Tohoku Math. J.
51 (1999), 433–446

A CONSTRUCTION OF K-CONTACT MANIFOLDS BY A FIBER JOIN

TSUTOMU YAMAZAKI

(Received March 21, 1997, revised June 21, 1999)

Abstract. In this paper we introduce a process of making a fiber join of regular K-contact manifolds and then construct some explicit examples of K-contact flows which generate contact transformations of a torus. We also discuss the equivalence of these examples.

1. Introduction. A contact flow φ_t is a flow which is generated by the Reeb vector field of a contact manifold (M, α). It preserves the contact form α and the contact plane field $\ker\alpha$. A contact flow φ_t is called a *K-contact flow* if there exists a metric g on M such that φ_t is an isometry. In this case the triple (M, α, g) is called a *K-contact manifold* ([2, 3]).

Suppose we are given a K-contact manifold (M, α, g). If M is compact, the closure of a K-contact flow $\{\varphi_t \mid t \in \boldsymbol{R}\}$ in the isometry group of (M, g) makes a compact connected abelian Lie group, hence isomorphic to T^k for some integer k. Clearly this action of the torus T^k also preserves α and g. Thus a compact K-contact manifold (M, α, g) has a T^k-action which preserves both α and g. We will see that this property of T^k-action on a contact manifold characterizes the "K-contactness" and k satisfies $1 \leq k \leq n+1$ when $\dim M = 2n+1$ (see Proposition 2.1). We call (M, α, g) with this T^k-action a K-contact manifold *of rank k*. A typical class of examples of K-contact manifolds of rank 1 is a family of *regular* K-contact manifolds (M, α, g). A regular contact manifold (M, α) consists of a pair of a principal S^1-bundle M over a symplectic manifold (W, ω) and a connection one-form α. A metric g is given by $g = \pi^* g_W \oplus (\alpha \otimes \alpha)$, where g_W is a Riemannian metric compatible with ω and π is the bundle projection $M \to W$ (see Example 2.4).

In this paper we will present a method of constructing a K-contact manifold of rank $k \geq 2$ out of K-contact manifolds of rank 1 by making use of join construction in topology.

Let $(M_0, \alpha_0, g_0), \dots, (M_n, \alpha_n, g_n)$ be regular K-contact manifolds and L_j an associated complex line bundle of $M_j \to W$ for each j $(j = 0, 1, \dots, n)$. From these we construct a K-contact manifold $(M_0 *_f \cdots *_f M_n, \beta_\lambda, g_\lambda)$ of rank $n+1$. Here $M_0 *_f \cdots *_f M_n$ is the unit sphere bundle $S(L_0 \oplus \cdots \oplus L_n)$ and β_λ is a contact form with a parameter $\lambda = (\lambda_0, \dots, \lambda_n) \in \boldsymbol{R}^{n+1}$. We call the resulted K-contact manifold a *fiber join* of $(M_0, \alpha_0, g_0), \dots, (M_n, \alpha_n, g_n)$.

Applying a fiber join construction to three dimensional regular K-contact manifolds, we obtain infinitely many distinct K-contact structures on $\Sigma_g \times S^{2n+1}$ and $\Sigma_g \tilde{\times} S^{2n+1}$ (Σ_g is a closed Riemannian surface of genus g) which are not T^{n+1}-equivariant. Namely, we obtain the following:

1991 *Mathematics Subject Classification.* Primary 53C12; Secondary 5315, 58F25.

THEOREM 4.5. *For $n \geq 1$ there exist infinitely many different K-contact equivalence classes of K-contact flows on $\Sigma_g \times S^{2n+1}$ and $\Sigma_g \tilde{\times} S^{2n+1}$.*

The author would like to express his gratitude to Professor Tadayoshi Mizutani and Professor Yoshihiko Mitsumatsu for their continuous encouragement and helpful discussions.

2. The torus action on K-contact manifold. A contact form on a $(2n+1)$-dimensional smooth manifold M is a one-form α such that $\alpha \wedge (d\alpha)^n$ is everywhere nonzero. The pair (M, α) is called a contact manifold. A contact form α determines a unique vector field Z on M such that $\alpha(Z) = 1, d\alpha(Z, X) = 0$ for any vector field X on M. We call Z and the flow φ_t generated by it the *Reeb vector field* and the *contact flow*, respectively. A $2n$-dimensional distribution D on M defined by $D := \ker \alpha$ is called a *contact plane field.* From the definition of α and D, it is obvious the two-form $d\alpha$ is non-degenerate on D. Namely, $d\alpha$ induces a symplectic structure on D. In this situation, it is well-known that there exists a positive definite metric g_T and an almost complex structure J on D such that $g_T(X, Y) = d\alpha(X, JY)$, $g_T(JX, JY) = g_T(X, Y)$ for all $X, Y \in \Gamma(D)$ (the set of smooth sections of a vector bundle D) (see [1]). The pair (g_T, J) is said to be *compatible* with $d\alpha$. We can extend g_T on D to whole TM by requiring $g_T(Z, X) = 0$ for any vector field X on M. Thus we get a Riemannian metric $g := g_T \oplus (\alpha \otimes \alpha)$ on M, which is called an *adapted metric* to the contact form α. Note that an adapted metric g is not unique, depending on the choice g_T.

Now we define a K-contact manifold.

DEFINITION. Let (M, α) be a contact manifold with the Reeb vector field Z. If there exists an adapted metric g to α on M such that Z is a Killing vector field with respect to g, that is,

$$L_Z g = 0, \tag{2.1}$$

then we call (M, α, g) a *K-contact manifold.* Here L_Z is the Lie differentiation in direction of Z.

We call α and g of a K-contact manifold (M, α, g) the *K-contact form* and the *K-contact metric*, respectively. We also call the flow φ_t generated by the Reeb vector field Z of the K-contact form α the *K-contact flow* of (M, α, g).

In general, a contact flow φ_t preserves the contact form α. This is because we have $L_Z\alpha = 0$ from the definition of the Reeb vector field Z. It follows that a K-contact manifold (M, α, g) has an $\boldsymbol{R}$-action induced by $\{\varphi_t \mid t \in \boldsymbol{R}\}$ which preserves both α and g.

The following proposition characterizes a K-contact manifold.

PROPOSITION 2.1. *Let (M, α) be a $(2n+1)$-dimensional contact manifold and φ_t the contact flow. If we assume that M is compact, then the following statements are equivalent.*

(1) *There exists an adapted metric g to α such that (M, α, g) is a K-contact manifold.*

(2) *There exist a torus T^k such that $1 \leq \dim(T^k) \leq n+1$, a smooth effective T^k-action $\{h_u \mid u \in T^k\}$ on M, and a homomorphism $\Psi : \boldsymbol{R} \to T^k$ with dense image such that $\varphi_t = h_{\Psi(t)}$.*

PROOF. We will prove (1) $\Rightarrow$ (2). Since M is compact, by Meyer-Steenrod theorem (see [7]), the isometry group $\mathrm{Isom}(M, g)$ of (M, g) is a compact Lie group. It follows that the closure of $\{\varphi_t \mid t \in \boldsymbol{R}\}$ in $\mathrm{Isom}(M, g)$ is a compact connected abelian Lie group, and hence is isomorphic to a torus T^k for some integer k.

