Tohoku Math. J.
52 (2000), 271–282

(1, 2)-SYMPLECTIC STRUCTURES ON FLAG MANIFOLDS

XIAOHUAN MO* AND CAIO J. C. NEGREIROS

(Received October 9, 1998, revised June 14, 1999)

Abstract. By using moving frames and directred digraphs, we study invariant (1,2)-symplectic structures on complex flag manifolds. Let F be a flag manifold with height $k-1$. We show that there is a k-dimensional family of invariant (1,2)-symplectic metrics of any parabolic structure on F. We also prove any invariant almost complex structure J on F with height 4 admits an invariant (1,2)-symplectic metric if and only if J is parabolic or integrable.

1. Introduction. Eells and Sampson proved in [9] that any holomorphic map between Kähler manifolds is harmonic and this was later generalized by Lichnerowicz as follows ([10]): Let $\phi : (M, g, J) \to (N, h, J^N)$ be a holomorphic map from a cosymplectic manifold to a (1,2) symplectic one. Then ϕ is harmonic. Note that a (1,2)-symplectic manifold is automatically cosymplectic. From this point of view, it is important for us to study the (1,2)-symplectic structures on almost Hermitian manifolds.

In this paper, we discuss the existence and explicit construction of (1,2)-symplectic structures on complex flag manifolds. By a result of Borel and Hirzebruch, there are $2^{\binom{N}{2}}$ invariant almost complex structures on the complex flag manifold $F = U(N)/U(n_1) \times \cdots \times U(n_k)$ ([3]). Here the number $k-1$ is called the height of F ([6]). Inspired by the existence of J_2-holomorphic maps from topological spheres to flag manifolds ([5]), we first concern ourselves with the parabolic structures on F. Notice that when $k = 3$, Eells and Salamon showed that any parabolic structure on F admits a (1,2)-symplectic metric ([8]). Also a result of Borel asserts that there is a $(k-1)$-dimensional family of invariant Kähler metrics for each invariant complex structure on F ([1,2]). In this paper, we show not only the existence but also an explicit construction of k-dimensional family of invariant (1,2)-symplectic metrics for each parabolic structure on F with height $k-1$.

Since a Kähler metric is automatically (1,2)-symplectic, a basic fact is that when $k = 3$, each invariant almost complex structure on F admits a (1,2)-symplectic metric. The second objective of this article is to show that there are two classes of almost complex structures (they have been described by using directed digraphs), and these structures do not admit an arbitrary invariant (1,2)-symplectic metric. In particular, we prove that when $k = 4$, an invariant almost complex structure J on F admits a (1,2)-symplectic metric if and only if J is integrable or parabolic. However, for $k = 5$, the corresponding result is false. In fact in Section 4 we explicitly construct a 5-dimensional family of (1,2)-symplectic metrics for an invariant almost

---

1991 *Mathematics Subject Classification.* Primary 53C55; Secondary 58E20, 05C20.

* Supported by the National Natural Science Foundation of China (19871001) and Fundação de Amparo à Pesquisa do Estado de Sao Paulo-FAPESP, Brazil (97/01108-5).

complex structure which is neither integrable nor parabolic. Finally, by applying these results, we obtain some new harmonic maps into flag manifolds.

2. Invariant almost Hermitian structures on $F(N)$. Without loss of generality, we may consider full complex flag manifolds

$$F = F(1, 1, \dots, 1; N) := F(N).$$

We denote by ω the Maurer-Cartan form of $U(N)$, that is,

$$\omega = (\omega_{i\bar{j}}) \tag{2.1}$$

and at the Lie algebra level, we write

$$\begin{aligned} u^*(N)^C &= \bigoplus_{i,j}(\operatorname{span}\{\omega_{i\bar{j}}\} \oplus \operatorname{span}\{\omega_{\bar{i}j}\}) \\ &= \left[\bigoplus_{i}(\operatorname{span}\{\omega_{i\bar{i}}\} \oplus \operatorname{span}\{\omega_{\bar{i}i}\})\right] \oplus \left[\bigoplus_{i\neq j}(\operatorname{span}\{\omega_{i\bar{j}}\} \oplus \operatorname{span}\{\omega_{\bar{i}j}\})\right] \\ &= (u(1) + \cdots + u(1))^{*C} \oplus \left(\bigoplus_{i\neq j} D_{ij}^C\right), \end{aligned} \tag{2.2}$$

where

$$D_{ij} = \operatorname{span}\{\operatorname{Re}\omega_{i\bar{j}}, \operatorname{Im}\omega_{\bar{i}j}\}. \tag{2.3}$$

Each real vector space D_{ij} has two invariant almost complex structures, with its $(1,0)$-type and $(0,1)$-type forms generated by $\omega_{i\bar{j}}$ and $\omega_{\bar{i}j}$, respectively. Borel and Hirzebruch ([3]) showed that there are $2^{\binom{N}{2}}$ $U(N)$-invariant almost complex structure J on $F(N)$ determined by the choice of one of these two structures in each D_{ij}. We see that such a choice defines a tournament $\mathcal{J}(J)$ with players $T = \{1, 2, \dots, N\}$. Indeed, the space of $(1,0)$-cotangent vectors at the identity coset, can be identified with

$$m_{1,0} = \operatorname*{span}_{i\to j}\{\omega_{i\bar{j}}\}, \tag{2.4}$$

where

$$\mathcal{J}(J) = \{i \to j; i, j = 1, \dots, N \text{ with } i \neq j\}. \tag{2.5}$$

