1) \\ q_1\mu_{2,1}+\mu_{2,j} & (i=1, j\geq 2) \\ (q_1+1)\mu_{2,1} & (i=1, j=1)\,. \end{cases}$$

Hence, we know that $v_{a,m}(n)=\psi((2, 1), (i+1, j))$. Thus by Definition-Lemma 5.8 (iii), we have $v_{a,m}(n)\circ v_{a,m}(n)\geq j$. As a result, we have $v_{a,m}(n)\circ v_{a,m}(n)(P_i+Q_i)\geq j(P_i+Q_i)=n$, and hence we are done in this case.

Case (II). Put $\eta=\min\{\iota\in I^{e}_{a.m}\mid \iota'\prec\iota$ for any $\iota'\in I^{e}_{a,m}$ such that $v_{a,m}(n')_{\iota'}\neq 0\}$. We claim:

$$v_{a,m}(n)=v_{a,m}(n')+\psi(\eta,(i+1,j))\,. \tag{13}$$

PROOF OF (13). We look at the relation of the λ- and μ-expansion between n' and n. Let $n'=\lambda_{1,j_1}+\sum_{\substack{3\le h\le i\\ h\text{ odd}}}(l_{h-1}\lambda_{h,1}+\lambda_{h,j_h})$ be the λ-expansion of n'. Then the λ-expansion of n is $n=\lambda_{1,j_1}+\sum_{\substack{3\le h\le i\\ h\text{ odd}}}(l_{h-1}\lambda_{h,1}+\lambda_{h,j_h})+j\lambda_{i+2,1}$. For the μ-expansion, we have two cases: (α) $\eta^r=(i',j_{i'})$ in which $j_{i'}\ge 2$, and (β) $\eta^r=(i',1)$. (The suffix "r" is the same as in Definition-Lemma 5.8 (i).) In the case (α), n' has the μ-expansion of the form $n'=\sum_{\substack{2\le h\le i'\\ h\text{ even}}}(l_{h-1}\mu_{h,1}+\mu_{h,j_h})$. Then the μ-expansion of n is

$$n=\begin{cases}
\sum_{\substack{2\le h\le i'-2\\ h\text{ even}}}(l_{h-1}\mu_{h,1}+\mu_{h,j_h})+l_{i'-1}\mu_{i',1}+\mu_{i',j_{i'}+1} & \\
\quad+\sum_{\substack{i'+2\le h\le i-1\\ h\text{ even}}}q_{h-1}\mu_{h,1}+(q_i-1)\mu_{i+1,1}+\mu_{i+1,j} & (j_{i'}\le q_{i'}-1,\ j\ge 2)\\
\sum_{\substack{2\le h\le i'-2\\ h\text{ even}}}(l_{h-1}\mu_{h,1}+\mu_{h,j_h})+l_{i'-1}\mu_{i',1}+\mu_{i',j_{i'}+1} & \\
\quad+\sum_{\substack{i'+2\le h\le i-1\\ h\text{ even}}}q_{h-1}\mu_{h,1}+q_i\mu_{i+1,1} & (j_{i'}\le q_{i'}-1,\ j=1)\\
\sum_{\substack{2\le h\le i'-2\\ h\text{ even}}}(l_{h-1}\mu_{h,1}+\mu_{h,j_h})+l_{i'-1}\mu_{i',1}+\mu_{i',0}+(q_{i'+1}+1)\mu_{i'+2,1} & \\
\quad+\sum_{\substack{i'+4\le h\le i-1\\ h\text{ even}}}q_{h-1}\mu_{h,1}+(q_i-1)\mu_{i+1,1}+\mu_{i+1,j} & (i'\le i-3,\ j_{i'}=q_{i'},\ j\ge 2)\\
\sum_{\substack{2\le h\le i'-2\\ h\text{ even}}}(l_{h-1}\mu_{h,1}+\mu_{h,j_h})+l_{i'-1}\mu_{i',1}+\mu_{i',0}+(q_{i'+1}+1)\mu_{i'+2,1} & \\
\quad+\sum_{\substack{i'+4\le h\le i-1\\ h\text{ even}}}q_{h-1}\mu_{h,1}+q_i\mu_{i+1,1} & (i'\le i-3,\ j_{i'}=q_{i'},\ j=1)\\
\sum_{\substack{2\le h\le i'-2\\ h\text{ even}}}(l_{h-1}\mu_{h,1}+\mu_{h,j_h})+l_{i-2}\mu_{i-1,1}+\mu_{i-1,0}+q_i\mu_{i+1,1}+\mu_{i+1,j} & \\
 & (i'=i-1,\ j_{i'}=q_{i'},\ j\ge 2)\\
\sum_{\substack{2\le h\le i'-2\\ h\text{ even}}}(l_{h-1}\mu_{h,1}+\mu_{h,j_h})+l_{i-2}\mu_{i-1,1}+\mu_{i-1,0}+(q_i+1)\mu_{i+1,1} & \\
 & (i'=i-1,\ j_{i'}=q_{i'},\ j=1)\,.
\end{cases}$$