We now prove $k \leq n+1$. Let $\Gamma(TM)$ be the Lie algebra of the vector field on M and V the Lie algebra determined by the image of the Lie algebra homomorphism $\mathrm{Lie}(T^k) \ni \xi \to d/dt\,\big|_{t=0} \exp(t\xi) \in \Gamma(TM)$. Here $\exp : \mathrm{Lie}(T^k) \to T^k$ is the exponential map. Let Z be the Reeb vector field of (M, α). We denote by $\boldsymbol{R}Z$ a trivial line bundle spanned by Z. By the isomorphism $TM \cong D \oplus \boldsymbol{R}Z$ we have a unique decomposition $X = \bar{X} + \alpha(X)Z$ for $X \in V$ and $\bar{X} \in \Gamma(D)$. From the fact that $\alpha(X)$ is a T^k-invariant function and $[X, Y] = 0$ for any $X, Y \in V$, we see that $[\bar{X}, \bar{Y}] = 0$ for any $X, Y \in V$. It follows that if we denote by $X_1, \dots, X_k$ the fundamental vector fields of T^k-action determined by a basis of the Lie algebra $\mathrm{Lie}(T^k)$, there is an open set U such that $\bar{X}_1, \dots, \bar{X}_k$ determine a $(k-1)$-dimensional integrable distribution on U tangent to D. It is well-known that the maximal dimension of integrable submanifolds of the contact distribution is n, so $k-1 \leq n$ and hence $k \leq n+1$.

We will prove (2) $\Rightarrow$ (1). From the fact that $\varphi_t^*\alpha = \alpha$ and the closure of $\{\varphi_t \mid t \in \boldsymbol{R}\}$ is isomorphic to T^k, we have $h_u^*\alpha = \alpha$ for all $u \in T^k$. Namely, α is invariant under the T^k-action, and so is $d\alpha$. In this case we can also take a positive definite metric g_T and an almost complex structure J, which is compatible with the symplectic form $d\alpha$ on D, to be invariant under this T^k-action (see [1, 15]). Thus we have a metric $g = g_T \oplus (\alpha \otimes \alpha)$ and it is invariant under the action of T^k. In particular, we have $L_Z g = 0$, and hence (M, α, g) is a K-contact manifold. q.e.d.

The property of T^k-action of Proposition 2.1 characterizes the "K-contactness". Namely, we may consider a K-contact manifold as a manifold which has an action of the torus T^k containing the contact flow as a dense image, and hence the action of T^k preserves both α and g.

DEFINITION. (M, α, g) is called a *K-contact manifold of rank k* if the closure of the K-contact flow $\{\varphi_t \mid t \in \boldsymbol{R}\}$ in $\mathrm{Isom}(M, g)$ is isomorphic to a k-dimensional torus T^k.

As a result of Proposition 2.1, we see that in the case of the contact flow on the compact contact manifold (M, α) there is no difference between an isometric flow and a Riemannian flow. Namely, we get the following:

COROLLARY 2.2 ([15]). *If a contact flow on a compact manifold is a Riemannian flow, then, (changing the transverse metric, if necessary), it is a K-contact flow.*

PROOF. Let (M, α) be a compact contact manifold with the Reeb vector field Z. Assume that a contact flow φ_t of Z is a Riemannian flow, that is, there exists a transverse metric $\tilde{g}_T$ to the contact flow φ_t (a positive definite metric on the contact plane field $\ker\alpha$) such that $L_Z\tilde{g}_T = 0$. Note that $\tilde{g}_T$ needs not to be compatible with the symplectic form $d\alpha$. Then Z is a Killing vector field with respect to a Riemannian metric $\tilde{g} = \tilde{g}_T \oplus (\alpha \otimes \alpha)$. So φ_t is an isometric flow. Since the closure of $\{\varphi_t \mid t \in \boldsymbol{R}\}$ in $\mathrm{Isom}(M, \tilde{g})$ is isomorphic to a torus, φ_t

satisfies the condition (2) in Proposition 2.1. Therefore there exists a K-contact metric g on M, and hence φ_t is a K-contact flow. q.e.d.

We will give two typical classes of examples of K-contact manifolds. They are needed for the construction in Section 3.

EXAMPLE 2.3 (($2n+1$)-dimensional K-contact manifold of rank $n+1$). Let $S^{2n+1} = \{z = (z_0, \dots, z_n) \in \boldsymbol{C}^{n+1} \mid \sum_{j=0}^{n} z_j \bar{z}_j = 1\}$ be a $(2n+1)$-dimensional unit sphere in complex $(n+1)$-space $\boldsymbol{C}^{n+1}$. We denote the polar coordinate of $\boldsymbol{C}^{n+1}$ by $(r_0, \theta_0, \dots, r_n, \theta_n)$. For rationally independent positive constants $\lambda_0, \dots, \lambda_n$, we take

$$\alpha_\lambda = \sqrt{-1}/2 \sum_{j=0}^{n} \lambda_j (z_j d\bar{z}_j - \bar{z}_j dz_j) = \sum_{j=0}^{n} \lambda_j r_j^2 d\theta_j \,. \tag{2.2}$$

Then it is easily seen that α_λ is the contact form on S^{2n+1} with the Reeb vector field

$$X_\lambda = \sqrt{-1} \sum_{j=0}^{n} 1/\lambda_j (z_j \partial/\partial z_j - \bar{z}_j \partial/\partial \bar{z}_j) \,. \tag{2.3}$$

Let φ_t^λ be the contact flow of X_λ and

$$\left(e^{\sqrt{-1}t_0}, \dots, e^{\sqrt{-1}t_n}\right) \cdot (z_0, \dots, z_n) = \left(e^{\sqrt{-1}t_0} z_0, \dots, e^{\sqrt{-1}t_n} z_n\right), \tag{2.4}$$

where $\left(e^{\sqrt{-1}t_0}, \dots, e^{\sqrt{-1}t_n}\right) \in T^{n+1} \subset (\boldsymbol{C}^*)^{n+1}$, be the standard T^{n+1}-action on S^{2n+1}. Then we have

$$\varphi_t^\lambda (z_0, \dots, z_n) = \left(e^{\sqrt{-1}(1/\lambda_0)t} z_0, \dots, e^{\sqrt{-1}(1/\lambda_n)t} z_n\right). \tag{2.5}$$

Since $\lambda_0, \dots, \lambda_n$ are rationally independent, the closure $\overline{\varphi_t^\lambda \cdot z}$ of the orbit $\varphi_t^\lambda \cdot z$ coincides with the orbit $T^{n+1} \cdot z$ for any $z \in S^{2n+1}$. Thus, by Proposition 2.1, there exists an adapted metric g_λ to α_λ such that $(S^{2n+1}, \alpha_\lambda, g_\lambda)$ is a K-contact manifold of rank $n+1$. Here g_λ is given by choosing a transverse metric g_T on $\ker \alpha_\lambda$ and setting $g_\lambda = g_T \oplus (\alpha_\lambda \otimes \alpha_\lambda)$.

We define a S^1-action on S^{2n+1} by

$$e^{\sqrt{-1}\theta} \cdot (z_0, z_1, \dots, z_n) = \left(e^{\sqrt{-1}\theta} z_0, e^{\sqrt{-1}\theta q_1} z_1, \dots, e^{\sqrt{-1}\theta q_n} \cdot z_n\right)$$

for $e^{\sqrt{-1}\theta} \in S^1 \subset \boldsymbol{C}^*$ and positive integers $q_1, \dots, q_n$. Choose an integer p such that p and each q_j are relatively prime, and consider the action restricted to $\{e^{2\pi k\sqrt{-1}\theta/p} \mid k = 0, 1, \dots, p-1\} \cong \boldsymbol{Z}/p\boldsymbol{Z}$ of the above S^1-action. Then $S^{2n+1}/(\boldsymbol{Z}/p\boldsymbol{Z})$ is also a K-contact manifold of rank $n+1$ with the K-contact form and the K-contact metric induced from S^{2n+1}.

