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ON REPRESENTABILITY OF THE SMOOTH EULER CLASS

Dedicated to Professor Mitsuyoshi Kato on his sixtieth birthday

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Abstract. The Euler class, which lies in the second cohomology of the group of orientation preserving homeomorphisms of the circle, is pulled back to the "smooth" Euler class in the cohomology of the group of orientation preserving smooth diffeomorphisms of the circle. Suppose a surface group Γ (of genus > 1) is a normal subgroup of a group G, so that we have an extension of $Q = G/\Gamma$ by Γ. We prove that if the canonical outer action of Q on Γ is finite, then there is a canonical second cohomology class of G restricting to the Euler class on Γ which is smoothly representable, that is, it is pulled back from the smooth Euler class by a representation from G to the group of diffeomorphisms. Also, we prove that if the above outer action is infinite, then any second cohomology class of G restricting to the Euler class on Γ is not smoothly representable.

1. Introduction and statement of results. Let $\mathrm{Diff}_+^\infty S^1$ denote the group of orientation preserving smooth (C^∞) diffeomorphisms of the circle. In this paper, we prove several results on the representability of the Euler class of the Eilenberg-MacLane cohomology $H^2(\mathrm{Diff}_+^\infty S^1; \boldsymbol{Z})$ of the group $\mathrm{Diff}_+^\infty S^1$. The Euler class lies in the cohomology of the group $\mathrm{Homeo}_+ S^1$ of orientation preserving homeomorphisms of the circle and it is pulled back to $H^2(\mathrm{Diff}_+^\infty S^1; \boldsymbol{Z})$ by the inclusion $\mathrm{Diff}_+^\infty S^1 \hookrightarrow \mathrm{Homeo}_+ S^1$. The Euler class in $H^2(\mathrm{Homeo}_+ S^1; \boldsymbol{Z})$ is defined as follows. We denote by $\widetilde{\mathrm{Homeo}}_+ S^1$ the universal covering group of $\mathrm{Homeo}_+ S^1$. Indeed, with respect to the identification $\boldsymbol{R}/\boldsymbol{Z} = S^1$, $\widetilde{\mathrm{Homeo}}_+ S^1$ is the group of orientation preserving homeomorphisms of the real line $\boldsymbol{R}$ each of which has period 1:

$$\widetilde{\mathrm{Homeo}}_+ S^1 = \{f \in \mathrm{Homeo}_+ \boldsymbol{R} \mid f(t+1) = f(t) + 1\}\,.$$

We fix a set theoretical section $\sigma : \mathrm{Homeo}_+ S^1 \to \widetilde{\mathrm{Homeo}}_+ S^1$. For a 2-simplex (f, g) of $\mathrm{Homeo}_+ S^1$ we define $\chi(f, g) = \sigma(fg)^{-1}\sigma(f)\sigma(g)$. Here, we identify each integer with the translation of $\boldsymbol{R}$ by the integer. Thus we have a 2-cochain χ which is in fact a 2-cocycle. This 2-cocycle χ represents the *Euler class* E and is called the *Euler cocycle*. The Euler class E does not depend on the choice of the section σ. Ghys [Gh1] showed that the Euler cocycle is a bounded cocycle.

Now, suppose that a group G is given. A second cohomology class $c \in H^2(G; \boldsymbol{Z})$ is said to be *representable* if there is a homomorhism $\varphi : G \to \mathrm{Homeo}_+ S^1$ such that $c = \varphi^* E$. In the

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case where G is the fundamental group of a manifold M with contractible universal covering, $c \in H^2(G; \mathbf{Z}) = H^2(M; \mathbf{Z})$ is representable if and only if c is the Euler class of a topological foliated circle bundle over M. It was shown by Ghys [Gh1] that if G is a countable discrete group, then $c \in H^2(G; \mathbf{Z})$ is representable if and only if its Gromov norm $|c|$ is less than or equal to 1/2 in $H^2(G; \boldsymbol{R})$. In [My] we showed that there is a foliated circle bundle which is not smoothable, namely, there is a homomorphism $\varphi : G \to \mathrm{Homeo}_+ S^1$ which never factors through $\mathrm{Diff}_+^\infty S^1$ even cohomological level. To be precise, the class $\varphi^* E$ does not come from the Euler class in $H^2(\mathrm{Diff}_+^\infty S^1; \mathbf{Z})$. We call the Euler class in $H^2(\mathrm{Diff}_+^\infty S^1; \mathbf{Z})$ the *smooth Euler class* and denote it by E^∞. The smooth Euler class E^∞ is represented by the restriction cocycle $\chi | \mathrm{Diff}_+^\infty S^1$. Also, we say that a class $c \in H^2(G; \mathbf{Z})$ is *smoothly representable* if there is a homomorphism $\varphi : G \to \mathrm{Diff}_+^\infty S^1$ such that $c = \varphi^* E^\infty$.

