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SINGULAR INVARIANT HYPERFUNCTIONS ON THE SPACE OF REAL SYMMETRIC MATRICES

Dedicated to Professor Takeshi Hirai on his sixtieth birthday

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Abstract. Singular invariant hyperfunctions on the space of real symmetric matrices of size n are discussed in this paper. We construct singular invariant hyperfunctions, i.e., invariant hyperfunctions whose supports are contained in the set of the points of rank strictly less than n, in terms of negative order coefficients of the Laurent expansions of the complex powers of the determinant function. In particular, we give an algorithm to determine the orders of poles of the complex powers of the determinant functions and the support of the singular hyperfunctions appearing in the principal part of the Laurent expansions of the complex powers.

Introduction. A complex power of a polynomial is an important material to study in contemporary mathematics. We often encounter integrals of complex powers of polynomials in various aspect; for example, zeta functions of various types, hypergeometric functions and their extensions, kernels of integral transformations and so on. There are many important problems we have to solve. In particular, the explicit calculation of the exact orders of poles and the principal parts of the Laurent expansions at the poles with respect to the power parameter is an essential problem.

In this paper, we study the microlocal structure of the complex power of the determinant function on the real symmetric matrix space, and compute the exact order of poles with respect to the power parameter (Theorem 2.2). Moreover, we determine the exact support of the principal part of the pole (Theorem 2.3).

By these theorems, we can construct a suitable basis of the space of singular invariant hyperfunctions on the space of $n \times n$ real symmetric matrices $V := \mathrm{Sym}_n(\boldsymbol{R})$. The hyperfunctions belonging to the basis are expressed by the coefficients of the Laurent expansion of $|\det(x)|^s$, the complex power of the determinant function. We estimate the exact order of the poles of $|\det(x)|^s$ and give the exact support of the negative-order coefficients of the Laurent expansion of $|\det(x)|^s$.

In Section 1, we introduce some notions and basic properties on the complex power function $P^{[\vec{a},s]}(x)$ on the space of real symmetric matrices. In the next section (Section 2), the main theorems are stated without proof. In Section 3, we explain principal symbols $\sigma_\Lambda(P^{[\vec{a},s]}(x))$ of the regular holonomic hyperfunction $P^{[\vec{a},s]}(x)$ on the Lagrangian subvariety

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Λ and the coefficient functions $c_i^{j,k}(\vec{a}, s)$ on the connected Lagrangian component $\Lambda_i^{j,k}$. We investigate some distinguished properties of the coefficient functions and give the recursion relation formula. They will play a crucial role in the proof of the main theorems. However, since the purpose of this article is to calculate the singular invariant hyperfunctions explicitly, we only give an outline of the principal symbol theory. For details, see Kashiwara [4], [5], Kashiwara-Kawai [6], Kashiwara-Shapira [9], and so on. In the last three sections (Section 4, Section 5, Section 6), the proof of the main theorems is given.

We can obtain the same results on similar matrix spaces, for example, the space of complex Hermitian matrices or quaternion Hermitian matrices. They will appear in the forthcoming article [14].

We list here some related works on this topic. Similar results has been obtained by Blind [1] and [2] by a functional analytic method. Gelfand and Shilov [3] is the first elementary text on the complex powers of polynomials. Raïs [15] treated invariant distributions from his original view point. Satake [16] and Satake-Faraut [17], Sato-Shintani [18] and Shintani [19] are the works on zeta functions associated to the symmetric matrix space, which is closely related to the hyperfunctions treated here.

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1. Complex powers of the determinant function. In this section, we explain our problem more precisely, introduce some notions and notations, and state some preliminary known results. They are well-known results and we omit the proof.

1.1. Some fundamental definitions. Let $\boldsymbol{V} := \mathrm{Sym}_n(\boldsymbol{R})$ be the space of $n \times n$ symmetric matrices over the real field $\boldsymbol{R}$ and let $GL_n(\boldsymbol{R})$ (resp. $SL_n(\boldsymbol{R})$) be the general (resp. special) linear group over $\boldsymbol{R}$. Then the real algebraic group $\boldsymbol{G} := GL_n(\boldsymbol{R})$ acts on the vector space $\boldsymbol{V}$ through the representation

$$\rho(g) : x \mapsto g \cdot x \cdot {}^t g, \tag{1}$$

with $x \in \boldsymbol{V}$ and $g \in \boldsymbol{G}$. We say that a hyperfunction $f(x)$ on $\boldsymbol{V}$ is *singular* if the support of $f(x)$ is contained in the set $\boldsymbol{S} := \{x \in \boldsymbol{V} \mid \det(x) = 0\}$. We call $\boldsymbol{S}$ a *singular set* of $\boldsymbol{V}$. In addition, if $f(x)$ is $SL_n(\boldsymbol{R})$-invariant, i.e., $f(g \cdot x) = f(x)$ for all $g \in SL_n(\boldsymbol{R})$, we call $f(x)$ a *singular invariant* hyperfunction on $\boldsymbol{V}$.

Put $P(x) := \det(x)$. Then $P(x)$ is an irreducible polynomial on $\boldsymbol{V}$, and is a relative invariant corresponding to the character $\det(g)^2$ with respect to the action of $\boldsymbol{G}$, i.e., $P(\rho(g) \cdot x) = \det(g)^2 P(x)$. The non-singular subset $\boldsymbol{V} - \boldsymbol{S}$ decomposes into $n+1$ open $\boldsymbol{G}$-orbits

$$\boldsymbol{V}_i := \{x \in \mathrm{Sym}_n(\boldsymbol{R}) | \mathrm{sgn}(x) = (i, n-i)\} \tag{2}$$

with $i = 0, 1, \ldots, n$. Here, $\mathrm{sgn}(x)$ for $x \in \mathrm{Sym}_n(\boldsymbol{R})$ is the signature of the quadratic form $q_x(\vec{v}) :={}^t \vec{v} \cdot x \cdot \vec{v}$ on $\vec{v} \in \boldsymbol{R}^n$. We let for a complex number $s \in \boldsymbol{C}$,

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & \text{if } x \in \boldsymbol{V}_i\,, \\ 0 & \text{if } x \notin \boldsymbol{V}_i\,. \end{cases} \tag{3}$$

Let $\mathscr{S}(\boldsymbol{V})$ be the space of rapidly decreasing smooth functions on $\boldsymbol{V}$. For $f(x) \in \mathscr{S}(\boldsymbol{V})$, the integral

$$Z_i(f, s) := \int_{\boldsymbol{V}} |P(x)|_i^s f(x) dx \,, \tag{4}$$

is convergent if the real part $\Re s$ of s is sufficiently large and is meromorphically extended to the whole complex plane. Thus we can regard $|P(x)|_i^s$ as a tempered distribution with a meromorphic parameter $s \in \boldsymbol{C}$. We consider a linear combination of the hyperfunctions $|P(x)|_i^s$

$$P^{[\vec{a},s]}(x) := \sum_{i=0}^{n} a_i \cdot |P(x)|_i^s \tag{5}$$

with $s \in \boldsymbol{C}$ and $\vec{a} := (a_0, a_1, \ldots, a_n) \in \boldsymbol{C}^{n+1}$. Then $P^{[\vec{a},s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \boldsymbol{C}$, and depends on $\vec{a} \in \boldsymbol{C}^{n+1}$ linearly.

REMARK 1.1. Hyperfunctions (or microfunctions) with a meromorphic parameter is defined as follows. Let D be a domain in $\boldsymbol{C}$. We say that $u(s, x)$ a hyperfunction (or a microfunction) with a holomorphic parameter $s \in D$ if it satisfies the Cauchy-Riemann equation with respect to s on D. We say that $u(s, x)$ a hyperfunction (or a microfunction) with a meromorphic parameter $s \in D$ if it is a hyperfunction (or a microfunction) with a holomorphic parameter $s \in D - K$ with a discrete subset K of D, and, for each $s_0 \in K$, there exists $m \in \boldsymbol{Z}_{>0}$ such that $(s - s_0)^m u(s, x)$ is holomorphic with respect to s near s_0. For the detail of the properties on hyperfunctions (or microfunctions) with a holomorphic parameter, see [7, those after Definition 3.8.4].

1.2. Basic properties and some known results on complex powers. The following theorem is easily proved by the general theory of b-functions. See, for example, [13].

THEOREM 1.1. 1. *$P^{[\vec{a},s]}(x)$ is holomorphic with respect to $s \in \boldsymbol{C}$ except for the poles at $s = -(k+1)/2$ with $k = 1, 2, \ldots$.*

2. *The possibly highest order of the pole of $P^{[\vec{a},s]}(x)$ at $s = -(k+1)/2$ is given by*

$$\begin{cases} \lfloor \frac{k+1}{2} \rfloor & (k = 1, 2, \ldots, n-1)\,, \\ \lfloor \frac{n}{2} \rfloor & (k = n, n+1, \ldots, \text{ and } k+n \text{ is odd})\,, \\ \lfloor \frac{n+1}{2} \rfloor & (k = n, n+1, \ldots, \text{ and } k+n \text{ is even})\,. \end{cases} \tag{6}$$

Here, $\lfloor x \rfloor$ means the floor of $x \in \boldsymbol{R}$, i.e., the largest integer which does not exceed x.

Any negative-order coefficient of a Laurent expansion of $P^{[\vec{a},s]}(x)$ is a singular invariant hyperfunction, since the integral

$$\int f(x) P^{[\vec{a},s]}(x) dx = \sum_{i=0}^{n} Z_i(f, s) \tag{7}$$

is an entire function with respect to $s \in \boldsymbol{C}$ if $f(x) \in C_0^\infty(\boldsymbol{V} - \boldsymbol{S})$, where $C_0^\infty(\boldsymbol{V} - \boldsymbol{S})$ is the space of compactly supported C^∞-functions on $\boldsymbol{V} - \boldsymbol{S}$. Conversely, we have the following proposition.

PROPOSITION 1.2 ([12], [13]). *Any singular invariant hyperfunction on V is given as a linear combination of some negative-order coefficients of Laurent expansions of $P^{[\vec{a},s]}(x)$ at various poles and for some $\vec{a} \in C^{n+1}$.*

PROOF. The prehomogeneous vector space

$$(G, V) := (GL_n(R), \mathrm{Sym}_n(R))$$

satisfies the sufficient conditions stated in [12] and [13]. One is the finite-orbit condition and the other is that the dimension of the space of relatively invariant hyperfunctions coincides with the number of open orbits. □

1.3. Orbit decomposition. The vector space V decomposes into a finite number of G-orbits;

$$V := \bigsqcup_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n-i}} S_i^j \tag{8}$$

where

$$S_i^j := \{x \in \mathrm{Sym}_n(R) \mid \mathrm{sgn}(x) = (j, n-i-j)\} \tag{9}$$

with integers $0 \leq i \leq n$ and $0 \leq j \leq n-i$. A G-orbit in S is called a *singular orbit*. The subset $S_i := \{x \in V \mid \mathrm{rank}(x) = n-i\}$ is the set of elements of rank $n-i$. It is easily seen that $S := \bigsqcup_{1 \leq i \leq n} S_i$ and $S_i = \bigsqcup_{0 \leq j \leq n-i} S_i^j$. Each singular orbit is a stratum which not only is a G-orbit but is an $SL_n(R)$-orbit. The strata $\{S_i^j\}_{1 \leq i \leq n, 0 \leq j \leq n-i}$ have the following closure inclusion relation

$$\overline{S_i^j} \supset S_{i+1}^{j-1} \cup S_{i+1}^j, \tag{10}$$

where $\overline{S_i^j}$ means the *closure* of the stratum S_i^j.

The support of a singular invariant hyperfunction is a closed set consisting of a union of some strata S_i^j. Since the support is a closed G-invariant subset, we can express the support of a singular invariant hyperfunction as a closure of a union of the highest rank strata, which is easily rewritten in terms of a union of singular orbits.

2. Statement of the main results.

In this section we state the main problems and results. When we give a complex $n+1$ dimensional vector $\vec{a} \in C^{n+1}$, we can determine the exact order of poles of $P^{[\vec{a},s]}(x)$ and the exact support of the hyperfunctions appearing in the principal part of the Laurent expansion. We shall give the statements of the theorems in this section without proof. The proof will be given in Section 5.

2.1. Main problem. When we consider complex powers of relatively invariant polynomials, we naturally ask the following questions.

PROBLEM 2.1. What are the principal parts of the Laurent expansion of $P^{[\vec{a},s]}(x)$ at poles? What are their exact orders of poles? What are the supports of negative-order coefficients of a Laurent expansion of $P^{[\vec{a},s]}(x)$ at poles?

In order to determine the exact order of the pole of $P^{[\vec{a},s]}(x)$ at $s = s_0$, we introduce the coefficient vectors

$$\boldsymbol{d}^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \dots, d_{n-k}^{(k)}[s_0]) \in ((\boldsymbol{C}^{n+1})^*)^{n-k+1} \tag{11}$$

with $k = 0, 1, \dots, n$. Here, $(\boldsymbol{C}^{n+1})^*$ means the dual vector space of $\boldsymbol{C}^{n+1}$. Each element of $\boldsymbol{d}^{(k)}[s_0]$ is a linear form on $\vec{a} \in \boldsymbol{C}^{n+1}$ depending on $s_0 \in \boldsymbol{C}$, i.e., a linear map from $\boldsymbol{C}^{n+1}$ to $\boldsymbol{C}$,

$$d_i^{(k)}[s_0] : \boldsymbol{C}^{n+1} \ni \vec{a} \mapsto \langle d_i^{(k)}[s_0], \vec{a} \rangle \in \boldsymbol{C}. \tag{12}$$

We denote

$$\langle \boldsymbol{d}^{(k)}[s_0], \vec{a} \rangle = (\langle d_0^{(k)}[s_0], \vec{a} \rangle, \langle d_1^{(k)}[s_0], \vec{a} \rangle, \dots, \langle d_{n-k}^{(k)}[s_0], \vec{a} \rangle) \in \boldsymbol{C}^{n-k+1}. \tag{13}$$

DEFINITION 2.1 (Coefficient vectors $\boldsymbol{d}^{(k)}[s_0]$). Let s_0 be a *half integer*, i.e., a rational number given by $q/2$ with an integer q. We define the *coefficient vectors* $\boldsymbol{d}^{(k)}[s_0]$ $(k = 0, 1, \dots, n)$ by induction in the following way.

1. First, we set

$$\boldsymbol{d}^{(0)}[s_0] := (d_0^{(0)}[s_0], d_1^{(0)}[s_0], \dots, d_n^{(0)}[s_0]) \tag{14}$$

such that $\langle d_i^{(0)}[s_0], \vec{a} \rangle := a_i$ for $i = 0, 1, \dots, n$. Next, we define $\boldsymbol{d}^{(1)}[s_0]$ by

$$\boldsymbol{d}^{(1)}[s_0] := (d_0^{(1)}[s_0], d_1^{(1)}[s_0], \dots, d_{n-1}^{(1)}[s_0]) \in ((\boldsymbol{C}^{n+1})^*)^n \tag{15}$$

with $d_j^{(1)}[s_0] := d_j^{(0)}[s_0] + \varepsilon[s_0] d_{j+1}^{(0)}[s_0]$. Here,

$$\varepsilon[s_0] := \begin{cases} 1 & \text{if } s_0 \text{ is a strict half integer}, \\ (-1)^{s_0+1} & \text{if } s_0 \text{ is an integer}. \end{cases} \tag{16}$$

A *strict half integer* means a rational number given by $q/2$ with an odd integer q.

2. Then, by induction on k, we define the coefficient vectors $\boldsymbol{d}^{(k)}[s_0]$ for $k = 0, 1, \dots, n$ by

$$\boldsymbol{d}^{(2l+1)}[s_0] := (d_0^{(2l+1)}[s_0], d_1^{(2l+1)}[s_0], \dots, d_{n-2l-1}^{(2l+1)}[s_0]) \in ((\boldsymbol{C}^{n+1})^*)^{n-2l} \tag{17}$$

with $d_j^{(2l+1)}[s_0] := d_j^{(2l-1)}[s_0] - d_{j+2}^{(2l-1)}[s_0]$ and

$$\boldsymbol{d}^{(2l)}[s_0] := (d_0^{(2l)}[s_0], d_1^{(2l)}[s_0], \dots, d_{n-2l}^{(2l)}[s_0]) \in ((\boldsymbol{C}^{n+1})^*)^{n-2l+1} \tag{18}$$

with $d_j^{(2l)}[s_0] := d_j^{(2l-2)}[s_0] + d_{j+2}^{(2l-2)}[s_0]$.

Then we have the following proposition.

