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ON THE EXCEPTIONALITY OF SOME SEMIPOLAR SETS OF TIME INHOMOGENEOUS MARKOV PROCESSES

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Abstract. For a Markov process associated with a not necessarily symmetric regular Dirichlet form, if the form satisfies the sector condition, then any semipolar sets are exceptional. On the other hand, in the case of the space-time Markov process associated with a family of time dependent Dirichlet forms, there exist non-exceptional semipolar sets. The main purpose of this paper is to show that any semipolar set $B = J \times \Gamma$ of the direct product type of a subset J of time and a subset Γ of space is exceptional if J has positive Lebesgue measure.

1. Introduction and preliminaries. Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. In this paper we assume that we are given a family of (not necessarily symmetric) regular Dirichlet forms $\{(E^{(\tau)}, F)\}_{\tau \geq 0}$ on $H = L^2(X; m)$ satisfying the following conditions:

(i) For any $\varphi, \psi \in F$, $E^{(\tau)}(\varphi, \psi)$ is a measurable function of τ.

(ii) For any $T \in [0, \infty)$, there exist positive constants $\lambda_1(T)$ and $\lambda_2(T)$ such that

$$\lambda_1(T) E^{(\tau)}(\varphi, \varphi) \leq E(\varphi, \varphi) \leq \lambda_2(T) E^{(\tau)}(\varphi, \varphi) \tag{1}$$

for all $\tau \in [0, T]$ and $\varphi \in F$, where $E = E^{(0)}$.

(iii) For any $T \in [0, \infty)$, there exists a positive constant $\Lambda(T)$ such that

$$E^{(\tau)}(\varphi, \varphi) \geq 0\,, \tag{2}$$

$$|E^{(\tau)}(\varphi, \psi)| \leq \Lambda(T) E^{(\tau)}(\varphi, \varphi)^{1/2} E^{(\tau)}(\psi, \psi)^{1/2} \tag{3}$$

for any $\tau \in [0, T]$ and $\varphi, \psi \in F$ (see [3], [5]).

Although it is not essential to treat our problem, we consider that $E^{(\tau)}$ is defined for any $\tau \in \boldsymbol{R}^1$ by putting $E^{(\tau)} = E$ for $\tau < 0$. Setting $Z = \boldsymbol{R}^1 \times X$ and $d\nu(\tau, x) = d\tau dm(x)$, let

$$\mathcal{H} = L^2(\boldsymbol{R}^1; H) = \{u(\tau, x) \mid u(\tau, \cdot) \in H, \|u\|_{\mathcal{H}} < \infty\}\,,$$

where

$$\|u\|_{\mathcal{H}}^2 = \int_{\boldsymbol{R}^1} \|u(\tau, \cdot)\|_H^2 d\tau = \int_Z u(\tau, x)^2 d\nu(\tau, x)\,.$$

Let $E_\alpha(\varphi, \psi) = E(\varphi, \psi) + \alpha(\varphi, \psi)_m$ and define the norn $\|\cdot\|_F$ on F by $\|\varphi\|_F = E_1(\varphi, \varphi)^{1/2}$. Using this norm, define $\mathcal{F} = L^2(\boldsymbol{R}^1; F)$ similarly. For $u, v \in \mathcal{F}$, let

$$\mathcal{A}(u, v) = \int_{\boldsymbol{R}^1} E^{(\tau)}(u(\tau, \cdot), v(\tau, \cdot)) d\tau\,.$$

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Identifying H with its dual space H', we consider as $F \subset H = H' \subset F'$. Then $\mathcal{F}' = L^2(\boldsymbol{R}^1; F')$ can be defined similarly and $\mathcal{F} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{F}'$. For $u \in \mathcal{F}'$, $du/d\tau$ is considered as a distribution sense derivative of F'-valued function $u(\tau, \cdot)$. Introduce the space $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ and a bilinear form $\mathcal{E}$ defined by

$$\mathcal{W} = \left\{ u \in \mathcal{F} \;\middle|\; \frac{du}{d\tau} \in \mathcal{F}' \right\},$$

$$\|u\|_{\mathcal{W}}^2 = \|u\|_{\mathcal{F}}^2 + \left\| \frac{du}{d\tau} \right\|_{\mathcal{F}'}^2,$$

