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A NON-EXISTENCE THEOREM OF PROPER HARMONIC MORPHISMS FROM WEAKLY ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract. We prove a non-existence theorem for proper harmonic morphisms from weakly asymptotically hyperbolic manifolds to hyperbolic manifolds which are C^2 up to the boundary at infinity.

1. Introduction. *Harmonic morphisms* between Riemannian manifolds are maps which pull back (local) harmonic functions to (local) harmonic functions. By a basic result of Fuglede and Ishihara ([4, 6]) a smooth map $\phi : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if it is harmonic and *horizontally* (*weakly*) *conformal*, in the sense that at any point $x \in M$ not contained in the critical set $C_\phi = \{x \in M \mid d\phi_x = 0\}$ of ϕ, the restriction of the differential $d\phi_x$ to the orthogonal complement

$$\{X \in T_x M \mid \langle X, Y \rangle = 0 \text{ for all } Y \in \operatorname{Ker} d\phi_x\}$$

of $\operatorname{Ker} d\phi_x$ is surjective and conformal onto the tangent space $T_{\phi(x)}N$. The interplay between the analytical condition (harmonicity) and the geometrical one (horizontally weak conformality) is a rich source of properties. See [2] for a general account.

In [11] the authors showed that, when $m > n$, there is no proper harmonic morphism between hyperbolic manifolds which is C^2 up to the boundary at infinity (also see [2]). This contrasts sharply with the situation for harmonic maps, where Li and Tam constructed proper harmonic maps between such manifolds by using C^1 maps between their boundaries at infinity ([7, 8, 12]). Recall that a map $\phi : M \to N$ is called *proper* if $\phi^{-1}(U) \cap M^0$ is compact for each compact subset U in N^0, where M^0 denotes the set of interior points of M.

Viewing hyperbolic spaces as complete non-compact Riemannian manifolds with boundary at infinity, their most natural generalization is weakly asymptotically hyperbolic manifolds, i.e., manifolds which are asymptotic to the hyperbolic space in a certain sense (see Definition 2.3). For instance, the Anti de Sitter-Schwarzschild space is a weakly asymptotically hyperbolic manifold (see Example 2.4). The study of weakly asymptotically hyperbolic manifolds has recently attracted a lot of attention [1, 14].

In this note we discuss proper harmonic morphisms from weakly asymptotically hyperbolic manifolds. We show that when $m > n$, there is no proper harmonic morphism from an

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$(m+1)$-dimensional weakly asymptotically hyperbolic manifold M to an $(n+1)$-dimensional hyperbolic manifold H^{n+1} which is C^2 up to the boundary of M (see Theorem 6.3), generalizing a result previously known only when the source manifold is a hyperbolic manifold.

2. Weakly asymptotically hyperbolic manifolds. Let M be a compact $(m + 1)$-dimensional manifold with boundary $X^m = \partial M$. If ξ is a C^3-smooth function on M satisfying $d\xi \neq 0$ on X^m, positive on the interior of M and zero on X^m, then ξ is called a *defining function.*

DEFINITION 2.1. A Riemannian manifold $(M, \bar{g})$ is called C^2*-conformally compact* if $g := \xi^2\bar{g}$ can be C^2 extended to the boundary of M, where ξ is a defining function ([14]).

Recently the study of conformally compact manifolds has attracted a lot of attention [3, 10, 13, 15]. Let $(M, \bar{g})$ be a conformally compact manifold with boundary X^m. The restriction of g to X^m gives rise to a metric on X^m. This metric changes by a conformal factor if the defining function is changed, so that X^m has a well-defined conformal structure. We call X^m with this induced conformal structure the *conformal infinity*. A straightforward computation (see [10]) shows that the sectional curvatures of $\bar{g}$ approach $-|d\xi|_g^2$ on X^m.

