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# DUALITY FOR A CLASS OF MINIMAL SURFACES IN $\boldsymbol{R}^{n+1}$

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**Abstract.** Much is known about the geometry of a minimal surface in Euclidean space whose Gauss map takes values on a linear subspace of the quadric hypersurface. We consider minimal surfaces whose Gauss maps take values on rational normal curves. These are the non-degenerate minimal surfaces with smallest possible Gaussian images. We show that the geometry of such a minimal surface may be understood in terms of an auxiliary holomorphic curve on the total space of a line bundle over the Gaussian image. This is related to classical osculation duality. Natural analogues in higher dimensions of Enneper's surface, Henneberg's surface and surfaces with Platonic symmetries are described in terms of algebraic curves.

**Introduction.** Let $M$ be a Riemann surface and suppose that $\phi : M \to \boldsymbol{R}^{n+1}$ is a branched minimal immersion. Locally at least, $\phi = \mathrm{Re}(\psi)$, where $\psi : M \to \boldsymbol{C}^{n+1}$ is a null holomorphic curve. This means that the Gauss map $\gamma_\psi = [d\psi/d\xi] : M \to \boldsymbol{P}_n$ takes values on the quadric hypersurface $Q_{n-1} = (z_0^2 + \cdots + z_n^2 = 0)$. The Grassmannian of oriented 2-planes in $\boldsymbol{R}^{n+1}$ may be identified with $Q_{n-1}$ and the Euclidean Gauss map $\gamma_\phi$ of $\phi$, which takes values in the former, is thus identified with $\gamma_\psi$ ([2]).

A natural approach to the study of minimal surfaces in $\boldsymbol{R}^{n+1}$ via their Gauss maps is to fix a curve $\mathcal{G}$ in $Q_{n-1}$, and consider the class of minimal surfaces whose Gauss maps take values on $\mathcal{G}$. Alternatively, one might simply stipulate some condition on the Gaussian image. Perhaps the first condition to consider is that $\gamma_\phi(M)$ lies on a linear subspace of $Q_{n-1}$. Lawson ([14]) showed that in even dimensions this is equivalent to the existence of an orthogonal complex structure on $\boldsymbol{R}^{2n}$, with respect to which $\phi(M)$ is a holomorphic curve. In general such conditions lead to the splitting of $\phi$ into a sum of a holomorphic curve and a branched minimal immersion in lower dimensions, see [4], [10] for further details.

In this paper we study the consequences of $\mathcal{G}$ being a *rational normal curve*: this is the next simplest condition to consider after linear constraints. This means that $\mathcal{G}$ is a full (i.e., $\mathcal{G}$ does not lie on a hyperplane), irreducible algebraic curve of degree $n$ in $\boldsymbol{P}_n$. Every such curve is rational, all such curves are projectively isomorphic. Our main result is that if $\gamma_\phi$ takes values on such a $\mathcal{G}$ (and is non-constant), then there exists a natural lift of $\gamma_\phi$ into the restriction of the hyperplane bundle of $\boldsymbol{P}_n$ to $\mathcal{G}$, from which the minimal surface can be recovered, see Section 3. (In general, it may be necessary to pass to the universal cover of $M$ to define this lift globally.) This fact derives from the existence of a correspondence between holomorphic curves in $\boldsymbol{C}^{n+1}$ whose Gauss maps take values on $\mathcal{G}$, and *free* holomorphic curves in the line bundle over $\mathcal{G}$. This might be viewed as a kind of 'twistor correspondence',

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where the infinitesimal constraint on one side is encoded into a global aspect on the other. It turns out that this correspondence is best understood in terms of classical osculation duality between curves in $\boldsymbol{P}_n$ and its dual $\boldsymbol{P}_n^*$. This is explained in Section 4.

This approach is most powerful in the case of algebraic minimal surfaces, i.e., surfaces that are the real parts of null meromorphic curves. In this case the lift describes an algebraic curve. 'Freeness' means that if we 'write down' an algebraic curve on the line bundle, then it generates an algebraic minimal surface in $\boldsymbol{R}^{n+1}$, whose total Gaussian curvature, end structure, symmetries and branch locus we can 'read off' the algebraic curve: this is discussed in Section 5. This facilitates the construction of interesting new examples of minimal surfaces in $\boldsymbol{R}^{n+1}$. For example, in Section 6 we describe higher dimensional analogues of Enneper's surface, Henneberg's surface and surfaces with Platonic symmetries.

These algebraic minimal surfaces have a natural *differential geometric* property. First recall that $\phi : M \to \boldsymbol{R}^{n+1}$ is said to be *non-degenerate* if $\gamma_\phi(M)$ is a full curve in $\boldsymbol{P}_n$. Now, the degree of a full algebraic curve in $\boldsymbol{P}_n$ is at least $n$. But the area of an algebraic curve in $\boldsymbol{P}_n$ is $2\pi$ times its degree. It follows that non-degenerate algebraic minimal surfaces in $\boldsymbol{R}^{n+1}$ with Gauss maps taking values on a rational normal curve are exactly the non-degenerate algebraic minimal surfaces with the smallest possible Gaussian images.

When $n = 2$, $Q_1$ is a rational normal curve in $\boldsymbol{P}_2$ and every minimal surface in $\boldsymbol{R}^3$ derives from a curve in the line bundle of degree 2 over $Q_1$. The 'freeness' of the curve underlies the *Weierstrass formulae in free form* of [23]. We describe analogous formulae in higher dimensions. This is done directly by a simple integration by parts in Section 2: in Section 3 their geometric meaning is explained.

This approach to minimal surfaces in $\boldsymbol{R}^3$ was indicated by Hitchin in [8]. In [20] we amplified this and explained how it relates to a classical construction of Lie which is described in [3]. This paper describes a generalization of the classical construction to higher dimensions. The key point is that there is an analogous construction when we generalize $Q_1$ to other rational normal curves. We do not expect to see such a simple picture when we generalize $Q_1$ to higher dimensional quadrics. (However there is a similar construction which applies to all minimal surfaces in $\boldsymbol{R}^4$, see [19], [22].)

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## 1. Preliminaries.

(1.1) In this section we introduce some terminology and notation. For basic facts concerning minimal surfaces in $\boldsymbol{R}^{n+1}$ and complex geometry we refer the reader to [10], [14], [15] and [6], respectively.

(1.2) For the sake of brevity we make the following

DEFINITION. A full curve $\psi : M \to \boldsymbol{C}^{n+1}$ which is such that the Gauss map $\gamma_\psi : M \to \boldsymbol{P}_n$ takes values on a rational normal curve $\mathcal{G}$ is referred to here as a *$\mathcal{G}$-curve.*

DEFINITION. If $\phi : M \to \boldsymbol{R}^{n+1}$ is a non-degenerate branched minimal immersion such that $\gamma_\phi : M \to \boldsymbol{P}_n$ takes values on a rational normal curve, then we say that $\phi$ describes a *$\Lambda$-surface.*

DEFINITION. By a *Calabi curve* $\psi : M \to \boldsymbol{C}^{n+1}$ we mean a holomorphic curve whose Gauss map $\gamma_\psi = [d\psi/d\xi] : M \to \boldsymbol{P}_n$ has the following property: away from its branch points, $\gamma_\psi$ induces a metric of constant Gaussian curvature from the Fubini-Study metric.

It follows immediately from a result of Calabi ([1]) that if $\psi$ is full, then $\gamma_\psi$ must take values on some projective isometry of $\mathcal{R}_n$, the image of $\rho_n : \boldsymbol{P}_1 \to \boldsymbol{P}_n$, where

$$\rho_n(\zeta) = \left[1, \sqrt{n}\zeta, \dots, \sqrt{\binom{n}{k}}\zeta^k, \dots, \zeta^n\right],$$

and furthermore, the induced Gaussian curvature equals $2/n$.

REMARKS. (i) The real part of a $\mathcal{G}$-curve $\psi : M \to \boldsymbol{C}^{n+1}$ describes a $\Lambda$-surface in $\boldsymbol{R}^{n+1}$ if and only if $\mathcal{G}$ lies on $Q_{n-1}$.

(ii) Full Calabi curves generate all $\mathcal{G}$-curves by linear transformation. This follows from the fact that every rational normal curve in $\boldsymbol{P}_n$ is projectively isomorphic to $\mathcal{R}_n$.

