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ELASTICAE WITH CONSTANT SLANT IN THE COMPLEX PROJECTIVE PLANE AND NEW EXAMPLES OF WILLMORE TORI IN FIVE SPHERES

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Abstract. We exhibit a reduction of variables criterion for the Willmore variational problem. It can be considered as an application of the Palais principle of symmetric criticality. Thus, via the Hopf map, we reduce the problem of finding Willmore tori (with a certain degree of symmetry) in the five sphere equipped with its standard conformal structure, to that for closed elasticae in the complex projective plane. Then, we succeed in obtaining the complete classification of elasticae with constant slant in this space. It essentially consists in three kinds of elasticae. Two of them correspond with torsion free elasticae. They lie into certain totally geodesic surfaces of the complex projective plane and their slants reach the extremal values. The third type gives a two-parameter family of helices, lying fully in this space. A nice closure condition, involving the rationality of one parameter, is obtained for these helices. Hence, we get three associated families of Willmore tori in the standard five sphere. They are Hopf map liftings of the above mentioned families of elasticae. The method also works for a one-parameter family of conformal structures on the five sphere, which defines a canonical deformation of the standard one.

1. Introduction. Let M be an immersed compact surface (throughout this paper surfaces are assumed to be compact) into a Riemannian manifold $\tilde{M}$. We denote by α and S the mean curvature function of M and the sectional curvature function of $\tilde{M}$ with respect to the tangent space of M, and define

$$\mathcal{W}(M)=\int_M(\alpha^2+S)dv\,.$$

This functional is an invariant under conformal changes of the metric of $\tilde{M}$ and the critical points of $\mathcal{W}$ are called *Willmore surfaces* ([6]).

Minimal surfaces of a sphere are obvious examples of Willmore surfaces in such a sphere. However, N. Ejiri [8], answering to a problem of J. L. Weiner [16], gave an example of a non-minimal Willmore flat torus in S^5. Later, U. Pinkall [15], using a nice description for the Hopf fibration of S^3 onto S^2 (both unit spheres), gave an infinite family of unstable non-minimal Willmore surfaces in S^3 which can be obtained

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as Hopf tori associated to certain closed elastic curves in $\boldsymbol{S}^2$ ([12]).

In [3], B. Y. Chen and the first author obtained a complete classification of the Willmore surfaces in any sphere which can be constructed in the corresponding Euclidean space using eigenfunctions associated with two different eigenvalues of the Laplacian (2-type surfaces, [7]), in contrast with minimal surfaces for which only one eigenvalue is needed. In particular, they gave an infinite family of Willmore tori with nonzero constant mean curvature, living fully in $\boldsymbol{S}^5$. This series includes the Ejiri torus.

In this paper we will exploit the usual Hopf fibration of the 5-dimensional unit sphere $\boldsymbol{S}^5$ onto the complex projective space, $\boldsymbol{CP}^2(4)$, of constant holomorphic sectional curvature 4, to obtain more examples of Willmore surfaces in $\boldsymbol{S}^5$.

First, we use the principle of symmetric criticality ([14]) to connect the Willmore variational problem for tori in $\boldsymbol{S}^5$ with the variational problem relative to elastic curves into $\boldsymbol{CP}^2(4)$. More precisely, we notice that the complete lift of any curve γ in $\boldsymbol{CP}^2(4)$ gives a flat cylinder, the *Hopf cylinder* associated with γ, N_γ in $\boldsymbol{S}^5$. Then, we prove that the *Hopf torus* N_γ of a closed curve γ in $\boldsymbol{CP}^2(4)$ is a Willmore torus in $\boldsymbol{S}^5$ if and only if γ is an elastica, with Lagrange multiplier $\lambda=4$, in $\boldsymbol{CP}^2(4)$ (see Theorem 1).

The Ejiri torus in $\boldsymbol{S}^5$ is the only 2-type Willmore torus in $\boldsymbol{S}^5$ which can be obtained as the Hopf torus of a curve γ in $\boldsymbol{CP}^2(4)$. Moreover, this curve is an elastic circle ($\lambda=4$) with Lagrangian osculating plane in $\boldsymbol{CP}^2(4)$ and so it lies as an elastica in some totally geodesic, Lagrangian real projective plane in $\boldsymbol{CP}^2(4)$.

We say that a curve γ in $\boldsymbol{CP}^2(4)$ has *constant slant* if the angle between the complex tangent plane and the osculating plane of γ is constant along γ. In particular, curves with osculating plane either holomorphic or Lagrangian have constant slant.

In this paper we obtain the complete classification of elasticae with constant slant in $\boldsymbol{CP}^2(4)$. It can be described as follows. There are three types of members in this family:

(1) Elasticae living in a totally geodesic, holomorphic surface $\boldsymbol{S}^2(4)$ in $\boldsymbol{CP}^2(4)$. This case corresponds with slant zero.

(2) Elasticae living in a totally geodesic, Lagrangian surface $\boldsymbol{RP}^2(1)$ in $\boldsymbol{CP}^2(4)$. This case corresponds with slant $\pi/2$.

(3) A two-parameter family of elastic helices living fully in $\boldsymbol{CP}^2(4)$. One parameter in this family is the Lagrange multiplier of the elastica.

We notice that the first two cases correspond with elasticae of zero torsion in $\boldsymbol{CP}^2(4)$.

We also consider closure conditions for curves of the third type, to obtain a rational one-parameter family of closed elastic helices in $\boldsymbol{CP}^2(4)$ (with Lagrange multiplier $\lambda=4$).

The existence of a closed elastica, even for an elastic energy functional with potential, has been proved in [9] (see also [10]). The solution given there is stable, namely it is a minimum, however it could be a geodesic. The helices obtained in our classification, in particular those which are closed, provide the first known examples of elasticae, in particular closed elasticae, in a space with no constant sectional curvature and of dimension greater than two (for elastic parallel in a surface of revolution see [4]).