EXAMPLE 2.2. Using the Poincaré model, an $(m + 1)$-dimensional hyperbolic space H^{m+1} is identified with the $(m + 1)$-dimensional unit ball B^{m+1} equipped with the metric $\bar{g} := 4(1 - |x|^2)^{-2}dx^2$. Then H^{m+1} is conformal compact with the conformal infinity $(S^m, [dx^2|_{S^m}])$, where $[dx^2|_{S^m}] := \{f dx^2|_{S^m} \mid f : S^m \to R^+\}$. If we take the defining function $\xi := (1 - |x|^2)/2$, then $g = dx^2$. It is easy to see that

$$|d\xi|_g^2 = \frac{1}{4}|d|x|^2|_g^2 = \Big|\sum_i x^i dx^i\Big|_g^2 = \sum_i (x^i)^2|dx^i|_g^2 = \sum_i (x^i)^2 .$$

Hence, for the sectional curvature we have

$$\mathrm{Sec}_{\bar{g}} H^{m+1} = -1 = -|d\xi|^2_{g|_{S^m}} .$$

DEFINITION 2.3. A conformally compact manifold $(M, \bar{g})$ is said to be *weakly asymptotically hyperbolic* if $|d\xi|_g^2 = 1$ on the boundary of M for some defining function ξ, where $g = \xi^2\bar{g}$.

Intuitively, for a weakly asymptotically hyperbolic manifold $(M, \bar{g})$, its sectional curvature approaches that of hyperbolic space at the boundary of M. To illustrate this definition, let us consider the following

EXAMPLE 2.4 ([14]). The Anti de Sitter-Schwarzchild space (ADS-Schwarzchild space for short) is the product manifold $]r_0, \infty[\times S^2$ with the metric

$$\bar{g} = \frac{dr^2}{1 + r^2 - c/r} + r^2 d\omega^2 ,$$

where $c > 0$ is a constant and r_0 is the zero of the function $1 + r^2 - c/r$. This manifold has two ends with the same asymptotic behavior, so we only analyze the end $r \to \infty$. We change coordinates by solving the following ODE:

$$\begin{cases} \dot{r}(t) = -\sinh^{-1}(t)\sqrt{1+r^2-c/r}\,, \\ r(0)=\infty\,. \end{cases}$$

Let $u(t) = r(t)\sinh(t)$. A straightforward computation shows that (see [14])

$$\begin{cases} \cosh(t)u - \sinh(t)\dot{u} = \sqrt{\sinh^2(t) + u^2 - c\sinh^3(t)/u}\,, \\ u(0) = 1\,. \end{cases}$$

In the new coordinates, we have

$$\bar{g} = \frac{(\dot{r}(t)dt)^2}{1+r^2-c/r} + \left[\frac{u}{\sinh(t)}\right]^2 d\omega^2 = \sinh^{-2}(t)[dt^2 + u(t)^2 d\omega^2]\,.$$

Taking the defining function $\xi := \sinh(t)$ yields $g = dt^2 + u(t)^2 d\omega^2$. A simple calculation shows that

$$|d\xi|_g^2 = |\cosh(t)dt|_g^2 = \cosh^2(t)|dt|_g^2 = \cosh^2(t)\,.$$

Letting $t \to 0$, we obtain

$$|d\xi|_g^2 = \cosh^2(0) = 1\,.$$

Therefore the ADS-Schwarzchild space is weakly asymptotically hyperbolic.

In general, we have the following (see [1, 14])

LEMMA 2.5. *Suppose that $(M,\ \bar{g})$ is a conformally compact manifold and $\bar{g}_0$ is a metric on X which represents the induced conformal structure. If $(M,\ \bar{g})$ is weakly asymptotically hyperbolic, then there is a unique defining function ρ in a collar neighborhood of $X = \partial M$, satisfying $\bar{g} = \sinh^{-2}(\rho)(d\rho^2 + \bar{g}_\rho)$ with $\bar{g}_\rho$ a ρ-dependent family of metrics on X such that $\bar{g}_\rho|_{\rho=0}$ is the given metric $\bar{g}_0$.*

