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A BOUNDARY UNIQUENESS THEOREM FOR SOBOLEV FUNCTIONS

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Abstract. We show that under a condition on the Dirichlet integral, Sobolev functions with zero boundary values at the points of a set of positive capacity are identically zero.

1. Main result. All zeros of nonconstant analytic functions of the complex plane are isolated by a basic topological property of analytic functions. Results on the size of the zero set of the radial limits or nontangential boundary values of analytic functions of the unit disk and some related classes of functions have been proved by several authors. We now briefly review such theorems.

Beurling [1] proved that for nonconstant analytic functions f of the unit disk B with $|f'|\in L^2(B)$ the radial limit values cannot be equal to zero on a set of positive capacity. Carleson [2, Thm. 4] found examples of nonconstant analytic functions f of B with $|f'|\in L^2(B)$ and with radial limit zero on a given closed set $E\subset\partial B$ of zero capacity, and also proved other results about such functions. Tsuji [9] proved that an analytic function f, having radial limits zero on a set of positive capacity, is identically zero provided that there exists $c>0$ with

$$I(\varepsilon)=\int_{B_\varepsilon}|f'|^2 dm\leq c\varepsilon^2 \tag{1}$$

for all $\varepsilon\in(0, 1/2)$, where $B_\varepsilon=\{z\in B:|f(z)|<\varepsilon\}$ and dm is an element of the Lebesgue measure on B. Jenkins [4] extended this result assuming that the integral in (1) has order $\varepsilon^2\log(1/\varepsilon)$ (see also Villamor [11] with $o(\varepsilon^2\log(1/\varepsilon))$). Recently, Koskela [5] has established this result for $ACL^p(B)$-functions u with

$$I(\varepsilon)=\int_{B_\varepsilon}|\nabla u|^p dm\leq C\varepsilon^p\left(\log\frac{1}{\varepsilon}\right)^{p-1}, \qquad 1<p\leq n, \tag{2}$$

for $\varepsilon\in(0, 1/2)$, where $B=\{x\in R^n:|x|<1\}$ and

$$B_\varepsilon=\{x\in B:|u(x)|<\varepsilon\}, \qquad \varepsilon>0.$$

For related results see Mizuta [8].

Let $\mathscr{B}\subset R^n$ be a bounded domain and u be a (continuous) $ACL^p(\mathscr{B})$-function. Fix

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$\varepsilon > 0$ and denote

$$I(\varepsilon) = \int_{\mathscr{B}_\varepsilon} |\nabla u|^p dm\,, \qquad \mathscr{B}_\varepsilon = \{x \in \mathscr{B} : |u(x)| < \varepsilon\}\,.$$

We prove the following result.

THEOREM. *Let $u \in ACL^p(\mathscr{B})$, $p > 1$, be a continuous function approaching zero on a set $E \subset \partial\mathscr{B}$ of positive p-capacity. If the integrals $I(\varepsilon)$ satisfy one of the conditions*:

$$\int_0 \left(\frac{1}{I'(\varepsilon)}\right)^{1/(p-1)} d\varepsilon = \infty\,; \tag{3}$$

or

$$\int_0 \left(\frac{\varepsilon}{I(\varepsilon)}\right)^{1/(p-1)} d\varepsilon = \infty\,; \tag{4}$$

or there exists a nonnegative function $f(\varepsilon)$ with conditions

$$I(\varepsilon) \leq \varepsilon^p (f(\varepsilon))^{p-1} \qquad \textit{for all} \quad \varepsilon \in (0, 1/2)\,, \tag{5}$$

and

$$\sum_{k=0}^{\infty} \frac{1}{f(2^{-k})} = \infty\,; \tag{6}$$

or

$$\liminf_{\varepsilon \to 0} \frac{I(\varepsilon)}{\varepsilon^p} < \infty\,; \tag{7}$$

then u is identically zero.

REMARK 1. As we will see from the proof of Theorem we have (5) & (6) $\Rightarrow$ (4) $\Rightarrow$ (3), (7) $\Rightarrow$ (4) $\Rightarrow$ (3). It is clear that (1) $\Rightarrow$ (7) and (2) $\Rightarrow$ (5) & (6).