REMARK. (1) The choice of g_T on $\ker \alpha_\lambda$ is not unique. However, for example, we can choose it to be the restriction of a Riemannian metric

$$2 \sum_{j=0}^{n} \lambda_j (dr_j \otimes dr_j + r_j^2 d\theta_j \otimes d\theta_j) \left(= \sqrt{-1} \sum_{j=0}^{n} \lambda_j dz_j \otimes d\bar{z}_j \right) \tag{2.6}$$

on $\boldsymbol{C}^{n+1}$ to $\ker \alpha_\lambda$.

(2) Take $\{\lambda_0, \dots, \lambda_n\}$ so that $\lambda_0, \dots, \lambda_n$ form a k-dimensional vector space over $\boldsymbol{Q}$. Then there exists a subgroup T^k of T^{n+1} and a T^k-action induced by (2.4) such that for $z \in S^{2n+1}$, the closure of the orbit $\varphi_t \cdot z$ coincides with the orbit $T^k \cdot z$. Thus we obtain a K-contact manifold $(S^{2n+1}, \alpha_\lambda, g_\lambda)$ of rank k. In particular, if we take $\lambda_0 = \cdots = \lambda_n = 1$, then we get the K-contact manifold of rank 1 such that a K-contact flow determines the Hopf S^1-fibration $S^{2n+1} \to \boldsymbol{C}P^n$.

EXAMPLE 2.4 (K-contact manifold of rank 1). Let (W, ω) be a symplectic manifold whose symplectic two-form determines a de Rham cohomology class contained in the image of $H^2(W; \boldsymbol{Z}) \to H^2(W; \boldsymbol{R})$. Then there exists a principal S^1-bundle $\pi : M \to W$ whose first Chern class is equal to $[\omega] \in H^2(M; \boldsymbol{Z})$ and a connection one-form η on M with the curvature form $d\eta = \pi^*\omega$ ([6]). Hence η is a contact form on M whose contact flow of arbitrary point is a principal S^1-orbit. It follows that by Proposition 2.1, there exists an adapted metric g to η such that (M, η, g) is a K-contact manifold of rank 1. Here g is given by $g = \pi^* g_W \oplus (\eta \otimes \eta)$, where g_W is a Riemannian metric compatible with ω on W. We call this K-contact manifold a *regular* K-contact manifold and its contact flow a regular K-contact flow. We also call the principal S^1-fibration $(M, \eta, g) \to (W, \omega)$ the *Boothby-Wang fibration* ([4]).

3. A fiber join of regular K-contact manifolds. In this section we will present a method of construction of a K-contact manifold of rank $n+1$ out of $(n+1)$-pieces of regular K-contact manifolds.

For $j = 0, 1, \dots, n$, let (M_j, η_j, g_j) be a $(2m+1)$-dimensional regular K-contact manifold, whose Boothby-Wang fibration $p_j : (M_j, \eta_j, g_j) \to (W, \omega_j)$ has the same base space W. Let L_j be the total space of the associated complex line bundle of $p_j : M_j \to W$. Then L_j carries a Hermitian metric h induced by a canonical Hermitian metric on $\boldsymbol{C}$. We denote the norm on L_j determined by h and its natural lift to the Whitney sum $L_0 \oplus \cdots \oplus L_n$ by the same letter $r_j : L_j \to \boldsymbol{R}$. In this situation we define a *fiber join* $M_0 *_f \cdots *_f M_n$ of $M_0, \dots, M_n$ to be the unit sphere bundle

$$S(L_0 \oplus \cdots \oplus L_n) = \left\{ v \in L_0 \oplus \cdots \oplus L_n \;\middle|\; \sum_{j=0}^{n} r_j(v)^2 = 1 \right\} \tag{3.1}$$

of $L_0 \oplus \cdots \oplus L_n$.

REMARK. In the above construction, we are actually taking the join of the fibers of $M_0, \dots, M_n$ over each point of W. Recall than $n+1$ times join $S^1 * \cdots * S^1 = S^{2n+1}$.

We will show that on $M_0 *_f \cdots *_f M_n$ there exist a K-contact form and its Reeb vector field, which are naturally induced from those of M_j's.

For this, we denote a polar coordinate and a real coordinate of $\boldsymbol{C}$ by $(\bar{r}_j, \theta_j)$ and (x_j, y_j), respectively. We also denote the Reeb vector field of η_j and its natural lift to $M_j \times \boldsymbol{C}$ by the same letter X_j. Similarly, we denote the natural lifts of the differential forms or vector fields on M_j and $\boldsymbol{C}$ (such as η_j) to $M_j \times \boldsymbol{C}$ by the same letter. Let L_j^0 be the complement of the zero section of L_j. Then we have the following:

LEMMA 3.1. (1) *For each j, the one-form $\eta_j - d\theta_j$ on $M_j \times (\boldsymbol{C} - \{0\})$ and the vector field $\tilde{Z}_j := 1/2\{X_j - (x_j\partial/\partial y_j - y_j\partial/\partial x_j)\}$ on $M_j \times \boldsymbol{C}$ are projectable. Namely, there exist a smooth one-form β_j on L_j^0 and a smooth vector field Z_j on L_j such that $pr_j^*(\beta_j) = \eta_j - d\theta_j$ and $(pr_j)_*(\tilde{Z}_j) = Z_j$ hold, where $pr_j : M_j \times \boldsymbol{C} \to L_j$ is the natural projection. Also β_j and Z_j satisfy the following*:

$$\beta_j(Z_j) = 1\,, \quad \beta_j(\partial/\partial r_j) = 0\,, \quad dr_j(Z_j) = 0\,, \quad d\beta_j = \pi_j^*(\omega_j)\,, \tag{3.2}$$

where $\pi_j : L_j \to W$ is the projection.

(2) *For each j, $r_j^2\beta_j$ and $d\beta_j$ extend to the S^1-invariant smooth one-form and two-form on L_j, respectively. The restriction of $2r_j dr_j \wedge \beta_j$ to the fibers of L_j is a nowhere zero two-form.*

(3) *Put*

$$H_j = \{X \in TL_j \mid \iota_X(2r_j dr_j \wedge \beta_j) = 0\}\,, \quad V_j = \{X \in TL_j \mid \iota_X d\beta_j = 0\}\,. \tag{3.3}$$

Then we have a direct sum decomposition $TL_j \cong H_j \oplus V_j$ and $H_j = \pi^*TW$.

PROOF. First we will prove (1) and (2). Let S^1 act in the standard fashion on $\boldsymbol{C}$. We consider the diagonal S^1-action on $M_j \times \boldsymbol{C}$. Then the one-form $\eta_j - d\theta_j$ is invariant by this S^1-action on $M_j \times \boldsymbol{C}$. We also have $(\eta_j - d\theta_j)(X_j + \partial/\partial\theta_j) = 0$, where $\partial/\partial\theta_j := x_j\partial/\partial y_j - y_j\partial/\partial x_j$. Namely, $\eta_j - d\theta_j$ is a basic form. Hence there exists a one-form β_j on L_j^0 such that $pr_j^*(\beta_j) = \eta_j - d\theta_j$. Moreover $r_j^2\beta_j$ is extended to whole L_j as a smooth one-form, since $\bar{r}_j^2 d\theta_j$ is extended to whole $M_j \times \boldsymbol{C}$.