Now we may define all invariant metrics on $F(N)$ (see [4]) by

$$ds_\Lambda^2 = \sum_{i,j} \lambda_{ij}\omega_{i\bar{j}} \otimes \omega_{\bar{i}j}, \tag{2.6}$$

where

$$\Lambda = (\lambda_{ij}) \tag{2.7}$$

is a real symmetric matrix such that

$$\lambda_{ij} \begin{cases} > 0 & \text{if } i \neq j, \\ = 0 & \text{if } i = j. \end{cases} \tag{2.8}$$

For an alternative description, see for example [11]. Note that (2.6)–(2.8) define an Hermitian metric on $F(N)$ for each invariant almost complex structure J, since

$$
\begin{aligned}
ds^2_\Lambda(JX, JY) &= \sum_{i,j} \lambda_{ij}\omega_{i\bar{j}}(JX)\omega_{\bar{i}j}(JY) \\
&= \sum_{i,j} \lambda_{ij}\varepsilon_{ij}\sqrt{-1}\omega_{i\bar{j}}(X)\varepsilon_{ij}(-\sqrt{-1})\omega_{\bar{i}j}(Y) = ds^2_\Lambda(X, Y)
\end{aligned}
\tag{2.9}
$$

for any vector fields X and Y, where

$$
\varepsilon_{ij} = \begin{cases} 1 & i \to j\,, \\ -1 & j \to i\,, \\ 0 & i = j\,. \end{cases}
\tag{2.10}
$$

It is clear that $\varepsilon := (\varepsilon_{ij})$ is anti-symmetric.

Let $\sum_N$ be the permutation group of N elements with identity e. For each $\tau \in \sum_N$, the Kähler form Ω, with respect to the $U(N)$-invariant almost Hermitian structure corresponding to a tournament $\mathcal{J}(J)$ (see (2.5)) and an invariant Hermitian metric ds^2_Λ (see (2.6)), is defined by

$$
\begin{aligned}
\Omega(X, Y) :=& ds^2_\Lambda(X, JY) \\
=& \sum_{ij} \lambda_{\tau(i)\tau(j)}\omega_{\tau(i)\overline{\tau(j)}}(X) J\omega_{\overline{\tau(i)}\tau(j)}(Y) \\
=& -\sqrt{-1}\left(\sum_{i<j} + \sum_{i>j}\right) \varepsilon_{\tau(i)\tau(j)}\lambda_{\tau(i)\tau(j)}[\omega_{\tau(i)\overline{\tau(j)}}(X)\omega_{\overline{\tau(i)}\tau(j)}(Y)] \\
=& -2\sqrt{-1}\sum_{i<j} \varepsilon_{\tau(i)\tau(j)}\lambda_{\tau(i)\tau(j)}\omega_{\tau(i)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)}(X, Y)\,,
\end{aligned}
\tag{2.11}
$$

where ε_{ij} is defined by (2.10). The Kähler form Ω is given by:

$$
\Omega = -2\sqrt{-1}\sum_{i<j} \mu_{\tau(i)\tau(j)}\omega_{\tau(i)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)}
\tag{2.12}
$$

for arbitrary $\tau \in \sum_N$, and

$$
\mu_{ij} := \varepsilon_{ij}\lambda_{ij}
\tag{2.13}
$$

satisfies that

$$
\mu_{ij} + \mu_{ji} = 0\,.
\tag{2.14}
$$

By differentiating (2.12) and using the Maurer-Cartan equations for $U(N)$, one deduces the following:

(2.15)
$$\begin{aligned}\frac{\sqrt{-1}}{2}d\Omega &= \sum_{i<j}\mu_{\tau(i)\tau(j)}\left[\sum_k \omega_{\tau(i)\overline{\tau(k)}}\wedge\omega_{\tau(k)\overline{\tau(j)}}\wedge\omega_{\overline{\tau(i)}\tau(j)}\right.\\ &\qquad\left.-\sum_k \omega_{\tau(i)\overline{\tau(j)}}\wedge\overline{\omega_{\tau(i)\overline{\tau(k)}}}\wedge\overline{\omega_{\tau(k)\overline{\tau(j)}}}\right]\\ &= 2\sqrt{-1}\sum_{i<j}\mu_{\tau(i)\tau(j)}\sum_{k\neq i,j}\mathrm{Im}(\omega_{\tau(i)\overline{\tau(k)}}\wedge\omega_{\tau(k)\overline{\tau(j)}}\wedge\omega_{\overline{\tau(i)}\tau(j)}).\end{aligned}$$