2. Sets of positive p-capacity on $\partial\mathscr{B}$. Fix an arbitrary domain $\mathscr{B} \subset R^n$. Let P, $Q \subset \mathscr{B}$ be relatively closed sets in $\mathscr{B}$ with $P \cap Q = \varnothing$. The p-capacity of the condenser $(P, Q; \mathscr{B})$ is defined by

$$\operatorname{cap}_p(P, Q; \mathscr{B}) = \inf_{v \in W} \int_{\mathscr{B}} |\nabla v|^p dm\,, \qquad p > 1\,,$$

where the infimum is taken over all functions v in the class

$$W(P, Q; \mathscr{B}) = \{v \in ACL^p(\mathscr{B}) : v|_P = 0,\ v|_Q = 1\}\,.$$

Here $ACL^p(\mathscr{B})$ denotes the class of all continuous functions v with the generalized (in

Sobolev's sense) derivatives $\partial v/\partial x_i$ $(i=1, 2, \ldots, n)$ satisfying

$$\int_{\mathscr{B}} |\nabla v|^p dm < \infty ,$$

where $\nabla v = (\partial v/\partial x_1, \partial v/\partial x_2, \ldots, \partial v/\partial x_n)$ stands for the formal gradient of the function v.

Let $E \subset \partial\mathscr{B}$ be a nonempty set and D, $\bar{D} \subset \mathscr{B}$, a bounded nonempty domain. We consider an open set $\mathcal{O} \subset \mathscr{B}$ such that $\mathcal{O} \cap D = \varnothing$, and assume that $\gamma \cap \mathcal{O} \neq \varnothing$ holds for every arc $\gamma \subset \mathscr{B}$ joining the domain D with the set E.

We say that $\operatorname{cap}_p E > 0$ if

$$\inf \operatorname{cap}_p(\bar{\mathcal{O}}, \bar{D}; \mathscr{B}) > 0 , \tag{8}$$

where the infimum is taken over all the sets $\mathcal{O}$ defined above.

Otherwise we say that $\operatorname{cap}_p E = 0$.

It is clear that the property (8) is independent of the choice of the domain D (see for example [12, Section 6]).

3. Proof of the Theorem. The proof of our Theorem is divided into four parts: A. (3) $\Rightarrow$ the statement, B. (4) $\Rightarrow$ (3), C. (5) & (6) $\Rightarrow$ (4), D. (7) $\Rightarrow$ (4).

Part A. (3) $\Rightarrow$ the statement. We assume that u is not identically zero, and fix a pair of sets $\mathscr{B}_\varepsilon$, $\mathscr{B}_\delta$, $0 < \varepsilon < \delta$ with $\delta < \sup_{x \in \mathscr{B}} |u(x)|$. The latter condition implies that $\mathscr{B}_\delta \neq \varnothing$. We consider the condenser $(\bar{\mathscr{B}}_\varepsilon, F; \mathscr{B})$ where $F = \bar{\mathscr{B}} \setminus \mathscr{B}_\delta$.

The set $E \subset \partial\mathscr{B}$ was assumed to be of positive p-capacity. Therefore

$$\liminf_{\varepsilon \to 0} \operatorname{cap}_p(\bar{\mathscr{B}}_\varepsilon, F; \mathscr{B}) > 0 . \tag{9}$$

Let $h: (0, \infty) \to R$ be a Lipschitz function such that

$$h(t)\big|_{t \le \varepsilon} = 0 , \qquad h(t)\big|_{t \ge \delta} = h(\delta) > 0 .$$

Then the function $v = h(|u(x)|)/h(\delta)$, $x \in \mathscr{B}$, belongs to the class $W(\bar{\mathscr{B}}_\varepsilon, F; \mathscr{B})$. Therefore

$$\liminf_{\varepsilon \to 0} \operatorname{cap}_p(\bar{\mathscr{B}}_\varepsilon, F; \mathscr{B}) > 0 . \tag{10}$$

From the definition of the p-capacity it follows that

$$h^p(\delta) \operatorname{cap}_p(\bar{\mathscr{B}}_\varepsilon, F; \mathscr{B}) \le \int_{\mathscr{B}} |\nabla v(x)|^p dx = \int_{\varepsilon < u(x) < \delta} |h'(u(x))|^p |\nabla u(x)|^p dx . \tag{11}$$

Here we used the chain rule [6], [13] and the convention that the product

$$|h'(u(x))|^p |\nabla u(x)|^p$$

is defined to be zero on the set $\{\nabla u = 0\}$ even if $h'(u(x))$ is undefined. We transform the right side of the equation (11). It has the form

$$\int_{\mathscr{B}} \Phi(x) |\nabla u(x)| dx ,$$

where

$$\Phi(x) = |h'(u(x))|^p |\nabla u(x)|^{p-1} , \qquad x \in \mathscr{B}_{\varepsilon,\delta} = \{x \in \mathscr{B} : \varepsilon < u(x) < \delta\} .$$