$$\mathcal{E}(u, v) = \begin{cases} -\left(\dfrac{du}{d\tau}, v\right) + \mathcal{A}(u, v), & u \in \mathcal{W}, v \in \mathcal{F} \\ \left(\dfrac{dv}{d\tau}, u\right) + \mathcal{A}(u, v), & u \in \mathcal{F}, v \in \mathcal{W}, \end{cases}$$

where $(du/d\tau, v)$ is the canonical coupling of $du/d\tau \in \mathcal{F}'$ and $v \in \mathcal{F}$. For any $f \in \mathcal{H}$, then there exist Markovian resolvents $G_\alpha f \in \mathcal{W}$ and $\hat{G}_\alpha f \in \mathcal{W}$ such that

$$\mathcal{E}_\alpha(G_\alpha f, u) = \mathcal{E}_\alpha(u, \hat{G}_\alpha f) = (f, u) \quad \text{for any } u \in \mathcal{F} \tag{4}$$

where $\mathcal{E}_\alpha = \mathcal{E} + \alpha(\cdot, \cdot)_\nu$ (see [6], [10]).

For a family of functions $\mathcal{K}$, we denote the family of non-negative (resp. compact support) functions of $\mathcal{K}$ by $\mathcal{K}^+$ (resp. $\mathcal{K}_0$). A function $u \in \mathcal{F}$ is said to be α-excessive if

$$\mathcal{E}_\alpha(u, v) \geq 0 \quad \text{for any } v \in C_0(Z) \cap \mathcal{W}^+.$$

Denote by $\mathcal{P}_\alpha$ the family of α-excessive functions in $\mathcal{F}$. Then $u \in \mathcal{F}^+$ belongs to $\mathcal{P}_\alpha$ if and only if $\beta G_{\alpha+\beta} u \leq u$ ν-a.e. for any $\beta > 0$. Given a function $h \in \mathcal{H}^+$, put

$$\mathcal{L}_h = \{u \in \mathcal{F} \mid u \geq h,\ \nu\text{-a.e.}\}.$$

In particular, put $\mathcal{L}_B = \mathcal{L}_{I_B}$. If $\mathcal{W} \cap \mathcal{L}_h \cap \mathcal{P}_\alpha$ is non-empty, then there exists a unique function $e_h^{(\alpha)} \in \mathcal{P}_\alpha$ such that $e_h^{(\alpha)} \leq u$ for all $u \in \mathcal{L}_h \cap \mathcal{P}_\alpha$. Further this function $e_h^{(\alpha)}$ is characterized as a minimal function of $\mathcal{L}_h \cap \mathcal{P}_\alpha$ satisfying

$$\mathcal{E}_\alpha(e_h^{(\alpha)}, v) \geq \mathcal{A}_\alpha(e_h^{(\alpha)}, e_h^{(\alpha)}) \tag{5}$$

for any $v \in \mathcal{W} \cap \mathcal{L}_h$. In particular, put $e_h = e_h^{(1)}$ and $e_B = e_{I_B}^{(1)}$. If $u \in \mathcal{W}$, then there exists a constant $\beta > 0$ such that

$$\mathcal{A}_1(e_u, e_u) \leq \beta \|u\|_{\mathcal{W}}^2 \tag{6}$$

(see [4], [7], [8], [10]).

For a relatively compact open set A of Z, define the capacity of A by

$$\mathrm{Cap}(A) = \mathcal{E}_1(e_A, v), \quad v \in \mathcal{W}, \quad v = 1\nu\text{-a.e. on } A. \tag{7}$$

It is independent of the choice of v and extended to any compact set B of Z by

$$\mathrm{Cap}(B) = \inf\{\mathrm{Cap}(A) \mid A \text{ is open} \supset B\}. \tag{8}$$

Further it is also extended to any Borel sets in a usual manner. A function u is called $\mathcal{E}$-quasi-conitinuous if there exists a decreasing sequence of open sets $\{A_n\}$ of Z such that $\lim_{n\to\infty} \mathrm{Cap}(A_n) = 0$ and $u|_{Z\setminus A_n}$ is continuous. Since (E, F) is a (non-symmetric) Dirichlet form, the E-capacity $\mathrm{C}(\Gamma)$ of an open set Γ of X can be defined by

$$\mathrm{C}(\Gamma) = \inf\{E_1(\varphi, \varphi) \mid \varphi \in F, \varphi \geq I_\Gamma\ m\text{-a.e.}\}\,. \tag{9}$$