(1.3) REMARKS. (i) If there exists $U \in U(n+1)$ such that $U\mathcal{R}_n \subset Q_{n-1}$, then the construction described here gives all branched minimal surfaces in $\boldsymbol{R}^{n+1}$ whose Gaussian image have constant curvature. Note that for $n = 3$, no such $U$ exists and consequently, if a minimal surface in $\boldsymbol{R}^4$ has Gaussian image with constant curvature $k$, then $k = 1$ or 2; 2/3 is not possible, see Section 5 of [10] and [18]. In [5] it is shown that such a $U$ exists for every even dimensional space with dimension $m \geq 10$. These minimal surfaces in $\boldsymbol{R}^m$ with Gaussian image of constant curvature $2/(m-1)$ derive naturally from curves in a line bundle of degree $m-1$ on $\boldsymbol{P}_1$.

(ii) The metric $ds^2_\psi$, induced by a Calabi curve $\psi : M \to \boldsymbol{C}^{n+1}$, satisfies the *Ricci-Lawson condition*, i.e., away from the (isolated) points where the Gaussian curvature $K_\psi = 0$, the metric $d\hat{s}^2_\psi = (-K_\psi)^{n/(n+2)} ds^2_\psi$ is flat. Note that this is an intrinsic condition. See [11], [13], [14], [18].

## 2. Weierstrass formulae.

(2.1) In this section we derive Weierstrass formulae in free form. We state the formulae for full Calabi curves; analogous formulae for $\Lambda$-surfaces follow immediately from the fact that any $\Lambda$-surface $\phi : M \to \boldsymbol{R}^{n+1}$ has a representation of the form: $\phi := \mathrm{Re}(T\psi)$, for some full Calabi curve $\psi : M \to \boldsymbol{C}^{n+1}$ and $T \in GL(n+1, \boldsymbol{C})$ such that $T\mathcal{R}_n \subset Q_{n-1}$. (Such $T$ exist and are easy to write down, see Section 6 for examples.)

(2.2) Let $M$ be a Riemann surface and suppose that $\psi : M \to \boldsymbol{C}^{n+1}$ is a full Calabi curve. From [1], there exists $U \in U(n+1)$ and a holomorphic differential $\eta$ on $M$ such that

$$\psi = U\tilde{\psi} = U \int \left(1, \sqrt{n}g, \dots, \sqrt{\binom{n}{k}}g^k, \dots, g^n\right)\eta,$$

where $g = \rho_n^{-1} \circ \gamma_{\tilde{\psi}}$. (The fullness assumption may be dropped here once the obvious modifications are made, cf. [10].)

Conversely, if $g : M \to \boldsymbol{C}$ is meromorphic and $\eta$ is a holomorphic differential on $M$ such that $\eta, g\eta, \dots, g^n\eta$ have no periods and whenever $g$ has a pole of order $m$ at $p \in M$, $\eta$ has a zero at $p$ of order at least $nm$, then $\psi$, defined as above, gives a Calabi curve $\psi : M \to \boldsymbol{C}^{n+1}$. (This is analogous to the usual Weierstrass formulae for null curves, cf. [14].)

(2.3) Now (locally) reparameterise $\tilde{\psi}$ by its Gauss map: i.e., suppose that $g^{-1}$ exists on an open set $V \subset \boldsymbol{P}_1$, and furthermore that $f : V \to \boldsymbol{C}$ holomorphic, satisfies

$$f^{(n+1)}(\zeta)d\zeta = \eta\,,$$

where $g(\xi) = \zeta$ and $f^{(n+1)}$ denotes the $(n+1)$-derivative of $f$ with respect to $\zeta$. Substituting $f^{(n+1)}$ into the above formula and changing the variable to $\zeta$, we integrate by parts to obtain:

$$\tilde{\psi}_k \circ g^{-1}(\zeta) = \sqrt{\binom{n}{k}} \sum_{l=0}^{k} (-1)^l \frac{k!}{(k-l)!} \zeta^{k-l} f^{(n-l)}(\zeta)\,,$$

where $\tilde{\psi} = (\tilde{\psi}_0, \dots, \tilde{\psi}_n)$.

REMARK. When $n = 2$, these formulae are equivalent to the integrated form of the Weierstrass representation formulae for minimal surfaces in $\boldsymbol{R}^3$, cf. [20].

### 3. Duality.

(3.1) Here we derive the correspondence mentioned in the introduction. Technically it is simpler to describe this by starting on the bundle side:

Let $\mathcal{L}_n$ be the total space of the holomorphic line bundle of degree $n$ over $\boldsymbol{P}_1$ and let $\mathcal{O}(n)$ denote the sheaf of germs of local holomorphic sections of $\pi : \mathcal{L}_n \to \boldsymbol{P}_1$. Recall that $\dim H^0(\boldsymbol{P}_1, \mathcal{O}(n)) = n + 1$.

DEFINITION. A global holomorphic section $\sigma \in H^0(\boldsymbol{P}_1, \mathcal{O}(n))$ that vanishes to order $n$ at some point of $\boldsymbol{P}_1$ is said to be *normal.*

The lines of normal sections comprise a curve of degree $n$, $\mathcal{A}_n \subset \boldsymbol{P}_n = \boldsymbol{P}(H^0(\boldsymbol{P}_1, \mathcal{O}(n)))$. The map $q : \boldsymbol{P}_1 \to \boldsymbol{P}_n$ given by $q(\zeta) = \{\sigma \in H^0(\boldsymbol{P}_1, \mathcal{O}(n)); \sigma \text{ vanishes to order } n \text{ at } \zeta\}$ gives a canonical identification of $\boldsymbol{P}_1$ with $\mathcal{A}_n$.

(3.2) Let $C(\mathcal{A}_n)$ denote the affine cone in $H^0(\boldsymbol{P}_1, \mathcal{O}(n))$ over $\mathcal{A}_n$. The hyperplane $\Pi_\zeta = \{\sigma \in H^0(\boldsymbol{P}_1, \mathcal{O}(n)); \sigma(\zeta) = 0\}$ intersects $C(\mathcal{A}_n)$ with multiplicity $n$ along $q(\zeta)$. This follows because if $\sigma$ vanishes at $\zeta$ then it cannot vanish to order $n$ elsewhere on $\boldsymbol{P}_1$. Such a hyperplane is said to be *normal.* Observe that a normal line lies on a unique normal hyperplane.

$\Pi = \bigcup_{\zeta \in \boldsymbol{P}_1} \Pi_\zeta$ is the kernel of the evaluation map

$$\boldsymbol{P}_1 \times H^0(\boldsymbol{P}_1, \mathcal{O}(n)) \to \mathcal{L}_n\,, \quad (\zeta, \sigma) \mapsto \sigma(\zeta)\,,$$

which is surjective, and hence there is the following isomorphism:

$$\mathcal{L}_n \simeq (\boldsymbol{P}_1 \times H^0(\boldsymbol{P}_1, \mathcal{O}(n)))/\Pi = \{\text{affine normal planes in } H^0(\boldsymbol{P}_1, \mathcal{O}(n))\}\,.$$

Observe that $l \in \mathcal{L}_n$ is dual to the affine normal plane $\Pi_l \subset H^0(\boldsymbol{P}_1, \mathcal{O}(n))$ of global sections that pass through $l$ and consequently, $l$ lies on the image of a global section $\sigma$ iff $\sigma$ lies on $\Pi_l$.

REMARK. For $n = 1$, the normality constraint is vacuous. For $n = 2$, normality=nullity.

(3.3) A full curve $\psi : M \to H^0(\boldsymbol{P}_1, \mathcal{O}(n))$ is an $\mathcal{A}_n$-curve if $(d\psi/d\xi)(\xi)$ is a normal section for each $\xi \in M$. Identifying $\mathcal{A}_n$ with $\boldsymbol{P}_1$ via $q$, and thus viewing the Gauss map $\gamma_\psi$ as a map to $\boldsymbol{P}_1$, for $\gamma_\psi$ non-constant,

$$\Gamma_\psi : M \to \mathcal{L}_n ,$$

given by $\Gamma_\psi(\xi) = \psi(\xi)(\gamma_\psi(\xi))$, is a globally defined lift of the Gauss map: we call it the *associated map* of $\psi$. $\Gamma_\psi(\xi)$ may be viewed as the (unique) affine normal plane, with normal direction $\gamma_\psi(\xi)$, that passes through the point $\psi(\xi) \in H^0(\boldsymbol{P}_1, \mathcal{O}(n))$.