PROPOSITION 2.1. *Let s_0 be a half integer. Then we have the following results.*

1. *There exists an even integer i_0 in $0 \le i_0 \le n+1$ such that*

$$\langle \boldsymbol{d}^{(i)}[s_0], \vec{a} \rangle \text{ is } \begin{cases} \neq 0 & \text{for all odd } i \text{ in } 0 \le i < i_0, \\ = 0 & \text{for all odd } i \text{ in } n \ge i > i_0. \end{cases} \tag{19}$$

2. *There exists an odd integer i_1 in $-1 \le i_1 \le n+1$ such that*

$$\langle \boldsymbol{d}^{(i)}[s_0], \vec{a} \rangle \text{ is } \begin{cases} \neq 0 & \text{for all even } i \text{ in } 0 \le i < i_1, \\ = 0 & \text{for all even } i \text{ in } n \ge i > i_1. \end{cases} \tag{20}$$

PROOF. Let s_0 be a half integer. For an integer i in $0 \leq i \leq n-2$ and $\vec{a} \in \boldsymbol{C}^{n+1}$, if $\langle \boldsymbol{d}^{(i)}[s_0], \vec{a} \rangle = 0$, then $\langle \boldsymbol{d}^{(i+2)}[s_0], \vec{a} \rangle = 0$ from the definition of $\boldsymbol{d}^{(i)}[s_0]$. In other words, if $\langle \boldsymbol{d}^{(i+2)}[s_0], \vec{a} \rangle \neq 0$, then $\langle \boldsymbol{d}^{(i)}[s_0], \vec{a} \rangle \neq 0$. Hence we have the result. □

2.2. Results on the poles of the complex power functions. Using the above mentioned vectors $\boldsymbol{d}^{(k)}[s_0]$, we can determine the exact orders of poles of $P^{[\vec{a},s]}(x)$.

THEOREM 2.2 (Exact orders of poles). *By using the coefficient vector $\boldsymbol{d}^{(k)}[s_0]$ defined in Definition* 2.1, *the exact orders of poles of $P^{[\vec{a},s]}(x)$ are computed by the following algorithm.*

1. *The exact order $P^{[\vec{a},s]}(x)$ at $s = -(2m+1)/2$ $(m = 1, 2, \dots)$ is given in terms of the coefficient vector $\boldsymbol{d}^{(2k)}[-(2m+1)/2]$.*

(a) *If $1 \leq m \leq n/2$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not greater than m.*

- *$P^{[\vec{a},s]}(x)$ is holomorphic if and only if $\langle \boldsymbol{d}^{(2)}[-(2m+1)/2], \vec{a} \rangle = 0$.*
- *For a fixed integer p in $1 \leq p < m$, $P^{[\vec{a},s]}(x)$ has pole of order p if and only if $\langle \boldsymbol{d}^{(2p+2)}[-(2m+1)/2], \vec{a} \rangle = 0$ and $\langle \boldsymbol{d}^{(2p)}[-(2m+1)/2], \vec{a} \rangle \neq 0$.*
- *$P^{[\vec{a},s]}(x)$ has pole of order m if and only if $\langle \boldsymbol{d}^{(2m)}[-(2m+1)/2], \vec{a} \rangle \neq 0$.*

(b) *If $m > n/2$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not greater than $n' := \lfloor n/2 \rfloor$.*

- *$P^{[\vec{a},s]}(x)$ is holomorphic if and only if $\langle \boldsymbol{d}^{(2)}[-(2m+1)/2], \vec{a} \rangle = 0$.*
- *For a fixed integer p in $1 \leq p < n'$, $P^{[\vec{a},s]}(x)$ has pole of order p if and only if $\langle \boldsymbol{d}^{(2p+2)}[-(2m+1)/2], \vec{a} \rangle = 0$ and $\langle \boldsymbol{d}^{(2p)}[-(2m+1)/2], \vec{a} \rangle \neq 0$.*
- *$P^{[\vec{a},s]}(x)$ has pole of order n' if and only if $\langle \boldsymbol{d}^{(n-1)}[-(2m+1)/2], \vec{a} \rangle \neq 0$ (when n is odd) or $\langle \boldsymbol{d}^{(n)}[-(2m+1)/2], \vec{a} \rangle \neq 0$ (when n is even).*

2. *We obtain the exact order at $s = -m$ $(m = 1, 2, \dots)$ in terms of the coefficient vectors $\boldsymbol{d}^{(2k+1)}[-m]$.*

(a) *If $1 \leq m \leq n/2$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not greater than m.*

- *$P^{[\vec{a},s]}(x)$ is holomorphic if and only if $\langle \boldsymbol{d}^{(1)}[-m], \vec{a} \rangle = 0$.*
- *For a fixed integer p in $1 \leq p < m$, $P^{[\vec{a},s]}(x)$ has pole of order p if and only if $\langle \boldsymbol{d}^{(2p+1)}[-m], \vec{a} \rangle = 0$ and $\langle \boldsymbol{d}^{(2p-1)}[-m], \vec{a} \rangle \neq 0$.*
- *$P^{[\vec{a},s]}(x)$ has pole of order m if and only if $\langle \boldsymbol{d}^{(2m-1)}[-m], \vec{a} \rangle \neq 0$.*

(b) *If $m > n/2$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not greater than $n' := \lfloor (n+1)/2 \rfloor$.*

- *$P^{[\vec{a},s]}(x)$ is holomorphic if and only if $\langle \boldsymbol{d}^{(1)}[-m], \vec{a} \rangle = 0$.*
- *For a fixed integer p in $1 \leq p < n'$, $P^{[\vec{a},s]}(x)$ has pole of order p if and only if $\langle \boldsymbol{d}^{(2p+1)}[-m], \vec{a} \rangle = 0$ and $\langle \boldsymbol{d}^{(2p-1)}[-m], \vec{a} \rangle \neq 0$.*
- *$P^{[\vec{a},s]}(x)$ has pole of order n' if and only if $\langle \boldsymbol{d}^{(n)}[-m], \vec{a} \rangle \neq 0$ (when n is odd) or $\langle \boldsymbol{d}^{(n-1)}[-m], \vec{a} \rangle \neq 0$ (when n is even).*

2.3. Results on the supports of the principal symbols. The exact support of $P^{[\vec{a},s]}(x)$ is given by the following theorem.

THEOREM 2.3 (Support of the singular invariant hyperfunctions). *Let q be a positive integer. Suppose that $P^{[\vec{a},s]}(x)$ has pole of order p at $s = -(q+1)/2$. Let*

$$P^{[\vec{a},s]}(x) = \sum_{w=-p}^{\infty} P_w^{[\vec{a},-(q+1)/2]}(x)\left(s + \frac{q+1}{2}\right)^w \tag{21}$$

be the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = -(q+1)/2$. The support of the Laurent expansion coefficients $P_w^{[\vec{a},-(q+1)/2]}(x)$ is contained in $\mathbf{S}$ if $w < 0$.

1. *Let q be an even positive integer. Then the support of $P_w^{[\vec{a},-(q+1)/2]}(x)$ for $w = -1, -2, \dots, -p$ is contained in the closure $\overline{S_{-2w}}$. More precisely, it is given by*

$$\operatorname{Supp}(P_w^{[\vec{a},-(q+1)/2]}(x)) = \overline{\bigcup_{j\in\{0\le j\le n+2w \mid \langle d_j^{(-2w)}[-(q+1)/2],\vec{a}\rangle\neq 0\}} S_{-2w}^j}\,. \tag{22}$$

2. *Let q be an odd positive integer. Then the support of $P_w^{[\vec{a},-(q+1)/2]}(x)$ for $w = -1, -2, \dots, -p$ is contained in the closure $\overline{S_{-2w-1}}$. More precisely, it is given by*

$$\operatorname{Supp}(P_w^{[\vec{a},-(q+1)/2]}(x)) = \overline{\bigcup_{j\in\{0\le j\le n+2w+1 \mid \langle d_j^{(-2w-1)}[-(q+1)/2],\vec{a}\rangle\neq 0\}} S_{-2w-1}^j}\,. \tag{23}$$

Here, Supp$(-)$ *means the support of the hyperfunction in* $(-)$.

3. Principal symbols of invariant hyperfunctions. In this section, we review the notion of principal symbols of simple holonomic microfunctions and coefficient functions with respect to the canonical basis of principal symbols. Our proof is on the line that we reduce the pole of the order and the support of the Laurent expansion coefficients of the hyperfunction of $P^{[\vec{a},s]}(x)$ to those of the microfunction $\mathrm{sp}(P^{[\vec{a},s]}(x))$. We adopt the manner that we calculate the coefficient functions (Definition 3.2) of $\mathrm{sp}(P^{[\vec{a},s]}(x))$ instead of dealing with the microfunction itself, since it is not easy to handle the microfunction directly. Proposition 3.7 and Proposition 3.8 guarantee that the calculation of the coefficient functions is equivalent to that of the microfunction. Lastly, in Proposition 3.9 and Proposition 3.10, we shall give recursion relations (59) and (60) among the coefficient functions. Then our problem is finally reduced to the estimate of the orders of poles of the coefficient functions, which are meromorphic functions in s explicitly computed by the recursion formula (59).

3.1. Microfunctions on the cotangent bundle. Let $\mathcal{B}_V$ be the sheaf of hyperfunctions on V and let $\mathcal{C}_V$ be the sheaf of microfunctions on the cotangent bundle T^*V of V. We have a natural isomorphism sp:

$$\mathrm{sp} : \mathcal{B}_V \xrightarrow{\sim} \pi_*(\mathcal{C}_V) \tag{24}$$

and an exact sequence

$$0 \to \mathcal{A}_V \to \mathcal{B}_V \to \pi_*(\mathcal{C}_V|_{T^*V-V}) \to 0\,. \tag{25}$$

Here, π is the projection map from the cotangent bundle T^*V to V and $\mathcal{A}_V$ is the sheaf of real analytic functions on V. By the isomorphism (24), we can regard a hyperfunction $f(x)$ on V as a microfunction $\mathrm{sp}(f(x))$ on T^*V. In this article, we often identify the hyperfunction

$f(x)$ on V with the microfunction $\mathrm{sp}(f(x))$ on T^*V through the isomorphism (24). We call the support $\mathrm{Supp}(\mathrm{sp}(f(x)))$ the *singular support* of $f(x)$.

REMARK 3.1. In this paper, the sheaf $\mathcal{C}_V$ means the sheaf of microfunctions on T^*V, not on $T^*V - V$. It was originally denoted by $\check{\mathcal{C}}_V$ when Sato introduced the notion of microfunction originally. Roughly speaking, the sheaf of microfunctions $\mathcal{C}_V$ on T^*V is the union of the sheaf of hyperfunctions $\mathcal{B}_V$ and the sheaf $\mathcal{C}_V|_{T^*V-V}$. When the notion of microfunction was introduced as a singular part of a hyperfunctions, it often meant the sheaf $\mathcal{C}_V|_{T^*V-V}$. However, in this article, we always mean by the sheaf $\mathcal{C}_V$ the one on the whole space T^*V.

3.2. Holonomic systems for relatively invariant hyperfunctions. We consider invariant hyperfunctions on V under the action of G as solutions to a holonomic system. Let $f(x)$ be a hyperfunction on V. We say that $f(x)$ is a χ^s*-invariant hyperfunction* if

$$f(\rho(g)x) = \chi(g)^s f(x) \tag{26}$$

for all $g \in G$ with $s \in C$ and $\chi(g) := \det(g)^2$. Then, it is a hyperfunction solution to the following system of linear differential equations $\mathcal{M}_s$ obtained by taking infinitesimal actions of G,

$$\mathcal{M}_s : \left(\left\langle d\rho(A)x, \frac{\partial}{\partial x}\right\rangle - s\delta\chi(A)\right)u(x) = 0 \quad \text{for all } A \in \mathfrak{G}. \tag{27}$$

Here, $\mathfrak{G}$ is the Lie algebra of G; $d\rho$ is the infinitesimal representation of ρ; $\delta\chi$ is the infinitesimal character of χ. The system of linear differential equation (27) is a regular holonomic system and hence the solution space is finite dimensional. See [13] for details.

The characteristic subvariety of the holonomic system (27) is denoted by $\mathrm{ch}(\mathcal{M}_s)$. It is given by

$$\mathrm{ch}(\mathcal{M}_s) := \{(x, y) \in T^*V \mid \langle d\rho(A)x, y\rangle = 0 \text{ for all } A \in \mathfrak{G}\}. \tag{28}$$

The characteristic variety has the following irreducible component decomposition,

$$\mathrm{ch}(\mathcal{M}_s) := \bigcup_{i=0}^{n} \Lambda_i \tag{29}$$

with $\Lambda_i = \overline{T^*_{S_i}V}$, where $T^*_{S_i}V$ stands for the conormal bundle of the orbit S_i of rank $n - i$. It is well-known that the singular support of the hyperfunction solution to $\mathcal{M}_s$ is contained in $\mathrm{ch}(\mathcal{M}_s)$.

REMARK 3.2. In this article, the singular support of a hyperfunction $f(x)$ means, by definition, the support of $\mathrm{sp}(f(x))$ in T^*V, and not in $T^*V - V$.

We denote by V^* the dual vector space of V. The cotangent bundle T^*V is naturally identified with the product space $V \times V^*$. Since the group G acts on V^* by the contragredient action, $V \times V^*$ admits a G-action. The characteristic variety $\mathrm{ch}(\mathcal{M}_s)$ is an invariant subset in $V \times V^*$, and it decomposes into a finite number of orbits. See [10, Proposition 1.1].

PROPOSITION 3.1. *The holonomic system $\mathcal{M}_s$ is simple on each Lagrangian subvariety Λ_i. The order of $\mathcal{M}_s$ on Λ_i is given by*

$$\operatorname{ord}_{\Lambda_i}(\mathcal{M}_s) = -is - \frac{i(i+1)}{4}. \tag{30}$$

The irreducible Lagrangian subvarieties Λ_i and Λ_{i+1} have an intersection of codimension one.

PROOF. The orders on Λ_i are calculated in [10]. The intersections of codimension one among Λ_i's are also given there. See the holonomy diagrams in [10]. $\square$

3.3. Principal symbols on simple Lagrangian subvarieties. Recall the definition of a principal symbol on a simple holonomic system defined in [10], which was originally defined by [4] and [5]. Let Λ be a non-singular Lagrangian subvariety and let $u(x)$ be a local section of a microfunction solution to a simple holonomic system $\mathcal{M}$ whose support is Λ. We denote by $\sigma_\Lambda(u)$ the principal symbol of $u(x)$ on Λ. It is a real analytic section of $\sqrt{|\Omega_\Lambda|}\otimes\sqrt{|\Omega_V|}^{-1}$ where $\sqrt{|\Omega_\Lambda|}$ and $\sqrt{|\Omega_V|}$ are the sheaves of half-volume elements on Λ and $\boldsymbol{V}$, respectively. For the precise definition, see [10, Definition 2.7] and also the definitions in [4] and [5]. As explained in [10], the map

$$\sigma_\Lambda : u \mapsto \sigma_\Lambda(u) \tag{31}$$

is a linear isomorphism from the space of microfunction solutions to the space of principal symbols of the holonomic system $\mathcal{M}_s$. In other words, there is a one-to-one correspondence between the local sections of the microfunction solution to $\mathcal{M}_s$ and those of its principal symbol.

When we consider a hyperfunction solution to the holonomic system $\mathcal{M}_s$, it suffices to handle the global section of the principal symbol on a open dense subset of $\operatorname{ch}(\mathcal{M}_s)$. We shall explain the meaning below. The final statement will be given in Proposition 3.7.

First, we introduce the open dense subset Λ_i° of Λ_i and consider the solutions on Λ_i°.

DEFINITION 3.1 (Open dense subset in Λ_i°). Let Λ_i be one of the irreducible components of $\operatorname{ch}(\mathcal{M}_s)$ defined in (29). We define the subset Λ_i° by

$$\Lambda_i^\circ := \Lambda_i - \bigcup_{i\neq j} \Lambda_j. \tag{32}$$

It is an open dense subset of Λ_i.

The open subset Λ_i° consists of several open connected components, each of which is a $\boldsymbol{G}$-orbit. Furthermore, Λ_i° is a non-singular algebraic subvariety and an open dense subset in Λ_i.

PROPOSITION 3.2. *The open set Λ_i° of Λ_i decomposes into the following $\boldsymbol{G}$-orbits*

$$\Lambda_i^\circ = \bigsqcup_{\substack{0\leq j\leq n-i\\ 0\leq k\leq i}} \Lambda_i^{j,k} \tag{33}$$

with

$$\Lambda_i^{j,k} := \boldsymbol{G} \cdot \left(\begin{pmatrix} I_{n-i}^{(j)} & \\ & 0_i \end{pmatrix}, \begin{pmatrix} 0_{n-i} & \\ & I_i^{(k)} \end{pmatrix} \right). \tag{34}$$

Here,

$$I_p^{(q)} := \begin{pmatrix} I_q & \\ & -I_{p-q} \end{pmatrix}$$

and I_p is the identity matrix of size p. Each orbit $\Lambda_i^{j,k}$ is a connected component in Λ_i°.