Since we have $L_{(X_j+\partial/\partial\theta_j)}(\tilde{Z}_j) = 0$ on $M \times \boldsymbol{C}$, we see that there exists a smooth vector field Z_j on L_j such that $d(pr_j)(\tilde{Z}_j(x)) = Z_j(pr_j(x))$ for all $x \in M_j \times \boldsymbol{C}$.

Next we verify the equations (3.2). The first three equations are obtained by direct calculations. Namely, $\beta_j(Z_j) = pr_j^*(\beta_j)(\tilde{Z}_j) = (\eta_j - d\theta_j)(\tilde{Z}_j) = 1$, $\beta_j(\partial/\partial r_j) = (\eta_j - d\theta_j)(\partial/\partial\bar{r}_j) = 0$, $dr_j(Z_j) = d\bar{r}_j(\tilde{Z}_j) = 0$. The equations $d\beta_j = \pi_j^*(\omega_j)$ follows from $d\eta_j = \tilde{p}_j^*\omega_j$, where $\tilde{p}_j : M_j \times \boldsymbol{C} \to W$.

By using the equations (3.2) and the Cartan formula $L_X = \iota_X d + d\iota_X$, we get $L_{Z_j}(r_j^2\beta_j) = 0$. Namely, $r_j^2\beta_j$ is a one-form on L_j which is invariant by the S^1-action determined by Z_j.

The two-form $pr_j^*(2r_j dr_j \wedge \beta_j) = 2\bar{r}_j d\bar{r}_j \wedge \eta_j - 2\bar{r}_j d\bar{r}_j \wedge d\theta_j$ is nowhere zero on the fibers of $M_j \times \boldsymbol{C} \to W$. Hence $2r_j dr_j \wedge \beta_j$ is also nowhere zero on the fibers of $L_j \to W$.

We will prove (3). Let $\tilde{H}_j$, $\tilde{V}_j$ be subbundles of $T(M_j \times \boldsymbol{C})$ defined by

$$\begin{cases} \tilde{H}_j := \{X \in T(M_j \times \boldsymbol{C}) \mid \eta_j(X) = 0, \iota_X(2\bar{r}_j d\bar{r}_j \wedge d\theta_j) = 0\}\,, \\ \tilde{V}_j := \{X \in T(M_j \times \boldsymbol{C}) \mid \iota_X d\eta_j = 0\}\,. \end{cases}$$

Then we have the direct sum decomposition $T(M_j \times \boldsymbol{C}) \cong \tilde{H}_j \oplus \tilde{V}_j$. By using equations $pr_j^*(2r_j dr_j \wedge \beta_j) = 2\bar{r}_j d\bar{r}_j \wedge \eta_j - 2\bar{r}_j d\bar{r}_j \wedge d\theta_j$ and $d\eta_j = pr_j^* d\beta_j$, this direct sum decomposition gives rise to the direct sum decomposition $TL_j \cong H_j \oplus V_j$. Here $H_j = \{X \in TL_j \mid \iota_X(2r_j dr_j \wedge \beta_j) = 0\}$, $V_j = \{X \in TL_j \mid \iota_X d\beta_j = 0\}$. q.e.d.

We extend each vector field Z_j on L_j to the one on $M_0 *_f \cdots *_f M_n$ as follows, and denote it by the same letter. By using the canonical projection $P_j : L_0 \times \cdots \times L_n \to L_j$ and the inclusion map $I : TL_j \to TL_0 \times \cdots \times TL_n$ defined by $I(w_j) = (0, \ldots, w_j, \ldots, 0)$ for $w_j \in TL_j$, we have $I \circ Z_j \circ P_j : L_0 \times \cdots \times L_n \to TL_0 \times \cdots \times TL_n$. Namely, the vector field Z_j is extended to the one on $L_0 \times \cdots \times L_n$. It is a vector field along the fiber of $L_0 \times \cdots \times L_n$ and preserves the norm $\sum_{j=0}^n r_j^2$. It follows that its restriction to $M_0 *_f \cdots *_f M_n$ is tangent to $M_0 *_f \cdots *_f M_n$, and hence Z_j is extended to the vector field on $M_0 *_f \cdots *_f M_n$.

We consider the pull back of the one-form $r_j^2 \beta_j$ on L_j by the composition map $M_0 *_f \cdots *_f M_n \to L_0 \oplus \cdots \oplus L_n \to L_j$, and denote it by the same letter.

THEOREM 3.2. *For $j = 0, 1, \ldots, n$, let (M_j, η_j, g_j) be a $(2m+1)$-dimensional regular K-contact manifold with the Boothby-Wang fibration $(M_j, \eta_j, g_j) \to (W, \omega_j)$. Let $\pi : M_0 *_f \cdots *_f M_n \to W$ be the projection. If $\sum_{j=0}^n \lambda_j r_j^2 \pi^* \omega_j$ is non-degenerate on $\pi^* TW$ for some non-zero constants $\lambda_0, \ldots, \lambda_n$, then we have*

(1) *the fiber join $M_0 *_f \cdots *_f M_n$ of $M_0, \ldots, M_n$ is a $(2m+2n+1)$-dimensional K-contact manifold with the K-contact form*

$$\beta_\lambda := \sum_{j=0}^n \lambda_j r_j^2 \beta_j \,. \tag{3.4}$$

Its Reeb vector field and a K-contact metric are given by

$$Z_\lambda := \sum_{j=0}^n 1/\lambda_j Z_j \,, \quad g_\lambda = g_T \oplus (\beta_\lambda \otimes \beta_\lambda) \,, \tag{3.5}$$

where g_T is a positive definite metric on $\ker \beta_\lambda$.

(2) *If we choose $\{\lambda_0, \ldots, \lambda_n\}$ so that $\lambda_0, \ldots, \lambda_n$ form a k-dimensional vector space over $\boldsymbol{Q}$, $(M_0 *_f \cdots *_f M_n, \beta_\lambda, g_\lambda)$ is a K-contact manifold of rank k. In particular, if $\lambda_0, \ldots, \lambda_n$ are rationally independent, then $(M_0 *_f \cdots *_f M_n, \beta_\lambda, g_\lambda)$ is a K-contact manifold of rank $n+1$.*

PROOF. First we prove that β_λ is a contact form on $M_0 *_f \cdots *_f M_n$. We put $R^2 = \sum_{j=0}^n r_j^2$. Since $d\beta_j = \pi^* \omega_j$, by a direct calculation, we have

$$2RdR \wedge \beta_\lambda \wedge (d\beta_\lambda)^{m+n} = \lambda_0 \cdots \lambda_n R^2 \cdot 2r_0 dr_0 \wedge \beta_0 \wedge \cdots \wedge 2r_n dr_n \wedge \beta_n \wedge \left(\sum_{j=0}^n \lambda_j r_j^2 \pi^* \omega_j \right)^m$$

on $L_0 \oplus \cdots \oplus L_n$.

By the assumption, $\lambda_0 \cdots \lambda_n \sum_{j=0}^n \lambda_j r_j^2 \pi^* \omega_j$ is non-degenerate on $\pi^* TW$ and clearly $R^2 \neq 0$ on $L^0 := L_0 \oplus \cdots \oplus L_n - \{\text{zero-section}\}$. From this together with Lemma 3.1, we see that $2RdR \wedge \beta_\lambda \wedge (d\beta_\lambda)^{m+n} \neq 0$ on L^0. It follows that we have $\beta_\lambda \wedge (d\beta_\lambda)^{m+n} \neq 0$ on $M_0 *_f \cdots *_f M_n$, that is, β_λ is a contact form on $M_0 *_f \cdots *_f M_n$. Its Reeb vector field is given by $Z_\lambda = \sum_{j=0}^n 1/\lambda_j Z_j$. This is because it holds that $\beta_\lambda(Z_\lambda) = R^2 = 1$ and $\iota_{Z_\lambda} d\beta_\lambda = \sum_{j=0}^n 2r_j dr_j = 0$ on $M_0 *_f \cdots *_f M_n$.