Hence we get

(2.16)
$$\begin{aligned}\frac{1}{4}d\Omega &= \left(\sum_{k<i<j}+\sum_{i<k<j}+\sum_{i<j<k}\right)\mu_{\tau(i)\tau(j)}\cdot\mathrm{Im}(\omega_{\tau(i)\overline{\tau(k)}}\wedge\omega_{\tau(k)\overline{\tau(j)}}\wedge\omega_{\overline{\tau(i)}\tau(j)})\\ &= \sum_{i<j<k} C_{\tau(i)\tau(j)\tau(k)}\Psi_{\tau(i)\tau(j)\tau(k)}\end{aligned}$$

where

(2.17)
$$C_{ijk} = \mu_{ij} - \mu_{ik} + \mu_{jk}$$

and

(2.18)
$$\Psi_{ijk} = \mathrm{Im}(\omega_{i\bar{j}}\wedge\omega_{\bar{i}k}\wedge\omega_{j\bar{k}}).$$

We denote by $\boldsymbol{C}^{p,q}$ the space of complex forms with degree (p,q) on $F(N)$. Then, for any i, j, k, we have either

(2.19)
$$\Psi_{ijk}\in\boldsymbol{C}^{0,3}\oplus\boldsymbol{C}^{3,0}$$

or

(2.20)
$$\Psi_{ijk}\in\boldsymbol{C}^{1,2}\oplus\boldsymbol{C}^{2,1}.$$

DEFINITION 2.1. An almost Hermitian structure (ds^2_Λ, J) is $(1,2)$-symplectic (resp. Kähler) if and only if

$$(d\Omega)^{1,2} = 0 \quad (\text{resp. } d\Omega = 0)$$

REMARK. For the root pattern criteria of the $(1,2)$-symplectic (quasi-Kähler in an alternative terminology due to Wolf and Gray) see [13, page 154, Theorem 9.15].

3. Parabolic invariant almost complex structures. An invariant almost complex structure J on $F(N)$ is called *parabolic* if there exists a permutation τ such that $\mathcal{J}(J)$ is given by

(3.1)
$$\tau(i)\to\tau(j)\Leftrightarrow i-j\in 2\boldsymbol{N}\quad\text{or}\quad j-i\in 2\boldsymbol{N}-1.$$

REMARK 3.1. For an equivalent description, see for example [5]. The main goal of this section is to show that for each parabolic almost complex structure on $F(N)$, there exists an N-dimensional family of almost Hermitian $(1,2)$-symplectic metrics and to write out their explicit formulas. More precisely, we have the following.

THEOREM 3.2. *Suppose that J is a parabolic invariant almost complex structure on $F(N)$ with the corresponding tournament given by* (3.1). *Then an invariant metric ds^2_Λ is $(1,2)$-symplectic with respect to J if and only if Λ satisfies that*

$$\lambda_{\tau(i)\tau(k)} = \begin{cases} a_i + a_{i+2} + \cdots + a_{k-2}\,, \\ \qquad\qquad \text{if} \quad k-i \in 2\mathbf{N}\,, \\ a_k + a_{k+2} + \cdots + a_{N-1} + a_1 + a_3 + \cdots + a_{i-2}\,, \\ \qquad\qquad \text{if} \quad i, N \in 2\mathbf{N}-1 \text{ and } k \in 2\mathbf{N}\,, \\ a_k + a_{k+2} + \cdots + a_N + a_2 + a_4 + \cdots + a_{i-2}\,, \\ \qquad\qquad \text{if} \quad N, k \in 2\mathbf{N}-1 \text{ and } i \in 2\mathbf{N}\,, \\ a_k + a_{k+2} + \cdots + a_{N-2} + a_{N-1} + a_1 + a_3 + \cdots + a_{i-2}\,, \\ \qquad\qquad \text{if} \quad N, k \in 2\mathbf{N} \text{ and } i \in 2\mathbf{N}-1\,, \\ a_k + a_{k+2} + \cdots + a_{N-3} + a_N + a_2 + a_4 + \cdots + a_{i-2}\,, \\ \qquad\qquad \text{if} \quad i, N \in 2\mathbf{N} \text{ and } k \in 2\mathbf{N}-1\,, \end{cases} \tag{3.2}$$

where $a_0 = a_N$, $a_{-1} = a_{N-1}$.

PROOF. For any $i < j < k$, from (2.4) and (2.8) it follows that

$$\Psi_{\tau(i)\tau(j)\tau(k)} \in \mathbf{C}^{1,2} \oplus \mathbf{C}^{2,1} \tag{3.3}$$

if and only if one of the following is true for any $i < j < k$:

(i) $j-i \in 2\mathbf{N}, k-j \in 2\mathbf{N}-1$,
(ii) $k-j \in 2\mathbf{N}, j-i \in 2\mathbf{N}-1$,
(iii) $j-i, k-j \in 2\mathbf{N}$.

The corresponding $C_{\tau(i)\tau(j)\tau(k)}$ vanishes if and only if, from (2.17):

(I) $\lambda_{\tau(j)\tau(k)} = \lambda_{\tau(i)\tau(j)} + \lambda_{\tau(i)\tau(k)}$,
(II) $\lambda_{\tau(i)\tau(j)} = \lambda_{\tau(i)\tau(k)} + \lambda_{\tau(j)\tau(k)}$,
(III) $\lambda_{\tau(i)\tau(k)} = \lambda_{\tau(i)\tau(j)} + \lambda_{\tau(j)\tau(k)}$,

respectively. It follows that ds^2_Λ is a $(1,2)$-symplectic metric with respect to J if and only if (I)–(III) hold, where $i < j < k$ satisfy (i)–(iii), respectivcely.