(3.4) It is not hard to show that if $\gamma_\psi$ is non-constant, then $\Gamma_\psi$ determines $\psi$. Let Spé($\mathcal{O}(n)$) denote the étalé space of $\mathcal{O}(n)$ (see [24] for definition). There is a (canonically defined) holomorphic map

$$\Psi_n : \text{Spé}(\mathcal{O}(n)) \to H^0(\boldsymbol{P}_1, \mathcal{O}(n)) ,$$

which is given on stalks by

$$\Psi_n : \mathcal{O}(n)_\zeta \to \mathcal{O}(n)_\zeta / (\mathcal{I}_\zeta^{n+1} \otimes \mathcal{O}(n)_\zeta) \xrightarrow{\sim} H^0(\boldsymbol{P}_1, \mathcal{O}(n)) ,$$

where $\mathcal{I}_\zeta$ is the ideal sheaf of holomorphic functions vanishing at $\zeta$.

(3.5) Let $\mathcal{G}_n \subset$ Spé($\mathcal{O}(n)$) denote the set of germs of global sections. The following is an immediate generalization of results described in [20]:

PROPOSITION. (i) *The holomorphic curve* $\Psi_n$ : Spé($\mathcal{O}(n)$) $\to H^0(\boldsymbol{P}_1, \mathcal{O}(n))$ *is an* $\mathcal{A}_n$*-curve, and its associated map, which is defined on* Spé($\mathcal{O}(n)$)$\backslash \mathcal{G}_n$, *is given by* $\Gamma_{\Psi_n}([\sigma]_\zeta) = \sigma(\zeta)$.

(ii) *If* $\psi$ *is an* $\mathcal{A}_n$*-curve, with* $\gamma_\psi$ *non-constant, then* $\psi|_{\tilde{M}} = \Psi_n \circ \Gamma_\psi^*$, *where* $\tilde{M} = \{\xi \in M$; *there exists some neighbourhood* $V$ *of* $\xi$ *such that* $\Gamma_\psi(V)$ *is transverse to the fibre* $\pi^{-1}(\gamma_\psi(\xi))\}$, *and* $\Gamma_\psi^* : \tilde{M} \to$ Spé($\mathcal{O}(n)$) *is the natural lift of* $\Gamma_\psi$ *over* $\tilde{M}$.

REMARK. $\Psi_n \circ \Gamma_\psi^*$ describes the curve of global sections that *osculate* $\Gamma_\psi(M)$.

(3.6) If an $\mathcal{A}_n$-curve $\psi : M \to H^0(\boldsymbol{P}_1, \mathcal{O}(n))$ has non-constant Gauss map, then locally trivializing $\mathcal{L}_n$ one can write, away from branch points of $\gamma_\psi$,

$$\Gamma_\psi \circ \gamma_\psi^{-1}(\zeta) = f(\zeta) \left( \frac{d}{d\zeta} \right)^{n/2} ,$$

where $f$ is a (locally defined) holomorphic function. Thus if we choose the basis $\beta_0, \ldots, \beta_n$ for $H^0(\boldsymbol{P}_1, \mathcal{O}(n))$, where

$$\beta_k(\zeta) = \frac{(-1)^k}{n!} \sqrt{\binom{n}{k}} \zeta^{n-k} \left( \frac{d}{d\zeta} \right)^{n/2} ,$$

then

$$f(\zeta)\left(\frac{d}{d\zeta}\right)^{n/2} = \psi_0 \circ \gamma_\psi^{-1}(\zeta_0)\beta_0(\zeta) + \cdots + \psi_n \circ \gamma_\psi^{-1}(\zeta_0)\beta_n(\zeta) + \mathcal{O}[(\zeta-\zeta_0)^{n+1}],$$

where, as in (2.3),

$$\psi_k \circ \gamma_\psi^{-1}(\zeta_0) = \sqrt{\binom{n}{k}} \sum_{l=0}^{k} (-1)^l \frac{k!}{(k-l)!} \zeta_0^{k-l} f^{(n-l)}(\zeta_0).$$

*This elucidates the geometric meaning of $f$ in the Weierstrass formulae of* 2.3: *$f$ is an implicit description of the associated map of the curve $\psi$.*

## 4. Duality from the 'compactified view'.

(4.1) In this section we recast the results of the previous section in terms of osculation duality.

Let $\tilde{\beta}_k = \pi^*\beta_k$ and let $\eta(\zeta,\eta) = \eta(d/d\zeta)^{n/2}$ denote the tautological section of $\pi^*\mathcal{O}(n) \to \mathcal{L}_n$. For $n \geq 1$, $H^0(\mathcal{L}_n, \mathcal{O}) = \boldsymbol{C}$ and hence the complete linear system $|H^0(\mathcal{L}_n, \pi^*\mathcal{O}(n))|$ is parameterized by $\boldsymbol{P}(H^0(\mathcal{L}_n, \pi^*\mathcal{O}(n)))$. Moreover, since $|H^0(\mathcal{L}_n, \pi^*\mathcal{O}(n))|$ is base point free it follows that there exists a holomorphic map $\iota = \iota_{\pi^*\mathcal{O}(n)} : \mathcal{L}_n \to \boldsymbol{P}(H^0(\mathcal{L}_n, \pi^*\mathcal{O}(n)))^*$, cf. Section 1.4 in [6].

(4.2) An elementary power series argument shows that $\{\tilde{\beta}_0, \ldots, \tilde{\beta}_n, \eta\}$ is a basis for $H^0(\mathcal{L}_n, \pi^*\mathcal{O}(n))$. With respect to this basis

$$\iota : \mathcal{L}_n \to \boldsymbol{P}(H^0(\mathcal{L}_n, \pi^*\mathcal{O}(n)))^* \simeq \boldsymbol{P}_{n+1}$$

is given by $\iota(\zeta,\eta) = [\beta_0(\zeta), \ldots, \beta_n(\zeta), \eta]$: thus observe that $\mathcal{L}_n$ is embedded and compactified to $\mathcal{C}(\mathcal{R}_n^\natural)$, the projective cone over $\mathcal{R}_n^\natural = [\beta_0(\zeta), \ldots, \beta_n(\zeta), 0]$.

$\mathcal{R}_n^\natural$ is the image of the zero section, has degree $n$ and lies on the hyperplane $H = (z_{n+1} = 0)$. The vertex of $\mathcal{C}(\mathcal{R}_n^\natural)$ is $\mathbf{v} = [0, \ldots, 0, 1]$: $\mathcal{C}(\mathcal{R}_n^\natural) = \mathcal{L}_n \cup \{\mathbf{v}\}$.

(4.3) Now fix $\zeta_0 \in \mathcal{R}_n^\natural$ and consider the hyperplanes of $\boldsymbol{P}(H^0(\mathcal{L}_n, \pi^*\mathcal{O}(n)))^*$ that intersect $\mathcal{R}_n^\natural$ at $\zeta_0$ with multiplicity $n$. Such an osculating hyperplane either cuts out on $\mathcal{C}(\mathcal{R}_n^\natural)$ the image of a normal section or, if it passes through $\mathbf{v}$, the image of the fibre through $\zeta_0$ (with multiplicity $n$). Accordingly, a hyperplane that osculates $\mathcal{R}_n^\natural$ will be called *normal.*

(4.4) Consider the 'dual' variety in $\boldsymbol{P}_{n+1}^* = \boldsymbol{P}(H^0(\mathcal{L}_n, \pi^*\mathcal{O}(n)))$, whose points parameterise the normal hyperplanes of $\boldsymbol{P}_{n+1}$. A hyperplane osculating $\mathcal{R}_n^\natural$ at $\zeta_0$ cuts out the zero divisor $(s_\lambda)$ of a global section of $\pi^*\mathcal{O}(n)$ of the form: $s_\lambda(\zeta,\eta) = (\lambda\eta + (\zeta-\zeta_0)^n)(d/d\zeta)^{n/2}$, for some $\lambda \in \boldsymbol{C}$.

To find the dual variety explicitly in coordinates we write:

$$a_0(\zeta_0,\lambda)\beta_0(\zeta) + \cdots + a_n(\zeta_0,\lambda)\beta_n(\zeta) + a_{n+1}(\zeta_0,\eta)\eta = (\lambda\eta + (\zeta-\zeta_0)^n/n!)\left(\frac{d}{d\zeta}\right)^{n/2},$$

which gives:

$$[a_0(\zeta_0,\lambda), \ldots, a_{n+1}(\zeta_0,\lambda)] = \left[1, \sqrt{n}\zeta_0, \ldots, \sqrt{\binom{n}{k}}\zeta_0^k, \ldots, \zeta_0^n, \lambda\right].$$

Thus the points corresponding to normal hyperplanes, together with (the vertex) $H^* = [0, \dots, 0, 1] \in \boldsymbol{P}^*_{n+1}$, comprise $\mathcal{C}(\mathcal{R}_n) \subset \boldsymbol{P}^*_{n+1}$, the projective cone over $\mathcal{R}_n \subset \mathbf{v}^* = (a_{n+1} = 0)$.