3.4. Canonical basis of principal symbols. We shall give a canonical basis of the principal symbol following [8] and [10]. Let Λ_i° be the open subset defined by Definition 3.1 and let $\Lambda_i^{j,k}$ be a connected component in Λ_i°. We define a non-zero real analytic section $\Omega_i^{j,k}(s)$ of $\sqrt{|\Omega_{\Lambda_i^{j,k}}|}$ by

$$\Omega_i^{j,k}(s) := \left| P_{\Lambda_i^{j,k}}(x, y) \right|^s \sqrt{|\omega_{\Lambda_i^{j,k}}(x, y)|}. \tag{35}$$

Here, we set

$$P_{\Lambda_i^{j,k}}(x, y) := P(\pi(x, y))/(\sigma(x, y))^{m_{\Lambda_i}} \big|_{\Lambda_i^{j,k}}, \tag{36}$$

$$\omega_{\Lambda_i^{j,k}}(x, y) := \frac{\pi^{-1}(|dx|) \wedge d\sigma(x, y)}{\sigma(x, y)^{\mu_{\Lambda_i}}} \Bigg/ d\sigma(x, y) \Bigg|_{\Lambda_i^{j,k}}, \tag{37}$$

where $\sigma := \sigma(x, y)$ is a function on $\boldsymbol{V} \times \boldsymbol{V}^*$ defined by $\sigma := \langle x, y \rangle / n$; π is the projection map from the subvariety

$$W := \{(x, y) \in T^*\boldsymbol{V} \,|\, \langle d\rho(A)x, y \rangle = 0 \text{ for all } A \in \mathfrak{G}_0\} \subset \boldsymbol{V} \times \boldsymbol{V}^* \tag{38}$$

to $\boldsymbol{V}$, when $\mathfrak{G}_0 := \{A \in \mathfrak{G} \,|\, \delta\chi(A) = 0\}$; m_{Λ_i} and μ_{Λ_i} are the constants such that $-m_{\Lambda_i}s - \mu_{\Lambda_i}/2$ is the order of $\mathcal{M}_s$ on Λ_i. In particular, $m_{\Lambda_i} = i$ and $\mu_{\Lambda_i} = i(i+1)/2$ in our case. The section $\Omega_i^{j,k}(s)$ depends on $s \in \boldsymbol{C}$ holomorphically.

PROPOSITION 3.3. *Let $u(s, x)$ be a microfunction solution with a complex parameter $s \in \boldsymbol{C}$ to the holonomic system $\mathcal{M}_s$ and let $\Lambda_i^{j,k}$ be a connected component in Λ_i°. Then we have*:

1. *The principal symbol $\sigma_{\Lambda_i^{j,k}}(u(s, x))$ is written as a constant multiple of the real analytic section of $\Omega_i^{j,k}(s)/\sqrt{|dx|}$ as follows*;

$$\sigma_{\Lambda_i^{j,k}}(u(s, x)) = c_i^{j,k}(s) \cdot \Omega_i^{j,k}(s)/\sqrt{|dx|}. \tag{39}$$

Here, $|dx|$ is a non-zero volume element on $\boldsymbol{V}$ defined by

$$|dx| := \left| \bigwedge_{1 \le i \le j \le n} dx_{ij} \right| \tag{40}$$

with

$$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{pmatrix} \in V.$$

Conversely, if the constant multiplication term $c_i^{j,k}(s)$ *is given on each* $\Lambda_i^{j,k}$, *then the corresponding microfunction solution* $u(s,x)$ *satisfying* (39) *is determined uniquely.*

2. $c_i^{j,k}(s)$ *is a holomorphic* (*resp. meromorphic*) *function in* $s \in \boldsymbol{C}$, *if and only if* $u(s,x)$ *depends on* $s \in \boldsymbol{C}$ *holomorphically* (*resp. meromorphically*).

PROOF. 1. This assertion is deduced from the definition of the principal symbol on prehomogeneous vector space. See [8] for details.

2. This is easily seen, since the isomorphisms sp in (24) and σ_Λ in (31) are $\boldsymbol{C}[s]$-linear, where $\boldsymbol{C}[s]$ is the polynomial ring of s, and are commutative with the Cauchy-Riemann operator $\partial/\partial\bar{s}$. In fact, we see that $\sigma_{\Lambda_i^{j,k}}(u(s,x))$ depends on s holomorphically (resp. meromorphically) if and only if $u(s,x)$ is a holomorphic (resp. meromorphic) function on $s \in \boldsymbol{C}$ because $\sigma_{\Lambda_i^{j,k}}(u(s,x))$ satisfies the Cauchy-Riemann equation with respect to $s \in \boldsymbol{C}$ if and only if $u(s,x)$ also satisfies it. Then, if $u(x,s)$ is holomorphic (resp. meromorphic) with respect to s, then $c_i^{j,k}(s)$ is a holomorphic (resp. meromorphic) function in $s \in \boldsymbol{C}$, since $\Omega_i^{j,k}(s)/\sqrt{|dx|}$ in (39) is non-zero and depends on $s \in \boldsymbol{C}$ holomorphically. □

3.5. Hyperfunction solutions and coefficient functions. In this paper we consider hyperfunction solutions to $\mathcal{M}_s$ of the form

$$P^{[\vec{a},s]}(x) := \sum_{i=0}^{n} a_i \cdot |P(x)|_i^s \tag{41}$$

with $\vec{a} = (a_0, a_1, \dots, a_n) \in \boldsymbol{C}^{n+1}$ introduced in (5). Since $P^{[\vec{a},s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \boldsymbol{C}$, the microfunction $\mathrm{sp}(P^{[\vec{a},s]}(x))$ and its principal symbols $\sigma_{\Lambda_i^{j,k}}(P^{[\vec{a},s]}(x))$ depend on $s \in \boldsymbol{C}$ meromorphically. In the particular case (39), we define the coefficient functions of $P^{[\vec{a},s]}(x)$ on the Lagrangian connected component $\Lambda_i^{j,k}$ as a function of $\vec{a}$ and s.

DEFINITION 3.2 (Coefficient functions). Let

$$\sigma_{\Lambda_i^{j,k}}(P^{[\vec{a},s]}(x)) := c_i^{j,k}(\vec{a},s)\Omega_i^{j,k}(s)/\sqrt{|dx|} \tag{42}$$

with $c_i^{j,k}(\vec{a},s)$ being a meromorphic function in $s \in \boldsymbol{C}$. We call $c_i^{j,k}(\vec{a},s)$ a *coefficient function* or simply a *coefficient* of $P^{[\vec{a},s]}(x)$ on $\Lambda_i^{j,k}$ with respect to the *canonical basis*

$$\Omega_i^{j,k}(s)/\sqrt{|dx|}\,. \tag{43}$$

Then the coefficient functions $c_i^{j,k}(\vec{a},s)$ depend on $\vec{a} \in \boldsymbol{C}^{n+1}$ linearly and on $s \in \boldsymbol{C}$ meromorphically.

PROPOSITION 3.4. *Let $P^{[\vec{a}_1,s]}(x)$ and $P^{[\vec{a}_2,s]}(x)$ be two hyperfunction solutions to the holonomic system $\mathcal{M}_s$. If their coefficient functions coincide on each $\Lambda_i^{j,k}$:*

$$c_i^{j,k}(\vec{a}_1, s) = c_i^{j,k}(\vec{a}_2, s)\,, \tag{44}$$

then we have $\vec{a}_1 = \vec{a}_2$. In other words, two hyperfunction solutions having the same coefficient functions on all $\Lambda_i^{j,k}$'s coincide with each other.

PROOF. Recall the following fact on the uniqueness of hyperfunction solutions to a holonomic system. It is proved by the same argument as in the main theorem in [11].

LEMMA 3.5. *Let $f_1(x)$ and $f_2(x)$ be two hyperfunctions whose singular supports are contained in $\bigcup_{i=0}^n \Lambda_i$, the characteristic variety of the holonomic system $\mathcal{M}_s$. If $\mathrm{sp}(f_1(x)) = \mathrm{sp}(f_2(x))$ on the open set $\bigcup_{i=0}^n \Lambda_i^\circ$, then $f_1(x)$ coincides with $f_2(x)$ as a hyperfunction on $\boldsymbol{V}$.*

Lemma 3.5 asserts that a microfunction solution to $\mathcal{M}_s$ is determined by the given data on $\bigcup_{i=0}^n \Lambda_i^\circ$. Therefore we only need to consider the microfunction solutions on $\bigcup_{i=0}^n \Lambda_i^\circ$ instead of on the whole characteristic variety $\mathrm{ch}(\mathcal{M}_s)$.

By Proposition 3.3, if (44) is satisfied, then $\mathrm{sp}(P^{[\vec{a}_1,s]}(x)) = \mathrm{sp}(P^{[\vec{a}_2,s]}(x))$ on each Lagrangian connected component $\Lambda_i^{j,k}$ and hence they coincide on the open set $\bigcup_{i=0}^n \Lambda_i^\circ$. Thus, from Lemma 3.5, we have $P^{[\vec{a}_1,s]}(x) = P^{[\vec{a}_2,s]}(x)$, which means $\vec{a}_1 = \vec{a}_2$. □

For a microfunction solution on each Lagrangian connected component $\Lambda_i^{j,k}$, we have the following equivalent conditions.

PROPOSITION 3.6. *The following three conditions are equivalent.*

1. *The microfunction $\mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\Lambda_i^{j,k}}$ has pole of order p at $s = s_0$.*
2. *The principal symbol $\sigma_{\Lambda_i^{j,k}}(\mathrm{sp}(P^{[\vec{a},s]}(x)))$ has pole of order p at $s = s_0$.*
3. *The coefficient $c_i^{j,k}(\vec{a}, s)$ has pole of order p at $s = s_0$.*

PROOF. By Proposition 3.3-2, the microfunction $\mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\Lambda_i^{j,k}}$, the principal symbol $\sigma_{\Lambda_i^{j,k}}(\mathrm{sp}(P^{[\vec{a},s]}(x)))$, and the coefficient $c_i^{j,k}(\vec{a}, s)$ are all meromorphic with respect to $s \in \boldsymbol{C}$, since $P^{[\vec{a},s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \boldsymbol{C}$.

The equivalence of the first two follows from the fact that the isomorphism σ_Λ in (31) is $\boldsymbol{C}[s]$-linear. In fact, from the general theory of principal symbols of simple microfunction (see [4]), a principal symbol of a non-zero simple microfunction is non-zero and a simple microfunction with a non-zero principal symbol is non-zero. Then the principal symbol $\sigma_{\Lambda_i^{j,k}}((s-s_0)^p \mathrm{sp}(P^{[\vec{a},s]}(x))) = (s-s_0)^p \sigma_{\Lambda_i^{j,k}}(\mathrm{sp}(P^{[\vec{a},s]}(x)))$ is non-zero at $s = s_0$ if and only if $(s-s_0)^p \mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\Lambda_i^{j,k}}$ is non-zero at $s = s_0$.

The equivalence of the second two follows from (39) and that $\Omega_i^{j,k}(s)/\sqrt{|dx|}$ is holomorphic and non-zero at all $s \in \boldsymbol{C}$. □

PROPOSITION 3.7. *The following three conditions are equivalent.*

1. *$P^{[\vec{a},s]}(x)$ has pole of order p at $s = s_0$.*

2. $\mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\bigcup_{i=1}^{n}\Lambda_i^\circ}$ *has pole of order p at* $s = s_0$.

3. *All the coefficient functions in* $\{c_i^{j,k}(\vec{a},s) \mid 0 \le i \le n, 0 \le j \le n-i, 0 \le k \le i\}$ *have pole of order not greater than p at* $s = s_0$ *and at least one coefficient of them has pole of order p at* $s = s_0$.

PROOF. the equivalence of 2 and 3 follows from Proposition 3.6, since

$$\bigcup_{i=0}^{n} \Lambda_i^\circ = \bigsqcup_{\substack{0\le i\le n\\ 0\le j\le n-i\\ 0\le k\le i}} \Lambda_i^{j,k}.$$

We now show that the condition 2 follows from the condition 1. If $P^{[\vec{a},s]}(x)$ has pole of order p at $s = s_0$, then $(s-s_0)^p P^{[\vec{a},s]}(x)$ is a non-zero holomorphic function at $s = s_0$ with respect to s. Then

$$\mathrm{sp}((s-s_0)^p P^{[\vec{a},s]}(x)) = (s-s_0)^p \mathrm{sp}(P^{[\vec{a},s]}(x))$$

is also holomorphic at $s=s_0$ and it is non-zero there. (Note that we consider the microfunction on the whole T^*V but not on $T^*V - V$. See Remark 3.1.) Since $(s-s_0)^p\mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\bigcup_{i=0}^{n}\Lambda_i^\circ}$ is holomorphic at $s = s_0$, $\mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\bigcup_{i=0}^{n}\Lambda_i^\circ}$ has pole of order not greater than p at $s = s_0$. If the order is strictly less than p, then $(s-s_0)^p\mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\bigcup_{i=0}^{n}\Lambda_i^\circ}\big|_{s=s_0}$ is a zero function. Then $(s-s_0)^p P^{[\vec{a},s]}(x)\big|_{s=s_0}$ is zero by Lemma 3.5. This is a contradiction. Therefore $\mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\bigcup_{i=0}^{n}\Lambda_i^\circ}$ has pole of order p at $s = s_0$. This means that the condition 2 follows from the condition 1.

We shall show that the condition 1 follows from the condition 2. If $\mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\bigcup_{i=0}^{n}\Lambda_i^\circ}$ has pole of order p at $s = s_0$, then

$$(s-s_0)^p\mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\bigcup_{i=0}^{n}\Lambda_i^\circ} = \mathrm{sp}((s-s_0)^p P^{[\vec{a},s]}(x))\big|_{\bigcup_{i=0}^{n}\Lambda_i^\circ}$$

is non-zero and holomorphic at $s = s_0$. We note that $(s-s_0)^p P^{[\vec{a},s]}(x)$ is a hyperfunction whose singular support is contained in $\bigcup_{i=0}^{n}\Lambda_i$. If $P^{[\vec{a},s]}(x)$ has pole of order $> p$, then the singular supports of the Laurent expansion coefficients of order $> p$ are contained in $\bigcup_{i=0}^{n}\Lambda_i - \bigcup_{i=0}^{n}\Lambda_i^\circ$ and hence it is empty by Lemma 3.5. Then the hyperfunction $P^{[\vec{a},s]}(x)$ has pole of order p at $s = s_0$. Therefore, $(s-s_0)^p P^{[\vec{a},s]}(x)$ is non-zero and holomorphic at $s = s_0$. Thus, $P^{[\vec{a},s]}(x)$ has pole of order p at $s = s_0$. This means that the condition 1 follows from the condition 2. $\square$

3.6. Laurent expansions of coefficient functions. We give the Laurent expansion coefficients of $P^{[\vec{a},s]}(x)$ and $c_i^{j,k}(\vec{a},s)$ in the following definitions.

DEFINITION 3.3. Suppose that the complex power function $P^{[\vec{a},s]}(x)$ has pole of order p at $s = s_0$. We give the *Laurent expansion* of $P^{[\vec{a},s]}(x)$ at $s = s_0$ by

$$P^{[\vec{a},s]}(x) = \sum_{w=-p}^{\infty} P_w^{[\vec{a},s_0]}(x)(s-s_0)^w. \tag{45}$$

Here,

$$P_w^{[\vec{a},s_0]}(x) \tag{46}$$

is the Laurent expansion coefficient of degree w of $P^{[\vec{a},s]}(x)$. For the coefficient $c_i^{j,k}(\vec{a},s)$, we give the *Laurent expansion* at $s = s_0$ by

$$c_i^{j,k}(\vec{a},s) = \sum_{w=-p}^{\infty} c_{i,(\vec{a},s_0),w}^{j,k}(s-s_0)^w\,. \tag{47}$$

Here,

$$c_{i,(\vec{a},s_0),w}^{j,k} \tag{48}$$

is the Laurent expansion coefficient of degree w of $c_i^{j,k}(\vec{a},s)$. Since the order of the pole of $c_i^{j,k}(\vec{a},s)$ at $s = s_0$ is not greater than p, some of the beginning Laurent expansion coefficients of (47) may be zero.

We can express the support of $P_w^{[\vec{a},s_0]}(x)$ in terms of the Laurent expansion coefficients of $c_i^{j,k}(\vec{a},s)$. Namely, we have the following proposition.

PROPOSITION 3.8. *Suppose that $P^{[\vec{a},s]}(x)$ has pole of order p at $s = s_0$. Let (45) be the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = s_0$. Then we have*

$$\mathrm{Supp}(P_w^{[\vec{a},s_0]}(x)) = \overline{\bigcup_{(i,j)\in L} S_i^j} \tag{49}$$

with $L := \{(i,j) \in \boldsymbol{Z}^2 \mid \mathrm{ord}_{s=s_0}(c_i^{j,k}(\vec{a},s)) \geq -w$ for some $k \in \boldsymbol{Z} \cap [0,i]\}$. Here, $[0,i]$ means the closed interval in $\boldsymbol{R}$ from 0 to i and $\mathrm{ord}_{s=s_0}(c_i^{j,k}(\vec{a},s))$ stands for the order of pole of $c_i^{j,k}(\vec{a},s)$ at $s = s_0$.