Let Σ be a closed orientable surface of genus greater than 1. We denote by Γ the fundamental group of Σ with respect to a certain base point. Let $e_\Gamma \in H^2(\Gamma; \mathbf{Z})$ be the Euler class of Σ, that is, e_Γ is the class such that $\langle e_\Gamma, [\Gamma] \rangle = 2 - 2\mathrm{genus}(\Sigma)$, where $[\Gamma]$ is the fundamental class in $H_2(\Gamma; \mathbf{Z}) = H_2(\Sigma; \mathbf{Z})$ and $\langle \cdot, \cdot \rangle$ denotes the natural pairing. Now, consider an extension of a group Q by Γ:

$$1 \to \Gamma \to G \to Q \to 1 \,.$$

Given such an extension, there naturally arises an outer action of Q on Γ by the conjugation with an element of G. More precisely, for any $q \in Q$, choose an element $\tilde{q} \in G$ which is mapped to q. Then conjugation by $\tilde{q}$, $\gamma \mapsto \tilde{q}\gamma\tilde{q}^{-1}$, defines an automorphism of Γ. Different choice of an element of G, which is mapped to q, determines another automorphism, and these automorphisms differ by an inner automorphism of Γ. Note that Γ is not Abelian. Thus we have an outer automorphism of Γ depending only on $q \in Q$. We denote this outer action by $\mu : Q \to \mathrm{Out}(\Gamma)$. Since Γ has the trivial center, an extension of a group Q by Γ inducing the given outer action, if any, is uniquely determined by the outer action μ up to equivalence. For details, we refer to [Mac]. Suppose an extension of a group Q by Γ

$$1 \to \Gamma \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$$

is given. Then we have the following results.

THEOREM I. *If the image of the outer action $\mu(Q)$ is a finite subgroup of* $\mathrm{Out}(\Gamma)$, *then either*

(i) *there is a canonical class $e \in H^2(G; \mathbf{Z})$ with $\iota^* e = e_\Gamma$, which is smoothly representable,*

or

(ii) *there are a subgroup $Q' \subset Q$ of index* 2 *and a canonical class $e' \in H^2(\pi^{-1}(Q'); \mathbf{Z})$ with $\iota^* e' = e_\Gamma$, which is smoothly representable.*

THEOREM II. *If $\mu(Q)$ is an infinite subgroup of* $\mathrm{Out}(\Gamma)$, *then no class $e \in H^2(G; \mathbf{Z})$ with $\iota^* e = e_\Gamma$ is smoothly representable.*

COROLLARY 1 ([My]). *Let $\Sigma \hookrightarrow M \to S^1$ be a smooth orientable surface bundle over S^1, whose monodromy diffeomorphism is not isotopic to any diffeomorphism of finite order. Then the Euler class $e \in H^2(M; \boldsymbol{Z})$ of the tangent bundle to the fibres of the surface bundle is representable but is not smoothly representable.*

COROLLARY 2. *Let $\Sigma \hookrightarrow M \to B$ be a smooth orientable surface bundle over a manifold B. Assume that the monodromy group of the surface bundle is finite and the second homotopy group $\pi_2(B)$ is trivial. Then the Euler class $e \in H^2(M; \boldsymbol{Z})$ of the tangent bundle to the fibres of the surface bundle is smoothly representable.*

In Section 2, we review relevant results on faithful representations and the mapping class group of a hyperbolic surface. The proofs of our theorems are given in Section 3.