We will show the K-contactness of $(M *_f \cdots *_f M_n, \beta_\lambda)$. Using the one-parameter group ϕ_t^j of Z_j, we define a T^{n+1}-action on $M_0 *_f \cdots *_f M_n$ by

$$(e^{\sqrt{-1}t_0}, \dots, e^{\sqrt{-1}t_n}) \cdot (v_0, \dots, v_n) = (\phi_{t_0}^0 v_0, \dots, \phi_{t_n}^n v_n), \tag{3.6}$$

where $(e^{\sqrt{-1}t_0}, \dots, e^{\sqrt{-1}t_n}) \in T^{n+1}$ and $v = (v_0, \dots, v_n) \in M_0 *_f \cdots *_f M_n \subset L_0 \oplus \cdots \oplus L_n$. Let ψ_t^λ be the contact flow of Z_λ. Then we have

$$\psi_t \cdot v = (\phi_{(1/\lambda_0)t}^0 v_0, \dots, \phi_{(1/\lambda_n)t}^n v_n). \tag{3.7}$$

Constants $\lambda_0, \dots, \lambda_n$ form a k-dimensional vector space over $\boldsymbol{Q}$. Thus there exists a subgroup T^k of T^{n+1} and a T^k-action induce by (3.6) such that, for any $v = (v_0, \dots, v_n) \in M_0 *_f \cdots *_f M_n$, the closure of the orbit $\psi_t^\lambda \cdot v$ coincides with the orbit $T^k \cdot v$. Hence, by Proposition 2.1, there exists an adapted metric g_λ to α_λ such that $(M_0 *_f \cdots *_f M_n, \beta_\lambda, g_\lambda)$ is a K-contact manifold of rank k. Here g_λ is given by choosing g_T on $\ker \beta_\lambda$ and setting $g_\lambda = g_T \oplus (\beta_\lambda \otimes \beta_\lambda)$. q.e.d.

Indeed, there exist sympletic forms ω_j, $j = 0, 1, \dots, n$, satisfying the condition of Theorem 3.2 that $\sum_{j=0}^n \lambda_j r_j^2 \pi^* \omega_j$ is non-degenerate on $\pi^* TW$. For example, let (W, ω) be a symplectic manifold and λ_j, c_j, $j = 0, 1, \dots, n$, be constants such that $\lambda_j c_j$ is positive for all j. Then taking ω_j, $j = 0, 1, \dots, n$, defined by $\omega_j = c_j \omega$, these satisfy the condition above.

REMARK. (1) As an example of g_T on $\ker \beta_\lambda$, we have the restiction of

$$2 \sum_{j=0}^n \lambda_j (dr_j \otimes dr_j + r_j^2 \beta_j \otimes \beta_j) + \sum_{j=0}^n \lambda_j r_j^2 \pi^* g_{W,\omega_j} \tag{3.8}$$

to $\ker \beta_\lambda$. Here g_{W,ω_j} is a Riemannian metric compatible with ω_j on W.

(2) For positive integers $q_1, \dots, q_n$ and $e^{\sqrt{-1}\theta} \in S^1 \subset \boldsymbol{C}^*$, we define the S^1-action on $M_0 *_f \cdots *_f M_n$ by

$$e^{\sqrt{-1}\theta} \cdot (v_0, v_1, \dots, v_n) = (\phi_\theta^0 v_0, \phi_{q_1\theta}^1 v_1, \dots, \phi_{q_n\theta}^n v_n), \tag{3.9}$$

where $(v_0, \dots, v_n) \in M_0 *_f \cdots *_f M_n$.

Let p be a positive integer such that p and q_j are relatively prime for all j. We consider the action restricted to $\{e^{2\pi k\sqrt{-1}/p} \mid k = 0, 1, \dots, p-1\} \cong \boldsymbol{Z}/p\boldsymbol{Z}$ of the S^1-action defined by (3.9). Then its quotient space $M_0 *_f \cdots *_f M_n/(\boldsymbol{Z}/p\boldsymbol{Z})$ is also a K-contact manifold of rank $n+1$ with K-contact form and K-contact metric induced from those on $M_0 *_f \cdots *_f M_n$.

(3) The unit sphere bundle $S(L_j)$ of L_j is a submanifold of $M_0 *_f \cdots *_f M_n$, which is diffeomorphic to M_j. As a metric g_T on $\ker \beta_\lambda$, take the one given by (3.8). Then $S(L_j)$ has a K-contact form $\lambda_j \beta_j$ and a K-contact metric $\lambda_j g_j$ which are given by the restriction of those on $M_0 *_f \cdots *_f M_n$ to $S(L_j)$. In this case $(S(L_j), \lambda_j \beta_j, \lambda_j g_j)$ is called a *K-contact submanifold* of $(M_0 *_f \cdots *_f M_n, \beta_\lambda, g_\lambda)$.

DEFINITION. A K-contact manifold $(M_0 *_f \cdots *_f M_n, \beta_\lambda, g_\lambda)$ is called the *fiber join of regular K-contact manifolds* $(M_0, \eta_0, g_0), \dots, (M_n, \eta_n, g_n)$.

Using the construction in Theorem 3.2, we obtain K-contact manifolds (M, α, g) of rank $n+1$ with no effective T^{n+2}-action which extends the $\boldsymbol{R}$-action induced by the K-contact flow and preserves α and g.

PROPOSITION 3.3. *Let (W, ω) be a symplectic manifold with no effective Hamiltonian S^1-action. Take symplectic forms $\omega_j = c_j\omega$ $(c_j > 0)$ in the construction of Theorem* 3.2. *Then $(M_0 *_f \cdots *_f M_n, \beta_\lambda, g_\lambda)$ has no effective T^{n+2} action which extends the T^{n+1}-action defined by* (3.6) *and preserves β_λ and g_λ.*

REMARK. An example of symplectic manifold which satisfies the condition above is negatively curved closed Kähler manifold. It has no torus action at all ([11]). It follows that starting from this manifold, we can actually construct K-contact manifolds as in Proposition 3.3.

PROOF. Suppose that β_λ is invariant under some effective T^{n+2}-action on $M_0 *_f \cdots *_f M_n$ which extends the T^{n+1}-action defined by (3.6). Let $(M_0 *_f \cdots *_f M_n \times \boldsymbol{R}_+, d(t\beta_\lambda))$ be the symplectization of $(M_0 *_f \cdots *_f M_n, \beta_\lambda)$, where $\boldsymbol{R}_+$ is the positive real line with coordinate t. We extend the T^{n+2}-action on $M_0 *_f \cdots *_f M_n$ to the one on $M_0 *_f \cdots *_f M_n \times \boldsymbol{R}_+$ such that it acts trivially on $\boldsymbol{R}_+$. Then this T^{n+2}-action is Hamiltonian. Its moment map μ is given by

$$\mu : M_0 *_f \cdots *_f M_n \times \boldsymbol{R}_+ \ni x \longmapsto -t(\beta_{\lambda x}(Z_{0x}), \dots , \beta_{\lambda x}(Z_{nx}), \beta_{\lambda x}(Y_x)) \in \boldsymbol{R}^{n+2},$$