PROOF. For a hyperfunction $f(x)$ on $\boldsymbol{V}$, we have

$$\mathrm{Supp}(f(x)) = \overline{\pi(\mathrm{Supp}(\mathrm{sp}(f(x))))}$$

by the isomorphism (24). Therefore, we have

$$\mathrm{Supp}(P_w^{[\vec{a},s_0]}(x)) = \overline{\pi(\mathrm{Supp}(\mathrm{sp}(P_w^{[\vec{a},s_0]}(x))))}\,. \tag{50}$$

Let q be an integer satisfying $-p \leq -q < +\infty$. If $\mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\Lambda_i^{j,k}}$ has pole of order q at $s = s_0$, then the pole of $c_i^{j,k}(\vec{a},s)$ at $s = s_0$ is of order q (Proposition 3.6). We have the Laurent expansion

$$\mathrm{sp}(P^{[\vec{a},s]}(x))\big|_{\Lambda_i^{j,k}} = \sum_{w=-q}^{\infty} \mathrm{sp}(P^{[\vec{a},s_0]}(x))\big|_{\Lambda_i^{j,k}} \cdot (s-s_0)^w \tag{51}$$

by (45). On the other hand, let

$$\sigma_{\Lambda_i^{j,k}}(\mathrm{sp}(P^{[\vec{a},s]}(x))) = \sum_{w=-q}^{\infty} \sigma_{i,(\vec{a},s_0),w}^{j,k} \cdot (s-s_0)^w \tag{52}$$

be the Laurent expansion of the principal symbol $\sigma_{\Lambda_i^{j,k}}(\mathrm{sp}(P^{[\vec{a},s]}(x)))$. Then we have

$$\sigma_{\Lambda_i^{j,k}}(\mathrm{sp}(P_w^{[\vec{a},s_0]}(x))) = \sigma_{i,(\vec{a},s_0),w}^{j,k} \tag{53}$$

for $-q \leq w < +\infty$.

Now we have the following Laurent expansions,

$$\begin{aligned} \sigma_{\Lambda_i^{j,k}}(P^{[\vec{a},s]}(x)) &= c_i^{j,k}(\vec{a},s)\Omega_i^{j,k}(s)/\sqrt{|dx|} \\ &= \sum_{w=-q}^{\infty} \sigma_{i,(\vec{a},s_0),w}^{j,k} \cdot (s-s_0)^w , \end{aligned} \tag{54}$$

$$c_i^{j,k}(\vec{a},s) = \sum_{u=-q}^{\infty} c_{i,(\vec{a},s_0),u}^{j,k} \cdot (s-s_0)^u , \tag{55}$$

$$\Omega_i^{j,k}(s) = \sum_{v=0}^{\infty} \Omega_{i,s_0,v}^{j,k} \cdot (s-s_0)^v . \tag{56}$$

Note that the Laurent expansion coefficients $\Omega_{i,s_0,0}^{j,k}, \Omega_{i,s_0,1}^{j,k}, \ldots$ in (56) are non-zero linearly independent half-volume forms on $\Lambda_i^{j,k}$. The proof of this fact is given in the following way. Let $f(x)$ be a non-zero, non-constant, real-valued and real analytic function on an open set of a real analytic manifold. Then, the Laurent expansion coefficients 1, $\log|f(x)|$, $(\log|f(x)|)^2, \ldots$ of the complex power $|f(x)|^s$ with respect to the complex variable s are linearly independent, since we may regard $|f(x)|$ as a real variable. By the definition (35), we have

$$\Omega_i^{j,k}(s) := |P_{\Lambda_i^{j,k}}(x,y)|^s \sqrt{|\omega_{\Lambda_i^{j,k}}(x,y)|} .$$

Put $f(x) := P_{\Lambda_i^{j,k}}(x,y)$. Then $f(x)$ is a non-zero, real-valued and real analytic function on the open set $\Lambda_i^{j,k}$ of the real analytic manifold Λ_i. $f(x)$ is a non-constant function, since it vanishes on the boundary $\overline{\Lambda_i^{j,k}} \cap \overline{\Lambda_{i+1}}$ if $i > 0$. If $i = 0$, then $f(x) = \det(x)$ and hence it is a non-constant function. Then, we see that the Laurent expansion coefficients $\Omega_{i,s_0,0}^{j,k}, \Omega_{i,s_0,1}^{j,k}, \ldots$ are non-zero linearly independent half-volume forms on $\Lambda_i^{j,k}$.

Then all the Laurent expansion coefficients

$$\sigma_{i,(\vec{a},s_0),w}^{j,k} \quad (-q \leq w \leq +\infty) \tag{57}$$

in (54) are non-zero if $c^{j,k}_{i,(\vec{a},s_0),-q} \neq 0$. This means that all the Laurent expansion coefficients of negative order of $\sigma_{\Lambda_i^{j,k}}(P^{[\vec{a},s]}(x))$ are not zero. Hence the support $\mathrm{Supp}(\mathrm{sp}(P_w^{[\vec{a},s_0]}(x)))$ contains $\Lambda_i^{j,k}$ if $-q \leq w < \infty$, which shows that

$$
\begin{aligned}
\mathrm{Supp}(P_w^{[\vec{a},s_0]}(x)) &= \overline{\pi(\mathrm{Supp}(\mathrm{sp}(P_w^{[\vec{a},s_0]}(x))))} \\
&= \overline{\pi\left(\bigcup_{(i,j,k)\in L''} \Lambda_i^{j,k}\right)} \\
&= \overline{\bigcup_{(i,j,k)\in L'} \pi(\Lambda_i^{j,k})} \\
&= \overline{\bigcup_{(i,j)\in L} S_i^j}\,,
\end{aligned} \tag{58}
$$

where

$$
\begin{aligned}
L'' &:= \{(i,j,k) \in \mathbf{Z}^3 \mid \mathrm{ord}_{s=s_0}(c_i^{j,k}(\vec{a},s)) \geq -w \text{ at } s = s_0\}\,, \\
L' &:= \{(i,j,k) \in \mathbf{Z}^3 \mid \mathrm{ord}_{s=s_0}(c_i^{j,k}(\vec{a},s)) \geq -w \text{ at } s = s_0\}\,, \\
L &:= \{(i,j) \in \mathbf{Z}^2 \mid \mathrm{ord}_{s=s_0}(c_i^{j,k}(\vec{a},s)) \geq -w \text{ for some } k \text{ in } [0,i] \cap \mathbf{Z} \text{ at } s = s_0\}\,.
\end{aligned}
$$

The equality between the first line and the second line of (58) is obtained from Lemma 3.5, since the singular support of each Laurent expansion coefficient $P_w^{[\vec{a},s_0]}(x)$ is contained in $\bigcup_{i=0}^n \Lambda_i$. □

REMARK 3.3. $\mathrm{sp}(P_w^{[\vec{a},s_0]}(x))$ may not be a simple microfunction. Then we cannot consider the principal symbol of $\mathrm{sp}(P_w^{[\vec{a},s_0]}(x))$ in the way that [4] or [5] have defined. By using the property that $\mathrm{sp}(P_w^{[\vec{a},s_0]}(x))$ is a regular holonomic microfunction, we can also give the definition of the principal symbol $\mathrm{sp}(P_w^{[\vec{a},s_0]}(x))$ directly, but we do not need to do it because, in our case, the principal symbol of $\mathrm{sp}(P_w^{[\vec{a},s_0]}(x))$ is obtained as a Laurent expansion coefficient of a simple microfunction with a meromorphic parameter s.

3.7. Relations of coefficient functions. We give the analytic relations (59) combining the coefficient functions of a hyperfunction solution to the holonomic system $\mathcal{M}_s$. By the formula (59), we can compute all the coefficients $c_i^{j,k}(\vec{a},s)$ $(0 \leq i \leq n, 0 \leq j \leq n-i, 0 \leq k \leq i)$ from the base coefficients $c_0^{j,0}(\vec{a},s) = a_j$ $(0 \leq j \leq n)$. The propositions obtained in this subsection enable us to estimate the order of poles of the coefficient functions. We use effectively the following two relations (59) and (60) in the proof of the main theorem after the next section together with Proposition 3.7 and Proposition 3.8.

PROPOSITION 3.9. *The coefficient functions on Λ_i° and Λ_{i+1}° have the following relation. These relations depend on $s \in \boldsymbol{C}$ meromorphically.*

$$
\begin{aligned}
\begin{bmatrix} c_{i+1}^{j,k+1}(\vec{a},s) \\ c_{i+1}^{j,k}(\vec{a},s) \end{bmatrix}
&= \frac{\Gamma\left(s+\frac{i+2}{2}\right)}{\sqrt{2\pi}}
\begin{bmatrix} \exp\left(-\frac{\pi}{2}\sqrt{-1}\left(s+\frac{i+2}{2}\right)\right) & \exp\left(+\frac{\pi}{2}\sqrt{-1}\left(s+\frac{i+2}{2}\right)\right) \\ \exp\left(+\frac{\pi}{2}\sqrt{-1}\left(s+\frac{i+2}{2}\right)\right) & \exp\left(-\frac{\pi}{2}\sqrt{-1}\left(s+\frac{i+2}{2}\right)\right) \end{bmatrix} \\
&\quad \times \begin{bmatrix} \exp\left(+\frac{\pi}{4}\sqrt{-1}(i-2k)\right) & 0 \\ 0 & \exp\left(-\frac{\pi}{4}\sqrt{-1}(i-2k)\right) \end{bmatrix} \\
&\quad \times \begin{bmatrix} c_i^{j+1,k}(\vec{a},s) \\ c_i^{j,k}(\vec{a},s) \end{bmatrix}.
\end{aligned}
\tag{59}
$$

See [10, Theorem 2.13]. The above relations are the case of $\mathrm{Sym}_n(\boldsymbol{R})$.

PROPOSITION 3.10. *The coefficient functions on Λ_i° and Λ_{i+2}° have the following relations.*

(60)

$$
\begin{aligned}
&\begin{bmatrix} c_{i+2}^{j,k+2}(\vec{a},s) \\ c_{i+2}^{j,k+1}(\vec{a},s) \\ c_{i+2}^{j,k}(\vec{a},s) \end{bmatrix}
= \frac{\Gamma\left(s+\frac{i+2}{2}\right)\Gamma\left(s+\frac{i+3}{2}\right)}{2\pi} \\
&\times \begin{bmatrix} +\sqrt{-1}\exp(-\pi\sqrt{-1}(s+k)) & 0 & -\sqrt{-1}\exp(+\pi\sqrt{-1}(s+k)) \\ \exp\left(\frac{1}{2}\pi\sqrt{-1}(i-2k)\right) & -2\cos\left(\frac{1}{2}\pi(2s+i)\right) & \exp\left(-\frac{1}{2}\pi\sqrt{-1}(i-2k)\right) \\ -\sqrt{-1}\exp(+\pi\sqrt{-1}(s-k+i)) & 0 & +\sqrt{-1}\exp(-\pi\sqrt{-1}(s-k+i)) \end{bmatrix} \\
&\times \begin{bmatrix} c_i^{j+2,k}(\vec{a},s) \\ c_i^{j+1,k}(\vec{a},s) \\ c_i^{j,k}(\vec{a},s) \end{bmatrix}.
\end{aligned}
$$

These relations depend on $s \in \boldsymbol{C}$ meromorphically.

We obtain these formulas by applying the relation formula (59) twice.

4. Estimates of the orders of the coefficient functions. By the recursion formula (59), we see that all the coefficient functions are meromorphic functions in $s \in \boldsymbol{C}$ and depend on $\vec{a} \in \boldsymbol{C}^{n+1}$ linearly and that they can be explicitly computed recursively. Since Proposition 3.7 and Proposition 3.8 claim that the calculation of the order of poles and the determination of the support of the Laurent expansion coefficients of $P^{[\vec{a},s]}(x)$ are reduced to those of the coefficient functions, the rest of the essential problem is the explicit computation of the coefficient functions.

In this paper, we give estimates for the orders of coefficient functions from above and below (Proposition 4.2 and Proposition 4.4, Proposition 4.9 and Proposition 4.11) instead of computing the closed forms of the coefficient functions. This may not be the best way for the proof, but we need not write the closed forms of the coefficient functions in our proof. The proof is carried out by inductions and we need some repeated arguments. In order to avoid repetitions, we shall give complete proof for the case of orders at strict half integers in the first subsection and give abbreviated proof for the case of orders at integers in the next subsection by skipping the parallel part of the proof.

4.1. Laurent expansions of the coefficient matrices. We define the coefficient matrices to estimate the coefficient functions. In the proof in this section the notation for the coefficient matrices turns out to be useful.

DEFINITION 4.1 (Coefficient matrix). 1. We define the *coefficient matrix* $\boldsymbol{c}_i^{\bullet,k}(\vec{a}, s)$ and $\boldsymbol{c}_i^{j,\bullet}(\vec{a}, s)$ as the $1 \times (n-i)$-matrix

$$\boldsymbol{c}_i^{\bullet,k}(\vec{a}, s) = (c_i^{0,k}(\vec{a}, s), c_i^{1,k}(\vec{a}, s), \dots, c_i^{n-i,k}(\vec{a}, s)) \tag{61}$$

and the $i \times 1$-matrix

$$\boldsymbol{c}_i^{j,\bullet}(\vec{a}, s) = {}^t(c_i^{j,0}(\vec{a}, s), c_i^{j,1}(\vec{a}, s), \dots, c_i^{j,i}(\vec{a}, s)), \tag{62}$$

respectively. The *coefficient matrix* $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ is defined to be the $i \times (n-i)$ matrix

$$\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s) = (c_i^{j,k}(\vec{a}, s))_{\substack{0\le k\le i\\ 0\le j\le n-i}}. \tag{63}$$

2. We define the *order of pole* of a coefficient matrix to be the maximum of the orders of the entries in the matrix. For example, the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is the maximum of the orders of the entries in $(c_i^{j,k}(\vec{a}, s))_{\substack{0\le k\le i\\ 0\le j\le n-i}}$.

Let p be the order of pole of $P^{[\vec{a},s]}(x)$ at $s = s_0$. Then the Laurent expansion of $\boldsymbol{c}_i^{\bullet,k}(\vec{a}, s)$, $\boldsymbol{c}_i^{j,\bullet}(\vec{a}, s)$ and $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ are given in the following form.

$$\boldsymbol{c}_i^{\bullet,k}(\vec{a}, s) = \sum_{w=-p}^{\infty} \boldsymbol{c}_{i,(\vec{a},s_0),w}^{\bullet,k}(s-s_0)^w, \tag{64}$$

$$\boldsymbol{c}_i^{j,\bullet}(\vec{a}, s) = \sum_{w=-p}^{\infty} \boldsymbol{c}_{i,(\vec{a},s_0),w}^{j,\bullet}(s-s_0)^w, \tag{65}$$

$$\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s) = \sum_{w=-p}^{\infty} \boldsymbol{c}_{i,(\vec{a},s_0),w}^{\bullet,\bullet}(s-s_0)^w. \tag{66}$$

Some of the beginning Laurent expansion coefficients may be zero in these Laurent expansions because the orders of poles of these coefficient functions are not greater than the order of $P^{[\vec{a},s]}(x)$.

PROPOSITION 4.1. *Let s_0 be a half integer satisfying $s_0 \le -1$ and let i_0 be an integer in $0 \le i_0 \le n-1$. We suppose that i_0 is even and s_0 is a strict half integer or that i_0 is odd and s_0 is an integer. Then $\boldsymbol{c}_{i_0}^{\bullet,\bullet}(\vec{a}, s)$ and $\boldsymbol{c}_{i_0+1}^{\bullet,\bullet}(\vec{a}, s)$ have poles of the same order at $s = s_0$.*

PROOF. Note that $s_0 + (i_0 + 2)/2$ is a strict half integer in both cases. We consider the relation (59) in a neighborhood of $s = s_0$. Then the relation matrix between

$$\begin{bmatrix} c_{i+1}^{j,k+1}(\vec{a}, s) \\ c_{i+1}^{j,k}(\vec{a}, s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_{i}^{j+1,k}(\vec{a}, s) \\ c_{i}^{j,k}(\vec{a}, s) \end{bmatrix}$$

depends on $s \in \boldsymbol{C}$ holomorphically and is invertible near $s = s_0$. The inverse matrix also depends on s holomorphically, and hence $\boldsymbol{c}_{i_0}^{\bullet,\bullet}(\vec{a}, s)$ and $\boldsymbol{c}_{i_0+1}^{\bullet,\bullet}(\vec{a}, s)$ have poles of the same order at $s = s_0$. □

4.2. The case at strict half integers.

PROPOSITION 4.2. *Let* $s_0 := -(2m + 1)/2$ $(m = 1, 2, \dots)$. *For a fixed integer* p *satisfying* $0 \le p \le m$ *and* $2p \le n$, *we suppose that* $\langle \boldsymbol{d}^{(2p)}[s_0], \vec{a}\rangle \ne 0$. *Then* $\boldsymbol{c}_{2q}^{\bullet,\bullet}(\vec{a}, s)$ *has pole of order* q *for* $q = 0, 1, \dots, p$ *at* $s = s_0$.

It is verified that $\boldsymbol{c}_{2q}^{\bullet,\bullet}(\vec{a}, s)$ has pole of order at most q for $q = 0, 1, \dots, p$ at $s = s_0$ from the relations (59). Then what we have to prove is that it has pole of order at least q for $q = 0, 1, \dots, p$ at $s = s_0$. For this purpose, we have only to prove the following Proposition 4.3, since we then have $\boldsymbol{c}_{2q,(\vec{a},s_0),-q}^{\bullet,0} \ne 0$ for $q = 0, 1, \dots, p$.

PROPOSITION 4.3. *Under the same condition as in Proposition* 4.2, *we have* $\boldsymbol{c}_{2q,(\vec{a},s_0),-q}^{\bullet,0} = (\mathrm{nzc})_q \times \langle \boldsymbol{d}^{(2q)}[s_0], \vec{a}\rangle$ *for* $q = 0, 1, \dots, p$, *where* $(\mathrm{nzc})_q$ *is a non-zero constant depending on* q.