Our main result can be summarized as follows:

There are infinitely many Willmore tori in S^5 which can be obtained by means of the Hopf fibration $\Pi: S^5 \to CP^2(4)$. This family includes the following three subfamilies:

(a) Given a closed elastic curve γ in $S^2(4)$ (a complex and totally geodesic surface in $CP^2(4)$), then $\Pi^{-1}(\gamma)$ is a Willmore torus living fully in some S^3 which is totally geodesic in S^5. This subfamily essentially coincides with that studied by Pinkall. Moreover, the Clifford torus in the above mentioned S^3 is the only constant mean curvature (actually minimal) surface obtained in this way.

(b) Given a closed non-geodesic elastica γ in $RP^2(1)$ (a Lagragian and totally geodesic surface in $CP^2(4)$), $\Pi^{-1}(\gamma)$ is a Willmore torus living fully in S^5. Moreover, the Ejiri torus in S^5 is the only constant mean curvature surface obtained in this way.

(c) Given a closed elastic helix γ in $CP^2(4)$, the Hopf torus $\Pi^{-1}(\gamma)$ is a Willmore torus of nonzero constant mean curvature living fully in S^5. This is a countably infinite family of tori.

In all the cases the Lagrange multiplier of the elasticae must be chosen to be $\lambda = 4$.

The last section of this paper is dedicated to extending the above mentioned results to other conformal structures on S^5, different to the standard one. We can carry out it, because the classification of elasticae with constant slant in $CP^2(4)$ does not essentially depend on the Lagrange multiplier. Therefore, we consider a one-parameter family of conformal structures $\{\mathscr{C}_t / t > 0\}$ on S^5. This is defined by deforming the standard metric of S^5 in such a way that, through the variation, the metrics are nicely projected via the Hopf mapping. In other words, we consider the conformal structures associated with the so-called canonical variation of the usual Hopf Riemannian submersion (see [1] for details).

The reduction of variables algorithm also works for the Willmore variational problems in $(S^5, \mathscr{C}_t)$ (see Theorem 4). This allows us to obtain families of Willmore tori into $(S^5, \mathscr{C}_t)$, which are Hopf tori shaped on elasticae in $CP^2(4)$ with Lagrange multiplier $\lambda = 4t^2$, $(t > 0)$. In particular, we can obtain Willmore tori N_γ into certain $(S^5, \mathscr{C}_t)$, such that $\mathscr{W}(N_\gamma) < 2\pi^2$. This fact contrasts with the Willmore conjecture, namely $\mathscr{W}(T) \geq 2\pi^2$ for any torus in $(S^5, \mathscr{C}_1)$.

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2. Hopf tori in the 5-sphere. We consider the space C^3 of three complex variables endowed with its usual complex structure $\bar{J}$ which can be defined as follows: We identity $z = (z_1, z_2, z_3) \in C^3$ with $(x_1, x_2, x_3, y_1, y_2, y_3) \in R^6$, where $z_k = x_k + \sqrt{-1}y_k$, $k = 1, 2, 3$. Then $\bar{J}_z = (-y_1, -y_2, -y_3, x_1, x_2, x_3)$. We give on $S^5 \subset C^3$ (the unit sphere), its usual contact structure (ξ, η, ϕ) (see [5] for details). In particular $\xi = -\bar{J}z$, where z denotes the position vector on points of $\mathbf{S}^5$, and so ξ is the unit tangent vector field to the fibers of the standard Hopf fibration $\Pi: S^5 \to CP^2(4)$. Here $CP^2(4)$ denotes the 2-dimensional complex projective space with the complex structure J obtained by restricting ϕ to the horizontal distribution $H_z = \langle \bar{J}z \rangle^\perp$, $z \in S^5$ and with the Fubini-Study metric of constant holomorphic sectional curvature 4. The following lemma collects some useful properties of this mapping.

LEMMA. *The following assertions hold*:

(1) *$\Pi: \boldsymbol{S}^5 \to \boldsymbol{CP}^2(4)$ is a Riemannian submersion with fibers being geodesics in $\boldsymbol{S}^5$.*

(2) *The natural action of $\boldsymbol{S}^1$ on $\boldsymbol{S}^5$ to obtain $\boldsymbol{CP}^2(4)$, via Π, as a space of orbits, is made up by isometries of $\boldsymbol{S}^5$.*

(3) *An immersed surface N in $\boldsymbol{S}^5$ is $\boldsymbol{S}^1$-invariant if and only if $N=\Pi^{-1}(\gamma)$, for some immersed curve γ in $\boldsymbol{CP}^2(4)$. In particular, if γ is closed, then $\Pi^{-1}(\gamma)$ is a torus, which is embedded if γ is free of self-intersections into $\boldsymbol{CP}^2(4)$.*

Let $\Gamma: M \to B$ be a harmonic submersion of semi-Riemannian manifolds M and B; that means Γ is a semi-Riemannian submersion and the fibers $\Gamma^{-1}(b)$, $b \in B$, are minimal submanifolds of M (for details on this subject see [1] or [13]). Given an immersed non null curve $\gamma: \boldsymbol{I} \subset \boldsymbol{R} \to B$ (we will always assume that γ is arclength parametrized), we consider the submanifold $N=\Gamma^{-1}(\gamma)$ in M. Denote by $X=\gamma'$ the unit tangent vector field of γ and by $\bar{X}$ its horizontal lift to M. If V_p is the tangent space to the fiber through $p \in M$, then $T_pN=\operatorname{Span}\{\bar{X}(p), V_p\}$ and the normal space of N at p, $T_p^{\perp}N$, is a horizontal subspace.

Let $\bar{\nabla}$ and ∇ be the Levi-Civita connections of M and B associated respectively with their semi-Riemannian metrics $\langle\, ,\rangle$ and $\langle\!\langle\, ,\rangle\!\rangle$. Also σ and h will denote the second fundamental forms of N and the fibers in M respectively. Notice that $\operatorname{trace}(h)=0$ because of the harmonicity of Γ. Now we use basic properties of semi-Riemannian submersions to have

$$\langle \sigma(\bar{X}, \bar{X}), \bar{\xi} \rangle = \langle\!\langle \nabla_X X, \xi \rangle\!\rangle \circ \Gamma ,$$

and

$$\langle \sigma(\eta, \eta), \bar{\xi} \rangle = \langle h(\eta, \eta), \bar{\xi} \rangle ,$$

where $\bar{\xi} \in T_p^{\perp}N$, $d\Gamma_p(\bar{\xi})=\xi$, and $\eta \in V_p$. Therefore if α and κ denote the mean curvature function of N in M and the curvature function of γ in B respectively, we obtain

$$\alpha^2(p)=\frac{1}{n^2}\kappa^2(\Gamma(p)) , \tag{1}$$

where n denotes the dimension of N.