(4.5) $\mathcal{R}_n^\natural$ is *dual* to $\mathcal{R}_n$ in the sense that for each $\zeta_0 \in \mathcal{R}_n^\natural$, there exists a unique normal hyperplane that 'osculates' $\mathcal{C}(\mathcal{R}_n^\natural)$ along $\iota(\pi^{-1}(\zeta_0))$, and thus passes through $\mathbf{v}$. This gives a point on $\mathcal{R}_n \subset \mathbf{v}^*$. Similarly, $\mathcal{R}_n^\natural$ can be recovered from $\mathcal{R}_n$.

Observe that $\mathcal{C}(\mathcal{R}_n^\natural) - \{\mathbf{v}\}$ parameterizes the set of hyperplanes of $\boldsymbol{P}^*_{n+1}$ that osculate $\mathcal{R}_n$, and $\mathbf{v}$ corresponds to the hyperplane that cuts out $\mathcal{R}_n$ on $\mathcal{C}(\mathcal{R}_n)$. Thus the construction is symmetric.

(4.6) The natural inclusion $H^0(\boldsymbol{P}_1, \mathcal{O}(n)) \to H^0(\mathcal{L}_n, \pi^*\mathcal{O}(n))$, given by $\sigma \to \pi^*\sigma$, gives $\boldsymbol{P}(H^0(\boldsymbol{P}_1, \mathcal{O}(n))) \simeq \mathbf{v}^*$, where $\mathcal{A}_n$ is identified with $\mathcal{R}_n$.

This gives the component 'at infinity' of the isomorphism

$$\boldsymbol{P}(H^0(\mathcal{L}_n, \pi^*\mathcal{O}(n))) \simeq H^0(\boldsymbol{P}_1, \mathcal{O}(n)) \cup \boldsymbol{P}(H^0(\boldsymbol{P}_1, \mathcal{O}(n))),$$

which, off $\mathbf{v}^*$, is given by observing that the zero divisor associated to a $\boldsymbol{C}^*$ of global sections of $\pi^*\mathcal{O}(n)$ over $\mathcal{L}_n$ is the image in $\mathcal{L}_n$ of a global section of $\mathcal{O}(n)$ over $\boldsymbol{P}_1$. Using the coordinates introduced above we rewrite this as: $\boldsymbol{P}^*_{n+1} = \boldsymbol{C}^{n+1} \cup \mathbf{v}^*$.

This isomorphism gives: $\mathcal{C}(\mathcal{R}_n) \simeq C(\mathcal{A}_n) \cup \mathcal{A}_n$.

REMARK. $\mathcal{R}_n$ parameterizes the set of normal lines through $H^*$, whereas $\mathcal{R}_n^\natural$ parameterizes the normal hyperplanes through $H^*$. The duality $\mathcal{R} \simeq \mathcal{R}_n^\natural$ reflects the fact that through the origin, each normal line in $H^0(\boldsymbol{P}_1, \mathcal{O}(n))$ lies on a unique normal hyperplane and conversely that a normal hyperplane contains a unique normal line.

(4.7) We are now in a position to describe the correspondence of Theorem 3.3 from the point of view of classical osculation duality between curves in $\boldsymbol{P}_{n+1}$ and $\boldsymbol{P}^*_{n+1}$, (see [6] for osculation duality). For, observe that a global section osculates a curve on $\mathcal{L}_n$ if and only if the hyperplane that cuts out the image of the global section on $\mathcal{C}(\mathcal{R}_n^\natural)$ osculates the image of the curve on $\mathcal{C}(\mathcal{R}_n^\natural) \subset \boldsymbol{P}_{n+1}$ in the classical sense.

From this point of view the map $\Psi_n$ described in (3.3) is just an 'intrinsic' description of the classical $n$th associated map of a curve lying on $\mathcal{C}(\mathcal{R}_n^\natural) \subset \boldsymbol{P}^*_{n+1}$. Moreover, the nature of osculation duality determines that the associated map of an $\mathcal{R}_n$-curve is just the (inverse) $n$th associated map. The fact that a curve is the compactification of an $\mathcal{R}_n$-curve in $\boldsymbol{C}^{n+1}$, is equivalent to the fact that its associated map takes values on $\mathcal{C}(\mathcal{R}_n^\natural)$.

Given a full holomorphic map $\Upsilon : M \to \boldsymbol{P}_{n+1}$, let $\Upsilon^* : M \to \boldsymbol{P}^*_{n+1}$ denote the $n$th associated map (with similiar notation for maps into the dual) so that $\Upsilon^{**} = \Upsilon$.

THEOREM. *If $\Upsilon : M \to \mathcal{C}(\mathcal{R}_n^\natural)$ is full, then $\Upsilon^* : M \to \boldsymbol{C}^{n+1} \cup \mathbf{v}^*$ is an $\mathcal{R}_n$-curve. Conversely, if $\psi : M \to \boldsymbol{C}^{n+1} \cup \mathbf{v}^*$ is an $\mathcal{R}_n$-curve, then $\psi^*$ is full and takes values on $\mathcal{C}(\mathcal{R}_n^\natural)$. Also, $\psi^{**} = \psi$.*

It follows that $\mathcal{R}_n$-curves in $\boldsymbol{C}^{n+1}$ are characterized by the fact that in $\boldsymbol{P}^*_{n+1} = \boldsymbol{C}^{n+1} \cup \mathbf{v}^*$, the hyperplanes of $\boldsymbol{P}^*_{n+1}$ that osculate them, osculate the curve on the hyperplane at infinity, $\mathbf{v}^*$, that is cut out by intersection with $\mathcal{C}(\mathcal{R}_n)$ (i.e., $\mathcal{R}_n$).

COROLLARY. *$\psi : M \to \boldsymbol{C}^{n+1}$ is a full Calabi curve with Gauss map taking values on $U(\mathcal{R}_n)$, for $U \in U(n+1)$, if and only if the osculating hyperplanes osculate the curve $U(\mathcal{R}_n)$ on the hyperplane at infinity, $\mathbf{v}^*$, that is cut out by intersection with $\mathcal{C}(U(\mathcal{R}_n))$.*

REMARK. Observe that the osculating affine hyperplane at a point on an $\mathcal{R}_n$-curve is the unique affine normal hyperplane that contains the tangent normal affine line to the curve at that point.

(4.8) REMARK. Blowing up the vertex of $\mathcal{C}(\mathcal{R}_n^\natural)$ gives the rational normal scroll

$$\mathcal{S}_n \simeq \boldsymbol{P}(\mathcal{L}_n \oplus \mathcal{L}_0)\,,$$

and $\boldsymbol{P}^*_{n+1}$ is thus identified with the linear system $|E_0|$, see [6] for details and notation. $\mathcal{R}_n^\natural$ determines a distinguished irreducible element of $|E_0|$ and normality is defined in $|E_0|$ with respect to that curve in the obvious way. Thus the above can be reformulated as:

THEOREM. *There exists a natural correspondence between 'full' algebraic curves on $\mathcal{S}_n$ and algebraic $\mathcal{R}_n$-curves in $|E_0|$.*

(Here 'full' means that the corresponding curve on $\mathcal{C}(\mathcal{R}_n^\natural)$ lies fully in $\boldsymbol{P}^*_{n+1}$. So the non-full curves on $\mathcal{S}_n$ are: the $(-n)$-curve, the fibres and curves in $|E_0|$. Similiar terminology applies to curves on $\mathcal{L}_n$.)

REMARK. It is clear that natural compactifications of moduli spaces for $\mathcal{R}_n$-curves are given by appropriate linear systems on $\mathcal{S}_n$. These linear systems are naturally labelled by the degree of the Gauss map and the class of the curve, cf. [20].

## 5. Algebraic $\Lambda$-surfaces in $\boldsymbol{R}^{n+1}$.

(5.1) The correspondence described 'intrinsically' in (3.3) and 'extrinsically' in (4.8) determines that the $\Lambda$-surface described by $\phi : M \to \boldsymbol{R}^{n+1}$ derives from the osculation of a holomorphic curve on $\mathcal{L}_n$. For, $\phi = \mathrm{Re}(T\psi)$, as in (2.1). In general though, it may be necessary to pass to the universal cover of $M$ to achieve this globally, because of the presence of non-vanishing imaginary periods. Here we discuss the algebraic case, i.e., $\Lambda$-surfaces in $\boldsymbol{R}^{n+1}$ that are *globally* the real parts of meromorphic curves in $\boldsymbol{C}^{n+1}$ whose Gauss maps takes values on a rational normal curve lying on $Q_{n-1}$. These are the non-degenerate algebraic minimal surfaces in $\boldsymbol{R}^{n+1}$ that have the smallest possible Gaussian images. Such surfaces derive from the osculation of *algebraic* curves on $\mathcal{L}_n$. In this section we show how the geometry of these minimal surfaces may be 'read off' their associated algebraic curves.