Put

$$a_j = \begin{cases} \lambda_{\tau(j)\tau(j+2)}\,, & \text{if} \quad j = 1, 2, \cdots, N-2\,, \\ \lambda_{\tau(1)\tau(N-1)}\,, & \text{if} \quad j = N-1 \in 2\mathbf{N}\,, \\ \lambda_{\tau(1)\tau(N)}\,, & \text{if} \quad j = N-1 \in 2\mathbf{N}-1\,, \\ \lambda_{\tau(2)\tau(N-1)}\,, & \text{if} \quad j = N \in 2\mathbf{N}\,, \\ \lambda_{\tau(2)\tau(N)}\,, & \text{if} \quad j = N \in 2\mathbf{N}-1\,. \end{cases} \tag{3.4}$$

Assume now that ds^2_Λ is a $(1,2)$-symplectic metric with respect to J. Then:

a) If $k - i \in 2\mathbf{N}$, then we have from (III)

$$\begin{aligned}\lambda_{\tau(i)\tau(k)} &= \lambda_{\tau(i)\tau(i+2)} + \lambda_{\tau(i+2)\tau(k)} \\ &= \lambda_{\tau(i)\tau(i+2)} + \lambda_{\tau(i+2)\tau(i+4)} + \lambda_{\tau(i+4)\tau(k)} \\ &= \lambda_{\tau(i)\tau(i+2)} + \lambda_{\tau(i+2)\tau(i+4)} + \cdots + \lambda_{\tau(k-2)\tau(k)} \\ &= a_i + a_{i+2} + \cdots + a_{k-2}\,.\end{aligned} \tag{3.5}$$

b) If $i, N \in 2\mathbf{N} - 1$, $k \in 2\mathbf{N}$, then

$$\begin{aligned}\lambda_{\tau(i)\tau(k)} &\overset{\text{(I)}}{=} \lambda_{\tau(1)\tau(i)} + \lambda_{\tau(1)\tau(k)} \\ &\overset{\text{(II)}}{=} \lambda_{\tau(1)\tau(i)} + \lambda_{\tau(1)\tau(N-1)} + \lambda_{\tau(k)\tau(N-1)} \\ &\underset{(4.5)}{\overset{(3.4)}{=}} a_1 + a_3 + \cdots + a_{i-2} + a_{N-1} + a_k + a_{k+2} + \cdots + a_{N-3}\,.\end{aligned} \tag{3.6}$$

c) If $N, k \in 2\mathbf{N} - 1$ and $i \in 2\mathbf{N}$, then

$$\begin{aligned}\lambda_{\tau(i)\tau(k)} &\overset{\text{(III)}}{=} \lambda_{\tau(i)\tau(N)} + \lambda_{\tau(k)\tau(N)} \\ &\overset{\text{(II)}}{=} \lambda_{\tau(2)\tau(i)} + \lambda_{\tau(2)\tau(N)} + \lambda_{\tau(k)\tau(N)} \\ &\underset{(3.6)}{\overset{(3.5)}{=}} a_2 + a_4 + \cdots + a_{i-2} + a_N + a_k + a_{k+2} + \cdots + a_{N-2}\,.\end{aligned} \tag{3.7}$$

d) When $N, k \in 2\mathbf{N}$ and $i \in 2\mathbf{N} - 1$, we have:

$$\begin{aligned}\lambda_{\tau(i)\tau(k)} &\overset{\text{(III)}}{=} \lambda_{\tau(i)\tau(N)} + \lambda_{\tau(k)\tau(N)} \\ &\overset{\text{(II)}}{=} \lambda_{\tau(1)\tau(i)} + \lambda_{\tau(1)\tau(N)} + \lambda_{\tau(k)\tau(N)} \\ &\underset{(3.6)}{\overset{(3.5)}{=}} a_1 + a_3 + \cdots + a_{i-2} + a_{N-1} + a_k + a_{k+2} + \cdots + a_{N-2}\,.\end{aligned} \tag{3.8}$$

e) If $i, N \in 2\mathbf{N}$ and $k \in 2\mathbf{N} - 1$, then

$$\begin{aligned}\lambda_{\tau(i)\tau(k)} &\overset{\text{(III)}}{=} \lambda_{\tau(2)\tau(i)} + \lambda_{\tau(2)\tau(k)} \\ &\overset{\text{(II)}}{=} \lambda_{\tau(2)\tau(i)} + \lambda_{\tau(2)\tau(N-1)} + \lambda_{\tau(k)\tau(N-1)} \\ &\underset{(3.6)}{\overset{(3.5)}{=}} a_2 + a_4 + \cdots + a_{i-2} + a_N + a_k + a_{k+2} + \cdots + a_{N-3}\,.\end{aligned} \tag{3.9}$$

Hence (3.2) holds.