Also the capacity $\mathrm{C}(\cdot)$ is extended to all Borel sets. Further, if we define the capacity $\mathrm{C}^{(\tau)}$ relative to $(E^{(\tau)}, F)$ similarly, then it follows from (1) that

$$(\lambda_1(T) \wedge 1)\,\mathrm{C}^{(\tau)}(K) \leq \mathrm{C}(K) \leq (\lambda_2(T) \vee 1)\,\mathrm{C}^{(\tau)}(K) \tag{10}$$

for any $\tau \in [0, T]$. The next result can be found in [6].

THEOREM 1.1. *There exists a pair of Hunt processes* $\mathbf{M} = (Z_t, P_z)$ *and* $\hat{\mathbf{M}} = (\hat{Z}_t, \hat{P}_z)$ *on* Z *such that, for any* $f \in \mathcal{H}$, *their resolvents* $R_\alpha f$ *and* $\hat{R}_\alpha f$ *are* $\mathcal{E}$*-q.c.modifications of* $G_\alpha f$ *and* $\hat{G}_\alpha f$, *respectively. Further, if we decompose as* $Z_t = (\tau_t, X_t)$ *and* $\hat{Z}_t = (\hat{\tau}_t, \hat{X}_t)$ *into the motions* τ_t, $\hat{\tau}_t$ *in* $\boldsymbol{R}^1$ *and* X_t, $\hat{X}_t$ *in* X, *then* τ_t *and* $\hat{\tau}_t$ *are the uniform motion to the right and the left respectively, that is,* $\tau_t = \tau_0 + t$ *and* $\hat{\tau}_t = \hat{\tau}_0 - t$.

It is well-known that, for a Hunt process Y_t on X associated with a non-symmetric Dirichlet form (E, F) satisfying the conditions (2) and (3), any semipolar set is E-exceptional, that is, of zero E-capacity (see [1], [2], [9]). However, since the space-time process Z_t moves to the time direction by uniform motion, the set of the form $\{\tau\} \times \Gamma$ with $m(\Gamma) > 0$ is semipolar (in fact, thin) but not $\mathcal{E}$-exceptional, that is, not of zero capacity relative to Cap. The main purpose of this paper is to show the equivalence of the semipolarity and $\mathcal{E}$-exceptionality of the set B of the form $B = J \times \Gamma$ with $|J| > 0$, where $|J|$ is the Lebesgue measure of J. Different from the process Y_t, the first hitting distribution p_B and the first entry distribution d_B of the set B relative to the process Z_t is not $\mathcal{E}$-quasi-continuous. But we can see that, for Lebesgue a.e. $\tau \in \boldsymbol{R}^1$, $d_B(\tau, \cdot)$ and $p_B(\tau, \cdot)$ are E-quasi-continuous and $d_B(\tau, \cdot) = p_B(\tau, \cdot)$ E-quasi-everywhere (E-q.e. in abbreviation). The proof of the main result is based upon this property.

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2. Equilibrium potentials and hitting probabilities. For a Borel set B of Z, define the first entry time D_B and the first hitting time σ_B by $D_B = \inf\{t \geq 0 \mid Z_t \in B\}$ and $\sigma_B = \{t > 0 \mid Z_t \in B\}$, respectively. Using these stopping times, their distridutions d_B and p_B are defined by

$$d_B(z) = E_z(e^{-D_B}) \quad \text{and} \quad p_B(z) = E_z(e^{-\sigma_B})\,,$$

respectively. Clearly, $D_B = \sigma_B$ and hence $d_B = p_B$ if B is open. The proof of the next lemma is similar to Stannat [10, Proposition I.3.7] but we shall present it for the readers convenience.

LEMMA 2.1. *For any relatively compact open set* B *of* Z, $\lim_{\alpha\to\infty} \alpha G_{\alpha+1} e_B = e_B$ *in* $\mathcal{F}$.