We call ρ the *special defining function* determined by $\bar{g}_0$. Set

$$\bar{g}_\rho = g_{\lambda\mu}(\rho,\ \eta)d\eta^\lambda d\eta^\mu$$

and

$$g = d\rho^2 + \bar{g}_\rho\,,$$

where η^λ are local coordinates in the collar neighborhood. Then

$$\bar{g} = \sinh^{-2}(\rho)(d\rho^2 + g_{\lambda\mu}(\rho,\ \eta)d\eta^\lambda d\eta^\mu) = \sinh^{-2}(\rho)g\,. \tag{1}$$

We rewrite g and $\bar{g}$ as

$$g := \sum_{i,j=0}^{m} g_{ij}dx^i \otimes dx^j\,, \quad \bar{g} := \sum_{i,j=0}^{m} \bar{g}_{ij}dx^i \otimes dx^j\,,$$

where

$$x^0 = \rho\,, \quad x^\lambda = \eta^\lambda\,, \quad \lambda = 1, 2, \ldots, m\,.$$

Then

$$g_{00} = 1\,, \quad g_{0\lambda} = 0\,, \quad \lambda = 1, 2, \ldots, m\,.$$

From (1), we have

$$\bar{g}_{ij} = \sinh^{-2}(\rho) g_{ij}\,, \quad \bar{g}^{ij} = \sinh^{2}(\rho) g^{ij}\,, \tag{2}$$

where $(g^{ij}) = (g_{ij})^{-1}$. It is easy to show that, from (2),

$$\frac{\partial \bar{g}_{ij}}{\partial x^k} = \begin{cases} \sinh^{-2}(\rho)\dfrac{\partial g_{ij}}{\partial \rho} + \dfrac{-2+\kappa(\rho)}{\sinh^3(\rho)} g_{ij} & \text{if } k = 0\,, \\ \sinh^{-2}(\rho)\dfrac{\partial g_{ij}}{\partial \eta^k} & \text{if } k \geq 1\,, \end{cases}$$

where

$$\lim_{\rho\to 0} \kappa(\rho) = 0\,.$$

By a direct calculation we have

$$\begin{aligned} \bar{\Gamma}^k_{ij} &:= \frac{1}{2}\bar{g}^{kl}\left(\frac{\partial \bar{g}_{li}}{\partial x^j} - \frac{\partial \bar{g}_{ij}}{\partial x^l} + \frac{\partial \bar{g}_{lj}}{\partial x^i}\right) \\ &= \begin{cases} \Gamma^k_{ij} - \dfrac{1+\zeta(\rho)}{\sinh(\rho)}\delta^k_j & \text{if } i = 0\,, \\ \Gamma^k_{ij} + \dfrac{1+\zeta(\rho)}{\sinh(\rho)}\delta^{k0} g_{ij} & \text{if } i,\, j \geq 1\,, \end{cases} \end{aligned} \tag{3}$$

where Γ^k_{ij}(resp. $\bar{\Gamma}^k_{ij}$) is the Christoffel symbol of g (resp. $\bar{g}$) and $\zeta(\rho) := -\kappa(\rho)/2$.

3. Hyperbolic spaces. We regard an $(n+1)$-dimensional hyperbolic space H^{n+1} as the Poincaré model with Riemannian metric

$$\bar{h} = \sum_{\alpha,\beta=0}^{n} \bar{h}_{\alpha\beta}\, dy^\alpha \otimes dy^\beta,$$

where

$$(\bar{h}_{\alpha\beta}) = \begin{bmatrix} 4/(1-r^2)^2 & 0 \\ 0 & (4r^2/(1-r^2)^2) h_{pq} \end{bmatrix}, \tag{4}$$

$$y^0 := r\,, \quad y^p := \theta^p\,, \quad \text{if } p \geq 1\,,$$

θ^β are local coordinates on S^n and $h_{pq} d\theta^p \otimes d\theta^q$ is the standard metric on S^n. Then