PROOF. We prove the following statement $(A)_p$ for $p = 0, 1, \dots$ by induction on p:

$$(A)_p : \begin{array}{l} \text{If } \langle \boldsymbol{d}^{(2p)}[s_0], \vec{a}\rangle \ne 0, \text{ then } \boldsymbol{c}_{2q,(\vec{a},s_0),-q}^{\bullet,0} = \\ (\mathrm{nzc})_q \times \langle \boldsymbol{d}^{(2q)}[s_0], \vec{a}\rangle \text{ for } q = 0, 1, \dots, p, \text{ where} \\ (\mathrm{nzc})_q \text{ is a non-zero constant depending on } q. \end{array} \tag{67}$$

When $p = 0$, $\langle \boldsymbol{d}^{(0)}[s_0], \vec{a}\rangle = \vec{a} \ne 0$ and $\boldsymbol{c}_{2q,(\vec{a},s_0),-q}^{\bullet,0} = \boldsymbol{c}_{0,(\vec{a},s_0),0}^{\bullet,0} = \vec{a}$ for all possible q. This means that $(A)_0$ is true.

Next we prove $(A)_{r+1}$ under the assumption $(A)_r$. Since $\langle \boldsymbol{d}^{(2r+2)}[s_0], \vec{a}\rangle \ne 0$ implies $\langle \boldsymbol{d}^{(2r)}[s_0], \vec{a}\rangle \ne 0$ (Proposition 2.1), we have $\boldsymbol{c}_{2q,(\vec{a},s_0),-q}^{\bullet,0} = (\mathrm{nzc})_q \times \langle \boldsymbol{d}^{(2q)}[s_0], \vec{a}\rangle$ for $q = 0, 1, \dots, r$ if $(A)_r$ is true. We have only to prove

$$\boldsymbol{c}_{2r+2,(\vec{a},s_0),-r-1}^{\bullet,0} = (\mathrm{nzc})_{r+1} \times \langle \boldsymbol{d}^{(2r+2)}[s_0], \vec{a}\rangle\,. \tag{68}$$

From (60),

$$\begin{aligned} c_{2r+2}^{j,0}(\vec{a}, s) =& \frac{1}{2\pi}\Gamma\left(s + \frac{2r+2}{2}\right)\Gamma\left(s + \frac{2r+3}{2}\right) \\ &\times \left(+ \sqrt{-1}\exp(-\pi\sqrt{-1}(s + 2r))c_{2r}^{j,0}(\vec{a}, s) \right. \\ &\left. - \sqrt{-1}\exp(+\pi\sqrt{-1}(s + 2r))c_{2r}^{j+2,0}(\vec{a}, s) \right). \end{aligned} \tag{69}$$

Note that $\Gamma(s + (2r + 3)/2)$ has pole of order 1 at $s = s_0$. Indeed, since $r + 1 \le p \le m$, $s_0 + (2r + 3)/2$ is a non-positive integer. Then $c_{2r+2}^{j,0}(\vec{a}, s)$ has pole of order $r + 1$ if $c_{2r}^{j,0}(\vec{a}, s)$

or $c_{2r}^{j+2,0}(\vec{a}, s)$ has pole of order r at $s = s_0$ and if

$$v_j^{(r)} := c^{j,0}_{2r,(\vec{a},s_0),-r} + c^{j+2,0}_{2r,(\vec{a},s_0),-r} \neq 0\,, \tag{70}$$

both of which assumptions are valid at least one j from the induction hypothesis $(A)_r$ and the hypothesis of $(A)_{r+1}$. This follows from that

$$+\sqrt{-1}\exp(-\pi\sqrt{-1}(s_0 + 2r)) = -\sqrt{-1}\exp(+\pi\sqrt{-1}(s_0 + 2r))\,,$$

since s_0 is a strict half integer.

By taking the Laurent expansion coefficient of degree $-r-1$ of (69), we have

$$c^{j,0}_{2r+2,(\vec{a},s_0),-r-1} = (\text{non-zero const.}) \times v_j^{(r)}\,. \tag{71}$$

Here (non-zero const.), a non-zero constant, does not depend on j. Then $c^{\bullet,0}_{2r+2}(\vec{a}, s)$ has pole of order $r+1$ if (70) is valid for at least one index j. On the other hand, by (70), (71) and the assumption $(A)_r$, we have

$$\begin{aligned}
&c^{\bullet,0}_{2r+2,(\vec{a},s_0),-r-1} \\
&\quad = (c^{0,0}_{2r+2,(\vec{a},s_0),-r-1}, c^{1,0}_{2r+2,(\vec{a},s_0),-r-1}, \dots, c^{n-2r-2,0}_{2r+2,(\vec{a},s_0),-r-1}) \\
&\quad = (\text{non-zero const.}) \times (v_0^{(r)}, v_1^{(r)}, \dots, v_{n-2r-2}^{(r)}) \\
&\quad = (\text{non-zero const.}) \times (\text{nzc})_r \\
&\qquad \times (\langle d_0^{(2r)}[s_0], \vec{a}\rangle + \langle d_2^{(2r)}[s_0], \vec{a}\rangle, \dots, \langle d_{n-2r-2}^{(2r)}[s_0], \vec{a}\rangle + \langle d_{n-2r}^{(2r)}[s_0], \vec{a}\rangle) \\
&\quad = (\text{non-zero const.}) \times (\text{nzc})_r \times \langle \boldsymbol{d}^{(2r+2)}[s_0], \vec{a}\rangle\,,
\end{aligned} \tag{72}$$

which implies (68) by putting

$$(\text{nzc})_{r+1} = (\text{non-zero const.}) \times (\text{nzc})_r\,.$$

Thus we have $(A)_{r+1}$. □

PROPOSITION 4.4. *Let* $s_0 := -(2m+1)/2$ $(m = 1, 2, \dots)$. *For a fixed integer* p *satisfying* $0 \le p \le m$ *and* $2p + 2 \le n$, *we suppose that* $\langle \boldsymbol{d}^{(2p+2)}[s_0], \vec{a}\rangle = 0$. *Then* $c^{\bullet,\bullet}_{2q}(\vec{a}, s)$ *has pole of order at most* p *for* $q = p, p+1, \dots$ *at* $s = s_0$.

PROOF. We prove the following $(B)_p$ for all $p = 0, 1, \dots$ by induction on p:

(73) $(B)_p$: If $\langle \boldsymbol{d}^{(2p+2)}[s_0], \vec{a}\rangle = 0$, then $\boldsymbol{c}^{\bullet,\bullet}_{2p+2f}(\vec{a}, s)$ has pole of order at most p at $s = s_0$ for $f = 0, 1, \dots$ satisfying $2p + 2f \le n$.

LEMMA 4.5. $(B)_0$ *is true. In other words, if* $\langle \boldsymbol{d}^{(2)}[s_0], \vec{a}\rangle = 0$, *then* $\boldsymbol{c}^{\bullet,\bullet}_{2f}(\vec{a}, s)$ *is holomorphic at* $s = s_0$ *for* $f = 0, 1, \dots$.

PROOF. We show that

$$\boldsymbol{c}^{j,\bullet}_{2f}(\vec{a}, s) = (-1)\boldsymbol{c}^{j+2,\bullet}_{2f}(\vec{a}, s) \quad \text{for all } j = 0, 1, \dots, n-2f-2\,, \tag{74}$$

and

$$\boldsymbol{c}^{j,\bullet}_{2f}(\vec{a}, s) \text{ is holomorphic at } s = s_0 \quad \text{for all } j = 0, 1, \dots, n-2f\,, \tag{75}$$

by induction on f for $f = 0, 1, \dots$ with $2f \le n$.

If $\langle \boldsymbol{d}^{(2)}[s_0], \vec{a} \rangle = 0$, then we have by definition

$$\langle d_j^{(2)}[s_0], \vec{a} \rangle = \langle d_j^{(0)}[s_0], \vec{a} \rangle + \langle d_{j+2}^{(0)}[s_0], \vec{a} \rangle = a_j + a_{j+2} = 0 .$$

Since $c_0^{j,0}(\vec{a}, s) = a_j$, this means that

$$c_0^{j,0}(\vec{a}, s) = -c_0^{j+2,0}(\vec{a}, s) . \tag{76}$$

The coefficient $c_0^{j,0}(\vec{a}, s)$ is a constant and hence is holomorphic. (74) and (75) are verified for $f = 0$. This is the first step of the induction on f.

Next we prove (74) and (75) for $f = q + 1$ under the assumption that (74) and (75) have been proved for $f = 0, 1, \dots, q$. Since the matrix relation in (60) does not depend on j, the property (74) for $f = q + 1$ follows from the property (74) for $f = q$. Indeed, we have

$$\begin{bmatrix} c_{2q}^{j+2,k}(\vec{a}, s) \\ c_{2q}^{j+1,k}(\vec{a}, s) \\ c_{2q}^{j,k}(\vec{a}, s) \end{bmatrix} = (-1) \times \begin{bmatrix} c_{2q}^{j+4,k}(\vec{a}, s) \\ c_{2q}^{j+3,k}(\vec{a}, s) \\ c_{2q}^{j+2,k}(\vec{a}, s) \end{bmatrix} \tag{77}$$

for all j by the induction hypothesis (74) for $f = q$. In addition, since the matrix in (60) does not depend on j, we have

$$\boldsymbol{c}_{2q+2}^{j,\bullet}(\vec{a}, s) = \begin{bmatrix} \vdots \\ c_{2q+2}^{j,k}(\vec{a}, s) \\ c_{2q+2}^{j,k+1}(\vec{a}, s) \\ c_{2q+2}^{j,k+2}(\vec{a}, s) \\ \vdots \end{bmatrix} = (-1) \times \begin{bmatrix} \vdots \\ c_{2q+2}^{j+2,k}(\vec{a}, s) \\ c_{2q+2}^{j+2,k+1}(\vec{a}, s) \\ c_{2q+2}^{j+2,k+2}(\vec{a}, s) \\ \vdots \end{bmatrix} = (-1) \times \boldsymbol{c}_{2q+2}^{j+2,\bullet}(\vec{a}, s) \tag{78}$$

for all j by using the relation (60). This means (74) for $f = q + 1$.

We see that the order of pole at $s = s_0$ of $\boldsymbol{c}_{2q+2}^{j,\bullet}(\vec{a}, s)$ may be greater by 1 than that of $\boldsymbol{c}_{2q}^{j,\bullet}(\vec{a}, s)$ by the gamma factor of the relation matrix in (60). However, the growing of the order of the pole is canceled from the property (74) for $f = q$. We shall prove it below.

For $c_{2q+2}^{j,k+2}(\vec{a}, s)$, we have by (60) with $i = 2q$

$$\begin{aligned} c_{2q+2}^{j,k+2}(\vec{a}, s) =& \frac{1}{2\pi} \times \Gamma\left(s + \frac{2q+2}{2}\right) \Gamma\left(s + \frac{2q+3}{2}\right) \sqrt{-1} \\ & \times (-\exp(-\pi\sqrt{-1}(s+k)) - \exp(+\pi\sqrt{-1}(s+k))) \\ & \times c_{2q}^{j,k}(\vec{a}, s) \end{aligned} \tag{79}$$

is holomorphic at $s = s_0$, since

$$\Gamma\left(s + \frac{2q+2}{2}\right) \Gamma\left(s + \frac{2q+3}{2}\right)$$

has pole of order 1, $(-\exp(-\pi\sqrt{-1}(s+k)) - \exp(+\pi\sqrt{-1}(s+k)))$ has zero of order 1, and $c_{2q}^{j,k}(\vec{a},s)$ is holomorphic, at $s=s_0$. For $c_{2q+2}^{j,k+1}(\vec{a},s)$ we have by (60) with $i=2q$

$$\begin{aligned}
&c_{2q+2}^{j,k+1}(\vec{a},s)\\
&=\frac{1}{2\pi}\Gamma\left(s+\frac{2q+2}{2}\right)\Gamma\left(s+\frac{2q+3}{2}\right)\\
&\quad\times\left(\left(-\exp\left(+\frac{\pi}{2}\sqrt{-1}(2q-2k)\right)+\exp\left(-\frac{\pi}{2}\sqrt{-1}(2q-2k)\right)\right)\right.\\
&\quad\left.\times c_{2q}^{j,k}(\vec{a},s)+\left(-2\cos\left(\frac{1}{2}\pi(2s+i)\right)\right)c_{2q}^{j+1,k}(\vec{a},s)\right)\\
&=\frac{1}{2\pi}\Gamma\left(s+\frac{2q+2}{2}\right)\Gamma\left(s+\frac{2q+3}{2}\right)\left(-2\cos\left(\frac{1}{2}\pi(2s+2q)\right)\right)c_{2q}^{j+1,k}(\vec{a},s)
\end{aligned}\tag{80}$$

is holomorphic at $s=s_0$. In fact, we see easily

$$\Gamma\left(s+\frac{2q+2}{2}\right)\Gamma\left(s+\frac{2q+3}{2}\right)$$

has pole of order 1 and $c_{2q}^{j,k}(\vec{a},s)$ is holomorphic, at $s=s_0$. On the other hand,

$$\left(-2\cos\left(\frac{1}{2}\pi(2s+2q)\right)\right)$$

has zero of order 1 at $s=s_0$, since s_0 is a strict half integer. Then $c_{2q+2}^{j,k+1}(\vec{a},s)$ is holomorphic at $s=s_0$.

For $c_{2q+2}^{j,k}(\vec{a},s)$, we have

$$\begin{aligned}
c_{2q+2}^{j,k}(\vec{a},s)=&\frac{1}{2\pi}\times\Gamma\left(s+\frac{2q+2}{2}\right)\Gamma\left(s+\frac{2q+3}{2}\right)\sqrt{-1}\\
&\times(+\exp(\pi\sqrt{-1}(s+k+2q))+\exp(-\pi\sqrt{-1}(s+k+2q)))\\
&\times c_{2q}^{j,k}(\vec{a},s)
\end{aligned}\tag{81}$$

is holomorphic at $s=s_0$, since

$$\Gamma\left(s+\frac{2q+2}{2}\right)\Gamma\left(s+\frac{2q+3}{2}\right)$$

has pole of order 1, $(+\exp(\pi\sqrt{-1}(s+k+2q))+\exp(-\pi\sqrt{-1}(s+k+2q)))$ has zero of order 1, and $c_{2q}^{j,k}(\vec{a},s)$ is holomorphic, at $s=s_0$.

Thus we see that $c_{2q+2}^{j,k+2}(\vec{a},s)$, $c_{2q+2}^{j,k+1}(\vec{a},s)$ and $c_{2q+2}^{j,k}(\vec{a},s)$ are all holomorphic at $s=s_0$ for all possible indices j and k if $c_{2q}^{j,k}(\vec{a},s)$ are all holomorphic at $s=s_0$ for all possible indices j,k. This means (75) with $f=q+1$. □

Thus we have verified $(B)_0$. This is the first step of the proof by induction of $(B)_p$ for all integers p. Next we proceed to the second step of the proof by induction. We prove $(B)_r$ under the assumption that we have verified $(B)_{r-1}$. Supposing that $(B)_{r-1}$ has been verified, the verification of $(B)_r$ is carried out by proving the following statement $(C)_f$ with the non-negative-integer index f, which is proved for all $f = 0, 1, \dots$ by induction on f.

$$(82) \qquad (C)_f : \quad \boldsymbol{c}^{\bullet,\bullet}_{2r+2f}(\vec{a}, s) \text{ has pole of order at most } r \text{ at } s = s_0 .$$

$(B)_r$ is equivalent to $(C)_f$ with $f = 0, 1, \dots$. The proposition $(C)_0$ has already been verified by the relations (60), since there appear Γ-functions with poles only r times in $c^{\bullet,\bullet}_{2r}(\vec{a}, s)$. Therefore, we have only to show that $(C)_f$ implies $(C)_{f+1}$ for the proof of $(B)_r$. We prove two lemmas Lemma 4.6 and Lemma 4.7, and then prove $(C)_{f+1}$ by Lemma 4.8 under the assumption $(C)_f$.