REMARK. We do not distinguish between $\mathcal{L}_n$ and its compactifications $\mathcal{C}(\mathcal{R}_n^\natural)$ and $\mathcal{S}_n$: by an 'algebraic curve' we mean an algebraic curve on $\mathcal{C}(\mathcal{R}_n^\natural)$.

(5.2) Suppose that $\mathcal{A} \subset \mathcal{C}(\mathcal{R}_n^\natural)$ is a full irreducible algebraic curve, with normalization $\chi : \tilde{\mathcal{A}} \to \mathcal{A}$. Let $\mathcal{A}' = \tilde{\mathcal{A}} - \chi^{*-1}(\mathbf{v}^*)$. $\chi^{*-1}(\mathbf{v}^*)$ is the set of poles of the Calabi curve $\chi^* : \mathcal{A}' \to \boldsymbol{C}^{n+1}$. Moreover, suppose that $T$ is chosen as in (2.1) and let $\phi = \mathrm{Re}(T\chi^*) : \mathcal{A}' \to \boldsymbol{R}^{n+1}$. $\phi$ is an algebraic $\Lambda$-surface whose Gauss map takes values on $T(\mathcal{R}_n^\natural) \subset Q_{n-1}$. $\chi^{*-1}(\mathbf{v}^*)$ is the set of *ends* of $\phi$.

(5.3) *Completeness.* Since $\phi$ is the real part of a null meromorphic curve in $\boldsymbol{C}^{n+1}$, it induces a complete semi-metric on $\mathcal{A}'$. ('Completeness' in this context means simply that every divergent path has infinite length, see [15].)

(5.4) *Gauss map.* The Euclidean Gauss map of $\phi$, $\gamma_\phi : \tilde{\mathcal{A}} \to G^+(2, \boldsymbol{R}^{n+1}) \simeq Q_{n-1}$, is identified with the Gauss map of $T\chi^*$ in the usual way, [14], and thus with $\gamma_{\chi^*}$, the Gauss map of $\chi^*$. The latter is given, via the isomorphism $\mathcal{R}_n^\natural \simeq \mathcal{R}_n$ of 4.5, by projection to the base $\boldsymbol{P}_1$ from $\mathcal{L}_n$, since $\pi \circ \Gamma_\chi^* = \gamma_{\chi^*}$, see 3.3.

Observe that $\gamma_{\chi^*}$ is defined over the ends of $\chi^*$. This is easy to see in the case of algebraic minimal surfaces but is also true for the wider class of finitely branched complete minimal surfaces of finite total Gaussian curvature, see [15].

Moreover, there is no problem defining the Gauss map at points passing through the vertex of $\mathcal{C}(\mathcal{R}_n^\natural)$. It is simply a matter of inspecting the corresponding curve on $\mathcal{S}_n$, or equivalently, on $\mathcal{L}_n$. Consequently, (blowing-up if necessary) we have:

$$\gamma_\phi = \pi\big|_{\mathcal{A}} \circ \chi \,.$$

(5.5) *Ends.* The ends of $\phi$ occur where $\mathcal{A}$ osculates a fibre. For, viewing the correspondence in its extrinsic formulation of (4.7), observe that if $\mathcal{A}$ osculates a fibre, this means that it osculates a hyperplane of $\boldsymbol{P}_{n+1}$ that passes through $\mathbf{v}$, and hence the corresponding point in $\boldsymbol{P}_{n+1}^*$ lies on $\mathbf{v}^*$, the hyperplane at infinity of $\boldsymbol{C}^{n+1}$.

In particular this applies if the curve passes through $\mathbf{v}$. At such a point the associated curve actually osculates $\mathbf{v}^*$, the hyperplane at infinity itself. We refer to such points as *vertex ends* of $\phi$. At non-vertex ends the associated curve osculates the completion of an affine hyperplane of $\boldsymbol{C}^{n+1}$ at infinity.

The number of vertex ends (counted with multiplicity) is given by $e_v = \mathcal{A} \cdot E_\infty$, where $E_\infty$ is the exceptional curve on $\mathcal{S}_n$. (Note that $\mathcal{A} \cdot E_0$ gives the *class* of the corresponding curve.)

Let $x = \chi(\xi) \neq \mathbf{v}$. Since $\mathcal{A}$ osculates the fibre through $x$, $\xi$ is a branch point of the Gauss map. (Off the branch locus of the Gauss map, the local inversion of the Gauss map gives the Weierstrass formula of (2.3).)

Suppose that the affine coordinate $\zeta$, in the base, is centered at $\pi(x)$. If $\xi$ is a branch point of $\gamma_\phi$, of ramification index $q$, then $q$ branches of $\mathcal{A}$ come together irreducibly at $x$. These branches of $\mathcal{A}$ have Puiseux series representations given by:

$$\chi \circ \gamma_\phi^{-1}(\zeta) = \sum_{i=p}^{\infty} a_i \zeta^{i/q} \,.$$

Substitution into the Weierstrass formula of (2.3) gives:

(i) If $p/q > n$, then $\phi(\xi) \in \boldsymbol{R}^{n+1}$ and for $q \geq 2$, $\xi$ is just a branch point of the Gauss map of the minimal surface.

(ii) If $p/q < n$, then $\xi$ is an end of $\phi$.

REMARK. This generalizes 240, [3].

Observe that if $x$ is a smooth point of $\mathcal{A}$ in the branch locus of $\gamma_\phi$, then $\mathcal{A}$ osculates the fibre and $\xi$ is an end.

(5.6) *Gaussian curvature.* On $\mathcal{S}_n$ there is the linear equivalence: $E_\infty \sim E_0 - nC$, where $C$ is any fibre. The degree of $\gamma_\phi$ is equal to the intersection number $\mathcal{A} \cdot C$. The total Gaussian curvature of the semi-metric on $\mathcal{A}'$ induced by $\phi$ is given by the area of its Gauss map, [14], consequently:

$$\int_{\mathcal{A}'} K = -2\pi n \mathcal{A} \cdot C\,.$$

Hence,

$$\begin{aligned}\int_{\mathcal{A}'} K &= -2\pi \mathcal{A} \cdot (E_0 - E_\infty)\\ &= -2\pi \deg(\mathcal{A}) + 2\pi e_v\,.\end{aligned}$$

In particular, if there are no vertex ends, then

$$\int_{\mathcal{A}'} K = -2\pi \deg(\mathcal{A})\,.$$

REMARK. $2\pi \deg(\mathcal{A})$ is the area of $\mathcal{A}$ in $\boldsymbol{P}_{n+1}$; it is 'transformed' into Gaussian curvature on the dual curve by osculation duality. The presence of vertex ends diminishes the absolute value of the total curvature on the dual curve and thus the total multiplicity $e_v$ might be viewed as Gaussian curvature 'lost at infinity'. Observe that non-vertex ends make no direct contribution to the above formula.

(5.7) *Genus.* $\mathcal{A} \subset \mathcal{S}_n$ exhibits $\mathcal{A}$ as a branched cover of $\boldsymbol{P}_1$. Thus the genus of $\mathcal{A}$ is given by the Riemann-Hurwitz formula.

(5.8) *Branch Points of the Induced Metric.* First suppose that $\xi_0$ is not a branch point of the Gauss map $\gamma_\phi : \mathcal{A}' \to \boldsymbol{P}_1$ and that $\zeta$ is an affine coordinate in $\boldsymbol{P}_1$ centred at $\gamma_\phi(\xi_0)$. $\mathcal{A}$ may be represented over a neighbourhood of $\gamma_\phi(\xi_0)$ as $f(\zeta) = \chi \circ \gamma_\phi^{-1}(\zeta)$. The metric on the corresponding Calabi curve is given by:

$$ds^2 = |f^{(n+1)}(\zeta)|^2(1 + |\zeta|^2)^n |d\zeta|^2\,.$$

Consequently, $f^{(n+1)}(\zeta_0) = 0$ if and only if $\xi_0$ is a branch point of $ds^2$. It is easy to see that $ds^2$ and $ds^2_\phi$ have the same branch locus, and thus the above give branch points in the metric induced by $\phi$.