$$(\bar{h}^{\alpha\beta}) = \begin{bmatrix} (1-r^2)^2/4 & 0 \\ 0 & ((1-r^2)^2/4r^2) h^{pq} \end{bmatrix} \tag{5}$$

where $(\bar{h}^{\alpha\beta}) = (\bar{h}_{\alpha\beta})^{-1}$. From (4) we have

$$\frac{\partial \bar{h}_{\alpha\beta}}{\partial y^\gamma} = \begin{cases} \dfrac{16r}{(1-r^2)^3} & \text{if } \alpha=\beta=\gamma=0\,, \\ 0 & \text{if } \alpha=0\,,\ \beta\geq 1 \text{ or } \alpha=\beta=0\,,\ \gamma\geq 1\,, \\ \dfrac{8r(1+r^2)}{(1-r^2)^3}h_{\alpha\beta} & \text{if } \alpha\,,\ \beta\geq 1\,,\ \gamma=0\,, \\ \dfrac{4r^2}{(1-r^2)^2}\dfrac{\partial h_{\alpha\beta}}{\partial\theta^\gamma} & \text{if } \alpha\,,\ \beta\,,\ \gamma\geq 1\,. \end{cases}$$

Then the Christoffel symbol of $\bar{h}$ are given by

$$\bar{\Gamma}^{\alpha*}_{\beta\gamma} = \begin{cases} \dfrac{2r}{1-r^2} & \text{if } \alpha=\beta=\gamma=0\,, \\ 0 & \text{if } \alpha=\beta=0\,,\ \gamma\geq 1 \text{ or } \alpha=0\,,\ \beta\,,\ \gamma\geq 1\,, \\ -\dfrac{r(1+r^2)}{1-r^2}h_{\beta\gamma} & \text{if } \alpha=0\,,\ \beta,\gamma\geq 1\,, \\ \dfrac{1+r^2}{r(1-r^2)}\delta^\alpha_\beta & \text{if } \alpha\,,\ \beta\geq 1\,,\ \gamma=0\,, \\ \Gamma^{\alpha*}_{\beta\gamma} & \text{if } \alpha\,,\ \beta\,,\ \gamma\geq 1\,. \end{cases} \tag{6}$$

where $\Gamma^{\alpha*}_{\beta\gamma}$ is the Christoffel symbol with respect to (h_{pq}).

4. Horizontally conformal maps. From now on we assume that $(M, \bar{g})$ is a weakly asymptotically hyperbolic manifold (see Section 2) and $(H^{n+1},\ \bar{h})$ is an $(n+1)$-dimensional hyperbolic space. Let $\phi : (M, \bar{g}) \to (H^{n+1}, \bar{h})$ be a horizontally conformal map. Then ϕ satisfies ([2, 16])

$$\bar{g}^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j} = \varrho^2(x)\bar{h}^{\alpha\beta} \tag{7}$$

for some function $\varrho : M \to [0, \infty)$ called the *dilation* of ϕ, where $\phi = (\phi^\alpha)$. By using (5) and (7), we have

$$\begin{aligned} \langle\nabla\phi^\alpha, \nabla\phi^\beta\rangle &:= g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j} \\ &= \begin{cases} \varrho^2\dfrac{(1-r^2)^2}{4\sinh^2(\rho)} & \text{if } \alpha=\beta=0\,, \\ 0 & \text{if } \alpha=0\,,\ \beta\geq 1\,, \\ \varrho^2\dfrac{(1-r^2)^2}{4r^2\sinh^2(\rho)}h^{\alpha\beta} & \text{if } \alpha\,,\ \beta\geq 1\,. \end{cases} \end{aligned} \tag{8}$$

It follows that

$$|\nabla\phi|^2 := g^{ij}\frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j} + r^2 g^{ij}\frac{\partial\theta^p}{\partial x^i}\frac{\partial\theta^q}{\partial x^j}h_{pq} = (n+1)\varrho^2\frac{(1-r^2)^2}{4\sinh^2(\rho)}\,. \tag{9}$$