where Y is the fundamental vector field determined by the action of the last factor S^1 of $T^{n+2} = T^{n+1} \times S^1$. Since μ is constant on any T^{n+2}-orbit ([1, Proposition 3.5.6]), the composition map $\bar{\mu} := pr \circ \mu : M_0 *_f \cdots *_f M_n \times \boldsymbol{R}_+ \to \boldsymbol{R}^{n+1}$ is also constant on it. Here pr is the projection to the first $n+1$ factor. Thus vector fields $Z_0, \dots , Z_n, Y$ are tangent to any regular level $\bar{\mu}^{-1}(\xi)$ of $\bar{\mu}$, and hence $\bar{\mu}^{-1}(\xi)$ has an effective T^{n+2}-action. Choosing $\xi = -(\lambda_0, \dots , \lambda_n)$ as a regular value of $\bar{\mu}$, $\bar{\mu}^{-1}(\xi)$ is a principal T^{n+1}-bundle over W with an effective T^{n+2}-action. It follows that the orbit space $\bar{\mu}^{-1}(\xi)/T^{n+1}$ is diffeomorphic to W and that it is a symplectic manifold $\left(W, \sum_{j=0}^n \lambda_j\omega_j\right)$ with an effective Hamiltonian S^1-action. From $\sum_{j=0}^n \lambda_j\omega_j = \left(\sum_{j=0}^n \lambda_j c_j\right)\omega$, we see that the symplectic manifold (W, ω) also has an effective Hamiltonian S^1-action. This contradicts the assumption. q.e.d.

4. The equivalence of K-contact manifolds. In this section we will study the following two equivalence classes among the K-contact flows of the compact connected K-contact manifolds of rank k. Let (M_1, α_1, g_1), (M_2, α_2, g_2) be two such manifolds with Reeb vector fields Z_1, Z_2. Let $\varphi_t^{(1)}, \varphi_t^{(2)}$ denote their K-contact flows, respectively.

DEFINITION. (a) Two K-contact flows $\varphi_t^{(1)}, \varphi_t^{(2)}$ are said to be *strictly equivalent* if there exists a diffeomorphism $\Phi : M_1 \to M_2$ such that $\Phi^*\alpha_2 = c\alpha_1$ for some positive constant c. (b) Two K-contact flows $\varphi_t^{(1)}, \varphi_t^{(2)}$ are said to be *K-contact equivalent* if there exists a T^k-equivalent *contact diffeomorphism* Φ between (M_1, α_1, g_1) and (M_2, α_2, g_2). Here a contact diffeomorphism implies that $\Phi^*\alpha_2 = f\alpha_1$ for some everywhere nonzero function f on M_1.

If $\varphi_t^{(1)}$, $\varphi_t^{(2)}$ are strictly equivalent, we have $d\Phi \circ Z_1(x) = cZ_2 \circ \Phi(x)$, and hence $\Phi \circ \varphi_t^{(1)}(x) = \varphi_{ct}^{(2)} \circ \Phi(x)$ for all x in M. Namely, after changing the parameter t of $\varphi_t^{(2)}$ into ct, there exists an $\boldsymbol{R}$-equivariant diffeomorphism Φ on M with respect to $\boldsymbol{R}$-actions induced by $\varphi_t^{(1)}$, $\varphi_{ct}^{(2)}$. From the definition, it is obvious if two K-contact flows are strictly equivalent, they are K-contact equivalent. The following two propositions show that the coverse is not always true.

PROPOSITION 4.1. *For any rationally independent read constants $\lambda = (\lambda_0, \dots, \lambda_n)$ and $\tilde{\lambda} = (\tilde{\lambda}_0, \dots, \tilde{\lambda}_n)$, the K-contact flows φ_t^λ, $\varphi_t^{\tilde{\lambda}}$ defined by* (2.5) *on S^{2n+1} are K-contact equivalent. Moreover, they are strictly equivalent if and only if λ coincides with $c\tilde{\lambda}$ as a set for some positive constant c.*

PROOF. First we will prove that φ_t^λ, $\varphi_t^{\tilde{\lambda}}$ are K-contact equivalent for any λ, $\tilde{\lambda}$. Consider a diffeomorphism $\Phi_{\lambda/\tilde{\lambda}} : S^{2n+1} \to S^{2n+1}$ defined by

$$\tag{4.1} \begin{aligned} \Phi_{\lambda/\tilde{\lambda}}(z_0, \dots, z_n) = \Bigg(& \left((\lambda_0/\tilde{\lambda}_0)^{1/2} \Big/ \left(\sum_{j=0}^{n} (\lambda_j/\tilde{\lambda}_j) z_j \bar{z}_j \right)^{1/2} \right) z_0, \dots, \\ & \left((\lambda_n/\tilde{\lambda}_n)^{1/2} \Big/ \left(\sum_{j=0}^{n} (\lambda_j/\tilde{\lambda}_j) z_j \bar{z}_j \right)^{1/2} \right) z_n \Bigg) . \end{aligned}$$

Then $\Phi_{\lambda/\tilde{\lambda}}$ is a T^{n+1}-equivariant diffeomorphism and $\Phi_{\lambda/\tilde{\lambda}}^* \alpha_{\tilde{\lambda}} = \left(\sum_{j=0}^n (\lambda_j/\tilde{\lambda}_j) z_j \bar{z}_j \right)^{-1} \alpha_\lambda$. Hence φ_t^λ, $\varphi_t^{\tilde{\lambda}}$ are K-contact equivariant.

We will prove the second statement. Assume that φ_t^λ and $\varphi_t^{\tilde{\lambda}}$ are strictly equivalent. Namely, there exists a diffeomorphism Φ such that $\Phi^* \alpha_\lambda = c\alpha_{\tilde{\lambda}}$ for some positive constant c. Then Φ is a $\boldsymbol{R}$-equivariant diffeomorphism with respect to $\boldsymbol{R}$-actions induced by φ_{ct}^λ and $\varphi_t^{\tilde{\lambda}}$. It follows that the set of isotropy groups $(\lambda_0/c)\boldsymbol{Z}, \dots, (\lambda_n/c)\boldsymbol{Z}$ of φ_{ct}^λ coincides with that of $\tilde{\lambda}_0\boldsymbol{Z}, \dots, \tilde{\lambda}_n\boldsymbol{Z}$ of $\varphi_t^{\tilde{\lambda}}$, where $\mu\boldsymbol{Z} = \{2\pi\mu k \mid k \in \boldsymbol{Z}\}$. Hence λ coincides with $c\tilde{\lambda}$ as a set. Conversely, assume that λ coincides with $c\tilde{\lambda}$ as a set for some positive constant c; $(\lambda_0, \dots, \lambda_n) = c(\tilde{\lambda}_{\sigma(0)}, \dots, \tilde{\lambda}_{\sigma(n)})$, where σ denote a permutation of $\{0, 1, \dots, n\}$. Consider a diffeomorphism $\Phi : S^{2n+1} \to S^{2n+1}$ defined by $\Phi(z_0, \dots, z_n) = (z_{\sigma(0)}, \dots, z_{\sigma(n)})$. Then we have $\Phi^* \alpha_\lambda = c\alpha_{\tilde{\lambda}}$. Hence φ_t^λ, $\varphi_t^{\tilde{\lambda}}$ are strictly equivalent. q.e.d.

A similar result holds for the contact flows of (3.7) in Section 3.

PROPOSITION 4.2. *For any rationally independent read constants $\lambda = (\lambda_0, \dots, \lambda_n)$ and $\tilde{\lambda} = (\tilde{\lambda}_0, \dots, \tilde{\lambda}_n)$, the K-contact flows ψ_t^λ, $\psi_t^{\tilde{\lambda}}$ defined by* (3.7) *on $M_0 *_f \cdots *_f M_n$ are K-contact equivalent. Moreover, they are strictly equivalent if and only if λ coincides with $c\tilde{\lambda}$ as a set for some positive constant c.*

PROOF. We only show that ψ_t^λ, $\psi_t^{\tilde{\lambda}}$ are K-contact equivalent for any λ, $\tilde{\lambda}$. (The second statement is proved by an argument similar to that in Proposition 4.1.)