We will derive a coarea formula for a general nonnegative measurable function Φ. First we suppose that Φ is a simple function, in other words there are k measurable sets $\mathscr{M}_1, \ldots, \mathscr{M}_k$, $\mathscr{M}_k \cap \mathscr{M}_l = \varnothing$, $k \neq l$, and numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$\Phi(x) = \sum_{i=1}^{k} \alpha_i \chi_{\mathscr{M}_i}(x) ,$$

where $\chi_{\mathscr{M}_i}$ is the characteristic function of a set $\mathscr{M}_i \subset \mathscr{B}_{\varepsilon,\delta}$. Then we have

$$\int_{\mathscr{B}_{\varepsilon,\delta}} \Phi(x) |\nabla u| dx = \sum_{i=1}^{k} \alpha_i \int_{\mathscr{M}_i} |\nabla u| dx .$$

Let $\mathscr{G} \subset \mathscr{B}_{\varepsilon,\delta}$ be an open set. The function u is a function of bounded variation $BV(\mathscr{G})$ (see Remark 5.1.2 [13]). We have from Theorem 5.4.4 [13]

$$\int_{\mathscr{G}} |\nabla u(x)| dx = \int_R \|\nabla \chi_{E_t}\|(\mathscr{G}) dt = \int_{\varepsilon}^{\delta} P(E_t, \mathscr{G}) dt ,$$

where

$$E_t = \{x \in \mathscr{B} : u(x) > t\}$$

and $P(E_t, \mathscr{G})$ is the perimeter of a set E_t in $\mathscr{G}$ for a.e. $t \in R$. We get from Theorem 5.8.1 [13]

$$P(E_t, \mathscr{G}) = H^{n-1}(\partial^* E_t \cap \mathscr{G}) ,$$

where H^{n-1} is the $(n-1)$-dimensional measure and the reduced boundary $\partial^* E_t$ is defined in 5.6.4 [13]. Therefore

$$\begin{aligned} \int_{\mathscr{B}_{\varepsilon,\delta}} \chi_{\mathscr{G}}(x) |\nabla u(x)| dx &= \int_{\mathscr{G}} |\nabla u(x)| dx \\ &= \int_{\varepsilon}^{\delta} H^{n-1}(\partial^* E_t \cap \mathscr{G}) dt \\ &= \int_{\varepsilon}^{\delta} \left(\int_{\partial^* E_t \cap \mathscr{B}} \chi_{\mathscr{G}}(x) dH^{n-1} \right) dt . \end{aligned}$$

Now, if $\mathscr{M} \subset \mathscr{B}_{\varepsilon,\delta}$ is measurable then there is a sequence of open subsets $\mathscr{G}_1 \supset \mathscr{G}_2 \supset \cdots$ of $\mathscr{B}_{\varepsilon,\delta}$ such that $\chi_{\mathscr{G}_j} \to \chi_{\mathscr{M}}$ a.e. Then using Lebesgue's dominated convergence theorem we derive that the preceding formula holds also for $\mathscr{M}$ in place of $\mathscr{G}$. Hence, for a simple function Φ,

$$\int_{\mathscr{B}_{\varepsilon,\delta}} \Phi(x)dx = \int_{\varepsilon}^{\delta} dt \int_{\partial^* E_t \cap \mathscr{B}} \Phi(x) dH^{n-1} ,$$

where

$$E_t = \{x \in \mathscr{B}_{\varepsilon,\delta} : u(x) > t\} .$$

Consider next the case when the function Φ is an arbitrary nonnegative measurable function. Then there is an increasing sequence $\{\Phi_l\}_{l=1,2,\cdots}$ of nonnegative simple functions which converge to $\Phi(x)$ in the domain $\mathscr{B}_{\varepsilon,\delta}$ ([3] p. 88). According to the well-known B. Levi theorem about integration of the increasing sequences of nonnegative measurable functions [3] §27,

$$\int_{\mathscr{B}_{\varepsilon,\delta}} \Phi |\nabla u| dx = \lim_k \int_{\mathscr{B}_{\varepsilon,\delta}} \Phi_k |\nabla u| dx = \lim_k \int_{\varepsilon}^{\delta} dt \int_{\partial^* E_t \cap \mathscr{B}} \Phi_k dH^{n-1} .$$

Using the Levi theorem for functions

$$\mathscr{F}_k(t) = \int_{\partial^* E_t \cap \mathscr{B}} \Phi_k(x) dH^{n-1} ,$$

and then for functions Φ_k, we complete the proof of the formula

$$\int_{\mathscr{B}_{\varepsilon,\delta}} \Phi(x) |\nabla u(x)| dx = \int_{\varepsilon}^{\delta} dt \int_{\partial^* E_t \cap \mathscr{B}} \Phi(x) dH^{n-1} .$$