Substituting (9) into (8), we get

$$\langle \nabla\phi^\alpha, \nabla\phi^\beta\rangle = \begin{cases} |\nabla\phi|^2/(n+1) & \text{if } \alpha=\beta=0\,, \\ 0 & \text{if } \alpha=0\,,\ \beta\geq 1\,, \\ \big(|\nabla\phi|^2/(n+1)r^2\big)\, h^{\alpha\beta} & \text{if } \alpha\,,\ \beta\geq 1\,. \end{cases} \tag{10}$$

5. Proper harmonic maps. By using (2), (3) and (6), we obtain the component of the tension field $\tau(\phi) :=$ trace $\nabla d\phi$ of ϕ in the direction r:

$$\begin{aligned} \tau_0(\phi) :&= \bar g^{ij}\left(\frac{\partial^2 r}{\partial x^i\partial x^j} - \bar\Gamma^k_{ij}\frac{\partial r}{\partial x^k} + \bar\Gamma^{0*}_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\right) \\ &= \sinh(\rho)\Bigg[\sinh(\rho)\Delta r + (1-m)(1+\zeta(\rho))\frac{\partial r}{\partial\rho} \\ &\quad + \frac{2r\sinh(\rho)}{1-r^2}|\nabla r|^2 - \frac{r\sinh(\rho)(1+r^2)}{1-r^2}h_{pq}\langle\nabla\theta^p, \nabla\theta^q\rangle\Bigg], \end{aligned} \tag{11}$$

where

$$\Delta r := g^{ij}\left(\frac{\partial^2 r}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial r}{\partial x^k}\right)$$

is the Laplacian operator on $(M,\ g)$ and

$$|\nabla r|^2 := \langle\nabla r, \nabla r\rangle\,,$$

$\langle\nabla\theta^p, \nabla\theta^q\rangle$ being defined by (8). Similarly, using (2), (3) and (6), we have the component of the tension field of ϕ in the direction θ^p:

$$\begin{aligned} \tau_p(\phi) :&= \bar g^{ij}\left(\frac{\partial^2\theta^p}{\partial x^i\partial x^j} - \bar\Gamma^k_{ij}\frac{\partial\theta^p}{\partial x^k} + \bar\Gamma^{p*}_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\right) \\ &= \sinh(\rho)\left[\sinh(\rho)\Delta\theta^p + (1-m)(1+\zeta(\rho))\frac{\partial\theta^p}{\partial\rho}\right] \\ &\quad + \sinh(\rho)\left[\sinh(\rho)\Gamma^p_{st}\langle\nabla\theta^s, \nabla\theta^t\rangle + 2\sinh(\rho)\frac{1+r^2}{r(1-r^2)}\langle\nabla r, \nabla\theta^p\rangle\right] \end{aligned} \tag{12}$$

where

$$\langle\nabla r, \nabla\theta^p\rangle := g^{ij}\frac{\partial r}{\partial x^i}\frac{\partial\theta^p}{\partial x^j}\,. \tag{13}$$

Recall that a map $\phi : M \to N$ between Riemannian manifolds is called *harmonic* if its tension field $\tau(\phi)$ vanishes identically. From (11) and (12) we have the following

LEMMA 5.1. *Let $(M, \bar g)$ an $(m+1)$-dimensional weakly asymptotically hyperbolic manifold and $\phi : (M, \bar g) \to (H^{n+1},\ \bar h)$ a C^2-smooth map. Then ϕ is harmonic if and only if*

$$\begin{aligned} &\sinh(\rho)\Delta r + (1-m)(1+\zeta(\rho))\frac{\partial r}{\partial\rho} \\ &\quad + \frac{2r\sinh(\rho)}{1-r^2}|\nabla r|^2 - \frac{r\sinh(\rho)(1+r^2)}{1-r^2}h_{pq}\langle\nabla\theta^p, \nabla\theta^q\rangle = 0 \end{aligned} \tag{14}$$

and

$$\begin{aligned}\sinh(\rho)\Delta\theta^p + (1-m)(1+\zeta(\rho))\frac{\partial\theta^p}{\partial\rho}\\ + \sinh(\rho)\Gamma^p_{st}\langle\nabla\theta^s, \nabla\theta^t\rangle + 2\sinh(\rho)\frac{1+r^2}{r(1-r^2)}\langle\nabla r, \nabla\theta^p\rangle = 0\end{aligned} \tag{15}$$

for $p = 1, \ldots, n$.