PROOF. First note that $\lim_{\alpha\to\infty} G_{\alpha+1}u = u$ in $\mathcal{F}$ for any $u = G_1 f$, $f \in \mathcal{H}^+$. In fact,

$$\begin{aligned}\mathcal{A}_1(\alpha G_{\alpha+1}u - u, \alpha G_{\alpha+1}u - u) &= \mathcal{E}_1(\alpha G_{\alpha+1}u - u, \alpha G_{\alpha+1}u - u)\\ &= (\alpha G_{\alpha+1} f - f, \alpha G_{\alpha+1}u - u) \to 0\,, \quad \alpha \to \infty\,.\end{aligned}$$

Since $\{G_1 f \mid f \in \mathcal{H}\}$ is dense in $\mathcal{F}$, by approximating e_B by a function u of the above form relative to $\mathcal{A}_1$, it suffices to show that $\mathcal{A}(\alpha G_\alpha e_B, \alpha G_\alpha e_B) \le k^2 \mathcal{A}(e_B, e_B)$ for some constant k. To show this, introduce a kernel K_α defined by

$$K_\alpha f(\tau, x) = \int_0^\infty e^{-\alpha t} f(\tau + t, x)dt = e^{\alpha\tau}\int_\tau^\infty e^{-\alpha s} f(s, x)ds\,.$$

Then it satisfies $(\partial/\partial\tau)K_\alpha f(\tau, x) = \alpha K_\alpha f(\tau, x) - f(\tau, x)$. In particular, $w_\alpha \equiv \alpha G_\alpha e_B - \alpha K_\alpha e_B \in \mathcal{W}$.

Since B is relatively compact, there exist $-\infty \le a < b < \infty$ such that $B \subset [a, b] \times X$. Noting that the function $h(\tau, x) = e^{\tau - a} I_{(-\infty,b]}(\tau)$ satisfies $e^{-t} p_t h(\tau, x) \le h(\tau, x)$, we can see that $e_B \wedge h$ is a 1-excessive function dominating I_B. Hence $e_B \wedge h = e_B$ and, in particular, $e_B(\tau, x) = 0$ for $\tau > b$. This further implies that $G_\alpha e_B(\tau, x)$ and $K_\alpha e_B(\tau, x)$ vanish for $\tau > b$. Therefore, we have

$$\begin{aligned}\mathcal{A}(w_\alpha, w_\alpha) &\le \mathcal{E}_\alpha(w_\alpha, w_\alpha)\\ &= \alpha\mathcal{E}_\alpha(G_\alpha e_B, w_\alpha) - \mathcal{A}(\alpha K_\alpha e_B, w_\alpha) - \alpha\left(\alpha K_\alpha e_B - \frac{\partial}{\partial\tau}K_\alpha e_B, w_\alpha\right)\\ &= -\mathcal{A}(\alpha K_\alpha e_B, w_\alpha) \le \Lambda(b)\mathcal{A}(\alpha K_\alpha e_B, \alpha K_\alpha e_B)^{1/2}\mathcal{A}(w_\alpha, w_\alpha)^{1/2}\,.\end{aligned}$$

This implies that $\mathcal{A}(w_\alpha, w_\alpha) \le \Lambda(b)^2 \mathcal{A}(\alpha K_\alpha e_B, \alpha K_\alpha e_B)$.

Since $E^{(\tau)}(\varphi, \varphi) \le (\lambda_2(b)/\lambda_1(b))E^{(s)}(\varphi, \varphi)$ for all $\varphi \in F$ and $\tau, s \le b$, by putting $e_B(t) = e_B(t, \cdot)$, we can also see that

$$\begin{aligned}&\mathcal{A}(\alpha K_\alpha e_B, \alpha K_\alpha e_B)\\ &= \alpha^2\int_{\boldsymbol{R}^1} d\tau\int_0^\infty dt\int_0^\infty ds e^{-\alpha t - \alpha s} E^{(\tau)}(e_B(t+\tau), e_B(s+\tau))\\ &\le \Lambda(b)\alpha^2\int_{-\infty}^b d\tau\int_\tau^b dt\int_\tau^b ds e^{\alpha(2\tau - t - s)} E^{(\tau)}(e_B(t), e_B(t))^{1/2} E^{(\tau)}(e_B(s), e_B(s))^{1/2}\\ &\le \Lambda(b)(\lambda_2(b)/\lambda_1(b))\alpha\int_{-\infty}^b d\tau\int_\tau^b e^{\alpha(\tau - s)} E^{(s)}(e_B(s), e_B(s))ds\\ &\le \Lambda(b)(\lambda_2(b)/\lambda_1(b))e^{\alpha b}\mathcal{A}(e_B, e_B)\,.\end{aligned}$$