For our special form of the function $\Phi(x)$

$$\Phi(x) = |h'(t)|^p |\nabla u(x)|^{p-1}$$

we obtain

$$(12) \qquad \int_{\varepsilon < u(x) < \delta} |h'(u(x))|^p |\nabla u(x)|^p dx = \int_{\varepsilon}^{\delta} |h'(t)|^p dt \int_{\partial^* E_t \cap \mathscr{B}} |\nabla u(x)|^{p-1} dH^{n-1} .$$

In the same way, we obtain

$$I(\varepsilon) = \int_0^{\varepsilon} dt \int_{\partial^* E_t \cap \mathscr{B}} |\nabla u(x)|^{p-1} dH^{n-1}$$

and

$$(13) \qquad I'(\varepsilon) = \int_{\partial^* E_{\varepsilon} \cap \mathscr{B}} |\nabla u(x)|^{p-1} dH^{n-1}$$

for almost all $\varepsilon > 0$.

We have from (11), (12) and (13)

(14) $$h^p(\delta)\,\mathrm{cap}_p(\bar{\mathscr{B}}_\varepsilon, F; \mathscr{B}) \le \int_\varepsilon^\delta |h'(t)|^p I'(t)dt \equiv U(h)\,.$$

We shall calculate the quantity

$$A = \min_h U(h)/h^p(\delta)\,.$$

Since by Hölder's inequality

$$h^p(\delta) \le \left(\int_\varepsilon^\delta |h'(t)|dt\right)^p \le \left(\int_\varepsilon^\delta |h'(t)|^p I'(t)dt\right)\left(\int_\varepsilon^\delta (I'(t))^{-1/(p-1)}dt\right)^{p-1},$$

we have

$$\left(\int_\varepsilon^\delta (I'(t))^{1/(1-p)}dt\right)^{1-p} \le A\,.$$

Let $\{\varphi_k\}$ be an increasing sequence of bounded nonnegative functions such that $\varphi_k(\tau) \nearrow (I'(\tau))^{1/(1-p)}$. Set

$$h_k(t) = \int_\varepsilon^t \varphi_k(\tau)d\tau\,, \qquad k = 1, 2, \ldots$$

for $t \in (\varepsilon, \delta]$ and $h_k(t) = 0$ for $t \le \varepsilon$, $h_k(t) = h_k(\delta)$ for $t \ge \delta$.

Then h_k is Lipschitz and we get from (14)

$$h_k^p(\delta)\mathrm{cap}_p(\bar{\mathscr{B}}_\varepsilon, F; \mathscr{B}) \le \int_\varepsilon^\delta I'(t)\varphi_k^p(t)dt \le \int_\varepsilon^\delta (I'(t))^{1/(1-p)}dt\,.$$

By Levi's theorem,

$$h_k^p(\delta) \nearrow \left(\int_\varepsilon^\delta (I'(t))^{1/(1-p)}dt\right)^p.$$

We get

$$\mathrm{cap}_p(\bar{\mathscr{B}}_\varepsilon, F; \mathscr{B}) \le \left(\int_\varepsilon^\delta (I'(t))^{1/(1-p)}dt\right)^{1-p}.$$

This inequality proves that the condition (3) contradicts (10). Therefore u is identically zero.

Part B. (4) $\Rightarrow$ (3). For all $0 < \varepsilon < \delta$ we have

(15) $$\left(\int_\varepsilon^\delta \left(\frac{t-\varepsilon}{I(t)}\right)^{1/(p-1)} dt\right)^p \le \int_\varepsilon^\delta \left(\frac{t-\varepsilon}{I(t)}\right)^{p/(p-1)} I'(t)dt\left(\int_\varepsilon^\delta (I'(t))^{1/(1-p)}dt\right)^{p-1}.$$

First we estimate

$$\int_\varepsilon^\delta \left(\frac{t-\varepsilon}{I(t)}\right)^{p/(p-1)} I'(t)dt = (1-p)\left(\frac{(t-\varepsilon)^p}{I(t)}\right)^{1/(p-1)}\Bigg|_\varepsilon^\delta + p\int_\varepsilon^\delta \left(\frac{t-\varepsilon}{I(t)}\right)^{1/(p-1)} dt$$

$$\leq p\int_\varepsilon^\delta \left(\frac{t-\varepsilon}{I(t)}\right)^{1/(p-1)} dt$$

and we get from (15)

$$\int_\varepsilon^\delta \left(\frac{t-\varepsilon}{I(t)}\right)^{1/(p-1)} dt \leq p^{1/(p-1)}\int_\varepsilon^\delta (I'(t))^{1/(1-p)}dt\,. \tag{16}$$