LEMMA 5.2. *Assume that* $\phi : (M, \bar{g}) \to (H^{n+1}, \bar{h})$ *is* C^2*-smooth up to the boundary and proper. Then at the boundary,* $\rho = 0$*, we have*

$$\frac{\partial r}{\partial\rho}\frac{\partial\theta^p}{\partial\rho} = 0\,, \quad p = 1, \ldots, n\,, \tag{16}$$

$$m\left(\frac{\partial r}{\partial\rho}\right)^2 = h_{pq}\langle\nabla\theta^p, \nabla\theta^q\rangle\,. \tag{17}$$

PROOF. From (15) we have

$$\Delta\theta^p + \frac{1-m}{\sinh(\rho)}(1+\zeta(\rho))\frac{\partial\theta^p}{\partial\rho} + \Gamma^p_{st}\langle\nabla\theta^s, \nabla\theta^t\rangle + 2\frac{1+r^2}{r(1-r^2)}\langle\nabla r, \nabla\theta^p\rangle = 0 \tag{18}$$

on M. That is,

$$\begin{aligned}(1-r)\Delta\theta^p + (r-1)\frac{m-1}{\sinh(\rho)}(1+\zeta(\rho))\frac{\partial\theta^p}{\partial\rho}\\ +(1-r)\Gamma^p_{st}\langle\nabla\theta^s, \nabla\theta^t\rangle + 2\frac{1+r^2}{r(1+r)}\langle\nabla r, \nabla\theta^p\rangle = 0\,.\end{aligned} \tag{19}$$

Since ϕ is proper, we get

$$\lim_{\rho\to 0} r = 1\,,$$

which implies that

$$r(0, \eta^1, \ldots, \eta^m) = 1\,. \tag{20}$$

It then follows that

$$\lim_{\rho\to 0}\frac{r-1}{\sinh(\rho)}(1+\zeta(\rho)) = \left.\frac{\partial(r-1)}{\partial\rho}\right|_{\rho=0} = \left.\frac{\partial r}{\partial\rho}\right|_{\rho=0} \tag{21}$$

by L'Hôspital's rule. Note that (M, g) is compact, so that both $\Delta\theta^p$ and $\Gamma^p_{st}\langle\nabla\theta^s, \nabla\theta^t\rangle$ are bounded. Then, at $\rho = 0$, (19) yields that

$$(m-1)\frac{\partial r}{\partial\rho}\frac{\partial\theta^p}{\partial\rho} + 2\langle\nabla r, \nabla\theta^p\rangle = 0\,. \tag{22}$$

From (20), we have

$$\left.\frac{\partial r}{\partial\eta^p}\right|_{\rho=0} = 0\,, \quad p = 1, \ldots, m\,. \tag{23}$$

Combining this with (13), we get

$$\langle \nabla r, \nabla \theta^p \rangle = g^{0j} \frac{\partial r}{\partial x^0} \frac{\partial \theta^p}{\partial x^j} = \frac{\partial r}{\partial \rho} \frac{\partial \theta^p}{\partial \rho} . \tag{24}$$

Substituting (23) into (22) gives (16). Also, from (23), we see

$$\lim_{\rho \to 0} |\nabla r|^2 = \lim_{\rho \to 0} g^{ij} \frac{\partial r}{\partial x^i} \frac{\partial r}{\partial x^j} = \left(\frac{\partial r}{\partial \rho} \right)^2 . \tag{25}$$

By using (14), we have

$$(1-r)\Delta r + (m-1)(1+\zeta(\rho)) \frac{r-1}{\sinh(\rho)} \frac{\partial r}{\partial \rho} + \frac{2r}{1+r} |\nabla r|^2 = \frac{r(1+r^2)}{1+r} h_{pq} \langle \nabla \theta^p, \nabla \theta^q \rangle . \tag{26}$$

By the same argument and using (21), (25) and (26), we obtain (17) at the boundary $\rho = 0$.