For a complex vector $\vec{a} := (a_0, a_1, \dots, a_n) \in \boldsymbol{C}^{n+1}$, we define $\vec{a}' \in \boldsymbol{C}^{n+1}$ and $\vec{b} \in \boldsymbol{C}^{n+1}$ by

$$(83) \qquad \vec{a}' := (0, 0, a_0, a_1, \dots, a_{n-2}) \quad \text{and} \quad \vec{b} := \vec{a} + \vec{a}' .$$

Then we have

$$(84) \qquad \boldsymbol{c}^{j,\bullet}_{2r+2f}(\vec{a}, s) = \boldsymbol{c}^{j+2,\bullet}_{2r+2f}(\vec{a}, s)$$

for all possible indices j, since the relation matrices (59) do not depend on j. Therefore, we have

$$(85) \qquad \begin{aligned} \boldsymbol{c}^{j,\bullet}_{2r+2f}(\vec{a}, s) &= \boldsymbol{c}^{j,\bullet}_{2r+2f}(\vec{b} - \vec{a}', s) = \boldsymbol{c}^{j,\bullet}_{2r+2f}(\vec{b}, s) - \boldsymbol{c}^{j,\bullet}_{2r+2f}(\vec{a}', s) \\ &= \boldsymbol{c}^{j,\bullet}_{2r+2f}(\vec{b}, s) - \boldsymbol{c}^{j-2,\bullet}_{2r+2f}(\vec{a}, s) . \end{aligned}$$

LEMMA 4.6. *$\langle d^{(2r)}_k[s_0], \vec{b}\rangle = 0$ for all $k \geq 2$.*

PROOF. For $k \geq 2$, we have

$$\begin{aligned} \langle d^{(2r)}_k[s_0], \vec{b}\rangle &= \langle d^{(2r)}_k[s_0], \vec{a} + \vec{a}'\rangle = \langle d^{(2r)}_k[s_0], \vec{a}\rangle + \langle d^{(2r)}_k[s_0], \vec{a}'\rangle \\ &= \langle d^{(2r)}_k[s_0], \vec{a}\rangle + \langle d^{(2r)}_{k-2}[s_0], \vec{a}\rangle \\ &= \langle d^{(2r+2)}_{k-2}[s_0], \vec{a}\rangle = 0 \end{aligned}$$

by the hypothesis in $(B)_r$. □

LEMMA 4.7. *For $j \geq 2$, the coefficient $\boldsymbol{c}^{j,\bullet}_{2r+2q}(\vec{b}, s)$ $(q \geq 0, 2r + 2q \leq n)$ has pole of order at most $r - 1$ at $s = s_0$.*

PROOF. We denote $\vec{b} = (b_0, b_1, b_2, \dots, b_n)$ and put $\vec{b}' := (b'_0, b'_1, b_2, \dots, b_n)$, where b'_0 and b'_1 are arbitrary numbers. Then we have

$$\langle d^{(2r)}_j[s_0], \vec{b}'\rangle = \langle d^{(2r)}_j[s_0], \vec{b}\rangle = 0$$

for all $j \geq 2$, since $\langle d^{(2r)}_j[s_0], \vec{b}'\rangle$ is determined independently of b'_0 and b'_1, and $\langle d^{(2r)}_j[s_0], \vec{b}\rangle$ is determined independently of b_0 and b_1. We see that

$$\boldsymbol{c}^{j,\bullet}_{2r}(\vec{b}, s) = \boldsymbol{c}^{j,\bullet}_{2r}(\vec{b}', s)$$

for all $j \geq 2$, since both sides are determined only by $(b_2, b_3, \dots, b_n) \in \boldsymbol{C}^{n-1}$. On the other hand, we may determine b_0' and b_1' so that

$$\langle d_0^{(2r)}[s_0], \vec{b}' \rangle = \langle d_1^{(2r)}[s_0], \vec{b}' \rangle = 0 .$$

Then we have $\langle \boldsymbol{d}^{(2r)}[s_0], \vec{b}' \rangle = 0$ by Lemma 4.6. By the induction hypothesis $(B)_{r-1}$, $\boldsymbol{c}_{2r+2q}^{j,\bullet}(\vec{b}', s) = \boldsymbol{c}_{2(r-1)+2(q+1)}^{j,\bullet}(\vec{b}', s)$ has pole of order at most $r-1$ for all j if $q \geq -1$. Therefore, $\boldsymbol{c}_{2r+2q}^{j,\bullet}(\vec{b}, s) = \boldsymbol{c}_{2r+2q}^{j,\bullet}(\vec{b}', s)$ has pole of order at most $r-1$ for all $j \geq 2$ and for all $q \geq 0$ with $2r + 2q \leq n$. □

LEMMA 4.8. *The coefficient* $\boldsymbol{c}_{2r+2f+2}^{\bullet,\bullet}(\vec{a}, s)$ *has pole of order at most* r *at* $s = s_0$. *Namely,* $(C)_{j+1}$ *is true.*

PROOF. Note that

$$\begin{bmatrix} c_{2r+2f}^{j+2,\bullet}(\vec{a}, s) \\ c_{2r+2f}^{j+1,\bullet}(\vec{a}, s) \\ c_{2r+2f}^{j,\bullet}(\vec{a}, s) \end{bmatrix} = \begin{bmatrix} c_{2r+2f}^{j+2,\bullet}(\vec{b}, s) - c_{2r+2f}^{j,\bullet}(\vec{a}, s) \\ c_{2r+2f}^{j+1,\bullet}(\vec{a}, s) \\ c_{2r+2f}^{j,\bullet}(\vec{a}, s) \end{bmatrix} \tag{86}$$

from (85). We can compute the orders of poles of the elements in

$$\begin{bmatrix} c_{2r+2f+2}^{j,k+2}(\vec{a}, s) \\ c_{2r+2f+2}^{j,k+1}(\vec{a}, s) \\ c_{2r+2f+2}^{j,k}(\vec{a}, s) \end{bmatrix} \tag{87}$$

from the orders of poles at $s = s_0$ of the elements in (86) by using the relation (60). In the following, we shall prove that the coefficients $c_{2r+2f+2}^{j,k+2}(\vec{a}, s)$, $c_{2r+2f+2}^{j,k+1}(\vec{a}, s)$ and $c_{2r+2f+2}^{j,k}(\vec{a}, s)$ have poles of order at most r by case-by-case calculation.

First, we consider the coefficient $c_{2r+2f+2}^{j,k+2}(\vec{a}, s)$. From the relation (60), we see that

$$\begin{aligned} &c_{2r+2f+2}^{j,k+2}(\vec{a}, s) \\ &\quad = \frac{1}{2\pi} \Gamma\left(s + \frac{2r+2f+2}{2}\right) \Gamma\left(s + \frac{2r+2f+3}{2}\right) \sqrt{-1} \\ &\qquad \times (-\exp(-\pi\sqrt{-1}(s+k)) - \exp(+\pi\sqrt{-1}(s+k))) c_{2r+2f}^{j,k}(\vec{a}, s) \\ &\qquad + \frac{1}{2\pi} \Gamma\left(s + \frac{2r+2f+2}{2}\right) \Gamma\left(s + \frac{2r+2f+3}{2}\right) \sqrt{-1} \\ &\qquad \times (-\exp(-\pi\sqrt{-1}(s+k))) c_{2r+2f}^{j+2,k}(\vec{b}, s) \end{aligned} \tag{88}$$

has pole of order at most r at $s = s_0$. Indeed, we see easily

$$\Gamma\left(s + \frac{2r+2f+2}{2}\right) \Gamma\left(s + \frac{2r+2f+3}{2}\right)$$

has pole of order 1 at $s = s_0$, $(-\exp(-\pi\sqrt{-1}(s+k)) - \exp(+\pi\sqrt{-1}(s+k)))$ has zero of order 1 at $s = s_0$, since s_0 is a strict half integer, $c_{2r+2f}^{j,k}(\vec{a}, s)$ has pole of order $\leq r$ at $s = s_0$ by the induction hypothesis $(C)_f$, $\exp(-\pi\sqrt{-1}(s+k))$ is holomorphic at $s = s_0$, and

$c^{j+2,k}_{2r+2f}(\vec{b}, s)$ has pole of order $\leq (r-1)$ at $s = s_0$ by Lemma 4.7. Then we have proved that $c^{j,k+2}_{2r+2f+2}(\vec{a}, s)$ has pole of order at most r at $s = s_0$.

Next, we consider the coefficient $c^{j,k+1}_{2r+2f+2}(\vec{a}, s)$. From the relation (60), we have

$$
\begin{aligned}
&c^{j,k+1}_{2r+2f+2}(\vec{a}, s)\\
&= \frac{1}{2\pi}\Gamma\left(s + \frac{2r+2f+2}{2}\right)\Gamma\left(s + \frac{2r+2f+3}{2}\right)\\
&\quad \times \left(-\exp\left(+\frac{\pi}{2}\sqrt{-1}(2r+2f-2k)\right) + \exp\left(-\frac{\pi}{2}\sqrt{-1}(2r+2f-2k)\right)\right)\\
&\quad \times c^{j,k}_{2r+2f}(\vec{a}, s)\\
&\quad + \frac{1}{2\pi}\Gamma\left(s + \frac{2r+2f+2}{2}\right)\Gamma\left(s + \frac{2r+2f+3}{2}\right)\\
&\quad \times \left(-2\cos\left(\frac{1}{2}\pi(2s+2r+2f)\right)\right) c^{j,k+1}_{2r+2f}(\vec{a}, s)\\
&\quad + \frac{1}{2\pi}\Gamma\left(s + \frac{2r+2f+2}{2}\right)\Gamma\left(s + \frac{2r+2f+3}{2}\right)\\
&\quad \times \exp\left(+\frac{\pi}{2}\sqrt{-1}(2r+2f-2k)\right) c^{j+2,k}_{2r+2f}(\vec{b}, s)\\
&= \frac{1}{2\pi}\Gamma\left(s + \frac{2r+2f+2}{2}\right)\Gamma\left(s + \frac{2r+2f+3}{2}\right)\\
&\quad \times \left(-2\cos\left(\frac{1}{2}\pi(2s+2r+2f)\right)\right) c^{j,k+1}_{2r+2f}(\vec{a}, s)\\
&\quad + \frac{1}{2\pi}\Gamma\left(s + \frac{2r+2f+2}{2}\right)\Gamma\left(s + \frac{2r+2f+3}{2}\right)\\
&\quad \times \exp\left(+\frac{\pi}{2}\sqrt{-1}(2r+2f-2k)\right) c^{j+2,k}_{2r+2f}(\vec{b}, s)
\end{aligned}
\tag{89}
$$

because

$$\left(-\exp\left(+\frac{\pi}{2}\sqrt{-1}(2r+2f-2k)\right) + \exp\left(-\frac{\pi}{2}\sqrt{-1}(2r+2f-2k)\right)\right) = 0,$$

since r, f, k are all integers. Then we see that $c^{j,k+1}_{2r+2f+2}(\vec{a}, s)$ has pole of order at most r at $s = s_0$. Indeed, we see easily

$$\Gamma\left(s + \frac{2r+2f+2}{2}\right)\Gamma\left(s + \frac{2r+2f+3}{2}\right)$$

has pole of order 1 at $s = s_0$, $-2\cos(\pi(2s+2r+2f)/2)$ has zero of order 1 at $s = s_0$, since s_0 is a strict half integer, $c^{j,k+1}_{2r+2f}(\vec{a}, s)$ has pole of order $\leq r$ at $s = s_0$ by the induction hypothesis $(C)_f$, $\exp(\pi\sqrt{-1}(2r+2f-2k)/2)$ is a constant, and $c^{j+2,k}_{2r+2f}(\vec{b}, s)$ has pole of order $\leq (r-1)$ at $s = s_0$ by Lemma 4.7, because s_0 is a strict half integer. Then we have proved that $c^{j,k+2}_{2r+2f+2}(\vec{a}, s)$ has pole of order at most r at $s = s_0$.

Lastly, we consider the coefficient $c^{j,k}_{2r+2f+2}(\vec{a}, s)$. From the relation (60), we see that

$$
\begin{aligned}
&c^{j,k}_{2r+2f+2}(\vec{a}, s)\\
&\quad= \frac{1}{2\pi}\Gamma\left(s+\frac{2r+2f+2}{2}\right)\Gamma\left(s+\frac{2r+2f+e}{2}\right)\sqrt{-1}\\
&\quad\times(+\exp(+\pi\sqrt{-1}(s-k+2r+2f))+\exp(+\pi\sqrt{-1}(s-k+2r+2f)))\\
&\quad\times c^{j,k}_{2r+2f}(\vec{a}, s)\\
&\quad+\frac{1}{2\pi}\Gamma\left(s+\frac{2r+2f+2}{2}\right)\Gamma\left(s+\frac{2r+2f+3}{2}\right)\sqrt{-1}\\
&\quad\times(-\exp(+\pi\sqrt{-1}(s-k+2r+2f)))c^{j+2,k}_{2r+2f}(\vec{b}, s)
\end{aligned}
\tag{90}
$$

has pole of order at most r at $s = s_0$. Indeed, we see easily

$$
\Gamma\left(s+\frac{2r+2f+2}{2}\right)\Gamma\left(s+\frac{2r+2f+3}{2}\right)
$$

has pole of order 1 at $s = s_0$, $(+\exp(+\pi\sqrt{-1}(s-k+2r+2f))+\exp(+\pi\sqrt{-1}(s-k+2r+2f)))$ has zero of order 1 at $s = s_0$, since s_0 is a strict half integer, $c^{j,k}_{2r+2f}(\vec{a}, s)$ has pole of order $\leq r$ at $s = s_0$ by the induction hypothesis $(C)_f$, $\exp(+\pi\sqrt{-1}(s-k+2r+2f))$ is holomorphic at $s = s_0$, and $c^{j+2,k}_{2r+2f}(\vec{b}, s)$ has pole of order $\leq (r-1)$ at $s = s_0$ by Lemma 4.7. Then we have proved that $c^{j,k}_{2r+2f+2}(\vec{a}, s)$ has pole of order at most r at $s = s_0$.

In the above arguments for the order of pole of $c^{j,k+2}_{2r+2f+2}(\vec{a}, s)$, $c^{j,k+1}_{2r+2f+2}(\vec{a}, s)$ and $c^{j,k}_{2r+2f+2}(\vec{a}, s)$, we can take the index k to be an arbitrary possible integer. Then $c^{j,k}_{2r+2f+2}(\vec{a}, s)$ has pole of order at most r at $s = s_0$ for all indices k in $0 \leq k \leq 2r+2f+2$. Thus we see that $c^{j,\bullet}_{2r+2f+2}(\vec{a}, s)$ has pole of order at most r at $s = s_0$. The proof above does not depend on the index j. Then $c^{\bullet,\bullet}_{2r+2f+2}(\vec{a}, s)$ has pole of order at most r at $s = s_0$. □

By Lemma 4.8, we see that $(C)_f$ implies $(C)_{f+1}$. Then $(C)_f$ is valid for all $f = 0, 1, \ldots$ by induction on f, which means that $(B)_r$ is valid. Thus we have $(B)_r$, which means that $(B)_{r-1}$ implies $(B)_r$. This is the second step of the proof by induction of $(B)_p$ for all $p = 0, 1, \ldots$. Therefore, by induction on p, we have $(B)_p$ for all p. This completes the proof of Proposition 4.4. □

4.3. The case at integers.

PROPOSITION 4.9. *Let* $s_0 := -m$ $(m = 1, 2, \ldots)$. *For a fixed integer* p *satisfying* $1 \leq p \leq m$ *and* $2p-1 \leq n$, *we suppose that* $\langle \boldsymbol{d}^{(2p-1)}[-m], \vec{a}\rangle \neq 0$. *Then* $\boldsymbol{c}^{\bullet,\bullet}_{2p-1}(\vec{a}, s)$ *has pole of order* q *for* $q = 1, \ldots, p$ *at* $s = s_0$.

We have only to prove the following Proposition 4.10 for the same reason as in the case Proposition 4.2.

PROPOSITION 4.10. *Under the same condition as in Proposition* 4.9, *we have* $\boldsymbol{c}^{\bullet,0}_{2q-1,(\vec{a},s_0),-q} = (\mathrm{nzc})_q \times \langle \boldsymbol{d}^{(2q-1)}[s_0], \vec{a}\rangle$ *for* $q = 1, 2, \dots, p$, *where* $(\mathrm{nzc})_q$ *is a non-zero constant depending on* q.

PROOF. We prove the following $(A)_p$ for $p = 1, 2, \dots$ by induction on p.

(91) $(A)_p$: If $\langle \boldsymbol{d}^{(2p-1)}[s_0], \vec{a}\rangle \neq 0$, then $\boldsymbol{c}^{\bullet,0}_{2q-1,(\vec{a},s_0),-q} = (\mathrm{nzc})_q \times \langle \boldsymbol{d}^{(2q-1)}[s_0], \vec{a}\rangle$ for $q = 1, 2, \dots, p$, where $(\mathrm{nzc})_q$ is a non-zero constant depending on q .

We shall prove $(A)_1$. From the assumption, we have $\langle \boldsymbol{d}^{(1)}[s_0], \vec{a}\rangle \neq 0$, hence there exists an integer j such that

$$\langle d^{(1)}_j[s_0], \vec{a}\rangle = a_j + (-1)^{s_0} a_{j+1} \neq 0 .$$

By the relation (59),

$$c^{j,0}_1(\vec{a}, s) = \frac{\Gamma(s+1)}{\sqrt{2\pi}} \exp\Big(\frac{\pi}{2}\sqrt{-1}(s+1)\Big) \times (\exp(\pi\sqrt{-1}(s+1))a_j + a_{j+1}) . \tag{92}$$

By taking the residue of (92), we have

$$c^{j,0}_{1,(\vec{a},s_0),-q} = (\text{non-zero const.}) \times ((-1)^{s_0} a_{j+1} + a_j) ,$$

where (non-zero const.) is a non-zero constant that does not depend on j. Then we have

$$\boldsymbol{c}^{\bullet,0}_{1,(\vec{a},s_0),-1} = (\text{non-zero const.}) \times \langle \boldsymbol{d}^{(1)}[s_0], \vec{a}\rangle .$$

This means that $(A)_1$ is true.