Note that these branch points in the induced metric are points where the curve $\mathcal{A} \subset \boldsymbol{P}_{n+1}$ is *hyperosculated* by the osculating hyperplane.

REMARK. This is interesting because such points often have intrinsic meaning. For instance, see (6.2) in [20], where they give the points of order 4 in the group structure of an elliptic curve. In the case of a canonical curve these are Weierstrass points on the Riemann surface, see (6.8).

If $\xi_0$ is a branch point of the Gauss map, then choosing a local coordinate $\xi$ around $\xi_0$ such that $\gamma_\phi(\xi) = \xi^q$ and writing $\chi(\xi) = (\xi^q, h(\xi))$, the metric on the corresponding Calabi curve is:

$$ds^2(\xi) = |q\xi^{q-1} f^{(n+1)}(\xi^q)|^2(1 + |\xi^q|^2)^n |d\xi|^2 ,$$

where $f(\zeta) = h \circ \gamma_\phi^{-1}(\zeta)$ (for a choice of inverse $\gamma_\phi^{-1}$) and $f^{(n+1)}$ denotes differentiation with respect to the affine coordinate in the base. So the induced metric is branched at such an $\xi_0$ iff $\xi_0^{q-1} f^{(n+1)}(\xi_0^q) = 0$. So if $h(\xi) \sim \xi^p$, where $p = nq + 1$, then $\phi$ is immersed at $\xi_0$, and if $p > nq + 1$, then $\xi_0$ is a branch point in the metric; otherwise $\xi_0$ is an end.

(5.9) *Symmetries.* An element of $SL(2, \boldsymbol{C})$ acts naturally as a bundle automorphism of $\mathcal{L}_n$ by differentiation of the induced Möbius transformation of $\boldsymbol{P}_1$ and tensoring. For

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

this is given in coordinates by $g_\star(\zeta, \eta) = ((a\zeta + b)/(c\zeta + d), (c\zeta + d)^{-n}\eta)$.

This induces the (unique up to isomorphism) $(n + 1)$-dimensional representation of $SL(2, \boldsymbol{C})$: $\tilde{g} : H^0(\boldsymbol{P}_1, \mathcal{O}(n)) \to H^0(\boldsymbol{P}_1, \mathcal{O}(n))$, where $\tilde{g}(\sigma) = g_\star^{-1} \circ \sigma \circ g$.

Viewing $H^0(\boldsymbol{P}_1, \mathcal{O}(n))$ as the $(n + 1)$-dimensional space of polynomials of degree $n$, this is simply $\tilde{g}(p(\zeta)) = (c\zeta + d)^n p((a\zeta + b)/(c\zeta + d))$. Note that this action preserves the affine cone in $H^0(\boldsymbol{P}_1, \mathcal{O}(n))$ over $\mathcal{R}_n$, cf. 3.8 in [20].

If $n$ is even (which we suppose for the rest of this section), then the identity element of $SL(2, \boldsymbol{C})$ acts as the identity on $H^0(\boldsymbol{P}_1, \mathcal{O}(n))$. The irreducible representation of $SU(2)$ thus obtained induces the unique $(n + 1)$-dimensional representation of $SO(3, \boldsymbol{R})$.

Now, suppose that $\mathcal{L}_n$ is endowed with a real structure $\tau$ such that the action of any $g \in SU(2)$ on $\mathcal{L}_n$ commutes with $\tau$. This means that $SU(2)$ preserves the real sections in $H^0(\boldsymbol{P}_1, \mathcal{O}(n))$ determined by $\tau$. (This follows because real sections are those that satisfy $\tau \circ \sigma \circ \alpha = \sigma$, where $\alpha$ is the (antiholomorphic) involution of $\boldsymbol{P}_1$ induced by $\tau$. For $g \in SU(2)$, commutativity with $\tau$ gives $\tau \circ g_\star^{-1} \circ \sigma \circ g \circ \alpha = g_\star^{-1} \circ \tau \circ \sigma \circ \alpha \circ g = g_\star^{-1} \circ \sigma \circ g$. So $\sigma$ is real if and only if $\tilde{\sigma}$ is real.) Thus we obtain the $(n + 1)$-dimensional irreducible *real* representation of $SO(3, \boldsymbol{R})$.

A curve described as a zero locus $P(\zeta, \eta) = 0$ in $\mathcal{L}_n$ is *invariant* under the action of $g \in SL(2, \boldsymbol{C})$ if $P(g_\star(\zeta, \eta)) = 0$ describes the same curve. Clearly a curve in $\mathcal{L}_n$ is invariant under the action of $g_\star$ if and only if the associated curve in $\boldsymbol{C}^{n+1}$ is invariant under the action of $\tilde{g}$.

Suppose that $G \subset SO(3, \boldsymbol{R})$ is the symmetry group of a regular polyhedron and $\tilde{G}$ the corresponding binary group. It follows from the above discussion that an algebraic curve on $\mathcal{L}_n$ which is invariant under the action of $\tilde{G}$ generates via osculation an algebraic $\Lambda$-surface in $\boldsymbol{R}^{n+1}$ which is invariant under $G$ acting through the unique $(n+1)$-dimensional representation as described above.

REMARK. For $n = 2$, this reduces to the adjoint representation of $SL(2, \boldsymbol{C})$, cf. [20]. This case was used by Goursat ([7]) in his prize winning work of 1887. His aim was to describe algebraic minimal surfaces in $\boldsymbol{R}^3$ that possess the symmetries of a regular polyhedron. See (6.8) for some new examples.

There has been much recent interest in the construction of *immersed* examples of such surfaces, see [17], [25].

## 6. Global Weierstrass formulae and examples.

(6.1) Examples of meromorphic Calabi curves in $\boldsymbol{C}^{n+1}$ and the associated algebraic $\Lambda$-surfaces in $\boldsymbol{R}^{n+1}$ can be constructed and studied by writing down a curve in $\boldsymbol{C}^2$ and considering its completion in $\mathcal{S}_n$. On the other hand, such a curve is described by a pair of meromorphic functions $(g, f) : M - D_\infty(g) \to \boldsymbol{C} \times \boldsymbol{P}_1$, where $D_\infty(g)$ denotes the divisor of poles of $g$. Working in the coordinates $\omega = -\zeta^{-1}$, $\mu = \zeta^{-n}\eta$ around the fibre over $\infty$ in $\mathcal{L}_n$, gives the equivalent description $(-1/g, g^{-n}f) : M - D_0(g) \to \boldsymbol{C} \times \boldsymbol{P}_1$, where $D_0(g)$ is the divisor of zeros of $g$.

Global Weierstrass formulae are obtained by considering

$$\begin{aligned} f(\xi) &= f(\xi_0) + (g(\xi) - g(\xi_0))\frac{df}{dg}(\xi_0) + \cdots + (g(\xi) - g(\xi_0))^n \frac{1}{n!}\frac{d^n f}{dg^n}(\xi_0) + \cdots \\ &= a_0(\xi_0) + a_1(\xi_0)g(\xi) + \cdots + a_n(\xi_0)g^n(\xi) + \cdots, \end{aligned}$$

where $df/dg = (df/d\xi)(dg/d\xi)^{-1}$, etc. Of course, the right hand side of the above expression does not converge at branch points of $g$, but the coefficients $a_0(\xi_0), \ldots, a_n(\xi_0)$, viewed as functions of $\xi_0$, determine a *globally* defined meromorphic map of $M$. This curve (after linear transformation) agrees with the Calabi curve determined by osculation on an open set and hence on all of $M$. Observe that it coincides with the curve determined by the corresponding expansion of $g^{-n}f$ with respect to $-1/g$. For example, when $n = 2$, we have, globally on $M$:

$$g^{-2} f \frac{d}{d(-1/g)} = f\frac{d}{dg},$$

where in a local coordinate $\xi$, $d/dg = (dg/d\xi)^{-1}(d/d\xi)$, and

$$\begin{aligned} \frac{1}{2}\frac{d^2(f/g^2)}{d(-1/g)^2} &= \frac{1}{2}g^2\frac{d^2 f}{dg^2} - g\frac{df}{dg} + f \\ \frac{d(f/g^2)}{d(-1/g)} + \frac{1}{g}\frac{d^2(f/g^2)}{d(-1/g)^2} &= -\frac{df}{dg} + g\frac{d^2 f}{dg^2} \\ \frac{1}{2}\frac{1}{g^2}\frac{d^2(f/g^2)}{d(-1/g)^2} + \frac{1}{g}\frac{d(f/g^2)}{d(-1/g)} + \frac{f}{g^2} &= \frac{1}{2}\frac{d^2 f}{dg^2}. \end{aligned}$$

These equations reflect the transformation of coefficients of a global section of $\mathcal{O}(2)$ that occurs when switching from $\zeta$ to $\omega = -\zeta^{-1}$ coordinates on $\boldsymbol{P}_1$.