Using these equalities, we can check the equality (13) in the case (α). We leave the case

(β) to the reader. ■

From the equality (13), we have

$$v_{a,m}(n)\circ v_{a,m}(n)=v_{a,m}(n')\circ v_{a,m}(n')+2v_{a,m}(n')\circ\psi(\eta,(i+1,j))+\psi(\eta,(i+1,j))\circ\psi(\eta,(i+1,j)).$$

The first term on the right hand side of the above equality is greater than or equal to $n'/(P_i+Q_i)$ by the induction hypothesis. The second is zero by Lemma 5.6, and the third is greater than or equal to j by Definition-Lemma 5.8 (iii). Thus we obtain the inequality (11) in this case.

Case (III). In this case, n' has the λ- and μ-expansions of the form

$$n'=\lambda_{1,j_1}+\sum_{\substack{3\le h\le i\\ h\,\text{odd}}}(l_{h-1}\lambda_{h,1}+\lambda_{h,j_h})=\sum_{\substack{2\le h\le i-1\\ h\,\text{even}}}(l_{h-1}\mu_{h,1}+\mu_{h,j_h})+l_i\mu_{i+1,1},$$

where $l_i>0$. Thus we know that the λ- and μ-expansions of n are

$$n=\lambda_{1,j_1}+\sum_{\substack{3\le h\le i\\ h\,\text{odd}}}(l_{h-1}\lambda_{h,1}+\lambda_{h,j_h})+j\lambda_{i+2,1}$$

$$=\sum_{\substack{2\le h\le i-1\\ h\,\text{even}}}(l_{h-1}\mu_{h,1}+\mu_{h,j_h})+(l_i-1)\mu_{i+1,1}+\mu_{i+1,j+1}.$$

Namely, we obtain $v_{a,m}(n)=v_{a,m}(n')+\varphi(i+1,j+1)$. Thus we obtain the inequality (11) by an argument similar to that in Case (II), using Definition-Lemma 5.7 (ii). ■

Next we shall prove Proposition 5.5. For this purpose, we introduce $\theta(i,j)\in \boldsymbol{L}_{a,m}$.

DEFINITION-LEMMA 5.9. (i) *For $(i,j)\in I_{a,m}$ such that $1\le i\le k-1$ and $1\le j\le q_i$, we define $\theta(i,j)\in\boldsymbol{L}_{a,m}$ by $\theta(i,j)=-\delta^{(i,q_i-j+1)}+\delta^{(i+2,1)}+j\delta^{(i+1,1)}$.*

(ii) *We have*

$$\sigma(\theta(i,j))=\tau(\theta(i,j))=j(P_{i-1}+Q_{i-1}),$$

$$\bar\sigma(\theta(i,j))=\bar\tau(\theta(i,j))=jP_{i-1},$$

$$\theta(i,j)\circ\theta(i,j)=j.$$

(iii) *Let n, i, j be positive integers such that $m-(P_{k-1}+Q_{k-1})\le n\le m-1$, $i\le k-1$, $j\le q_i$ and $i'(n)\le i-1$. Then $v_{a,m}(n)\circ\theta(i,j)=j$.*

PROOF. We can check (ii) and (iii) by direct computations. As for (iii), when i is even, note that n has the λ- and μ-expansions of the form

$$n=\lambda_{1,j_1}+\sum_{\substack{3\le h\le i-1\\ h\,\text{odd}}}(l_{h-1}\lambda_{h,1}+\lambda_{h,j_h})+l_i\lambda_{i+1,1}+\sum_{\substack{i+3\le h\le k-1\\ h\,\text{odd}}}q_{h-1}\lambda_{h,1}+q_k\lambda_{k+1,1}$$

$$=\sum_{\substack{2\le h\le i-2\\ h\,\text{even}}}(l_{h-1}\mu_{h,1}+\mu_{h,j_h})+l_{i-1}\mu_{i,1}+\sum_{\substack{i+2\le h\le k-2\\ h\,\text{even}}}q_{h-1}\mu_{h,1}+q_{k-1}\mu_{k,1}+\mu_{k,q_k}.$$

The details are left to the reader. ■

Finally, we prove Proposition 5.5, thereby completing the proof of Theorem 5.2.