Conversely, assume that $\Lambda = (\lambda_{ij})$ satisfies (3.2). Then

i) If $j - i \in 2\mathbf{N}$ and $k - j \in 2\mathbf{N} - 1$, then

$$
\begin{aligned}
&\lambda_{\tau(i)\tau(j)} + \lambda_{\tau(i)\tau(k)} \\
&\qquad = a_i + a_{i+2} + \cdots + a_{j-2} \\
&\qquad + \begin{cases}
a_k + a_{k+2} + \cdots + a_{N-1} + a_1 + a_3 + \cdots + a_{i-2} & (i, N \in 2\mathbf{N} - 1) \\
a_k + a_{k+2} + \cdots + a_{N-2} + a_{N-1} + a_1 + a_3 + \cdots + a_{i-2} & (i \in 2\mathbf{N} - 1, N \in 2\mathbf{N}) \\
a_k + a_{k+2} + \cdots + a_N + a_2 + a_4 + \cdots + a_{i-2} & (i \in 2\mathbf{N}, N \in 2\mathbf{N} - 1) \\
a_k + a_{k+2} + \cdots + a_{N-3} + a_N + a_2 + a_4 + \cdots + a_{i-2} & (i, N \in 2\mathbf{N})
\end{cases} \\
&\qquad = \begin{cases}
a_k + a_{k+2} + \cdots + a_{N-1} + a_1 + a_3 + \cdots + a_{i-2} \\
a_k + a_{k+2} + \cdots + a_{N-2} + a_{N-1} + a_1 + a_3 + \cdots + a_{i-2} \\
a_k + a_{k+2} + \cdots + a_N + a_2 + a_4 + \cdots + a_{i-2} \\
a_k + a_{k+2} + \cdots + a_{N-3} + a_N + a_2 + a_4 + \cdots + a_{i-2}
\end{cases} \\
&\qquad = \lambda_{\tau(j)\tau(k)} .
\end{aligned}
$$

ii) If $j - i \in 2\mathbf{N} - 1$ and $k - j \in 2\mathbf{N}$, then

$$
\begin{aligned}
&\lambda_{\tau(i)\tau(k)} + \lambda_{\tau(j)\tau(k)} \\
&\qquad = \begin{cases}
a_k + a_{k+2} + \cdots + a_{N-2} + a_{N-1} + a_1 + a_3 + \cdots + a_{i-2} & (i, N \in 2\mathbf{N} - 1) \\
a_k + a_{k+2} + \cdots + a_{N-2} + a_{N-1} + a_1 + a_3 + \cdots + a_{i-2} & (i \in 2\mathbf{N} - 1, N \in 2\mathbf{N}) \\
a_k + a_{k+2} + \cdots + a_N + a_2 + a_4 + \cdots + a_{i-2} & (i \in 2\mathbf{N}, N \in 2\mathbf{N} - 1) \\
a_k + a_{k+2} + \cdots + a_{N-3} + a_N + a_2 + a_4 + \cdots + a_{i-2} & (i, N \in 2\mathbf{N})
\end{cases} \\
&\qquad\quad + a_j + a_{j+2} + \cdots + a_{k+2} \\
&\qquad = \begin{cases}
a_j + a_{j+2} + \cdots + a_{N-1} + a_1 + a_3 + \cdots + a_{i-2} \\
a_j + a_{j+2} + \cdots + a_{N-2} + a_{N-1} + a_1 + a_3 + \cdots + a_{i-2} \\
a_j + a_{j+2} + \cdots + a_N + a_2 + a_4 + \cdots + a_{i-2} \\
a_j + a_{j+2} + \cdots + a_{N-3} + a_N + a_2 + a_4 + \cdots + a_{i-2}
\end{cases} \\
&\qquad = \lambda_{\tau(i)\tau(j)} .
\end{aligned}
$$

iii) If $j - i$ and $k - j \in 2\mathbf{N}$, then

$$
\begin{aligned}
\lambda_{\tau(i)\tau(j)} + \lambda_{\tau(j)\tau(k)} &= (a_i + a_{i+2} + \cdots + a_{j-2}) + (a_j + a_{j+2} + \cdots + a_{k-2}) \\
&= a_i + a_{i+2} + \cdots + a_{k-2} = \lambda_{\tau(i)\tau(k)} .
\end{aligned}
$$

Hence ds^2_Λ is an almost Hermitian $(1, 2)$-symplectic metric with respect to J.

4. Almost complex structures without invariant $(1,2)$-symplectic metrics. Since the Kähler condition implies the $(1,2)$-symplectic one, any invariant almost complex structure has a $(1,2)$-symplectic metric for $F(3)$. A natural question is then the following: "Is there a $(1,2)$-symplectic metric for any $U(N)$-invariant almost complex structure on $F(N)$?" Concerning this question, we have:

THEOREM 4.1. *Suppose J is a $U(N)$-invariant almost complex structure whose associated digraph contains configurations of Figure 1. type. Then J does not admit any left-invariant $(1,2)$-symplectic metric.*

PROOF. If the tournament $\mathcal{J}(J)$ contains (i), then we can mark this 4-subtournament as in the Figure 2 for some permutation $\tau \in \sum_n$. Suppose that ds^2_Λ is $(1,2)$-simplectic with respect to J. Since

$$\omega_{\tau(1)\overline{\tau(2)}},\ \omega_{\overline{\tau(1)}\tau(3)},\ \omega_{\overline{\tau(1)}\tau(4)},\ \omega_{\overline{\tau(2)}\tau(3)},\ \omega_{\tau(2)\overline{\tau(4)}},\ \omega_{\tau(3)\overline{\tau(4)}}$$

are $(1,0)$-forms, we have

$$\begin{cases} C_{\tau(1)\tau(2)\tau(3)} = 0\,, \\ C_{\tau(1)\tau(3)\tau(4)} = 0\,, \\ C_{\tau(2)\tau(3)\tau(4)} = 0\,. \end{cases} \tag{4.1}$$