Hence

$$\mathcal{A}(\alpha G_\alpha e_B, \alpha G_\alpha e_B)^{1/2} \le \mathcal{A}(\alpha K_\alpha e_B, \alpha K_\alpha e_B)^{1/2} + \mathcal{A}(w_\alpha, w_\alpha)^{1/2} \le k\mathcal{A}(e_B, e_B)^{1/2}$$

with $k = (1 + \Lambda(b))\sqrt{\Lambda(b)}(\lambda_2(b)/\lambda_1(b))^{1/2} e^{\alpha b/2}$.

LEMMA 2.2. *For any relatively compact open set* B, $d_B = e_B\ \nu - a.e.$

PROOF. Let h_B be a measurable version of e_B such that $h_B = 1$ on B. Then $p_t h_B$ is uniquely determined ν-a.e. independently of the choice of the version h_B. In fact, if $g = 0$

ν-a.e., then from $\langle \nu, p_t|g|\rangle = \langle \hat{p}_t 1, |g|\rangle_\nu = 0$, it follows that $p_t g = 0$ ν-a.e. Since $h_B \geq e^{-t} p_t h_B$ ν-a.e., there exists a ν-negligible set N such that

$$h_B(z) \geq e^{-t} p_t h_B(z) \quad \text{for any } \ z \notin N \quad \text{and } \ \text{any } t \in \boldsymbol{Q}_+ ,$$

where $\boldsymbol{Q}_+$ is the family of positive rational numbers. Hence, $\{e^{-t} h_B(Z_t)\}_{t \in \boldsymbol{Q}_+}$ is a positive supermartingale relative to $P_z, z \notin N$. For a finite subset S of $\boldsymbol{Q}_+$, if we put $D = \inf\{t \in S \mid Z_t \in B\}$, it then follows that $h_B(z) \geq E_z(e^{-D}),\ z \notin N$. Letting $S \uparrow \boldsymbol{Q}_+$, it follows that $h_B(z) \geq E_z(e^{-D_B}),\ z \notin N$. Therefore $e_B \geq d_B$ ν-a.e.

To show the converse inequality, first note that $-(dw/d\tau, h) \leq \mathcal{A}_1(h, w)$ for any 1-excessive function $h \in \mathcal{F}$ and $w \in \mathcal{W}^+$. Since d_B is 1-excessive and dominates I_B, to get that $e_B \leq d_B$ ν-a.e., it suffices to show that $d_B \in \mathcal{F}$. This further follows from the boundedness of $\mathcal{A}_1(\alpha G_{\alpha+1} d_B, \alpha G_{\alpha+1} d_B)$ relative to α. We may assume that $B \subset [0, b] \times X$ for some $b < \infty$. In view of $e_B \geq d_B$, we have

$$\begin{aligned}
\mathcal{A}_1(\alpha G_{\alpha+1} d_B, \alpha G_{\alpha+1} d_B) &= \mathcal{E}_1(\alpha G_{\alpha+1} d_B, \alpha G_{\alpha+1} d_B) \leq \mathcal{E}_1(\alpha G_{\alpha+1} d_B, e_B) \\
&= -\alpha \left(\frac{d}{d\tau} G_{\alpha+1} d_B, e_B \right) + \alpha \mathcal{A}_1\left(G_{\alpha+1} d_B, e_B\right) \\
&\leq \alpha \mathcal{A}_1(e_B, G_{\alpha+1} d_B) + \alpha \mathcal{A}_1(G_{\alpha+1} d_B, e_B) \\
&\leq 2\Lambda(b) \mathcal{A}_1(\alpha G_{\alpha+1} d_B, \alpha G_{\alpha+1} d_B)^{1/2} \mathcal{A}_1(e_B, e_B)^{1/2} .
\end{aligned}$$

Therefore $\mathcal{A}_1(\alpha G_{\alpha+1} d_B, \alpha G_{\alpha+1} d_B) \leq 4\Lambda(b)^2 \mathcal{A}_1(e_B, e_B)$.