PROOF OF PROPOSITION 5.5. We use induction on $i'(n)$. If $i'(n)=0$, then $n=m-1$ and its λ- and μ-expansions are

$$m-1=\lambda_{1,0}+\sum_{\substack{3\le h\le k-1\\ h\text{ odd}}}(q_{h-1}\lambda_{h,1}+\lambda_{h,0})+q_k\lambda_{k+1,1}$$

$$=\sum_{\substack{2\le h\le k-2\\ h\text{ even}}}(q_{h-1}\mu_{h,1}+\mu_{h,0})+q_{k-1}\mu_{k,1}+\mu_{k,q_k}\,.$$

From these we obtain $v_{a,m}(m-1)\circ v_{a,m}(m-1)=\sum_{1\le h\le k}q_h$. Thus we are done in this case.

Let i be a positive integer and assume that the inequality (12) holds for all n's such that $i'(n)<i$. We shall show that (12) holds for n such that $i'(n)=i$ under this induction hypothesis.

Let n be an integer such that $i'(n)=i$. Put $j:=j(n)$ and $n':=n+j(P_{i-1}+Q_{i-1})$. Then $i(n')<i$. In this situation we claim

$$v_{a,m}(n)=v_{a,m}(n')-\theta(i,j)\,. \tag{14}$$

PROOF OF (14). We only treat the case where i is odd, since the argument for even i is the same. First, since $m-(P_{i-1}+Q_{i-1})\le n'\le m-1$, n' has the λ-expansion of the form

$$n'=\lambda_{1,j_1}+\sum_{\substack{3\le h\le i-2\\ h\text{ odd}}}(l_{h-1}\lambda_{h,1}+\lambda_{h,j_h})+l_{i-1}\lambda_{i,1}+\lambda_{i,0}+\sum_{\substack{i+2\le h\le k-1\\ h\text{ odd}}}q_{h-1}\lambda_{h,1}+q_k\lambda_{k+1,1}\,.$$

Thus we know that the λ-expansion of n is

$$n=\begin{cases}\lambda_{1,j_1}+\sum_{\substack{3\le h\le i-2\\ h\text{ odd}}}(l_{h-1}\lambda_{h,1}+\lambda_{h,j_h})+l_{i-1}\lambda_{i,1}+\lambda_{i,q_i-j+1}\\ \qquad+(q_{i+1}-1)\lambda_{i+2,1}+\sum_{\substack{i+4\le h\le k\\ h\text{ odd}}}q_{h-1}\lambda_{h,1}+q_k\lambda_{k+1,1} & (j\le q_i-1)\\ \lambda_{1,j_1}+\sum_{\substack{3\le h\le i-2\\ h\text{ odd}}}(l_{h-1}\lambda_{h,1}+\lambda_{h,j_h})+(l_{i-1}+1)\lambda_{i,1}\\ \qquad+(q_{i+2}-1)\lambda_{i+2,1}+\sum_{\substack{i+4\le h\le k\\ h\text{ odd}}}q_{h-1}\lambda_{h,1}+q_k\lambda_{k+1,1} & (j=q_i)\,.\end{cases}$$

Next, n' has the μ-expansion of the form

$$n'=\sum_{\substack{2\le h\le i-1\\ h\text{ even}}}(l_{h-1}\mu_{h,1}+\mu_{h,j_h})+l_i\mu_{i+1,1}+\sum_{\substack{i+3\le h\le k-2\\ h\text{ even}}}q_{h-1}\mu_{h,1}+q_{k-1}\mu_{k,1}+\mu_{k,q_k}\,,$$

where $l_i \geq q_i - 1$. We can easily check that $l_i = q_i$ if $j = q_i$, and hence we know that the μ-expansion of n is

$$n = \sum_{\substack{2 \leq h \leq i-1 \\ h \text{ even}}} (l_{h-1}\mu_{h,1} + \mu_{h,j_h}) + (l_i - j)\mu_{i+1,1} + \sum_{\substack{i+3 \leq h \leq k-2 \\ h \text{ even}}} q_{h-1}\mu_{h,1} + q_{k-1}\mu_{k,1} + \mu_{k,q_k}.$$

We can check the equality (14) by the above equalities. ■

From the equality (14) and Definition-Lemma 5.9, we obtain

$$v_{a,m}(n) \circ v_{a,m}(n) = v_{a,m}(n') \circ v_{a,m}(n') - j. \tag{15}$$

By the induction hypothesis, we have

$$v_{a,m}(n') \circ v_{a,m}(n') \geq \sum_{i'(n') \leq h \leq k} q_h - j(n') \geq \sum_{i(n')+1 \leq h \leq k} q_h. \tag{16}$$

From (15) and (16) we obtain the inequality (12). ■

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