From (2.10), (4.1) is equivalent to

$$\begin{cases} \lambda_{\tau(1)\tau(2)} + \lambda_{\tau(1)\tau(3)} - \lambda_{\tau(2)\tau(3)} = 0\,, \\ -\lambda_{\tau(1)\tau(3)} + \lambda_{\tau(1)\tau(4)} + \lambda_{\tau(3)\tau(4)} = 0\,, \\ -\lambda_{\tau(2)\tau(3)} - \lambda_{\tau(2)\tau(4)} + \lambda_{\tau(3)\tau(4)} = 0\,. \end{cases} \tag{4.2}$$

(i) (ii)

FIGURE 1.

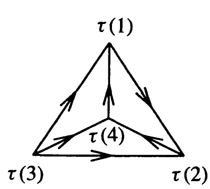

FIGURE 2.

Hence we have

$$\begin{aligned}\lambda_{\tau(2)\tau(3)} &= \lambda_{\tau(1)\tau(2)} + \lambda_{\tau(1)\tau(3)} \\ &= \lambda_{\tau(1)\tau(2)} + \lambda_{\tau(1)\tau(4)} + \lambda_{\tau(3)\tau(4)} \\ &= \lambda_{\tau(1)\tau(2)} + \lambda_{\tau(1)\tau(4)} + \lambda_{\tau(2)\tau(3)} + \lambda_{\tau(2)\tau(4)},\end{aligned}$$

which implies that

$$\lambda_{\tau(1)\tau(2)} + \lambda_{\tau(1)\tau(4)} + \lambda_{\tau(2)\tau(3)} = 0.$$

Therefore, by using (2.8), we derive a contradiction. In a similar manner we can prove the theorem for the type (ii). Q.E.D.

If we use Figure 3 (which is taken from [12]), we see all the isomorphism classes of a 4-tournament (see [12, page 87] for more details). Clearly, (i) is canonical, (ii) and (iii) are listed in Theorem 4.1, and (iv) is parabolic. Combining Theorems 3.2 and 4.1 with Borel's result ([1,2]), we have

THEOREM 4.2. *An almost complex structure on $F(4)$ is integrable* (*resp. parabolic*) *if and only if it admits a symplectic* (*resp. non-symplectic* $(1,2)$*-symplectic*) *invariant metric.*

Also, combining Theorem 3.2 with Borel's result ([1, 2]), we have

PROPOSITION 4.3. *Tournaments arising from integrable or parabolic almost complex structure contain no configurations of type* (i) *and* (ii) *in Theorem* 4.1.

From Figure 3 the converse of Proposition 4.3 is true if $N = 4$. Nevertheless, the following result shows that the converse is false in general.

PROPOSITION 4.4. *There is an almost complex structure J in $F(5)$ such that*:

(a) *J is neither integrable nor parabolic*;

(b) *$\mathcal{J}(J)$ contains no configuration as in Theorem* 4.1;

(c) *J has a* 5*-dimensional family of* $(1,2)$*-symplectic metrics.*

PROOF. Consider the almost complex structure J on $F(5)$ such that $\mathcal{J}(J)$ is defined by the tournament in Figure 4. Then it is easy to see that the score vector (i.e., the number of games that each player won) of $\mathcal{J}(J)$ is $(1,1,2,3,3)$. On the other hand, integrable (resp. parabolic) almost complex structures have score vector $(0,1,2,3,4)$ (resp. $(2,2,2,2,2)$). Furthermore, isomorphic tournaments have the same score vector. So J is neither integrable nor parabolic.

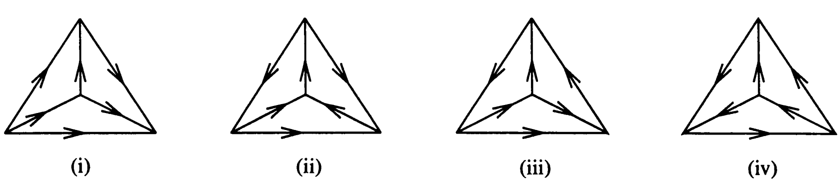

FIGURE 3.

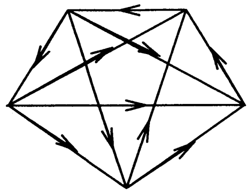

FIGURE 4.