2. Faithful representations of a hyperbolic surface and the Nielsen realization problem. Since we assume that the genus of the surface Σ is greater than 1, the surface Σ admits a hyperbolic metric, that is, a metric of constant negative curvature -1. Suppose a hyperbolic metric ρ on the surface Σ is given. Then it induces a faithful representation $\varphi : \Gamma \to PSL(2, \boldsymbol{R})$ of the fundamental group Γ of Σ. As the group of (orientation-preserving) isometries, $PSL(2, \boldsymbol{R})$ acts on the hyperbolic plane $\boldsymbol{H}^2$, and the action naturally extends to the circle at infinity, so that $PSL(2, \boldsymbol{R})$ can be considered as a subgroup of $\mathrm{Diff}_+^\infty S^1$. It is clear that $e_\Gamma = \varphi^* E^\infty$. Indeed, this faithful representation is the extreme case admitted by the Milner-Wood inequality (see [Mil], [W], [My]). Conversely, Ghys' rigidity theorem asserts that in this extreme case the Euler class characterizes the conjugate class of the representation. Two representations $\psi_1, \psi_2 : \Gamma \to \mathrm{Diff}_+^r S^1$ are said to be C^r *conjugate* if there exists $f \in \mathrm{Diff}_+^r S^1$ such that $\psi_1(\gamma) = f \circ \psi_2(\gamma) \circ f^{-1}$ for any $\gamma \in \Gamma$.

THEOREM 1 ([Gh2]). *Suppose that $\psi : \Gamma \to \mathrm{Diff}_+^r S^1$ $(3 \le r \le \infty)$ is a homomorphism with $\varphi^* E^\infty = e_\Gamma$. Then there exists an injective homomorphism $\varphi : \Gamma \to PSL(2, \boldsymbol{R})$ whose image is a discrete subgroup such that ψ is C^r conjugate to φ.*

Next, let $\mathrm{Diff}^\infty \Sigma$ (resp. $\mathrm{Diff}_+^\infty \Sigma$) denote the group of smooth diffeomorphisms (resp. orientation preserving smooth diffeomorphisms) of Σ. If we fix a hyperbolic metric ρ on Σ, we have the group of isometries $\mathrm{Isom}(\Sigma, \rho)$ (resp. $\mathrm{Isom}_+(\Sigma, \rho)$) of the hyperbolic surface as a subgroup of $\mathrm{Diff}^\infty \Sigma$ (resp. $\mathrm{Diff}_+^\infty \Sigma$). The following is well known.

LEMMA 1. *The group* $\mathrm{Isom}(\Sigma, \rho)$ *is finite.*

Let $\mathcal{M}$ denote the mapping class group of Σ. The group $\mathcal{M}$ can be defined as the group of connected components of $\mathrm{Diff}_+^\infty \Sigma$ with C^∞-topology, that is, $\mathcal{M} = \pi_0(\mathrm{Diff}_+^\infty \Sigma)$. It is well-known that $\pi_0(\mathrm{Diff}^\infty \Sigma)$ is isomorphic to $\mathrm{Out}(\Gamma)$. Therefore, we may consider the group $\mathcal{M}$ is a subgroup of $\mathrm{Out}(\Gamma)$ of index 2. The following is Kerckhoff's solution to the celebrate Nielsen realization problem. We denote by $\pi : \mathrm{Diff}^\infty \Sigma \to \pi_0(\mathrm{Diff}^\infty \Sigma)$ the natural quotient homomorphism.

THEOREM 2 ([Ke]). *For any finite subgroup F of $\pi_0(\mathrm{Diff}^\infty \Sigma)$, there exists a hyperbolic metric ρ on Σ such that* $\mathrm{Isom}(\Sigma, \rho) \subset \mathrm{Diff}^\infty \Sigma$ *is isomorphic to F and* $\pi(\mathrm{Isom}(\Sigma, \rho)) = F$.

The following are key lemmas for the proof of Theorem II. Suppose that $\tilde{\Sigma} \to \Sigma$ is the universal covering of Σ. Then $\tilde{\Sigma}$ is diffeomorphic to the open unit disk in $\boldsymbol{C}$. It is well-known that for any $f \in \mathrm{Diff}^\infty_+ \Sigma$, a lifted diffeomorphism $\tilde{f} : \tilde{\Sigma} \to \tilde{\Sigma}$ naturally extends to the circle at infinity $S^1_\infty \approx \partial \boldsymbol{D}$. We refer to [CB] for the details. We denote this extension (and its restriction) to S^1_∞ by $\tilde{f}|S^1_\infty$. Then we have the following homotopy invariance of the induced action on the circle at infinity.