3. Willmore tori in the standard 5-sphere. In this section we obtain a nice connection between the Willmore variational problem in $\boldsymbol{S}^5$ and the elastica variational problem for curves into $\boldsymbol{CP}^2(4)$. Our next result can be regarded as an example of reduction of variables for the Willmore variational problem. The chief point to get it is the principle of symmetric criticality ([14]).

THEOREM 1. *Let γ be a closed immersed curve in $\boldsymbol{CP}^2(4)$ with curvature function κ. Then $N_\gamma = \Pi^{-1}(\gamma)$ is a Willmore torus in $\boldsymbol{S}^5$ if and only if γ is a critical point of the elastic functional $\mathscr{F}(\gamma) = \int_\gamma (\kappa^2 + 4) ds$.*

PROOF. Let $\mathscr{N}$ be the smooth manifold of immersions, φ, of a genus one compact surface N into $\boldsymbol{S}^5$. The Willmore functional $\mathscr{W} : \mathscr{N} \to \boldsymbol{R}$ is given by

$$\mathscr{W}(\varphi) = \int_N (\alpha_\varphi^2 + 1) dv_\varphi ,$$

where α_φ and dv_φ denote the mean curvature function of φ and the volume element of the induced metric by φ on N, respectively. It is obvious that $\mathscr{W}$ is invariant under the usual $\boldsymbol{S}^1$-action on $\boldsymbol{S}^5$, that is, $\mathscr{W}(\varphi) = \mathscr{W}(e^{i\theta} \cdot \varphi)$ for any $\theta \in \boldsymbol{R}$.

We denote by $\mathscr{N}_{\boldsymbol{S}^1}$ the submanifold of $\mathscr{N}$ made up of immersions which are $\boldsymbol{S}^1$-invariants. Then $\mathscr{N}_{\boldsymbol{S}^1}$ can be identified with the set of Hopf tori in $\boldsymbol{S}^5$, that is,

$$\mathscr{N}_{\boldsymbol{S}^1} = \{N_\gamma \mid \gamma \text{ is an immersed closed curve in } \boldsymbol{CP}^2(4)\} .$$

We also write Σ and $\Sigma_{\boldsymbol{S}^1}$ to name the spaces of critical points of $\mathscr{W}$ (Willmore tori), and $\mathscr{W}$ restricted to $\mathscr{N}_{\boldsymbol{S}^1}$, respectively. Now, the principle of symmetric criticality ([14]) may be applied to obtain

$$\Sigma \cap \mathscr{N}_{\boldsymbol{S}^1} = \Sigma_{\boldsymbol{S}^1} .$$

Therefore, we get all Willmore-Hopf tori, by computing the critical points of $\mathscr{W}$ restricted to $\mathscr{N}_{\boldsymbol{S}^1}$. Then, we have

$$\alpha^2(\Pi^{-1}(p)) = \frac{1}{4} \kappa^2(p) , \tag{2}$$

where κ is the curvature function of γ in $\boldsymbol{CP}^2(4)$ and $p \in \gamma$.

Hence

$$\mathscr{W}(N_\gamma) = \int_F (\alpha_\varphi^2 + 1) ds dt ,$$

where F denotes a fundamental region for the covering $f : \boldsymbol{R}^2 \to N$. Thus, we get

$$\mathscr{W}(N_\gamma) = \frac{\pi}{2} \int_\gamma (\kappa^2(s) + 4) ds , \tag{3}$$

and this concludes the proof of the statement. □

Consequently, a way to construct nontrivial (non-minimal) examples of Willmore tori in $\boldsymbol{S}^5$ is to take Hopf tori shaped on non-geodesic extremal curves, γ, for the functional $\mathscr{F}$, that is, on elastic curves in $\boldsymbol{CP}^2(4)$ with Lagrange multiplier $\lambda = 4$. This is similar to the method used by U. Pinkall [15] to get non-minimal Willmore tori in S^3. However, one might propose the converse way: Start from non-minimal Willmore tori in S^5 (because we know a big family, [3]), and then look for those which are Hopf tori to get non-geodesic elastic curves in $\boldsymbol{CP}^2(4)$. This is the aim of the next section.

4. 2-type Willmore surfaces in the standard 5-sphere. If we pay attention to the well-known spectral behaviour of the position vector in $\boldsymbol{R}^6$ of the minimal surfaces in $\boldsymbol{S}^5$, it seems natural to ask for Willmore surfaces in $\boldsymbol{S}^5$ which can be constructed in $\boldsymbol{R}^6$ using eigenfunctions of the Laplacian associated with exactly two different eigenvalues (that means 2-type surfaces, [7]). We start this section with a brief description of the method used in [3], to get examples of non-minimal Willmore surfaces in the sphere. In particular, in this paper, we are interested in the case of the 5-sphere.

We consider a lattice $\Lambda=\{(2n\pi u,\ 2m\pi v+2n\pi w)/n,\ m\in\boldsymbol{Z}\}$ in the Euclidean plane $\boldsymbol{R}^2$, where u, v and w are real numbers with $u, v>0$. By computing its dual lattice, we obtain the spectrum of the flat torus $T_{uvw}=\boldsymbol{R}^2/\Lambda$,

$$\left\{\left(\frac{h}{u}-\frac{kw}{uv}\right)^2+\frac{k^2}{v^2}\,\middle|\, h, k\in\boldsymbol{Z}\right\}. \tag{4}$$

For any nonzero real number ε and two natural numbers h and k, satisfying $\varepsilon\neq 2hk^2/(k^2-2h^2)$, we choose Λ as follows:

$$u=\frac{\sqrt{3}\,\varepsilon k}{A},\quad v=\frac{k}{A},\quad w=\frac{(h-k)\varepsilon}{A},$$

where $A=(2\varepsilon^2+k^2)^{1/2}$.