There are five 4-subtournaments in $\mathcal{J}(J)$. The number of 3-cycles in them is 0 or 2. However, the diagrams in Theorem 4.1 have only one 3-cycle. So we have (b) of Proposition 4.4. Owing to the definition of J and (2.16), we have

$$\frac{1}{4}[(d\Omega)^{1,2} + (d\Omega)^{2,1}] = C_{124}\Psi_{124} + C_{125}\Psi_{125} + C_{134}\Psi_{134} + C_{135}\Psi_{135} + C_{145}\Psi_{145} + C_{245}\Psi_{245} + C_{345}\Psi_{345}\,.$$

Together with (2.10), (2.13) and (2.17), we see that ds^2_Λ is (1, 2)-symplectic if and only if $(\lambda_{ij}) = \Lambda$ satisfies:

$$\begin{cases} \lambda_{24} = \lambda_{12} + \lambda_{14} = \lambda_{25} + \lambda_{45}\,, \\ \lambda_{25} = \lambda_{12} + \lambda_{15}\,, \\ \lambda_{13} = \lambda_{14} + \lambda_{34} = \lambda_{15} + \lambda_{35}\,, \\ \lambda_{14} = \lambda_{15} + \lambda_{45}\,, \\ \lambda_{35} = \lambda_{34} + \lambda_{45}\,. \end{cases}$$

It has the following solution:

$$\begin{pmatrix} 0 & \lambda_1 & \lambda_2+\lambda_4+\lambda_5 & \lambda_2+\lambda_5 & \lambda_2 \\ \lambda_1 & 0 & \lambda_3 & \lambda_1+\lambda_4+\lambda_5 & \lambda_1+\lambda_2 \\ \lambda_2+\lambda_4+\lambda_5 & \lambda_3 & 0 & \lambda_4 & \lambda_4+\lambda_5 \\ \lambda_2+\lambda_5 & \lambda_1+\lambda_4+\lambda_5 & \lambda_4 & 0 & \lambda_5 \\ \lambda_2 & \lambda_1+\lambda_2 & \lambda_4+\lambda_5 & \lambda_5 & 0 \end{pmatrix},$$

which implies (c).

5. Harmonic maps into flag manifolds. In this section, we construct new examples of harmonic maps into flag manifolds by using the following

LEMMA 5.1 ([10]). *Let $\phi : (M, g) \to (N, h)$ be a $\pm$-holomorphic map between almost Hermitian manifolds where M is cosymplectic and N is* (1, 2)*-symplectic. Then ϕ is harmonic.*

THEOREM 5.2. *Let $\phi : S^2 \to G_r(C^N)$ be a harmonic map. Then there exists a flag manifold $F = F(n_1, \dots, n_k; N)$ and a harmonic map $\Psi : S^2 \to (F, ds^2_\Lambda)$ such that either ϕ or $\phi^\perp$ is given by $\pi_e \circ \Psi$, where $\Lambda = (\lambda_{ij})$ is given in* (3.2) *(take τ =identity), $k \le 2r+1$ is odd and $\phi^\perp$ is the orthogonal complement of ϕ with respect to the standard Hermitian metric.*

PROOF. From Proposition (2.15), (2.16), Theorem (3.9) and Corollary (3.3) in [5], there exist $k(\in 2N-1) \leq 2r+1$, $F = F(n_1, \dots, n_k; N)$ and a holomorphic map $\Psi : S^2 \to (F, J_2)$ such that either ϕ or $\phi^{\perp}$ is given by $\pi_e \circ \Psi$, where J_2 the canonical parabolic almost complex structure and $\pi_e : F \to G_r(\boldsymbol{C}^N)$ is a homogeneous Riemannian fibration with respect to the identity permutation e. Now our conclusion can be obtained as an immediate consequence of Theorem 3.2 and Lemma 5.1.

REMARK 5.3. It is clear that we can extend Theorem 5.2 to any nilconformal harmonic map of order k from a connected Riemann surface (see [5] for details).

Now we are in a position to construct new harmonic maps between flag manifolds. Define a homogeneous fibration

$$\pi : F(n_1, \dots, n_k, N) \to F(n_1 + n_k, n_2, \dots, n_{k-1}, N)$$

by

$$\pi(E_1, \dots, E_k) = (E_1 \oplus E_k, E_2, \dots, E_{k-1}).$$

Then π is holomorphic with respect to the parabolic the structure J_2 ([5]). Combining this with Theorem 3.2 and Lemma 5.1, one gets the following

PROPOSITION 5.4. *Let $k \in 2N$. Then π is harmonic with respect to all* $(1, 2)$*-symplectic metrics of J_2 given in* (3.2).

REMARK 5.5. It is easy to see that the argument about $(1, 2)$-symplectic structures for any $F(n_1, \dots, n_k; N) = U(N)/U(n_1) \times \cdots \times U(n_k)$ is similar to that for $F(k)$. For example, for $F(1, 1, 2; 4)$ the family of invariant metrics on it can be described as follows:

$$\begin{aligned} ds_\Lambda^2 &= \lambda_1(\omega_{1\bar{2}}\omega_{\bar{1}2} + \omega_{2\bar{1}}\omega_{\bar{2}1}) + \lambda_2(\omega_{1\bar{3}}\omega_{\bar{1}3} + \omega_{3\bar{1}}\omega_{\bar{3}1} + \omega_{1\bar{4}}\omega_{\bar{1}4} + \omega_{4\bar{1}}\omega_{\bar{4}1}) \\ &+ \lambda_3(\omega_{2\bar{3}}\omega_{\bar{2}3} + \omega_{3\bar{2}}\omega_{\bar{3}2} + \omega_{2\bar{4}}\omega_{\bar{2}4} + \omega_{4\bar{2}}\omega_{\bar{4}2}). \end{aligned}$$