LEMMA 2.3. *Let $\{B_n\}$ be a decreasing sequence of relatively compact open sets of Z such that* $\mathrm{Cap}(B_n) \downarrow 0$. *Then there exists a Lebesgue negligible subset L_1 of $\boldsymbol{R}^1$ and a subsequence $\{n_k\}$ satisfying* $\mathrm{C}(B_{n_k}^{(\tau)}) \downarrow 0$ *for any $\tau \notin L_1$, where $B^{(\tau)} = \{x \in X \mid (\tau, x) \in B\}$.*

PROOF. By virtue of (5), (7) and Lemma 2.2, since

$$\int_{\boldsymbol{R}^1} E_1^{(\tau)}(d_{B_n}(\tau, \cdot), d_{B_n}(\tau, \cdot)) d\tau = \mathcal{A}_1(d_{B_n}, d_{B_n}) \leq \mathrm{Cap}(B_n) \to 0, \quad n \to \infty ,$$

there exist a subsequence $\{n_k\}$ and a Lebesgue negligible set L_1 such that

$$\lim_{k \to \infty} E_1^{(\tau)}(d_{B_{n_k}}(\tau, \cdot), d_{B_{n_k}}(\tau, \cdot)) = 0 \quad \text{for any } \ \tau \notin L_1 . \tag{11}$$

In view of (1), we can replace $E_1^{(\tau)}$ by E_1 in (11). Further, since $d_{B_{n_k}} \in \mathcal{F}$ and $d_{B_{n_k}} = 1$ ν-a.e. on B_{n_k}, we may assume that $d_{B_{n_k}}(\tau, \cdot) \in F$ and $d_{B_{n_k}}(\tau, \cdot) = 1$ m-a.e. on $B_{n_k}^{(\tau)}$ for $\tau \notin L_1$. Then (9) implies that $C(B_{n_k}^{(\tau)}) \to 0$ as $k \to \infty$ for any $\tau \notin L_1$.

LEMMA 2.4. *Let B be a relatively compact open set of Z. Then there exists a subset L_2 of $\boldsymbol{R}^1$ of zero Lebesgue measure such that $d_B(\tau, \cdot)$ is E-quasi-continuous for any $\tau \notin L_2$.*

PROOF. In view of Lemma 2.1 and Lemma 2.2, since $\alpha R_{\alpha+1} d_B$ converges to d_B relative to $\mathcal{A}_1$, there exist a sequence $\alpha_k \uparrow \infty$ and a Lebesgue negligible set L_3 such that

$$\lim_{k \to \infty} E_1\left(\alpha_k R_{\alpha_k+1} d_B(\tau, \cdot) - d_B(\tau, \cdot), \alpha_k R_{\alpha_k+1} d_B(\tau, \cdot) - d_B(\tau, \cdot)\right) = 0$$

for all $\tau \notin L_3$. Then, by choosing a subsequence $\{\beta_k\}$ of $\{\alpha_k\}$, $\beta_k R_{\beta_k+1} d_B(\tau, \cdot)$ converges E-quasi-uniformly for any $\tau \notin L_3$ (see [2, Theorem 2.1.4]). The limit coincides with $d_B(\tau, \cdot)$ because $\lim_{\alpha\to\infty} \alpha R_{\alpha+1} d_B = p_B = d_B$ everywhere. Since $\beta_k R_{\beta_k+1} d_B$ is $\mathcal{E}$-quasi-continuous, for any relatively compact open set U of Z, there exists a sequence of relatively compact open subsets $\{N_n\}$ of U such that $\mathrm{Cap}(N_n) \to 0$ and $\beta_k R_{\beta_k+1} d_B$ is continuous on $U \setminus N_n$ for any k. By virtue of Lemma 2.3, then there exist a subsequence n_ℓ and a Lebesgue negligible set L_4 such that, for $\tau \notin L_4$, $C(N_{n_\ell}^{(\tau)}) \to 0$ and $\beta_k R_{\beta_k+1} d_B(\tau, \cdot)$ is continuous on $U^{(\tau)} \setminus N_{n_\ell}^{(\tau)}$ for any k and ℓ. Putting $L_2 = L_3 \cup L_4$, we get that $d_B(\tau, \cdot)$ is E-quasi-continuous on $U^{(\tau)}$ for any $\tau \notin L_2$. Since U is arbitrary, the assertion holds.