Now we define an isometric immersion $y:\boldsymbol{R}^2\to\boldsymbol{S}^5\subset\boldsymbol{C}^3$ by

$$y(s,t)=\left(v\cos\left(\frac{t}{v}\right)e^{i\varepsilon s/u};\ v\sin\left(\frac{t}{v}\right)e^{i\varepsilon s/u};\ (1-v^2)^{1/2}e^{iks/u}\right), \tag{5}$$

which induces an isometric immersion x from T_{uvw} into $\boldsymbol{S}^5$.

It was proved in [3] that x defines a Willmore surface in $\boldsymbol{S}^5$. Notice also that the center of mass of x coincides with the center of $\boldsymbol{S}^5$ (in this sense we say that x is of mass-symmetric) and it is constructed in $\boldsymbol{C}^3$ using eigenfunctions of the Laplacian of T_{uvw} associated with two eigenvalues, namely $(\varepsilon/u)^2+1/v^2$ and k^2/u^2. Also x has nonzero constant mean curvature, which is given by

$$\alpha^2=\left(1-\frac{\mu_1}{2}\right)\left(\frac{\mu_2}{2}-1\right),$$

where μ_1 and μ_2 are respectively the minimum and the maximum of both eigenvalues involved in the 2-type nature.

In particular it is not difficult to see that $x: T_{uvw}\to\boldsymbol{S}^5$ is the Hopf torus on a certain closed curve in $\boldsymbol{CP}^2(4)$ if and only if $\varepsilon=k$. In this case Λ is the rectangular lattice generated by $(2\pi\varepsilon, 0)$ and $(0, 2\pi/\sqrt{3})$. The covering y of the (s,t)-plane onto this

Willmore-Hopf torus in S^5 is given by

$$y(s,t)=\frac{\sqrt{3}}{3}e^{is}(\cos\sqrt{3}\,t;\sin\sqrt{3}\,t;\sqrt{2})\,,$$

and its Π-projection gives the curve

$$\gamma(t)=\Pi\left(\frac{\sqrt{3}}{3}(\cos\sqrt{3}\,t;\sin\sqrt{3}\,t;\sqrt{2})\right),$$

which is a closed elastica ($\lambda=4$) in $\boldsymbol{CP}^2(4)$. γ lies fully in a real projective plane $\boldsymbol{RP}^2(1)$ with Gaussian curvature 1, and is totally geodesic and Lagrangian in $\boldsymbol{CP}^2(4)$. Notice also that the osculating plane of γ is a Lagrangian plane in $\boldsymbol{CP}^2(4)$ everywhere.

5. Elastic curves in the complex projective plane. As usual, by torsion we mean the second curvature of a curve in a Riemannian manifold. It will be denoted by τ.

It is well-known that if we have a curve γ in a real space form, say Q, and τ vanishes identically, then we can integrate the distribution defined by the osculating plane along γ in Q to show that γ actually lies in a totally geodesic surface of Q. This is not true in general. For instance, in $\boldsymbol{CP}^2(4)$ we can find torsion-free curves which do not lie in any totally geodesic submanifold of $\boldsymbol{CP}^2(4)$. In order to establish this fact, it will be enough to have torsion-free curves with osculating plane neither holomorphic nor Lagrangian. Examples of this kind of curves in $\boldsymbol{CP}^2(4)$ can be given, for example, by considerating appropriate solutions of the later (32).

Let $\gamma: \boldsymbol{I}\subset\boldsymbol{R}\to\boldsymbol{CP}^2(4)$ be an arclength-parametrized curve and denote by $\{T,\xi_2,\xi_3,\xi_4\}$ a Frenet reference along γ. We recall the standard Frenet equations of γ, where ∇ will denote the Levi-Civita connection of $\boldsymbol{CP}^2(4)$:

$$\nabla_T T=\kappa\xi_2\,, \tag{6}$$

$$\nabla_T\xi_2=-\kappa T-\tau\xi_3\,, \tag{7}$$

$$\nabla_T\xi_3=\tau\xi_2+\delta\xi_4\,, \tag{8}$$

$$\nabla_T\xi_4=-\delta\xi_3\,. \tag{9}$$

It should be noticed that τ is determined up to sign by (7) and (8). Throughout this paper, we take a choice for sign of τ and then determine ξ_3. The same can be done for ξ_4.

If γ is an elastica in $\boldsymbol{CP}^2(4)$, then it satisfies the corresponding Euler equation ([11])

$$2\nabla_T^3 T+\nabla_T[(3\kappa^2-\lambda)T]+2R(\nabla_T T,T)T=0\,, \tag{10}$$

where R denotes the curvature tensor field of $\boldsymbol{CP}^2(4)$ and λ is some real number which works as a Lagrange multiplier associated with this variational problem.

REMARK 1. It is obvious that $JT=\cos\phi_2\xi_2+\cos\phi_3\xi_3+\cos\phi_4\xi_4$ with $\cos^2\phi_2+$

$\cos^2\phi_3+\cos^2\phi_4=1$. In particular, ϕ_2 is the angle between the complex tangent plane $\mathrm{Span}\{T, JT\}$ and the osculating plane $\mathrm{Span}\{T, \xi_2\}$. A curve γ is said to be of constant slant if ϕ_2 is constant along γ.