LEMMA 2. *Let h_1 and h_2 be homotopic homeomorphisms of Σ. Suppose a lift $\tilde{h_1} : \tilde{\Sigma} \to \tilde{\Sigma}$ is given. Then there exists a lift $\tilde{h_2} : \tilde{\Sigma} \to \tilde{\Sigma}$ of h_2 such that $\tilde{h_1}|S^1_\infty = \tilde{h_2}|S^1_\infty$.*

For the proof, we refer to [CB]. The following lemma is a characterization of the smooth action on S^1_∞ induced from a diffeomorphism of Σ. See also [I].

LEMMA 3. *The homeomorphism $\tilde{f} \mid S^1_\infty$ is a smooth diffeomorphism if and only if f is isotopic to an isometry with respect to a hyperbolic metric on Σ.*

PROOF. The action on S^1_∞ induced from an isometry is automatically smooth. Thus, with Lemma 2, "if part" is clear. Conversely, suppose that an arbitrary hyperbolic metric ρ on Σ is given and f is not isotopic to an isometry of (Σ, ρ). For any loop ℓ in Σ, there is a unique closed geodesic, denoted by $\hat{\ell}$, which is freely homotopic to ℓ. By the assumption on f, there is a closed geodesic g in (Σ, ρ) such that $\mathrm{length}(g) \neq \mathrm{length}(\widehat{f(g)})$. Let γ and $f_*\gamma$ denote the isometries corresponding to g and $\widehat{f(g)}$ respectively, by the faithful representation. In other words, γ is a hyperbolic translation with its axis a lift $\tilde{g}$ of g, and $f_*\gamma$ is that with its axis a lift of $\widehat{f(g)}$.

Suppose $x \in S^1_\infty$ is a fixed point of γ. Then it is easy to see that $\mathrm{length}(g) = \log \gamma'(x)$ and $\mathrm{length}(\widehat{f(g)}) = \log(f_*\gamma)'(\tilde{f}(x))$, where $\tilde{f}$ is the lift of f such that $f_*\gamma = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$ on S^1_∞. We claim that if $\tilde{f}$ is differentiable at x, then $\tilde{f}'(x) = 0$. Assume, to the contrary, that $\tilde{f}'(x) \neq 0$. Then, since $\tilde{f} \circ \gamma = f_*\gamma \circ \tilde{f}$ on S^1_∞, we have

$$\begin{aligned}
\exp(\mathrm{length}(g)) &= \gamma'(x) \\
&= (\tilde{f}^{-1} \circ f_*\gamma \circ \tilde{f})'(x) \\
&= (\tilde{f}^{-1})'(\tilde{f}(x)) \cdot (f_*\gamma)'(\tilde{f}(x)) \cdot \tilde{f}'(x) \\
&= (f_*\gamma)'(\tilde{f}(x)) \\
&= \exp(\mathrm{length}(\widehat{f(g)}))\,,
\end{aligned}$$

which is a contradiction. □

LEMMA 4. *Any infinite subgroup H of* $\mathrm{Out}(\Gamma)$ *has an element of infinite order.*

PROOF. Suppose that every element of H is of finite order. Then, since H is infinite, we have an increasing sequence of finite subgroups $H_1 \subset H_2 \subset \cdots \subset H$ such that

$H = \bigcup_{k=1}^{\infty} H_k$. $\mathrm{Out}(\Gamma) \cong \pi_0(\mathrm{Diff}^{\infty}\Sigma)$ acts properly discontinuously on $\mathcal{T}$, the Teichmüller space of all hyperbolic metrics on Σ. By Kerckhoff's theorem (Theorem 2), every subgroup H_k acting on $\mathcal{T}$ fixes some point in $\mathcal{T}$. Therefore H fixes a point in the Thurston boundary $\partial\mathcal{T}$ of the Teichmüller space $\mathcal{T}$ (for the details, we refer to [Th] and [FLP]). This implies that a representative $f \in \mathrm{Diff}^{\infty}\Sigma$ of any element of H (considered as a subgroup of $\pi_0(\mathrm{Diff}^{\infty}\Sigma)$) preserves the same (arational) measured foliation $(\mathcal{F}, \mu)$ on Σ up to isotopy (preserving the measured foliation): $f(\mathcal{F}, \mu) = (\mathcal{F}, \mu)$. However, it is easy to see that the group of isotopy classes of diffeomorphisms preserving an arational measured foliation is finite (see exposé 9 in [FLP]). This contradicts the assumption that H is infinite. □

3. Proof of Theorems I and II. Now suppose that an extension of Q by Γ

$$1 \to \Gamma \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$$

is given. First we give the proof of Theorem I.