3. The main result. Now we are in a position to prove the main result of this paper that any semipolar set B of the form $B = J \times \Gamma$ with $|J| > 0$ is $\mathcal{E}$-exceptional. The essential step is the following theorem.

THEOREM 3.1. *For any compact set K of Z, there exists a Lebesgue negligible set L such that $d_K(\tau, \cdot) = p_K(\tau, \cdot)$ E-q.e. for any $\tau \notin L$.*

PROOF. Let $\{B_n\}$ be a decreasing sequence of relatively compact open sets such that $\bigcap_n B_n = K$. Then $\lim_{n\to\infty} d_{B_n}(z) = d_K(z)$ for all $z \in Z$. Further, since $\mathcal{A}_1(d_{B_n}, d_{B_n}) \le \mathrm{Cap}(B_n)$ is bounded, there exists a subsequence of d_{B_n} such that its Cesàro means $\{u_k\}$ converges in $\mathcal{F}$ and $d_K \in \mathcal{F}$. Then, similarly to the proof of Lemma 2.4, by choosing a subsequence of $\{u_k\}$ if necessary, there exists a Lebesgue negligible set L_5 such that $\lim_{k\to\infty} u_k(\tau, \cdot) = d_K(\tau, \cdot)$ E-quasi-uniformly for any $\tau \notin L_5$. In particular, $d_K(\tau, \cdot)$ is E-quasi-continuous. Since $d_K \in \mathcal{F}$, we can see from Lemma 2.1 that $\alpha R_{\alpha+1} d_K$ converges to d_K in $\mathcal{F}$ as $\alpha \to \infty$ and hence, there exist a Lebesgue negligible set L_6 and a subsequence $\alpha_n \uparrow \infty$ such that $\lim_{n\to\infty} \alpha_n R_{\alpha_n+1} d_K(\tau, \cdot) = d_K(\tau, \cdot)$ E-q.e. for any $\tau \notin L_6$. On the other hand, since $\lim_{\alpha\to\infty} \alpha R_{\alpha+1} d_K(z) = p_K(z)$, we get the assertion.

LEMMA 3.1. *For any compact set Γ of X, $\mathrm{C}(\Gamma) = 0$ if and only if $\mathrm{Cap}(J \times \Gamma) = 0$ for some $J \subset [0, \infty)$ with $|J| > 0$.*

PROOF. Suppose that $\mathrm{C}(\Gamma) = 0$. Then there exists a sequence $\varphi_n \in F$ such that $\varphi_n = 1$ m-a.e. on a relatively compact neighbourhood Γ_n of Γ and $\lim_{n\to\infty} E_1(\varphi_n, \varphi_n) = 0$. We may assume that J is a compact set contained in a finite open interval (a, b). Take a non-negative smooth function ξ supported by $[0, T]$ such that $\xi = 1$ on $(a, b) \subset [0, T]$ for some $T < \infty$. Then $w_n(\tau, x) = \xi(\tau)\varphi_n(x) \in \mathcal{W}$, $w_n = 1$ ν-a.e. on $B_n = (a, b) \times \Gamma_n$ and

$$\|w_n\|_{\mathcal{W}}^2 \le k_1 \|\varphi_n\|_F^2 \int_{\boldsymbol{R}^1} \{(\xi'(\tau))^2 + (\xi(\tau))^2\} d\tau \ \to 0\,, \quad n \to \infty\,,$$

for some constant k_1. Noting that $\|e_{B_n}\|_{\mathcal{F}}^2 \le (1 + \lambda_2(T))\, \mathcal{A}_1(e_{B_n}, e_{B_n})$, we have from (5) that