A straightforward computation involving (6)–(9), (10) and the well-known expression for R show that γ is an elastica of $\boldsymbol{CP}^2(4)$ if and only if $\{\kappa, \tau, \delta\}$ are solutions of

$$0=2\kappa''+\kappa^3-2\kappa\tau^2+(6\cos^2\phi_2+2-\lambda)\kappa\,, \tag{11}$$

$$0=-(2\tau\kappa'+\kappa\tau')+3\kappa\cos\phi_2\cos\phi_3\,, \tag{12}$$

$$0=-\kappa\tau\delta+3\kappa\cos\phi_2\cos\phi_4\,. \tag{13}$$

PROPOSITION 1. *Let γ be a non-geodesic elastica of $\boldsymbol{CP}^2(4)$. Then its torsion τ vanishes identically if and only if the osculating plane of γ is either holomorphic or Lagrangian.*

PROOF. If τ vanishes identically, then we use (6) and (7) in (10) to obtain

$$(3\kappa\cos\phi_2)JT=\left[-\kappa''-\frac{1}{2}\kappa^3+\left(\frac{\lambda}{2}-1\right)\kappa\right]\xi_2\,,$$

which obviously proves that the osculating plane of γ must be either holomorphic or totally real. Conversely, suppose the osculating plane is Lagragian, then $\phi_2=\pi/2$ and so it is constant. Then $\tau\cos\phi_3=0$. If τ does not vanish, then $\cos\phi_3=0$. This allows us to take $JT=\xi_4$. Now (13) gives $\delta=0$ and therefore the Frenet equations show that $\kappa=0$, which is impossible because γ is not a geodesic of $\boldsymbol{CP}^2(4)$. If $\phi_2=0$, then $JT=\xi_2$ and so, we use once more the Frenet equations to obtain $\tau=0$. It should be noticed that in this second case, we do not need the elasticity of γ to prove the converse. □

PROPOSITION 2. *Let γ be an elastica in $\boldsymbol{CP}^2(4)$. Then the following holds.*

(1) *The osculating plane of γ is holomorphic everywhere in $\boldsymbol{CP}^2(4)$ if and only if γ lies, as an elastica, in some $\boldsymbol{S}^2(4)$ which is complex and totally geodesic in $\boldsymbol{CP}^2(4)$.*

(2) *The osculating plane of γ is Lagrangian everywhere in $\boldsymbol{CP}^2(4)$ if and only if γ lies, as an elastica, in some $\boldsymbol{RP}^2(1)$ which is Lagrangian and totally geodesic in $\boldsymbol{CP}^2(4)$.*

PROOF. The "only if" part in both cases is obvious. Conversely, if the osculating plane is holomorphic (resp. Lagrangian) everywhere, then from Proposition 1, we know that $\tau=0$ and so the normal subbundle $\nu=\mathrm{Span}\{\xi_3, \xi_4\}$ is a holomorphic (resp. Lagrangian) subbundle which is parallel in the normal bundle of γ in $\boldsymbol{CP}^2(4)$. Also it is a totally geodesic subbundle (that is, it is made up of totally geodesic directions) and so we can reduce complex (resp. real) codimension to prove that γ lies in a complex (resp. Lagrangian) totally geodesic surface of $\boldsymbol{CP}^2(4)$. □

REMARK 2. It is known that a helix in a Riemannian manifold is a curve which has all the curvatures constant. The next result shows that an elastica with constant slant in $\boldsymbol{CP}^2(4)$ is either a torsion-free elastica or a helix in $\boldsymbol{CP}^2(4)$. By this result and according to (3), we will restrict ourselves to the case $\lambda = 4$. However, an analogous statement can be stated for any value of the Lagrange multiplier λ.

PROPOSITION 3. *Let γ be an elastic with constant slant, say ϕ_2, into $\boldsymbol{CP}^2(4)$. Then either*:

(1) *τ vanishes identically or,*

(2) *$\cos^2\phi_2 \in (0, 2/3)$ and γ is a helix in $\boldsymbol{CP}^2(4)$ with curvatures,*

$$\kappa = \frac{2}{\sqrt{2 - 3\cos^2\phi_2}}\,, \tag{14}$$

$$\tau = -\frac{3\sin 2\phi_2}{2\sqrt{2 - 3\cos^2\phi_2}}\,, \tag{15}$$

$$\delta = -\sqrt{2 - 3\cos^2\phi_2}\,. \tag{16}$$

PROOF. Suppose γ is an elastica with constant slant $\phi_2 \in (0, \pi/2)$. Since τ does not vanish, then $\phi_3 = \pi/2$ and so ϕ_4 is also constant. After standard manipulations and the appropriate choice of ξ_3, we can obtain

$$(\kappa + \delta)\cos\phi_4 + \tau\cos\phi_2 = 0\,, \tag{17}$$

which combined with (12) and (13) gives

$$A\cos\phi_4\kappa^3 + \cos\phi_2 A^2 + 3\cos\phi_2\cos^2\phi_4\kappa^4 = 0\,, \tag{18}$$

where $A = \tau\kappa^2$, which is a non-zero constant because of (12). Consequently, all the coefficients appearing in (18) are constant, and so γ is a helix. To compute κ, τ and δ in terms of the slant ϕ_2, we combine (11), (13) and (17). $\square$

The standard Frenet equations (6)–(9) of a curve γ into $\boldsymbol{CP}^2(4)$ are useful, for example, in defining the concept of helices. To our purposes, however, we have need of a different reference frame along the curve γ, which involves the complex structure J of $\boldsymbol{CP}^2(4)$. One way to describe this frame is to begin by lifting the curve γ in $\boldsymbol{CP}^2(4)$ to a horizontal curve $Y(s)$ in $\boldsymbol{S}^5$. It should be noticed that this lifting is not unique, but different lifts are all of the form $e^{ir}\cdot Y(s)$, r being a constant. We also observe that $Y(s)$ is parametrized by arclength too, because Π is a Riemannian submersion. The tangent vector $T(s) = \gamma'(s)$ lifts to $\bar{T}(s) = Y'(s)$. Now we may uniquely choose a vector field $U(s)$ along $\gamma(s)$, orthogonal to $T(s)$, so that its horizontal lift $\bar{U}(s)$ gives the third component in a special unitary frame $\sigma(s) = \{Y(s), \bar{T}(s), \bar{U}(s)\}$ in $\boldsymbol{C}^3$. In other words, $\sigma(s)$ is a lifting of the curve $\gamma(s)$ to a curve in $SU(3)$. It is not difficult to see that σ satisfies the following differential equation:

$$\frac{d\sigma(s)}{ds}=\sigma(s)\cdot A(s)\,, \tag{19}$$

where $A(s)$ is a matrix in $su(3)$. Since the curve $Y(s)$ is horizontal, then $A(s)$ must have the form:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & \kappa_1 i & -\kappa_2+\kappa_3 i \\ 0 & \kappa_2+\kappa_3 i & -\kappa_1 i \end{pmatrix}.$$