Next for arbitrary ε, $2\varepsilon<\delta$, we obtain

$$\int_\varepsilon^\delta \left(\frac{t-\varepsilon}{I(t)}\right)^{1/(p-1)} dt = \int_\varepsilon^\delta \left(1-\frac{\varepsilon}{t}\right)^{1/(p-1)}\left(\frac{t}{I(t)}\right)^{1/(p-1)} dt$$

$$\geq \int_{2\varepsilon}^\delta \left(1-\frac{\varepsilon}{t}\right)^{1/(p-1)}\left(\frac{t}{I(t)}\right)^{1/(p-1)} dt \geq \left(\frac{1}{2}\right)^{1/(p-1)}\int_{2\varepsilon}^\delta \left(\frac{t}{I(t)}\right)^{1/(p-1)} dt$$

and for fixed δ the condition

$$\lim_{\varepsilon\to 0}\int_{2\varepsilon}^\delta \left(\frac{t}{I(t)}\right)^{1/(p-1)} dt = \infty$$

implies by (16)

$$\lim_{\varepsilon\to 0}\int_\varepsilon^\delta (I'(t))^{1/(1-p)}dt = \infty\,.$$

Part C. (5) & (6) $\Rightarrow$ (4). We have

$$\int_0^1 \left(\frac{\varepsilon}{I(\varepsilon)}\right)^{1/(p-1)} d\varepsilon = \sum_{k=0}^\infty \int_{2^{-k-1}}^{2^{-k}} \left(\frac{\varepsilon}{I(\varepsilon)}\right)^{1/(p-1)} d\varepsilon \geq \sum_{k=0}^\infty I^{1/(1-p)}(2^{-k})\int_{2^{-k-1}}^{2^{-k}} \varepsilon^{1/(p-1)}d\varepsilon$$

$$= \frac{p-1}{p}\sum_{k=0}^\infty (2^{-kp/(p-1)} - 2^{-(k+1)p/(p-1)})I^{1/(1-p)}(2^{-k})$$

$$= \frac{p-1}{p}(1-2^{-p/(p-1)})\sum_{k=0}^\infty (2^{kp}I(2^{-k}))^{1/(1-p)}\,.$$

Therefore, we get from (6)

$$\int_0^1 \left(\frac{\varepsilon}{I(\varepsilon)}\right)^{1/(p-1)} d\varepsilon \geq \frac{p-1}{p}(1-2^{-p/(p-1)})\sum_{k=0}^\infty \frac{1}{f(2^{-k})} = \infty\,.$$

Part D. (7) $\Rightarrow$ (4). Let $t_k\to 0$ be a sequence with $2t_{k+1}<t_k$ such that

$$I(t_k)\leq Ct_k^p\,, \qquad C\equiv\text{const}>0\,, \quad k=1,2,\ldots\,.$$

Then

$$\begin{aligned}\int_0^{t_1}\left(\frac{t}{I(t)}\right)^{1/(p-1)}dt &= \sum_{k=1}^{\infty}\int_{t_{k+1}}^{t_k}\left(\frac{t}{I(t)}\right)^{1/(p-1)}dt \\ &\geq \sum_{k=1}^{\infty}\left(\frac{1}{Ct_k^p}\right)^{1/(p-1)}\int_{t_{k+1}}^{t_k} t^{1/(p-1)}dt \\ &= \frac{p-1}{p}C^{1/(1-p)}\sum_{k=1}^{\infty} t_k^{p/(1-p)}(t_k^{p/(p-1)}-t_{k+1}^{p/(p-1)}) \\ &= \frac{p-1}{p}C^{1/(1-p)}\sum_{k=1}^{\infty}\left(1-\left(\frac{t_{k+1}}{t_k}\right)^{p/(p-1)}\right) \\ &\geq \frac{p-1}{p}C^{1/(1-p)}\sum_{k=1}^{\infty}\left(1-\left(\frac{1}{2}\right)^{p/(p-1)}\right)=\infty .\end{aligned}$$

REMARK 2. The case of $ACL^p(\mathscr{B})$-functions is not interesting if $p>n$ and the domain $\mathscr{B}$ is the unit ball $B=B(0,1)\subset R^n$. Then an arbitrary nonempty set $E\subset\partial B=S(0,1)$ has for $p>n$ a positive p-capacity.

However, for every $p>n$ there exist bounded domains $\mathscr{B}\subset R^n$, having nonempty sets $E\subset\partial\mathscr{B}$ of vanishing p-capacity. For example, see [7, p. 226–229, 5.1].

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