We define the bundle automorphism $\Psi_{\lambda/\tilde\lambda}$ of $L_0 \oplus \cdots \oplus L_n$ by

$$\begin{aligned}\Psi_{\lambda/\tilde\lambda}(v_0, \dots, v_n) = \Bigg(&\left((\lambda_0/\tilde\lambda_0)^{1/2} \Big/ \left(\sum_{j=0}^{n}(\lambda_j/\tilde\lambda_j) r_j(v_j)^2\right)^{1/2}\right) v_0, \dots, \\ &\left((\lambda_n/\tilde\lambda_j)^{1/2} \Big/ \left(\sum_{j=0}^{n}(\lambda_j/\tilde\lambda_j) r_j(v_j)^2\right)^{1/2}\right) v_n\Bigg)\end{aligned} \tag{4.2}$$

for $(v_0, \dots, v_n) \in L_0 \oplus \cdots \oplus L_n$. Then this preserves the norm $\sum_{j=0}^{n} r_j^2$. Thus we have its restriction to $M_0 *_f \cdots *_f M_n$. It is a T^{n+1}-equivariant diffeomorphism and $\Psi^*_{\lambda/\tilde\lambda}\beta_{\tilde\lambda} = \left(\sum_{j=0}^{n}(\lambda_j/\tilde\lambda_j) r_j^2\right)^{-1}\beta_\lambda$. Therefore ψ_t^λ and $\psi_t^{\tilde\lambda}$ are K-contact equivalent. q.e.d.

In [13], Takahashi showed that there exists a deformation of the K-contact flow on a manifold as follows. Let (M, α, g) be a K-contact manifold with Reeb vector field Z. Let V be a vector field on M which satisfies the following three conditions:

$$L_V g = 0\,, \quad [V, Z] = 0\,, \quad 1 + \alpha(V) > 0\,. \tag{4.3}$$

Consider a one-form $\tilde\alpha$ and a Riemannian metric $\tilde g$ defined by

$$\tilde\alpha = (1 + \alpha(V))^{-1}\alpha\,, \quad \tilde g = (1 + \alpha(V))^{-1} g_T \oplus (\tilde\alpha \otimes \tilde\alpha)\,, \tag{4.4}$$

where g_T is the restriction of g to $\ker\alpha$. Then we have following:

THEOREM 4.3 ([13]). *$(M, \tilde\alpha, \tilde g)$ is a K-contact manifold with Reeb vector field $Z + V$.*

K-contact flows φ_t^λ, ψ_t^λ in Propositions 4.1 and 4.2 are both strictly equivalent to the ones obtained by the above deformation out of the K-contact flow of the K-contact manifold of rank 1, which we shall see as follows.

The K-contact flow φ_t^λ of $(S^{2n+1}, \alpha_\lambda, g_\lambda)$ is strictly equivalent to the one obtained by deforming the Reeb vector field $Z_\varepsilon = \sqrt{-1}\sum_{j=0}^{n}(z_j\partial/\partial z_j - \bar z_j\partial/\partial \bar z_j)$ of $(S^{2n+1}, \alpha_\varepsilon, g_\varepsilon)$, where $\varepsilon = (1, \dots, 1)$. Indeed, for μ_j satisfying $1 + \mu_j = \lambda_j$, take $V = \sqrt{-1}\sum_{j=0}^{n}\mu_j(z_j\partial/\partial z_j - \bar z_j\partial/\partial \bar z_j)$ and consider $\tilde\alpha_\varepsilon$ and $\tilde g_\varepsilon$ defined by (4.3). Then we have $\tilde\alpha_\varepsilon = \left(1 + \sum_{j=0}^{n}\mu_j z_j \bar z_j\right)^{-1}\alpha_\varepsilon$ and $\Phi_\lambda^*\tilde\alpha_\varepsilon = \alpha_\lambda$, where Φ_λ is a diffeomorphism defined by (4.1).

In the same way, the K-contact flow ψ_t^λ of $(M_0 *_f \cdots *_f M_n, \beta_\lambda, g_\lambda)$ is strictly equivalent to the one obtained by deforming the Reeb vector field $Z_\varepsilon = \sum_{j=0}^{n} Z_j$ of $(M_0 *_f \cdots *_f M_n, \beta_\varepsilon, g_\varepsilon)$, where $\varepsilon = (1, \dots, 1)$. In this case we take $V = \sum_{j=0}^{n}\mu_j Z_j$, where μ_j is the same as the above one.

In general, we apply the deformation in Theorem 4.3 for the following situation. Let (M, α, g) be a K-contact manifold of rank 1. We assume that there exists the T^k-action preserving α and g which satisfies the following three conditions; (1) $k \geq 2$, (2) T^k contains the K-contact flow of (M, α, g), and (3) there is no T^{k+1}-action which extends this T^k-action. Then the Reeb vector field Z takes the form $Z = \sum_{j=0}^{k-1}(\xi_M)_j$, where $(\xi_M)_j$ is the vector field defined by $(\xi_M)_j(x) = d/dt\left.\right|_{t=0}\exp(t\xi_j)\cdot x$ at $x \in M$ for a basis $\xi_0, \dots, \xi_{k-1}$ of $\mathrm{Lie}(T^k)$.

Let $\lambda_0, \dots, \lambda_{k-1}$ be positive constants such that $\lambda_0, \dots, \lambda_{k-1}$ form a r-dimensional vector space over $\boldsymbol{Q}$, where $1 \leq r \leq k$. We take a vector field $V = \sum_{j=0}^{k-1} \lambda_j (\xi_M)_j$ and consider $\tilde{\alpha}$ and $\tilde{g}$ defined by (4.3). Then we have the following:

COROLLARY 4.4. *$(M, \tilde{\alpha}, \tilde{g})$ is a K-contact manifold of rank r with Reeb vector field $Z + V$ and is T^k-equivariantly contact diffeomorphic to (M, α, g).*

PROOF. The identity map gives a T^k-equivariant diffeomorphism between (M, α, g) and $(M, \tilde{\alpha}, \tilde{g})$. q.e.d.

We will show that there exist K-contact flows which are not K-contact equivalent. They are not obtained by the deformation in Corollary 4.4 out of the same K-contact flow of the K-contact manifold of rank 1.

Let Σ_g be the closed Riemann surface of genus g and $\Sigma_g \tilde{\times} S^{2n+1}$ be the non-trivial S^{2n+1}-bundle over Σ_g. Then our main theorem is the following:

THEOREM 4.5. *For $n \geq 1$ there exist infinitely many different K-contact equivalence classes of K-contact flows on $\Sigma_g \times S^{2n+1}$ and $\Sigma_g \tilde{\times} S^{2n+1}$.*

We will first consider the K-contact equivalence for K-contact flows of K-contact manifolds of rank $n+1$ we constructed in Theorem 3.2.