Next we have to prove $(A)_{r+1}$ under the assumption $(A)_r$. Since $\langle \boldsymbol{d}^{(2r+1)}[s_0], \vec{a}\rangle \neq 0$ implies $\langle \boldsymbol{d}^{(2r-1)}[s_0], \vec{a}\rangle \neq 0$ (Proposition 2.1), we have $\boldsymbol{c}^{\bullet,0}_{2q-1,(\vec{a},s_0),-q} = (\mathrm{nzc})_q \times \langle \boldsymbol{d}^{(2q-1)}[s_0], \vec{a}\rangle$ for $q = 1, 2, \dots, r$. Then we have only to prove

$$\boldsymbol{c}^{\bullet,0}_{2r+1,(\vec{a},s_0),-r-1} = (\mathrm{nzc})_{r+1} \times \langle \boldsymbol{d}^{(2r+1)}[s_0], \vec{a}\rangle . \tag{93}$$

However, the proof of this equation is almost the same as that for the equation (68). We do not repeat the proof. □

PROPOSITION 4.11. *Let* $s_0 := -m$ $(m = 1, 2, \dots)$. *For a fixed integer* q *satisfying* $0 \le p \le m$ *and* $2p + 1 \le n$, *we suppose that* $\langle \boldsymbol{d}^{(2p+1)}[-m], \vec{a}\rangle = 0$. *Then* $\boldsymbol{c}^{\bullet,\bullet}_{2q-1}(\vec{a}, s)$ *has pole of order at most* p *for* $q = p, p+1, \dots$ *at* $s = s_0$. *Here, we consider* $\boldsymbol{c}^{\bullet,\bullet}_0(\vec{a}, s)$ *instead of* $\boldsymbol{c}^{\bullet,\bullet}_{-1}(\vec{a}, s)$ *when* $p = q = 0$.

PROOF. We shall prove this proposition by showing the following $(B)_p$ for all $p = 0, 1, \dots$ by induction on p.

(94) $(B)_p$: If $\langle \boldsymbol{d}^{(2p+1)}[s_0], \vec{a}\rangle = 0$, then $c_{2p-1+2f}^{\bullet,\bullet}(\vec{a}, s)$ has pole of order at most p at $s = s_0$ for $f = 0, 1, \dots$. Here, we are considering $\boldsymbol{c}_0^{\bullet,\bullet}(\vec{a}, s)$ instead of $\boldsymbol{c}_{-1}^{\bullet,\bullet}(\vec{a}, s)$ when $p = f = 0$.

LEMMA 4.12. *$(B)_0$ is true. In other words, if $\langle \boldsymbol{d}^{(1)}[s_0], \vec{a}\rangle = 0$, then $\boldsymbol{c}_0^{\bullet,\bullet}(\vec{a}, s)$ is holomorphic at $s = s_0$ and $\boldsymbol{c}_{2f-1}^{\bullet,\bullet}(\vec{a}, s)$ is holomorphic at $s = s_0$ for $f = 1, 2, \dots$.*

PROOF. $\boldsymbol{c}_0^{\bullet,\bullet}(\vec{a}, s)$ is holomorphic at $s = s_0$ clearly, since $c_0^{j,0}(\vec{a}, s) = a_j$.

For the proof of holomorphy of $\boldsymbol{c}_{2f-1}^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ for $f = 1, 2, \dots$, we have only to show that

$$\boldsymbol{c}_{2p+1}^{j,\bullet}(\vec{a}, s) = \boldsymbol{c}_{2p+1}^{j+2,\bullet}(\vec{a}, s) \tag{95}$$

and that

$$\boldsymbol{c}_{2p+1}^{j,\bullet}(\vec{a}, s) \text{ is holomorphic at } s = s_0 \tag{96}$$

by induction on p for $p = 0, 1, \dots$. The proof is almost the same as Lemma 4.5 except the first step of the induction.

Now we give the proof for $p = 0$. If $\langle \boldsymbol{d}^{(1)}[s_0], \vec{a}\rangle = 0$, then we have

$$\langle d_j^{(1)}[s_0], \vec{a}\rangle = \langle d_j^{(0)}[s_0], \vec{a}\rangle + (-1)^{s_0+1}\langle d_{j+1}^{(0)}[s_0], \vec{a}\rangle = a_j + (-1)^{s_0+1}a_{j+1} = 0\,.$$

Since $c_0^{j,0}(\vec{a}, s) = a_j$,

$$c_0^{j,0}(\vec{a}, s) = (-1)^{s_0}c_0^{j+1,0}(\vec{a}, s)\,. \tag{97}$$

Then we have

$$\begin{bmatrix} c_0^{j+1,0}(\vec{a}, s) \\ c_0^{j,0}(\vec{a}, s) \end{bmatrix} = (-1)^{s_0} \times \begin{bmatrix} c_0^{j+2,0}(\vec{a}, s) \\ c_0^{j+1,0}(\vec{a}, s) \end{bmatrix} \tag{98}$$

for all j. Since the matrix in (59) does not depend on j, we have

$$\boldsymbol{c}_1^{j,\bullet}(\vec{a}, s) = \begin{bmatrix} c_1^{j,1}(\vec{a}, s) \\ c_1^{j,0}(\vec{a}, s) \end{bmatrix} = (-1)^{s_0}\begin{bmatrix} c_1^{j+1,1}(\vec{a}, s) \\ c_1^{j+1,0}(\vec{a}, s) \end{bmatrix} = (-1)^{s_0} \times \boldsymbol{c}_1^{j+1,\bullet}(\vec{a}, s) \tag{99}$$

by using the relation (59). This means

$$\boldsymbol{c}_1^{j,\bullet}(\vec{a}, s) = \boldsymbol{c}_1^{j+2,\bullet}(\vec{a}, s) \quad \text{for all } j\,. \tag{100}$$

The relation (97) shows that $\boldsymbol{c}_1^{\bullet,\bullet}(\vec{a}, s)$ are holomorphic at $s = s_0$ through the relation (59). Indeed, by the relation (59), we have

$$\begin{aligned} c_1^{j,1}(\vec{a}, s) =& \frac{\Gamma(s+1)}{\sqrt{2\pi}} \exp\Big(-\frac{\pi}{2}\sqrt{-1}(s+1)\Big) \\ &\times (c_0^{j,0}(\vec{a}, s) + \exp(\pi\sqrt{-1}(s+1))c_0^{j+1,0}(\vec{a}, s)) \end{aligned} \tag{101}$$

and

$$(102)\qquad c_1^{j,0}(\vec{a},s) = \frac{\Gamma(s+1)}{\sqrt{2\pi}}\exp\left(\frac{\pi}{2}\sqrt{-1}(s+1)\right) \times (\exp(\pi\sqrt{-1}(s+1))c_0^{j,0}(\vec{a},s) + c_0^{j+1,0}(\vec{a},s))$$

for all j. Then, by (97), $c_1^{j,1}(\vec{a},s)$ and $c_1^{j,0}(\vec{a},s)$ are holomorphic at $s = s_0$ for all j. This means that $\boldsymbol{c}_1^{j,\bullet}(\vec{a},s)$ is holomorphic at $s = s_0$ for all j. Thus the case of $p = 0$ has been proved. □

Thus we complete the proof of $(B)_0$

Next we have to prove $(B)_r$ under the assumption that we have verified $(B)_{r-1}$. However, this induction procedure is almost the same as the proof of Proposition 4.4. We have only to set the vector $\vec{b} := \vec{a} - \vec{a}'$ instead of $\vec{b} := \vec{a} + \vec{a}'$ in the proof of Proposition 4.4. The argument may be complicated but is completely parallel with the one given in the proof of Proposition 4.4. □

4.4. Exact orders of the coefficient functions. Now we can describe all the orders of poles of coefficient functions as a corollary to Theorem 2.2. The following Corollary 4.13 plays an important role in the proof of Theorem 2.3.

COROLLARY 4.13 (Exact orders of the coefficient functions). *The exact orders of poles of the coefficient functions* $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)$ *are determined by the following rule.*

1. *Let* s_0 *be a strict half integer not greater than* -1. *Then there is an integer* p *such that the orders of pole at* $s = s_0$ *of* $\boldsymbol{c}_{2q}^{\bullet,\bullet}(\vec{a},s)$ *and* $\boldsymbol{c}_{2q+1}^{\bullet,\bullet}(\vec{a},s)$ *are* q *for* q *with* $0 \le q \le p$, *and the order of pole of* $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)$ *is not greater than* p *for* i *with* $2p+2 \le i$.
2. *Let* s_0 *be an integer greater than* -1. *Then there is an integer* p *such that the orders of pole at* $s = s_0$ *of* $\boldsymbol{c}_{2q-1}^{\bullet,\bullet}(\vec{a},s)$ *and* $\boldsymbol{c}_{2q}^{\bullet,\bullet}(\vec{a},s)$ *are* q *for* q *with* $0 \le q \le p$, *and the order of pole of* $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)$ *is not greater than* p *for* i *with* $2p+1 \le i$. *Here we supose* $\boldsymbol{c}_{-1}^{\bullet,\bullet}(\vec{a},s) = 0$.

PROOF. We prove the first assertion. By Corollary 2.1, there exists an integer p such that $\langle \boldsymbol{d}^{(2p)}[s_0], \vec{a}\rangle \neq 0$ and $\langle \boldsymbol{d}^{(2p+2)}[s_0], \vec{a}\rangle = 0$. By Propositions 4.2 and 4.4, we have the result for all $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)$ with even i. On the other hand, by Proposition 4.1, we see that $\boldsymbol{c}_{2q+1}^{\bullet,\bullet}(\vec{a},s)$ has pole of the same order as $\boldsymbol{c}_{2q}^{\bullet,\bullet}(\vec{a},s)$ at $s = s_0$ for all integers q. Then we have the result for all $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)$ with arbitrary i.

The second assertion is proved in the same way by using Propositions 4.9 and 4.11 instead of Propositions 4.2 and 4.4. □

5. Proof of the theorem on the exact order. We shall give the proof of Theorem 2.2 in full detail for the case 1,(a) and 2,(a). Since the proof for other cases is almost the same, we explain only essential points instead of giving proof. We have proved the estimate of the order of the poles in the preceding section. What we have to do here is to point out which proposition should be applied for the proof of each case.

5.1. Proof of the case of 1(a). The case 1(a) consists of three assertions. We do not have to prove the converses obviously. If we establish all the statements, then the converses are automatically true, since all the possible cases are proved.

LEMMA 5.1. *Let* $s_0 := -(2m+1)/2$ $(m = 1, 2, \dots)$. *If* $\langle \boldsymbol{d}^{(2)}[-(2m+1)/2], \vec{a}\rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic at* $s = s_0$.

PROOF. By applying Proposition 4.4 in the case of $p = 0$, all the coefficients $\boldsymbol{c}_{2q}^{\bullet,\bullet}(\vec{a}, s)$ $(q = 0, 1, \dots)$ are holomorphic at $s = s_0$. Then, by Proposition 4.1, all the coefficients $\boldsymbol{c}_{2q+1}^{\bullet,\bullet}(\vec{a}, s)$ $(q = 0, 1, \dots)$ are holomorphic at $s = s_0$. Therefore, all the coefficients $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ $(0 \le i \le n)$ are holomorphic at $s = s_0$. Thus, by Proposition 3.7, $P^{[\vec{a},s]}(x)$ is holomorphic at $s = s_0$.

LEMMA 5.2. *Let* $s_0 := -(2m+1)/2$ $(m = 1, 2, \dots)$. *For a fixed integer* p *satisfying* $1 \le p < m$ *and* $2p \le n$, *if* $\langle \boldsymbol{d}^{(2p+2)}[-(2m+1)/2], \vec{a}\rangle = 0$ *and* $\langle \boldsymbol{d}^{(2p)}[-(2m+1)/2], \vec{a}\rangle \ne 0$, *then* $P^{[\vec{a},s]}(x)$ *has pole of order* p *at* $s = s_0$.

PROOF. From the conditions $\langle \boldsymbol{d}^{(2p)}[-(2m+1)/2], \vec{a}\rangle \ne 0$ and $\langle \boldsymbol{d}^{(2p+2)}[-(2m+1)/2], \vec{a}\rangle = 0$, all the coefficients $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ with even i have poles of order at most p, and the coefficient $\boldsymbol{c}_{2p}^{\bullet,\bullet}(\vec{a}, s)$ has pole of order p by Proposition 4.2 and 4.4. On the other hand, by Proposition 4.1, $\boldsymbol{c}_{2i+1}^{j,\bullet}(\vec{a}, s)$ has pole of the same order as $\boldsymbol{c}_{2i}^{j,\bullet}(\vec{a}, s)$ at $s = s_0$ for all integers i. Then all the coefficients $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ $(0 \le i \le n)$ have pole of order at most p, and at least one of them has pole of order p. Thus, by Proposition 3.7, $P^{[\vec{a},s]}(x)$ has pole of order p at $s = s_0$. □

LEMMA 5.3. *Let* $s_0 := -(2m+1)/2$ $(m = 1, 2, \dots)$ *and suppose that* $2m \le n$. *If* $\langle \boldsymbol{d}^{(2m)}[-(2m+1)/2], \vec{a}\rangle \ne 0$, *then* $P^{[\vec{a},s]}(x)$ *has pole of order* m *at* $s = s_0$.

PROOF. Since $\langle \boldsymbol{d}^{(2m)}[-(2m+1)/2], \vec{a}\rangle \ne 0$, the coefficient $\boldsymbol{c}_{2m}^{\bullet,\bullet}(\vec{a}, s)$ has pole of order m by Proposition 4.2 in the case $p = m$. On the other hand, $P^{[\vec{a},s]}(x)$ has pole of order at most m by Theorem 1.1, since we are considering the poles at $s = -(2m+1)/2$ with $m \le n/2$. Then all the coefficients $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ $(0 \le i \le n)$ have pole of order at most m, and at least one of them has pole of order m. Thus, by Proposition 3.7, $P^{[\vec{a},s]}(x)$ has pole of order m at $s = s_0$.

By Lemmas 5.1, 5.2 and 5.3, we complete the proof of Theorem 2.2,1(a).

5.2. Outline of the proof of the case of 1(b). The proof of the case of 1(b) is almost parallel with that of the case of 1(a). We only need Lemmas 5.1 and 5.2, and Proposition 4.2, since the conditions for other lemmas and propositions are invalid. The proof is valid without modification.

5.3. Proof of the case of 2(a). The case 2(a) can be proved almost in the same way as case 1(a). However, we need some modifications of the lemmas and the propositions together with the proof.

LEMMA 5.4. *Let* $s_0 := -m$ $(m = 1, 2, \dots)$. *If* $\langle \boldsymbol{d}^{(1)}[-m], \vec{a} \rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic at* $s = s_0$.

PROOF. Clearly, the coefficients $\boldsymbol{c}_0^{\bullet,\bullet}(\vec{a}, s) = \vec{a}$ is holomorphic at $s = s_0$. By Proposition 4.11 of the case of $p = 0$, all the coefficients $\boldsymbol{c}_{2q-1}^{\bullet,\bullet}(\vec{a}, s)$ $(q = 1, 2, \dots)$ are holomorphic at $s = s_0$. Then, by Proposition 4.1, all the coefficients $\boldsymbol{c}_{2q}^{\bullet,\bullet}(\vec{a}, s)$ $(q = 1, 2, \dots)$ are holomorphic at $s = s_0$. Therefore, all the coefficients $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ $(0 \le i \le n)$ are holomorphic at $s = s_0$. Thus, by Proposition 3.7, $P^{[\vec{a},s]}(x)$ is holomorphic at $s = s_0$. □

LEMMA 5.5. *Let* $s_0 := -m$ $(m = 1, 2, \dots)$. *For a fixed integer* p *satisfying* $1 \le p < m$ *and* $2p - 1 \le n$, *if* $\langle \boldsymbol{d}^{(2p+1)}[-m], \vec{a} \rangle = 0$ *and* $\langle \boldsymbol{d}^{(2p-1)}[-m], \vec{a} \rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has pole of order* p *at* $s = s_0$.

PROOF. From the conditions $\langle \boldsymbol{d}^{(2p-1)}[-m], \vec{a} \rangle \neq 0$ and $\langle \boldsymbol{d}^{(2p+1)}[-m], \vec{a} \rangle = 0$, all the coefficients $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ with odd i have pole of order at most p, and the coefficient $\boldsymbol{c}_{2p-1}^{\bullet,\bullet}(\vec{a}, s)$ has pole of order p by Propositions 4.9 and 4.11. On the other hand, by Proposition 4.1, $\boldsymbol{c}_{2i-1}^{\bullet,\bullet}(\vec{a}, s)$ has pole of the same order as $\boldsymbol{c}_{2i+1}^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ for all integers i. Then all the coefficients $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ $(0 \le i \le n)$ have pole of order at most p, and at least one of them has pole of order p. Thus, by Proposition 3.7, $P^{[\vec{a},s]}(x)$ has pole of order p at $s = s_0$. □

LEMMA 5.6. *Let* $s_0 := -m$ $(m = 1, 2, \dots)$ *and suppose that* $2m - 1 \le n$. *If* $\langle \boldsymbol{d}^{(2m-1)}[-m], \vec{a} \rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has pole of order* m *at* $s = s_0$.