Now we consider an invariant almost complex structure on $F(1, 1, 2; 4)$. We define ε_i $(i = 1, 2, 3)$ by

$$\varepsilon_1 = \begin{cases} 1 & \text{if } \omega_{1\bar{2}} \text{ is a } (1,0)-\text{form}, \\ -1 & \text{if } \omega_{1\bar{2}} \text{ is a } (0,1)-\text{form}, \end{cases}$$
$$\varepsilon_2 = \begin{cases} 1 & \text{if } \omega_{1\bar{3}} \text{ and } \omega_{1\bar{4}} \text{ are } (1,0)-\text{forms}, \\ -1 & \text{if } \omega_{1\bar{3}} \text{ and } \omega_{1\bar{4}} \text{ are } (0,1)-\text{forms}, \end{cases}$$
$$\varepsilon_3 = \begin{cases} 1 & \text{if } \omega_{2\bar{3}} \text{ and } \omega_{2\bar{4}} \text{ are } (1,0)-\text{forms}, \\ -1 & \text{if } \omega_{2\bar{3}} \text{ and } \omega_{2\bar{4}} \text{ are } (0,1)-\text{forms}. \end{cases}$$

Then each choice $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ determines an invariant almost complex structure so that the associated Kähler form is given by

$$\Omega = -2\sqrt{-1}[\mu_1(\omega_{1\bar{2}} \wedge \omega_{\bar{1}2}) + \mu_2(\omega_{1\bar{3}} \wedge \omega_{\bar{1}3} + \omega_{1\bar{4}} \wedge \omega_{\bar{1}4}) + \mu_3(\omega_{2\bar{3}} \wedge \omega_{\bar{2}3} + \omega_{2\bar{4}} \wedge \omega_{\bar{2}4})],$$

where

$$\mu_j = \varepsilon_j \lambda_j.$$

Hence it is easy to show that

$$\frac{1}{4}d\Omega = (\mu_1 - \mu_2 + \mu_3)\,\mathrm{Im}[(\omega_{1\bar{3}} \wedge \omega_{3\bar{2}} + \omega_{1\bar{4}} \wedge \omega_{4\bar{2}}) \wedge \omega_{\bar{1}2}], \tag{5.1}$$

which is very similar to (2.16), where τ is the identity permutation and $N = 3$.

ACKNOWLEDGMENTS. The second author wishes to express his gratitude to Professor Karen Uhlenbeck for her immense support throughout these years. The authors are grateful to the referee for the useful comments.

REFERENCES

[1] A. BESSE, Einstein Manifolds, Ergeb. Math. Grenzgeb. (3) 10, Springer Verlag, Berlin, Heidelberg, New York, 1987.

[2] A. BOREL, Kählerian coset spaces of semi-simple Lie groups, Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 1147–1151.

[3] A. BOREL AND F. HIRZEBRUCH, Characteristic classes and homogeneous spaces I, Amer. J. Math. 80 (1958), 458–538.

[4] M. BLACK, Harmonic maps into homogeneous spaces, Pitman Res. Notes Math. Ser. 255, Longman, Harlow, 1991.

[5] F. E. BURSTALL, A twistor description of harmonic maps of a 2-sphere into a Grassmannian, Math. Ann. 274 (1986), 61–74.

[6] F. E. BURSTALL AND J. H. RAWNSLEY, Twistor theory for Riemannian symmetric spaces, with applications to harmonic maps of Riemann surfaces, Lecture Notes in Math. 1424, Springer-Verlag, Berlin 1990.

[7] F. E. BURSTALL AND S. SALAMON, Tournaments, Flags and Harmonic maps, Math. Ann. 277 (1987), 249–265.

[8] J. EELLS AND S. SALAMON, Twistorial constructions of harmonic maps of surfaces into four-manifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 12 (1985), 589–640.

[9] J. EELLS AND J. H. SAMPSON, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964), 109–160.

[10] A. LICHNEROWICZ, Applications harmoniques et variétes Kählériennes, Sympos. Math. III, Bologna, (1970), 341–402.

[11] C. J. C. NEGREIROS, Harmonic maps from compact Riemann surfaces into flag manifolds, Thesis, University of Chicago (1987) or Indiana Univ. Math. J. 37 (1988), 617–636.

[12] K. B. REID AND L. W. BEINEKE, Tournaments, In: Select topics in graph theory, (Eds. L. W. Beineke and R. J. Wilson), Academic Press, London-New York, 1978.

[13] J. A. WOLF AND A. GRAY, Homogeneous spaces defined by Lie group automorphisms. II, J. Differential Geom. 2 (1968), 115–159.

SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING 100871
PEOPLE'S REPUBLIC OF CHINA

E-mail address: moxh@pku.edu.cn

DEPARTAMENTO DE MATHEMÁTICA
INSTITUTO DE MATEMATICA, ESTATÍSTICA E
COMPUTAÇÃO CIENTÍFICA, CP: 6065
UNIVERSIDADE ESTADUAL DE CAMPINAS
13083-970, CAMPINAS-SP
BRAZIL

E-mail address: caione@ime.unicamp.br