Let $(M_0, \eta_0, g_0), \dots, (M_0, \eta_n, g_n)$ and $(\tilde{M}_0, \tilde{\eta}_0, \tilde{g}_0), \dots, (\tilde{M}_n, \tilde{\eta}_n, \tilde{g})$ be two sets of regular K-contact manifolds whose Boothby-Wang fibrations have the same base space. Then the images $S(L_0), \dots, S(L_n)$ of $M_0, \dots, M_n$ in $M_0 *_f \cdots *_f M_n$ and the images $S(\tilde{L}_0), \dots, S(\tilde{L}_n)$ of $\tilde{M}_0, \dots, \tilde{M}_n$ in $\tilde{M}_0 *_f \cdots *_f \tilde{M}_n$ are two sets of points whose isotropy groups are isomorphic to T^n (see Remark (3) of Theorem 3.2). Hence if there exists a T^{n+1}-equivariant diffeomorphism Φ between $M_0 *_f \cdots *_f M_n$ and $\tilde{M}_0 *_f \cdots *_f \tilde{M}_n$, $S(L_0), \dots, S(L_n)$ are mapped to $S(\tilde{L}_j), \dots, S(\tilde{L}_n)$ by Φ such that (changing the order of suffix, if necessary), $S(L_j)$ is T^{n+1}-equivariantly diffeomorphic to $S(\tilde{L}_j)$ for all j. Thus M_j is S^1-equivariantly diffeomorphic to $\tilde{M}_j$ for all j. From the definition, it is obvious that K-contact flows on regular K-contact manifolds are K-contact equivalent if and only if they are isomorphic to each other as principal S^1-bundles. Therefore we have the following:

LEMMA 4.6. *If K-contact flows of $(M_0 *_f \cdots *_f M_n, \beta_\lambda, g_\lambda)$ and $(\tilde{M}_0 *_f \cdots *_f \tilde{M}_n, \tilde{\beta}_\lambda, \tilde{g}_\lambda)$ are K-contact equivalent, then K-contact flows of (M_j, η_j, g_j) and $(\tilde{M}_j, \tilde{\eta}_j, \tilde{g}_j)$ are K-contact equivalent for all j (changing the order of suffix, if necessary).*

Let $(M_0, \eta_0, g_0), \dots, (M_n, \eta_n, g_n)$, $(n \geq 1)$, be three-dimensional regular K-contact manifolds, whose Boothby-Wang fibration have the same closed Riemann surface Σ_g of genus g as base spaces. Then there are only two diffeomorphism classes of $M_0 *_f \cdots *_f M_n$, because $M_0 *_f \cdots *_f M_n$ is the S^{2n+1}-bundle over Σ_g and they are classified by the second Stiefel-Whitney class of the bundle (see [8], Proposition 1.12). More precisely, we have the following:

PROPOSITION 4.7. *Let $M_0, \dots, M_n$ be as above. Then the fiber join $M_0 *_f \cdots *_f M_n$ is diffeomorphic to $\Sigma_g \times S^{2n+1}$ if $\sum_{j=0}^n w_j$ is even class, and to $\Sigma_g \tilde{\times} S^{2n+1}$* if *$\sum_{j=0}^n w_j$ is* *odd class. Here* $\Sigma_g \tilde{\times} S^{2n+1}$ *is the non-trivial* S^{2n+1}*-bundle over* Σ_g *and* w_j *is the second Stiefel-Whitney class associated with* M_j.

PROOF OF THEOREM 4.5. Since $H^2(\Sigma_g; \mathbf{Z}) \cong \mathbf{Z}$, there exist infinitely many different isomorphism classes of principal S^1-bundles over Σ_g. From this result together with Lemma 4.6 and Proposition 4.7, we obtain Theorem 4.5. q.e.d.

Finally, we discuss some related problems.

Let φ_t be a non-singular flow generated by a vector field Z on a manifold M. Let $\boldsymbol{R}Z$ be the trivial line bundle spanned by Z and D the smooth codimension one distribution on M transverse to $\boldsymbol{R}Z$. Then φ_t is said to be *transversely symplectic Riemannian* flow if there exist a symplectic structure ω and a positive definite metric g_T on D such that $L_Z\omega = 0$, $L_Z g_T = 0$. From the definition, it is obvious that a K-contact flow of a K-contact manifold (M, α, g) is such a flow. In this case, D is a contact plane field $\ker\alpha$ and a symplectic structure on it is given by $d\alpha$. In [10], Molino suggested the following problem:

PROBLEM 1. Classify the transversely symplectic Reimannian flows on closed connected 5-manifolds.

The case of $n = 1$ in Theorem 4.5 gives examples of such flows. Further examples are given by introducing a surgery along a closed K-contact flow in [16].

We have the following problems related to Theorem 4.4.

PROBLEM 2. Are there different K-contact flows on a sphere bundle over the symplectic manifold W such that $\dim W \geq 4$?

The author does not know whether there exists a symplectic manifold W such that the isomorphism classes of the sphere bundle over W are finite and $\dim W \geq 4$.

PROBLEM 3. Are there K-contact flows of K-contact manifolds of rank $n+1$ on a $(2n+1)$-dimensional manifold which are not K-contact equivalent to each other?

By the fiber join of regular K-contact manifolds, it is impossible to construct the $(2n+1)$-dimensional K-contact manifold of rank $n+1$.

REFERENCES

[1] M. AUDIN, The Topology of Torus Action on Symplectic Manifolds, Progr. Math. 93, Birkhäuser, Basel, 1991.

[2] A. BANYAGA AND P. MOLINO, Geometric des formes de contact completement integrables de type torique, preprint.

[3] D. BLAIR, Contact Manifolds in Riemannian Geometry, Lecture Notes in Math. 509, Springer-Verlag, Berlin, 1976.

[4] W. BOOTHBY AND H. C. WANG, On contact manifolds, Ann. of Math. 68 (1958), 721–734.

[5] Y. CARRIERE, Variations on riemannian flows, Progr. Math. 73, Birkhäuser, Basel, (1988), 217–234.

[6] S. KOBAYASHI, Principal fiber bundles with 1-dimensional toroidal group, Tohoku Math. J. 8 (1956), 29–45.

[7] S. KOBAYASHI, Transformation Groups in Differential Geometry, Ergeb. Math. Grenzgeb. Band 70, Springer-Verlag, Berline, Heidelberg, New York, 1970.

[8] B. LAWSON, JR. AND M. MICHELSOHN, Spin Geometry, Princeton Math. Ser. 38, Princeton Univ. Press, Princeton, 1989.

[9] P. MOLINO, Reduction symplectique et feuiletages riemanniens; moment structural et theoreme de convexite, in Seminaire Gaston Darboux de geometrie et topologie differentielle 1987–1988, Universite des sciences et techniques du Languedoc, 1988, 11–25.

[10] P. MOLINO, Orbit-like foliation, Proceeding of Geometric study of foliation held in Tokyo 1993 (Tadayoshi Mizutani et al. eds.), World Scientific, Singapore, 1994, 97–119.

[11] P. MOLINO, Riemannian Foliations, Progr. Math. 73, Birkhäuser, Basel, 1988.

[12] K. ONO, Some remarks on group actions in symplectic geometry, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 35 (1988), 431–437.

[13] T. TAKAHASHI, Deformations of sasakian structures and its application to the Brieskorn manifolds, Tohoku Math. J. 30 (1978), 37–43.

[14] P. RUKIMBIRA, The dimension of leaf closures of K-contact flows, Ann. Global Anal. Geom. 12 (1994), 103–108.

[15] P. RUKIMBIRA, Vertical sectional curvature and K-contactness, J. Geom. 53 (1995), 163–166.

[16] T. YAMAZAKI, On a surgery of K-contact manifolds, preprint.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
SAITAMA UNIVERSITY
SHIMO-OKUBO 255
URAWA, SAITAMA 338–8570
JAPAN

E-mail address: tyamazak@rimath.saitama-u.ac.jp