6. A non-existence theorem. A smooth map $f : P \to Q$ between Riemannian manifold is called a *harmonic morphism* if for any harmonic function $\psi : U \to \boldsymbol{R}$ defined on an open subset U of Q with $f^{-1}(U)$ non-empty, $\psi \circ f : f^{-1}(U) \to \boldsymbol{R}$ is a harmonic function. The reader is referred to [2] for a detailed account of harmonic morphisms. Harmonic morphisms can be characterized as follows:

THEOREM 6.1 ([4, 6]). *A map $\phi : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally* (*weakly*) *conformal harmonic map.*

LEMMA 6.2. *Let $(M, \bar{g})$ be an $(m+1)$-dimensional weakly asymptotically hyperbolic manifold and $\phi : (M, \bar{g}) \to (H^{n+1}, \bar{h})$ a harmonic morphism which is proper and C^2-smooth up to the boundary of M. Then $|\tilde{\nabla}\bar{\phi}|^2 \equiv 0$ on the boundary ∂M, where*

$$|\tilde{\nabla}\bar{\phi}|^2 := r^2 \sum_{\lambda,\mu=1}^{m} \sum_{p,q=1}^{n} g^{\lambda\mu} \frac{\partial \theta^p}{\partial \eta^\lambda} \frac{\partial \theta^q}{\partial \eta^\mu} h_{pq}$$

and $\bar{\phi} := (\phi^1, \dots, \phi^n)$.

PROOF. Suppose there exist $x_0 \in \partial M$ such that $|\tilde{\nabla}\bar{\phi}|^2 (x_0) \neq 0$. From (17), we get

$$\begin{aligned} m \left(\frac{\partial r}{\partial \rho} \right)^2 (x_0) &= h_{pq} \langle \nabla \theta^p, \nabla \theta^q \rangle (x_0) \\ &= h_{pq} \frac{\partial \theta^p}{\partial \rho} \frac{\partial \theta^q}{\partial \rho} (x_0) + |\tilde{\nabla}\bar{\phi}|^2 (x_0) \\ &\geq |\tilde{\nabla}\bar{\phi}|^2 (x_0) \neq 0 . \end{aligned} \tag{27}$$

Combining this with (16), we have

$$\frac{\partial \theta^p}{\partial \rho} (x_0) = 0 , \quad p = 1, \dots, n . \tag{28}$$

Set

$$|\tilde{\nabla} r|^2 := \sum_{\lambda,\mu=1}^{m} g^{\lambda\mu} \frac{\partial r}{\partial \eta^\lambda} \frac{\partial r}{\partial \eta^\mu} .$$

Then it follows from (23) that

$$|\tilde{\nabla} r|^2 = 0\,.$$

Hence, at x_0 we have

$$\begin{aligned}|\nabla\theta^p|^2 :&= g^{ij}\frac{\partial\theta^p}{\partial x^i}\frac{\partial\theta^p}{\partial x^j}\\ &= \left(\frac{\partial\theta^p}{\partial\rho}\right)^2 + \sum_{\lambda,\mu=1}^{m} g^{\lambda\mu}\frac{\partial\theta^p}{\partial\eta^\lambda}\frac{\partial\theta^p}{\partial\eta^\mu} = \sum_{\lambda,\mu=1}^{m} g^{\lambda\mu}\frac{\partial\theta^p}{\partial\eta^\lambda}\frac{\partial\theta^p}{\partial\eta^\mu}\end{aligned}$$