It is easy to write down appropriate transformations for arbitrary $n$ to generate global Weierstrass formulae for algebraic $\Lambda$-surfaces in $\boldsymbol{R}^{n+1}$, but, since they are not canonical, we limit ourselves to illustrating the procedure in 3, 4 and 5 dimensions. Let $f^{(k)}$ denote $d^k f/dg^k$.

(6.2) Global formulae for algebraic minimal surfaces in $\boldsymbol{R}^3$ are given by (the classical Weierstrass formulae):

$$\phi_1 = \mathrm{Re}\left(\frac{1}{2}(1-g^2)\frac{d^2f}{dg^2} + g\frac{df}{dg} - f\right)$$
$$\phi_2 = \mathrm{Re}\left(\frac{i}{2}(1+g^2)\frac{d^2f}{dg^2} - ig\frac{df}{dg} + if\right)$$
$$\phi_3 = \mathrm{Re}\left(g\frac{d^2f}{dg^2} - \frac{df}{dg}\right).$$

See [20].

(6.3) A full meromorphic Calabi curve in $\boldsymbol{C}^4$ may, after unitary transformation, be brought into the following form:

$$\psi_0 = f^{(3)}, \quad \psi_1 = \sqrt{3}(gf^{(3)} - f^{(2)}), \quad \psi_2 = \sqrt{3}(g^2f^{(3)} - 2gf^{(2)} + 2f^{(1)}),$$
$$\psi_3 = g^3f^{(3)} - 3g^2f^{(2)} + 6gf^{(1)} - 6f,$$

where $f = f_\psi = (1/6)(-\psi_3 + \sqrt{3}g\psi_2 - \sqrt{3}g^2\psi_1 + g^3\psi_0)$, and $g$ gives the Gauss map of $\psi$.

These Calabi curves satisfy $3\psi_0'\psi_3' - \psi_1'\psi_2' = 0$ and consequently $\omega : M \to \boldsymbol{C}^4$ given by:

$$\omega_0 = \frac{\sqrt{3}}{2}(\psi_0 + \psi_3), \quad \omega_1 = i\frac{\sqrt{3}}{2}(\psi_0 - \psi_3), \quad \omega_2 = \frac{1}{2}(\psi_1 - \psi_2), \quad \omega_3 = \frac{i}{2}(\psi_1 + \psi_2)$$

satisfies $(\omega_0')^2 + \cdots + (\omega_3')^2 = 0$. Hence substitution of meromorphic functions $g$, $f$, such that $f$ is not a polynomial in $g$ of degree $\leq 3$, into the following formulae gives an algebraic $\Lambda$-surface in $\boldsymbol{R}^4$:

$$\phi_0 = \mathrm{Re}((1+g^3)f^{(3)} - 3g^2f^{(2)} + 6gf^{(1)} - 6f)$$
$$\phi_1 = \mathrm{Re}(i\{(1-g^3)f^{(3)} + 3g^2f^{(2)} - 6gf^{(1)} + 6f\})$$
$$\phi_2 = \mathrm{Re}((g-g^2)f^{(3)} - (1-2g)f^{(2)} - 2gf^{(1)})$$
$$\phi_3 = \mathrm{Re}(i\{(g+g^2)f^{(3)} - (1+2g)f^{(2)} + 2gf^{(1)}\}).$$

REMARK. There exist Weierstrass formulae in integrated form for general null curves in $\boldsymbol{C}^4$, see [19], [22]. For integral formulae see [10].

(6.4) Full Calabi curves $\psi : M \to \boldsymbol{C}^5$ satisfy $2\psi_0'\psi_4' + \psi_1'\psi_3' - (\psi_2')^2 = 0$ and hence $\omega : M \to \boldsymbol{C}^5$ given by:

$$\begin{aligned}
\omega_0 &= \frac{1}{\sqrt{2}}(\psi_0 + \psi_4) \\
\omega_1 &= \frac{-i}{\sqrt{2}}(\psi_0 - \psi_4) \\
\omega_2 &= \frac{1}{2}(\psi_1 + \psi_3) \\
\omega_3 &= \frac{-i}{2}(\psi_1 - \psi_3) \\
\omega_4 &= -i\psi_2
\end{aligned}$$

satisfies $(\omega_0')^2 + \cdots + (\omega_4')^2 = 0$. Hence substitution of meromorphic functions $g$, $f$, such that $f$ is not a polynomial in $g$ of degree $\leq 4$, into the following formulae gives an algebraic $\Lambda$-surface in $\boldsymbol{R}^5$:

$$\begin{aligned}
\phi_0 &= \mathrm{Re}\left(\frac{1}{\sqrt{2}}\{(1+g^4)f^{(4)} - 4g^3 f^{(3)} + 12g^2 f^{(2)} - 24gf^{(1)} + 24f\}\right) \\
\phi_1 &= \mathrm{Re}\left(\frac{-i}{\sqrt{2}}\{(1-g^4)f^{(4)} + 4g^3 f^{(3)} - 12g^2 f^{(2)} + 24gf^{(1)} - 24f\}\right) \\
\phi_2 &= \mathrm{Re}((g+g^3)f^{(4)} - (1+3g^2)f^{(3)} + 6gf^{(2)} - 6f^{(1)}) \\
\phi_3 &= \mathrm{Re}(-i\{(g-g^3)f^{(4)} + (1-3g^2)f^{(3)} - 6gf^{(2)} + 6f^{(1)}\}) \\
\phi_4 &= \mathrm{Re}(-i\sqrt{6}\{g^2 f^{(4)} - 2gf^{(3)} + 2f^{(2)}\}).
\end{aligned}$$

(6.5) *Higher Dimensional Enneper Surfaces.* The examples described in Theorem 3 of [2], which give complete immersed minimal surfaces with total Gaussian curvature $-2\pi n$ in $\boldsymbol{R}^{n+1}$, are generated by osculation of the curve $\eta = \zeta^{n+1}$ in $\mathcal{L}_n$. In dimension 3 this gives Enneper's surface. These give the 'simplest' non-trivial immersed examples in each dimension: for any power less than $n+1$, osculation gives a constant map. Any power greater than $n+1$ gives a minimal surface with branch points. (Note that the addition of a polynomial of degree $\leq n$ simply results in the translation of the corresponding minimal surface in $\boldsymbol{R}^{n+1}$.)

(6.6) *Higher Order Enneper Surfaces.* The rational examples described above and those in (6.1) of [20] may be generalized as follows. Let $p, q \in \boldsymbol{N}$ be coprime with $p+q \geq 3$. $\mathcal{C}_{p,q,n}$, the curve in $\mathcal{C}(\mathcal{R}_n^\natural)$ obtained by completing the curve in $\boldsymbol{C}^2$ given by $\eta^q = \zeta^p$, is irreducible and rational with normalization given by extending $u \mapsto (u^q, u^p)$.

For $p < nq + 1$, osculation of $\mathcal{C}_{p,q,n}$ yields complete minimal immersions of $\boldsymbol{C}^*$ into $\boldsymbol{R}^{n+1}$ with total curvature $-2\pi nq$.

For $p > nq + 1$, osculation gives one ended examples $\boldsymbol{C} \to \boldsymbol{R}^{n+1}$, with a branch point at 0 and total curvature $-2\pi nq$.

$p = nq + 1$, is a 'critical' case. It gives one ended immersions into $\boldsymbol{R}^{n+1}$, with total curvature $-2\pi nq$. This case reduces to (6.5) when $q = 1$. The $n = 2$ examples were discussed in [20] and from another point of view in [12].

(6.7) *Higher Dimensional Henneberg Surfaces.* The curve $\mathcal{H}_{n+1}$ in $\mathcal{L}_n$ described by $\eta = \zeta^{n+1} + (-1)^{n+1}\zeta^{-1}$ is the sum of two curves which are dual to Enneper type surfaces. In the coordinates $\omega = -\zeta^{-1}$, $\mu = \zeta^{-n}\eta$; $\eta = \zeta^{n+1}$ is the curve $\mu = \omega^{-1}$, see (5.9). (The resulting 'addition' of minimal surfaces was described, from a different point of view, in [16] for surfaces in $\boldsymbol{R}^3$, see also [21].)