This equation can be projected down to $\boldsymbol{CP}^2(4)$. Hence the new frame along $\gamma(s)$ is $\{T, JT, U, JU\}(s)$ and its associated equations are:

$$\nabla_T T=\kappa_1 JT+\kappa_2 U+\kappa_3 JU\,, \tag{20}$$

$$\nabla_T JT=-\kappa_1 T-\kappa_3 U+\kappa_2 JU\,, \tag{21}$$

$$\nabla_T U=-\kappa_2 T+\kappa_3 JT-\kappa_1 JU\,, \tag{22}$$

$$\nabla_T JU=-\kappa_3 T-\kappa_2 JT+\kappa_1 U\,. \tag{23}$$

Notice that $\kappa_1^2+\kappa_2^2+\kappa_3^2=\kappa^2$. In this setting, it turns out that $\gamma(s)$ is an elastica of $\boldsymbol{CP}^2(4)$ if and only if $\{\kappa_1, \kappa_2, \kappa_3\}(s)$ are solutions of

$$0=\kappa_1''+\kappa_1\left[4-\frac{\lambda}{2}+\frac{1}{2}(\kappa_1^2+\kappa_2^2+\kappa_3^2)\right]+\kappa_3\kappa_2'-\kappa_2\kappa_3'\,, \tag{24}$$

$$0=\kappa_2''+\kappa_2\left[1-\frac{\lambda}{2}+\frac{1}{2}(\kappa_1^2+\kappa_2^2+\kappa_3^2)\right]+\kappa_1\kappa_3'-\kappa_3\kappa_1'\,, \tag{25}$$

$$0=\kappa_3''+\kappa_3\left[1-\frac{\lambda}{2}+\frac{1}{2}(\kappa_1^2+\kappa_2^2+\kappa_3^2)\right]+\kappa_2\kappa_1'-\kappa_1\kappa_2'\,. \tag{26}$$

From now on we will assume that elasticae are not geodesics.

6. Elastic helices in the complex projective plane. In this section, we are going to study elasticae with constant slant, $\phi_2\in(0, \pi/2)$, into $\boldsymbol{CP}^2(4)$. From Proposition 3, we already know that they are helices in $\boldsymbol{CP}^2(4)$. The following proposition gives a partial converse of this fact and so a characterization of elastic helices in $\boldsymbol{CP}^2(4)$.

PROPOSITION 4. *Let $\gamma(s)$ be an elastica of $\boldsymbol{CP}^2(4)$. Then $\gamma(s)$ is a helix in $\boldsymbol{CP}^2(4)$ if and only if its curvature and slant are both constant.*

PROOF. The "if" part is contained in Proposition 3. To prove the converse, we first multiply equation (24) by κ_1', equation (25) by κ_2' and equation (26) by κ_3'. Then we take the sum to obtain

(27)
$$0=(\kappa_1'\kappa_1''+\kappa_2'\kappa_2''+\kappa_3'\kappa_3'')+\frac{1}{2}(\kappa^2)'\left[1-\frac{\lambda}{2}+\frac{1}{2}\kappa^2\right]+3\kappa_1\kappa_1' .$$

Next, we study the relationship between complex and regular Frenet equations of $\gamma(s)$ in $\boldsymbol{CP}^2(4)$. In particular, we compute $\nabla_T^2 T$ in both settings to obtain:

(28)
$$(\kappa_1')^2+(\kappa_2')^2+(\kappa_3')^2=(\kappa')^2+\kappa^2\tau^2 .$$

If κ and τ are constant, then we combine (27) and (28) to obtain the constancy of κ_1 too. Finally, we observe that $\cos\phi_2=\kappa_1/\kappa$, which completes the proof. □

REMARK 3. It should be noticed that, in the course of the proof of the last proposition, we have shown the following claim: Elasticae in $\boldsymbol{CP}^2(4)$ whose curvature and torsion are both constant have constant slant.

Next, we proceed to measure how big the family of elastic helices into $\boldsymbol{CP}^2(4)$ is. Therefore, we suppose that $\gamma(s)$ is an elastic helix in $\boldsymbol{CP}^2(4)$, with slant $\phi_2=\phi$. From $\kappa_1^2+\kappa_2^2+\kappa_3^2=\kappa^2$, it is clear that we can find a function $\psi(s)$, along $\gamma(s)$, such that

(29)
$$\kappa_1=\kappa\cos\phi , \quad \kappa_2=\kappa\sin\phi\cos\psi(s) , \quad \kappa_3=\kappa\sin\phi\sin\psi(s) .$$

We put $\omega=\psi'$ and then combine (28) with (29) to get

$$\tau^2=\omega^2\sin^2\phi ,$$

which proves that ω is also constant.

Since γ satisfies the equations of an elastica, we may substitute these into equations (24)–(26). The resulting equations are dependent and can be simplified as

(30)
$$3C=\omega S^2+\omega C^2-\omega^2 C ,$$

(31)
$$\frac{\lambda}{2}-1=C\omega-\omega^2+\frac{1}{2}(C^2+S^2) ,$$

where C and S are constant given by

$$\kappa_1=C , \quad \kappa_2=S\cos\omega s , \quad \kappa_3=S\sin\omega s .$$

The differential equation (19), giving the lift $\sigma(s)$ of γ to $SU(3)$, may now be written as

$$\sigma'(s)=\sigma(s)\cdot\begin{pmatrix} 0 & -1 & 0 \\ 1 & Ci & -Se^{-i\omega s} \\ 0 & Se^{i\omega s} & -Ci \end{pmatrix}.$$

We define a new curve in $SU(3)$, say $\tilde{\sigma}(s)=(\tilde{Y}, \tilde{Z}, \tilde{U})(s)$, by

$$\tilde{\sigma}(s)=\sigma(s)\cdot\begin{pmatrix} e^{-i\omega s/3} & 0 & 0 \\ 0 & e^{-i\omega s/3} & 0 \\ 0 & 0 & e^{2i\omega s/3} \end{pmatrix}.$$