$$
\begin{aligned}
\mathrm{Cap}(B_n) &= \mathcal{E}_1(e_{B_n}, w_n) \le \|e_{B_n}\|_{\mathcal{F}} \|w_n\|_{\mathcal{W}} \\
&\le \sqrt{1+\lambda_2(T)}\mathcal{A}_1(e_{B_n}, e_{B_n})^{1/2} \|w_n\|_{\mathcal{W}} \le \sqrt{1+\lambda_2(T)}\mathcal{E}_1(e_{B_n}, w_n)^{1/2} \|w_n\|_{\mathcal{W}} \\
&= \sqrt{1+\lambda_2(T)}\mathrm{Cap}(B_n)^{1/2} \|w_n\|_{\mathcal{W}}\,.
\end{aligned}
$$

Hence $\mathrm{Cap}(B_n) \le (1+\lambda_2(T))\, \|w_n\|^2_{\mathcal{W}}$ and which implies that

$$
\mathrm{Cap}(J \times \Gamma) \le \mathrm{Cap}(B_n) \le (1+\lambda_2(T))\, \|w_n\|^2_{\mathcal{W}} \ \to 0\,, \quad n \to \infty\,.
$$

Conversely, suppose that $\mathrm{Cap}(J \times \Gamma) = 0$ for some J with $|J| > 0$. By virtue of Lemma 2.3, then there exists a decreasing sequence of relatively compact open sets $\Gamma_n \supset \Gamma$ such that $\mathrm{C}(\Gamma_n) \downarrow 0$ which implies that $\mathrm{C}(\Gamma) = 0$.

THEOREM 3.2. *Let $B = J \times \Gamma$ for some J with $|J| > 0$ and $\Gamma \subset X$. If B is semipolar relative to* **M**, *then it is $\mathcal{E}$-exceptional.*

PROOF. It suffices to assume that B is thin, J and Γ are compact subsets of $[0,\infty)$ and X, respectively. By virtue of Theorem 3.1, there exists a Lebesgue negligible subset L such that $d_B(\tau,\cdot) = p_B(\tau,\cdot)$ E-q.e. for any $\tau \notin L$. Since B is thin, $p_B(\tau,x) < 1$ for any $(\tau,x) \in B$. On the other hand, since $d_B(\tau,x) = 1$ for $(\tau,x) \in B$, it follows that Γ is of zero E-capacity. Hence, from Lemma 3.1, B is of zero $\mathcal{E}$-capacity.

REFERENCES

[1] P. J. FITZSIMMONS, Markov processes and nonsymmetric Dirichlet forms without regularity, J. Funct. Anal. 85 (1989), 287–306.

[2] M. FUKUSHIMA, Y. OSHIMA AND M. TAKEDA, Dirichlet forms and symmetric Markov processes, de Gruyter Stud. in Math. 19, Walter de Gruyter & Co., Berlin, 1994.

[3] Z. M. MA AND M. RÖCKNER, Introduction to the theory of (non-symmetric) Dirichlet forms, Universitext, Springer-Verlag, Berlin, 1992.

[4] F. MIGNOT AND J. P. PUEL, Inequations d'evolution paraboliques avec convexes dépendant du temps, Applications aux inequations quasi-variationelles d'evolution, Arch. Rational Mech. Anal. 64 (1977), 59–91.

[5] Y. OSHIMA, Lecture on Dirichlet spaces, Universität Erlangen-Nürnberg, 1988.

[6] Y. OSHIMA, On a constraction of Markov processes associated with time dependent Dirichlet spaces, Forum Math. 4 (1992), 395–415.

[7] Y. OSHIMA, A short introduction to the general theory of Dirichlet forms, Lecture Note in Universität Erlangen-Nürnberg, 1994.

[8] M. PIERRE, Représentant précis d'un potentiel parabolique, Séminaire Théorie du Potentiel, Université Paris VI, Lecture Notes in Math. 814, Springer-Verlag, Berlin, 1980.

[9] M. L. SILVERSTEIN, The sector condition implies that semipolar sets are quasi-polar, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 41 (1977/78), 13–33.

[10] W. STANNAT, The theory of generalized Dirichlet forms and its applications in analysis and stochastics, Mem. Amer. Math. Soc. 142 (1999).

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