Osculation of this curve gives an $(n+1)$-dimensional analogue of the classical Henneberg surface in $\boldsymbol{R}^3$, which is given by $n = 2$. These surfaces have two (vertex) ends, branch points at the $(n+2)$-th roots of unity and Gaussian curvature $-2n\pi$. Recall that the Henneberg branched minimal immersion factors through a once punctured $\boldsymbol{RP}^2$ to give a branched minimally immersed Möbius strip with one end of total curvature $-2\pi$. It is easy to see how to generalize this feature also, at least to odd dimensions. To illustrate this we now discuss a five dimensional example.

A full Calabi curve $\psi : M \to \boldsymbol{C}^5$ satisfies $2\psi_0'\psi_4' + \psi_1'\psi_3' - (\psi_2')^2 = 0$. So the curve $\theta : M \to \boldsymbol{C}^5$, given by $\theta_0 = i\psi_0$, $\theta_1 = \psi_1$, $\theta_2 = \psi_2$, $\theta_3 = \psi_3$, $\theta_4 = i\psi_4$, satisfies $2\theta_0'\theta_4' - \theta_1'\theta_3' + (\theta_2')^2 = 0$. We now define $\omega : M \to \boldsymbol{C}^5$ by setting:

$$
\begin{aligned}
\omega_0 &= \frac{1}{\sqrt{2}}(\theta_0 + \theta_4)\\
\omega_1 &= \frac{i}{\sqrt{2}}(\theta_4 - \theta_0)\\
\omega_2 &= \frac{1}{2}(\theta_1 - \theta_3)\\
\omega_3 &= \frac{-i}{2}(\theta_1 + \theta_3)\\
\omega_4 &= \theta_2 .
\end{aligned}
$$

Since $(\omega_0')^2 + \cdots + (\omega_4')^2 = 2\theta_0'\theta_4' - \theta_1'\theta_3' + (\theta_2')^2$, it follows that $\omega$ is a null curve.

Consider the real structure $\tau_4 : \mathcal{L}_1 \to \mathcal{L}_4$ given by $(\zeta, \eta) \mapsto (-\bar{\zeta}^{-1}, \bar{\zeta}^{-4}\bar{\eta})$. It is easy to check that the real sections of $\mathcal{O}(4)$, i.e., those invariant under $\tau_4$, are of the form $a\beta_0 + b\beta_1 + c\beta_2 + d\beta_3 + e\beta_4$, where $a = \bar{e}$, $b = -\bar{d}$ and $c = \bar{c}$. It follows that $T : \boldsymbol{C}^5 \to H^0(\boldsymbol{P}_1, \mathcal{O}(4))$ given by $T(z_0, \dots, z_4) = ((z_0 + iz_1)/\sqrt{2})\beta_0 + (z_2 + iz_3)\beta_1 + z_4\beta_2 + (-z_2 + iz_3)\beta_3 + ((z_0 - iz_1)/\sqrt{2})\beta_4$, takes $\boldsymbol{R}^5 \subset \boldsymbol{C}^5$ to the real global sections determined by $\tau_4$. Observe that $\omega = T^{-1}\theta$.

The curve $\mathcal{H}_5$, described by $g(\zeta) = \zeta$, $f(\zeta) = \zeta^5 - \zeta^{-1}$ in $\mathcal{L}_4$, is $\tau_4$-invariant. Note that $\tau_4$ induces the antipodal map $\alpha$ on $\boldsymbol{P}_1$. Now, since a global section $\sigma$ osculates a $\tau_4$-invariant curve $\mathcal{C}$ at a point $p$ if and only if $\tau_4 \circ \sigma \circ \alpha$ osculates $\tau_4(\mathcal{C})$ at $\tau_4(p)$, it follows from the above that $\omega(\tau_4(p)) = \bar{\omega}(p)$ for all $p \in \mathcal{H}_5$. Consequently, $\phi(p) = (\omega(p) + \bar{\omega}(p))/2 = (\omega(p) + \omega(\tau_4(p)))/2$. It follows that $\phi(\tau_4(p)) = \phi(p)$ for all $p \in \mathcal{H}_5$.

Using $\zeta$ as a coordinate on $\mathcal{H}_5$, we have for $\phi = (\omega + \bar{\omega})/2$, that $\phi(-\bar{\zeta}^{-1}) = \phi(\zeta)$. Thus $\phi$ factors through $\boldsymbol{RP}^2$ to give a branched minimally immersed Möbius strip, with one end, into $\boldsymbol{R}^5$. The branch points at the sixth roots of unity descend in antipodal pairs to give three branch points. Thus we obtain a nondegenerate branched minimally immersed Möbius strip in $\boldsymbol{R}^5$ with total curvature $-4\pi$. The algebraic formulae can easily be found from the above.

(6.8) *Platonic Surfaces.* Following the discussion in (5.7), consider the compactifications in $\mathcal{L}_2$ of the following curves:

$$\eta^3 + ia\zeta(\zeta^4 - 1) = 0, \tag{1}$$

$$\eta^4 + a(\zeta^8 + 14\zeta^4 + 1) = 0, \tag{2}$$

$$\eta^6 + a\zeta(\zeta^{10} + 11\zeta^5 - 1) = 0, \tag{3}$$

where $a \in \boldsymbol{R} - \{0\}$. These were described in [9], following Klein, via the theory of invariant bilinear forms and polynomials on $\boldsymbol{P}_1$. (1) is invariant under the binary tetrahedral group, (2) is invariant under the binary octahedral group and (3) is invariant under the binary icosahedral group. Each of these curves is smooth in $\mathcal{L}_2$.

Via osculation duality:

(1) gives a one parameter family of algebraic minimal surface in $\boldsymbol{R}^3$ with tetrahedral symmetries. They are genus 4 with 6 (non-vertex) ends and total curvature $-12\pi$. Viewed as a curve in $\boldsymbol{P}_3$, (1) gives a canonical curve. Consequently the points of hyperosculation, which, away from ends, give branch points in the metric on the corresponding minimal surface, are Weierstrass points. Each of the ends, where the curve osculates a hyperplane cutting out a fibre of the cone in $\boldsymbol{P}_3$, gives a point of hyperosculation.

(2) gives a one parameter family of algebraic minimal surfaces in $\boldsymbol{R}^3$ with octahedral symmetries. They are genus 9 with 8 (non-vertex) ends and total curvature $-16\pi$.

(3) gives a one parameter family of algebraic minimal surfaces in $\boldsymbol{R}^3$ with icosahedral symmetries. They are genus 25 with 12 (non-vertex) ends and total curvature $-24\pi$.

In each case the curve is invariant under the real structure $\tau_2(\zeta, \eta) = (-\bar{\zeta}^{-1}, \bar{\zeta}^{-2}\bar{\eta})$. Thus the corresponding branched minimal immersion factors through the curve quotiented by the action of $\tau_2$. This shows that the corresponding surfaces in $\boldsymbol{R}^3$ really have half the number of ends, branch points and total curvature one calculates on the original curve. Compare the Klein bottle examples in (6.2) of [20].

(6.9) Let us compactify the curves described in (6.8) to obtain (singular) algebraic curves on $\mathcal{L}_4$.

Observe that the action of $SU(2)$ on $\mathcal{L}_4$ commutes with $\tau_4$. Hence following (5.9), we obtain from these curves, algebraic $\Lambda$-surfaces in $\boldsymbol{R}^5$ which are invariant under the action of the corresponding $G \subset SO(3, \boldsymbol{R})$ acting through the real 5-dimensional representation described in (5.9). Observe that, since these curves are not $\tau_4$-invariant, these branched minimal immersions do not factor.

The additional twisting in the bundle renders the curves singular in the fibre over infinity. Inspection reveals that these singular points give branch points in the Gauss maps of the corresponding minimal surfaces in $\boldsymbol{R}^5$, not ends. (Namely, the Gauss map onto the rational normal curve in the quadric $Q_3$ has a branch point.) Bearing this in mind, observe that via osculation duality:

(1) gives a one parameter family of algebraic minimal surface in $\boldsymbol{R}^5$ with tetrahedral symmetries. They are genus 4 with 5 (non-vertex) ends and total curvature $-24\pi$.

(2) gives a one parameter family of algebraic minimal surfaces in $\boldsymbol{R}^5$ with octahedral symmetries. They are genus 9 with 7 (non-vertex) ends and total curvature $-32\pi$.

(3) gives a one parameter family of algebraic minimal surfaces in $\boldsymbol{R}^5$ with icosahedral symmetries. They are genus 25 with 11 (non-vertex) ends and total curvature $-48\pi$.

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