We observe that $\tilde{Y}(s)=e^{-i\omega s/3}\cdot Y(s)$ and so $\Pi(\tilde{Y}(s))=\Pi(Y(s))=\gamma(s)$. This means that $\tilde{Y}(s)$ is a lift of $\gamma(s)$, although it is not a horizontal curve. The advantage of using $\tilde{\sigma}(s)$ is that it satisfies the following system of differential equations with constant coefficients:

$$\tilde{\sigma}'(s)=\tilde{\sigma}(s)\cdot M\,, \tag{32}$$

where $M\in su(3)$ is given by

$$M=\begin{pmatrix} -i\omega/3 & -1 & 0 \\ 1 & (C-\omega/3)i & -S \\ 0 & S & (2\omega/3-C)i \end{pmatrix}.$$

In particular, we have shown the following claim: *Every elastic helix in $\boldsymbol{CP}^2(4)$ is the image, under the natural projection, of a one-parameter subgroup of $SU(3)$.*

Since we are primarily interested in the case $\lambda=4$ (see §3), we substitute this value into the equations (30) and (31), and rewrite the resulting equations as:

$$\omega(S^2+C^2)=C(3+\omega^2)\,,$$
$$S^2+C^2=2+2\omega^2-2C\omega\,.$$

These equations are simplified to a single relation

$$C=\frac{2\omega}{3}\,. \tag{33}$$

The other constants can also be expressed nicely in terms of ω:

$$\kappa^2=2+\frac{2\omega^2}{3}\,, \tag{34}$$

$$\tau^2=\frac{\omega^2(\omega^2+9)}{3(\omega^2+3)}\,, \tag{35}$$

$$\delta=\frac{2\omega^2}{3}-\kappa\,, \tag{36}$$

$$S^2=2+\frac{2\omega^2}{9}\,. \tag{37}$$

Therefore, we have proved the following.

THEOREM 2. *There exists a two-parameter family of elastic helices, living fully into $\boldsymbol{CP}^2(4)$. If we choose one parameter to be $\lambda=4$, then for any choice of $\omega\neq 0$, we have an elastic helix in $\boldsymbol{CP}^2(4)$ with curvatures and slant given by the formulas* (34)–(37). *In particular, we must have $\kappa^2>2$ for the curvature of such a curve and* $\cos^2\phi=2\omega^2/(9+3\omega^2)<2/3$ *for its slant.*

7. Proof of the main statement. Certainly, there are infinitely many elastic closed curves in $\boldsymbol{CP}^2(4)$ with osculating plane being Lagrangian (or holomorphic). In fact, there are infinitely many non-geodesic closed curves in $\boldsymbol{RP}^2(1)$ that are critical points for the functional $\oint(\kappa^2+4)ds$; they can be obtained using results of [12]. Then we regard $\boldsymbol{RP}^2(1)$ as a Lagrangian, totally geodesic surface in $\boldsymbol{CP}^2(4)$. The above mentioned curves generate Hopf tori which lie fully in $\boldsymbol{S}^5$ to give infinitely many immersed Willmore surfaces (tori) on $\boldsymbol{S}^5$. It is also clear that the only Hopf torus obtained in this way having constant nonzero mean curvature is the Ejiri torus. Also, notice that we can use the infinitely many elastic closed curves of $\boldsymbol{S}^2(4)$ (the 2-sphere of Gaussian curvature 4, [12]) and regard them as complex and totally geodesic surfaces in $\boldsymbol{CP}^2(4)$, to produce, by pulling back these curves via Π, the associated Hof tori in $\boldsymbol{S}^5$ which are also Willmore surfaces in $\boldsymbol{S}^5$. Actually, they lie fully in some $\boldsymbol{S}^3$ which is totally geodesic in $\boldsymbol{S}^5$. This second family of Willmore tori was essentially obtained in [15] and it includes the Clifford torus as the unique minimal member.

It should be noticed that the Clifford torus in some $\boldsymbol{S}^3$, being totally geodesic in $\boldsymbol{S}^5$, can be also obtained in the Lagrangian case, if we allow the closed elastica in $\boldsymbol{RP}^2(1)$ to be a geodesic, because $\boldsymbol{RP}^2(1)\cap \boldsymbol{S}^2(4)$ is just a geodesic of both $\boldsymbol{RP}^2(1)$ and $\boldsymbol{S}^2(4)$.

It remains then to examine the elastic helices of $\boldsymbol{CP}^2(4)$, obtained in the last section, and determine which one among them is closed. It suffices, of course, that if the lifted curve $\tilde{\sigma}(s)$ is a closed curve in $SU(3)$. Therefore we must find a positive number, say L, so that $\tilde{\sigma}(s+L)=\tilde{\sigma}(s)$. Since $\tilde{\sigma}(s)=e^{sM}$ is a one-parameter subgroup of $SU(3)$, this reduces to the problem of finding $L>0$ such that the eigenvalues of $L\cdot M$ are all integer multiples of $2\pi i$. Let t_1, t_2 and t_3 be the eigenvalues of M. Since M is in $su(3)$, we have $t_1+t_2+t_3=0$. It follows that the required condition for the roots is that t_2/t_1 be rational.

The characteristic polynomial of M is

$$\chi_M(t)=t^3+\frac{9+\omega^2}{3}t+i\frac{2(9+\omega^2)\omega}{27}=0\,.$$

Replacing t by ir reduces this equation to

$$r^3-\frac{9+\omega^2}{3}r-\frac{2(9+\omega^2)\omega}{27}=0\,.$$

The roots of this equation turn out to be

$$r_1=\frac{\sqrt{\omega^2+9}}{3}\cos\left(\frac{\theta}{3}\right),$$

$$r_2=\frac{\sqrt{\omega^2+9}}{3}\cos\left(\frac{\theta+2\pi}{3}\right),$$

$$r_3 = \frac{\sqrt{\omega^2+9}}{3}\cos\left(\frac{\theta+4\pi}{3}\right),$$

where

$$\cos\theta = \frac{-\omega}{\sqrt{\omega^2+9}}, \quad \sin\theta = \frac{3}{\sqrt{\omega^2+9}}.$$