PROOF. Since $\langle \boldsymbol{d}^{(2m-1)}[-m], \vec{a} \rangle \neq 0$, the coefficient $\boldsymbol{c}_{2m-1}^{\bullet,\bullet}(\vec{a}, s)$ has pole of order m by Proposition 4.9 in the case $p = m$. On the other hand, $P^{[\vec{a},s]}(x)$ has pole of order at most m by Theorem 1.1, since we are considering the poles at $s = -m$ with $m \le (n + 1)/2$. Then all the coefficients $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ $(i = 1, 2, \dots, n)$ have pole of order at most m, and at least one of them has pole of order m. Thus, by Proposition 3.7, $P^{[\vec{a},s]}(x)$ has pole of order m at $s = s_0$. □

By Lemmas 5.4, 5.5 and 5.6, we complete the proof of Theorem 2.2,2(a).

5.4. Outline of the proof of the case of 2(b). The proof of the case of 2(b) is almost parallel with that of the case of 2(a). We only need Lemma 5.4 and Lemma 5.5, and Proposition 4.9 for the same reason of the proof of the case 1(b).

6. Proof of the theorem on the exact support. In this section we shall give a proof of Theorem 2.3 as an application of Propositon 3.8. We shall determine the support of the Laurent expansion coefficients of $P^{[\vec{a},s]}(x)$ by applying Proposition 3.8. This is nothing but estimating the support of the Laurent expansion coefficients of the microfunction $\mathrm{sp}(P^{[\vec{a},s]}(x))$. The support of the Laurent expansion coefficients of $P^{[\vec{a},s]}(x)$ is the projection image of the support of the Laurent expansion coefficients of $\mathrm{sp}(P^{[\vec{a},s]}(x))$ by the map $\pi : T^*\boldsymbol{V} \mapsto \boldsymbol{V}$. Then the lower dimensional strata may be contained in the closure of the higher dimensional strata. In the proof below, we have to extract the highest dimensional strata necessary for the exact determination of the support.

6.1. Preliminaries for the proof. In the proof, we let $s_0 := -(q+1)/2$. By Proposition 3.8, we have (49):

$$\operatorname{Supp}(P_w^{[\vec{a},s_0]}(x)) = \overline{\bigcup_{(i,j)\in L} S_i^j} \tag{103}$$

with $L := \{(i,j) \in \mathbf{Z}^2 \mid \operatorname{ord}_{s=s_0}(c_i^{j,k}(\vec{a},s)) \geq -w \text{ for some } k \in \mathbf{Z} \cap [0,i]\}$. Therefore, we have only to calculate the orders of the coefficient functions $c_i^{j,k}(\vec{a},s)$ at $s = s_0$. Since we have supposed that $P^{[\vec{a},s]}(x)$ has pole of order p at $s = s_0$ in this proof, we have

$$p \leq \left\lfloor \frac{q+1}{2} \right\rfloor \leq \frac{q+1}{2} = -s_0 \tag{104}$$

by (6). Let

$$U(s_0,p) := \left\{ (i,j) \in \mathbf{Z}^2 \;\middle|\; \begin{array}{l} c_i^{j,k}(\vec{a},s) \text{ has pole of order } \geq p \text{ for some} \\ k \text{ with } 0 \leq k \leq i \text{ at } s = s_0 \end{array} \right\}. \tag{105}$$

6.2. The case at strict half integers. First, suppose that q is an even integer. Namely, s_0 is a strict half integer.

LEMMA 6.1. *Suppose that* $0 \leq i < -2w$. *Then we have*

$$(i,j) \notin U(s_0,-w).$$

PROOF. By Proposition 4.2, $c_i^{\bullet,\bullet}(\vec{a},s)$ has pole of order strictly less than $-w$ at $s = s_0$ if $0 \leq i < -2w$. □

On the other hand, if $0 \leq i_0 < -2w$, then for all j_0, we have

$$S_{i_0}^{j_0} \cap \overline{\bigcup_{\substack{i \geq -2w \\ 0 \leq j \leq n-i}} S_i^j} = \emptyset.$$

Thus, if $0 \leq i_0 < -2w$, then for all j_0, we have

$$S_{i_0}^{j_0} \cap \overline{\bigcup_{(i,j)\in U(s_0,-w)} S_i^j} = \emptyset. \tag{106}$$

LEMMA 6.2. *If* $i_0 \geq -2w$ *and* $(i_0,j_0) \in U(s_0,-w)$, *then*

$$S_{i_0}^{j_0} \subset \overline{\bigcup_{\substack{(i,j)\in U(s_0,-w) \\ i=-2w}} S_i^j}. \tag{107}$$

PROOF. Suppose that there exists an integer i_0 with $i_0 \geq -2w$ and an integer j_0 in $0 \leq j_0 \leq n - i_0$ such that $\boldsymbol{c}_{i_0}^{j_0,\bullet}(\vec{a},s)$ has pole of order $\geq -w$. If i_0 is odd, then $\boldsymbol{c}_{i_0-1}^{j_0,\bullet}(\vec{a},s)$ and $\boldsymbol{c}_{i_0-1}^{j_0+1,\bullet}(\vec{a},s)$ have poles of the same order as $\boldsymbol{c}_{i_0}^{j_0,\bullet}(\vec{a},s)$ by Proposition 4.1. Thus we may assume from the beginning that there exists an even integer $i_0 \geq -2w$ and an integer j_0 in $0 \leq j_0 \leq n - i_0$ such that $\boldsymbol{c}_{i_0}^{j_0,\bullet}(\vec{a},s)$ has pole of order $\geq -w$. We prove that there exists an integer j_1 in $j_0 \leq j_1 \leq j_0 + (i_0 + 2w)$ such that $\boldsymbol{c}_{-2w}^{j_1,\bullet}(\vec{a},s)$ has pole of order $\geq -w$ at $s = s_0$.

In order to prove this, we deduce contradiction by assuming that $\boldsymbol{c}^{j_1,\bullet}_{-2w}(\vec{a}, s)$ has pole of order $< -w$ at $s = s_0$ for all integers j_1 with $j_0 \leq j_1 \leq j_0 + (i_0 + 2w)$.

We define the coefficient functions $c'^{j,k}_i(\vec{a}, s)$ with the renumbered index j by setting

$$c'^{j,k}_i(\vec{a}, s) := c^{j+j_0,k}_i(\vec{a}, s) \tag{108}$$

for integers i, j, k with $0 \leq i \leq i_0$, $0 \leq j \leq i_0 - i$ and $0 \leq k \leq i$. Then, the coefficient vectors $\boldsymbol{c}'^{j,\bullet}_i(\vec{a}, s)$ and $\boldsymbol{c}'^{\bullet,\bullet}_i(\vec{a}, s)$ are defined in the same manner as in the definitions of $\boldsymbol{c}^{j,\bullet}_i(\vec{a}, s)$ and $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a}, s)$ and so on. Note that the set of coefficients

$$\boldsymbol{c}'^{\bullet,\bullet}_i(\vec{a}, s) = \{c'^{j,k}_i(\vec{a}, s)\}_{\substack{0 \leq i \leq i_0 \\ 0 \leq j \leq i_0 - i \\ 0 \leq k \leq i}} \tag{109}$$

satisfy the relations (59), since the relation matrices in (59) do not depend on the index j.

Then we can apply Corollary 4.13,1 to the set of coefficients (109). If

$$\boldsymbol{c}'^{\bullet,\bullet}_{-2w}(\vec{a}, s) := \{\boldsymbol{c}^{j,\bullet}_{-2w}(\vec{a}, s)\}_{j_0 \leq j \leq j_0 + i_0 + 2w}$$

has pole of order strictly less than $-w$, then $\boldsymbol{c}'^{\bullet,\bullet}_i(\vec{a}, s)$ with $-2w \leq i$ has pole of order strictly less than $-w$ at $s = s_0$. Then $\boldsymbol{c}'^{\bullet,\bullet}_{i_0}(\vec{a}, s) = \boldsymbol{c}^{j_0,\bullet}_{i_0}(\vec{a}, s)$ has pole of order strictly less than $-w$ at $s = s_0$, a contradiction. Then at least one of the coefficients in $\{\boldsymbol{c}^{j,\bullet}_{-2w}(\vec{a}, s)\}_{j_0 \leq j \leq j_0 + i_0 + 2w}$ has pole of order $-w$ at $s = s_0$.

Thus, if there exists an integer $i_0 \geq -2w$ and an integer j_0 with $0 \leq j_0 \leq n - i_0$ such that $\boldsymbol{c}^{j_0,\bullet}_{i_0}(\vec{a}, s)$ has pole of order $\geq -w$, then there exists an integer j_1 with $j_0 \leq j_1 \leq j_0 + i_0 + 2w$ such that $\boldsymbol{c}^{j_1,\bullet}_{-2w}(\vec{a}, s)$ has pole of order $\geq -w$ at $s = s_0$. Then, we have $\boldsymbol{S}^{j_0}_{i_0} \subset \overline{\boldsymbol{S}^{j_1}_{-2w}}$ by (10), and $(-2w, j_1) \in U(s_0 - w)$. □

Therefore, we have

$$\overline{\bigcup_{\substack{(i,j) \in U(s_0, -w) \\ i \geq -2w}} \boldsymbol{S}^j_i} \subset \overline{\bigcup_{\substack{(i,j) \in U(s_0, -w) \\ i = -2w}} \boldsymbol{S}^j_i},$$

and hence

$$\overline{\bigcup_{\substack{(i,j) \in U(s_0, -w) \\ i \geq -2w}} \boldsymbol{S}^j_i} = \overline{\bigcup_{\substack{(i,j) \in U(s_0, -w) \\ i = -2w}} \boldsymbol{S}^j_i}.$$

Thus, by (106), we obtain

$$\overline{\bigcup_{(i,j) \in U(s_0, -w)} \boldsymbol{S}^j_i} = \overline{\bigcup_{\substack{(i,j) \in U(s_0, -w) \\ i = -2w}} \boldsymbol{S}^j_i}. \tag{110}$$

LEMMA 6.3.

$$\{(i, j) \in U(s_0, -w) \mid i = -2w\} = \{(-2w, j) \in \boldsymbol{Z}^2 \mid \langle d^{(-2w)}_j[s_0], \vec{a}\rangle \neq 0\}. \tag{111}$$

PROOF. By substituting $q := -w$ in Proposition 4.3, the index $(-2w, j)$ belonging to the set on the right-hand-side of (111) satisfies $c^{j,0}_{-2w,(\vec{a},s_0),+w} \neq 0$. Then $\boldsymbol{c}^{j,\bullet}_{-2w}(\vec{a}, s)$ has pole of order $-w$ at $s = s_0$ and hence the index $(-2w, j)$ belongs to the set on the left-hand-side. The converse is also true. Then we have (111). $\square$

Therefore, when q is an even integer,

$$\begin{aligned}
\mathrm{Supp}(P_w^{[\vec{a},s_0]}(x)) &= \overline{\bigcup_{(i,j)\in U(s_0,-w)} S_i^j} \qquad \text{(by Proposition 3.8)}\,, \\
&= \overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w}} S_i^j} \qquad \text{(by (110))}\,, \\
&= \overline{\bigcup_{\substack{0\le j\le n+2w\\ \langle d_j^{(-2w)}[s_0],\vec{a}\rangle\neq 0}} S_{-2w}^j} \qquad \text{(by (111))}\,.
\end{aligned}$$

This means the result (22).

6.3. The case at integers. Secondly, suppose that q is an odd integer. Namely, s_0 is an integer.

LEMMA 6.4. *Suppose that* $0 \le i < -2w - 1$. *Then we have*

$$(i, j) \notin U(s_0, -w)\,.$$

This is proved in the same way as in Lemma 6.1. We use Proposition 4.9 instead of Proposition 4.2.

On the other hand, if $0 \le i_0 < -2w - 1$, then for all j_0 we have

$$S_{i_0}^{j_0} \cap \overline{\bigcup_{(i,j)\in U(s_0,-w)} S_i^j} = \emptyset\,. \tag{112}$$

LEMMA 6.5. *If* $i_0 \ge -2w - 1$ *and* $(i_0, j_0) \in U(s_0, -w)$, *then*

$$S_{i_0}^{j_0} \subset \overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w-1}} S_i^j}\,. \tag{113}$$

This is proved in the same way as in Lemma 6.2. We use Proposition 4.1,2 and Corollary 4.13,2 instead of Proposition 4.1,1 and Corollary 4.13,1.

Therefore, we obtain

$$\overline{\bigcup_{(i,j)\in U(s_0,-w)} S_i^j} = \overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w-1}} S_i^j}\,, \tag{114}$$

in the same way as in the proof of the case q even.

LEMMA 6.6.

$$\begin{aligned}
&\{(i, j) \in U(s_0, -w) \,\big|\, i = -2w - 1\} \\
&\qquad = \{(-2w - 1, j) \in \mathbf{Z}^2 \mid \langle d_j^{(-2w-1)}[s_0], \vec{a}\rangle \neq 0\}\,.
\end{aligned} \tag{115}$$

This is proved in the same way as in Lemma 6.3 by using Proposition 4.10 instead of Proposition 4.3.

Therefore, when q is an even integer,

$$
\begin{aligned}
\mathrm{Supp}(P_w^{[\vec{a},s_0]}(x)) &= \overline{\bigcup_{(i,j)\in U(s_0,-w)} S_i^j} \qquad \text{(by Proposition 3.8)}, \\
&= \overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w-1}} S_i^j} \qquad \text{(by (114))}, \\
&= \overline{\bigcup_{\substack{0\le j\le n+2w+1\\ \langle d_j^{(-2w-1)}[s_0],\vec{a}\rangle\neq 0}} S_{-2w-1}^j} \qquad \text{(by (115))}.
\end{aligned}
$$

This means the result (23).

Thus, we complete the proof of Theorem 2.3.

REFERENCES

[1] B. BLIND, Analyse de Fourier sur une algèbre de Jordan, Université de Nancy 1, 1991.

[2] B. BLIND, Distributions homogènes sur une algèbre de Jordan, Bull. Soc. Math. France 125 (1997), 493–528.

[3] I. M. GELFAND AND G. E. SHILOV, Generalized Functions—properties and operations, Generalized Functions, Vol. 1, Academic Press, New York and London, 1964.

[4] M. KASHIWARA, Microlocal calculus of simple microfunctions, Complex Analysis and Algebraic Geometry, Papers in honor of Professor K. Kodaira (W. L. Baily, Jr. and T. Shioda, eds.), Iwanami Shoten Publishers, 1977, pp. 369–374.

[5] M. KASHIWARA, Microlocal Analysis, Proceedings of the International Congress of Mathematicians, Helsinki, 1978 (Olli Lehto, ed.), Academia Scientiarum, 1980, pp. 139–150.

[6] M. KASHIWARA AND T. KAWAI, Microlocal Analysis, Publ. Res. Inst. Math. Sci., Kyoto Univ. 19 (1983), 1003–1032.

[7] M. KASHIWARA, T. KAWAI AND T. KIMURA, Daisuukaisekigaku no Kiso (Foundations of Algebraic Analysis), Kinokuniya, Tokyo, 1980 (Japanese), The English translation was published by Princeton University Press in 1985.

[8] M. KASHIWARA AND T. MIWA, Microlocal calculus of and Fourier transforms of relative invariants of prehomogeneous vector spaces, Sûrikaisekikenkyûsho Kôkyûroku 283 (1974), 60–147.

[9] M. KASHIWARA AND P. SCHAPIRA, Sheaves on Manifolds, Grundlehren Math. Wiss., Vol. 292, Springer-Verlag, 1990.

[10] M. MURO, Microlocal analysis and calculations on some relatively invariant hyperfunctions related to zeta functions associated with the vector spaces of quadratic forms, Publ. Res. Inst. Math. Sci., Kyoto Univ. 22 (1986), 395–463.

[11] M. MURO, On uniqueness of hyperfunction solutions of holonomic systems, Ark. Mat. 26 (1988), 305–314.

[12] M. MURO, Singular invariant tempered distributions on regular prehomogeneous vector spaces, J. Funct. Anal. 76 (1988), 317–345.

[13] M. MURO, Invariant hyperfunctions on regular prehomogeneous vector spaces of commutative parabolic type, Tôhoku Math. J. 42 (1990), 163–193.

[14] M. MURO, Singular Invariant Hyperfunctions on the space of Complex and Quaternion Hermitian matrices, to appear in J. Math. Soc. Japan.

[15] M. RAÏS, Distributions homogènes sur des espaces de matrices, Bull. Soc. Math. France 30 (1972), 5–109.

[16] I. SATAKE, On zeta functions associated with self-dual homogeneous cone, Number theory and Related Topics (Bombay) (S. Raghavan, ed.), Tata Inst. Fund. Res. Stud. Math., 12, Tata Inst. Fund. Res., Bombay, 1989, pp. 177–193.

[17] I. SATAKE AND J. FARAUT, The functional equations of zeta distributions associated with formally real Jordan algebras, Tôhoku Math. J. 36 (1984), 469–482.

[18] M. SATO AND T. SHINTANI, On zeta functions associated with prehomogeneous vector spaces, Ann. of Math. (2) 100 (1974), 131–170.

[19] T. SHINTANI, On zeta functions associated with the vector spaces of quadratic forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975), 25–65.

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