from (28) and

$$\begin{aligned}|\nabla\phi|^2(x_0) :&= g^{ij}\frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j} + r^2 g^{ij}\frac{\partial\theta^p}{\partial x^i}\frac{\partial\theta^p}{\partial x^j}h_{pq}\\ &= \left(\frac{\partial r}{\partial\rho}\right)^2 + \sum_{\lambda,\mu=1}^{m} g^{\lambda\mu}\frac{\partial r}{\partial\eta^\lambda}\frac{\partial r}{\partial\eta^\mu}\\ &\quad + r^2\sum\frac{\partial\theta^p}{\partial\rho}\frac{\partial\theta^q}{\partial\rho}h_{pq} + r^2\sum_{\lambda,\mu=1}^{m} g^{\lambda\mu}\frac{\partial\theta^p}{\partial\eta^\lambda}\frac{\partial\theta^q}{\partial\eta^\mu}h_{pq}\\ &= \left(\frac{\partial r}{\partial\rho}\right)^2 + |\tilde{\nabla}\bar{\phi}|^2 = \frac{m+1}{m}|\tilde{\nabla}\bar{\phi}|^2(x_0)\end{aligned}\tag{29}$$

from (27) and (28). On the other hand, since ϕ is horizontally conformal, $\phi = (\phi^0, \ldots, \phi^n)$ satisfies (10). Combining this (8), (28) and (29), we get

$$\begin{aligned}|\tilde{\nabla}\bar{\phi}|^2 &= \sum_{\lambda,\mu=1}^{m} g^{\lambda\mu}\frac{\partial\theta^p}{\partial\eta^\lambda}\frac{\partial\theta^q}{\partial\eta^\mu}h_{pq} = \sum g^{ij}\frac{\partial\theta^p}{\partial x^i}\frac{\partial\theta^p}{\partial x^j}h_{pq}\\ &= h_{pq}\langle\nabla\theta^p, \nabla\theta^q\rangle = \frac{|\nabla\phi|^2}{n+1}h^{pq}h_{pq}\\ &= \frac{n}{n+1}|\nabla\phi|^2 = \frac{n}{n+1}\frac{m+1}{m}|\tilde{\nabla}\bar{\phi}|^2\,.\end{aligned}$$

Hence, $|\tilde{\nabla}\bar{\phi}|^2(x_0) = 0$, since $m > n$. This contradicts to our assumption.

THEOREM 6.3. *Let m and n be positive integers with $m > n$ and $(M, \bar{g})$ an $(m+1)$-dimensional weakly asymptotically hyperbolic manifold. Then there exists no proper harmonic morphism from $(M, \bar{g})$ to $(H^{n+1}, \bar{h})$ which is C^2 up to the boundary at infinity of M.*

PROOF. Let $\phi : (M, \bar{g}) \to (H^{n+1}, \bar{h})$ be a proper harmonic map. By Lemma 6.2 we have $\tilde{\nabla}\bar{\phi} \equiv 0$ at the boundary ∂M, i.e., the image of ∂M under ϕ is a point, say q. Put $\phi(O) = O'$, where O is a interior point of M. Since ϕ is C^2-smooth to the boundary of M, there exists a convex closed subset K of H^{n+1} with ∂K being totally geodesic hypersurface, such that $O' \in H^{n+1}\backslash K$ and $\phi(\partial B_\varepsilon) \subset K$ for sufficient large ε, where B_ε is the geodesic ball with radius ε and center O ([11]). Since ϕ is a harmonic morphism, the composite function

$f := h \circ \phi$ is a subharmonic function, where the function h is defined by $h(y) = d(y, K)$ (eg., [5, p. 510]). By the maximum principle and $f|_{\partial B_\varepsilon} = 0$, we know $f|_{B_\varepsilon} = 0$ ([5]). Thus

$$0 = f(O) = d(O', K)\,,$$

which is a contradiction and completes the proof of the theorem.

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