We put all this information together to obtain a rational one-parameter family of closed elastic helices into $\boldsymbol{CP}^2(4)$. More precisely, we have:

THEOREM 3. *Let q be a rational number with $1<q<3$. Define an angle θ, $\pi/2<\theta<\pi$, by $q=\sqrt{3}\tan\theta/3$. Then for $\omega=-3\cot\theta$, the corresponding elastic helix γ, with $\lambda=4$, in $\boldsymbol{CP}^2(4)$ is closed. Moreover, the Hopf torus shaped on γ is a Willmore torus of constant mean curvature and full in $\boldsymbol{S}^5$.*

8. Willmore tori in non-standard conformal structures on the 5-sphere. In Section 3, we have obtained a method to get Willmore tori into $\boldsymbol{S}^5$ as Hopf tori over 4-elasticae into $\boldsymbol{CP}^2(4)$ (see Theorem 1). Of course, in that statement $\boldsymbol{S}^5$ means the 5-sphere endowed with its standard conformal structure. Hence, the three subfamilies of Willmore tori exhibited in the main statement (see the introduction) correspond with this conformal structure on $\boldsymbol{S}^5$.

In this section, we extend the above mentioned results to a one-parameter family of conformal structures on $\boldsymbol{S}^5$. This family is defined to be a deformation of the standard conformal structure. We start by introducing the canonical variation of the Riemannian submersion $\Pi:\boldsymbol{S}^5\to\boldsymbol{CP}^2(4)$, (see [1]). Let $\bar{g}$ and g be the standard metric of constant curvature one on $\boldsymbol{S}^5$ and the Fubini-Study metric of constant holomorphic sectional curvature 4 on $\boldsymbol{CP}^2(4)$, respectively. The horizontal distribution H on $\boldsymbol{S}^5$ (see Section 2) defines a connection on the principal fibre bundle $\boldsymbol{S}^5(\boldsymbol{CP}^2(4), \boldsymbol{S}^1)$, where $\boldsymbol{S}^1$ works as the structure group (it is a *circle bundle*), whose connection 1-form is denoted by Ω. We also put dr^2 to name the usual metric on $\boldsymbol{S}^1$. With these ingredients, we can define the following one-parameter family of Riemannian metrics on $\boldsymbol{S}^5$, which will be called the canonical variation of $\Pi:(\boldsymbol{S}^5,\bar{g})\to(\boldsymbol{CP}^2(4),g)$ (or simply the canonical variation of $\bar{g}$)

$$\bar{G}=\{\bar{g}_t=\Pi^*(g)+t^2\Omega^*(dr^2)\,|\,t>0\}.$$

It is clear that $\bar{g}_1=\bar{g}$.

These metrics are examples of a kind of metrics which are known as *bundle like* metrics (also as Kaluza-Klein metrics). They have fine properties, some of which are listed as follows:

1. For any $t>0$, $\Pi:(\boldsymbol{S}^5,\bar{g}_t)\to(\boldsymbol{CP}^2(4),g)$ is a Riemannian submersion with geodesic fibers, isometric to $\boldsymbol{S}^1(t)$ (the circle of radius t).

2. The natural action of $\boldsymbol{S}^1$ on $\boldsymbol{S}^5$, to obtain $\boldsymbol{CP}^2(4)$ as the orbit space, is made up through isometries of $(\boldsymbol{S}^5, \bar{g}_t)$ for all $t>0$.

3. The mean curvature function α of $N_\gamma = \Pi^1(\gamma)$ into $(\boldsymbol{S}^5, \bar{g}_t)$ does not depend on t and so it is given by the formula (1).

4. Every $(\boldsymbol{S}^5, \bar{g}_t)$ has constant scalar curvature.

5. These metrics define a family of conformal structures $\{\mathscr{C}_t = [\bar{g}_t]/t>0\}$ on $\boldsymbol{S}^5$, which are pairwise distinct.

THEOREM 4. *Let γ be a closed immersed curve in $\boldsymbol{CP}^2(4)$ with curvature function κ. Then $N_\gamma = \Pi^{-1}(\gamma)$ is a Willmore torus in $(\boldsymbol{S}^5, \mathscr{C}_t)$ if and only if γ is a critical point of the elastic functional $\mathscr{F}^t(\gamma) = \int_\gamma (\kappa^2 + 4t^2)ds$.*

PROOF. We can reproduce the proof of Theorem 1 step by step, with only one difference. Now the term S appearing in the integrand of $\mathscr{W}$ depends on t, and we name it by S^t. To compute it, we use an argument similar to that used in [2]. First, we denote by T^t and A^t the O'Neill invariants of the corresponding submersion. We notice that T^t vanishes identically (because these Riemannian submersion have geodesic fibers). Since S^t is the sectional curvature in $(\boldsymbol{S}^5, \bar{g}_t)$ of a mixed (also called *vertizontal*, [17]) section, we have

$$S^t = \bar{g}_t(A^t_{\bar{X}} U, A^t_{\bar{X}} U),$$

where $\bar{X}$ denotes the horizontal lift of $X = \gamma'$ and $U = \xi/t$ (see §2), that is, U is a $\bar{g}_t$-unit vertical vector field. Then one can prove that $A^t_{\bar{X}} U = t \cdot i\bar{X}$, and so $S^t = t^2$.

Consequently, the Willmore functional of $(\boldsymbol{S}^5, \mathscr{C}_t)$ on N_γ is

$$\mathscr{W}(N_\gamma) = \frac{\pi t}{2} \int_\gamma (\kappa^2 + 4t^2)ds.$$

We can use this theorem and the complete classification of elasticae with constant slant in $\boldsymbol{CP}^2(4)$ to obtain three families of Willmore tori in $(\boldsymbol{S}^5, \mathscr{C}_t)$ for any $t>0$. In particular, we have

COROLLARY. *For any real number $t>0$, there exists a rational one-parameter family of Willmore tori into $(\boldsymbol{S}^5, \mathscr{C}_t)$. They are obtained as Hopf tori shaped on $4t^2$-elastic closed helices in $\boldsymbol{CP}^2(4)$ and so have constant mean curvature in $(\boldsymbol{S}^5, \bar{g}_t)$.*

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