PROOF OF THEOREM I. Since $\pi_0(\mathrm{Diff}^{\infty}\Sigma) \cong \mathrm{Out}(\Gamma)$ and $\mu(Q)$ is a finite subgroup of $\mathrm{Out}(\Gamma)$, by Kerckhoff's theorem (Theorem 2) we have a hyperbolic metric ρ on Σ such that the outer action $\mu : Q \to \mathrm{Out}(\Gamma)$ lifts to $\tilde{\mu} : Q \to \mathrm{Isom}(\Sigma, \rho) \subset \mathrm{Diff}^{\infty}\Sigma$. We assume that $\tilde{\mu}(Q) \subset \mathrm{Diff}^{\infty}_{+}\Sigma$. Otherwise, set $Q' = \tilde{\mu}^{-1}(\mathrm{Isom}(\Sigma, \rho) \cap \mathrm{Diff}^{\infty}_{+}\Sigma)$ and then replacing Q with Q' in the following, we obtain (ii) of the theorem. From now on, we identify Γ with the image of the faithful representation $\Gamma \to PSL(2, \boldsymbol{R})$ with respect to the hyperbolic metric ρ. Consider an extension

$$\mathcal{E} : 1 \to \Gamma \to N(\Gamma) \to \mathrm{Isom}(\Sigma, \rho) \to 1\,,$$

where $N(\Gamma)$ denotes the normalizer of Γ in $PSL(2, \boldsymbol{R})$. Let $\tilde{\mu}^*\mathcal{E}$ be an extension obtained by pulling back $\mathcal{E}$ by the homomorpism $\tilde{\mu} : Q \to \mathrm{Isom}(\Sigma, \rho)$. Then, since $\mu(Q)$ is isomorphic to $\mathrm{Isom}(\Sigma, \rho)$ and both of the outer actions are identical, the extension $\tilde{\mu}^*\mathcal{E}$ is isomorphic to the extension

$$1 \to \Gamma \to G \to Q \to 1\,.$$

Here, recall that an extension of Q by Γ is determined by its outer action up to equivalence. Therefore, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & \Gamma & \to & G & \to & Q & \to & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & \Gamma & \to & N(\Gamma) & \to & \mathrm{Isom}(\Sigma, \rho) & \to & 1\,. \end{array}$$

Thus, we have a homomorphism $G \to N(\Gamma)$ induced from $\tilde{\mu} : Q \to \mathrm{Isom}(\Sigma, \rho)$, which is an epimorphism. Since $N(\Gamma)$ is a subgroup of $PSL(2, \boldsymbol{R})$ and it acts on the circle at infinity S^1_{∞}, we have the desired conclusion. □

For the proof of Theorem II, we need the following.

LEMMA 5. *Let $\varphi : G \to \mathrm{Diff}^{\infty}_{+} S^1$ be a homomorphism such that $\iota^*\varphi^*E^{\infty} = e_{\Gamma}$. If there is $q \in Q$ such that $\mu(q)$ is an element of infinite order in* $\mathrm{Out}(\Gamma)$, *then, for any element $\tilde{q} \in G$ such that $\pi(\tilde{q}) = q \in Q$, $\varphi(\tilde{q})$ is also an element of infinite order in* $\mathrm{Diff}^{\infty}_{+} S^1$.

PROOF. By the assumption $\iota^*\varphi^*E^{\infty} = e_{\Gamma}$, applying Ghy's rigidity theorem (Theorem 1), we can assume that $\varphi \circ \iota : \Gamma \to \mathrm{Diff}^{\infty}_{+} S^1$ is an injective homomorphism onto a discrete subgroup of $PSL(2, \boldsymbol{R})$ by conjugating φ with a C^{∞} diffeomorphism of the circle, if necessary. For any $q \in Q$, choose an element $\tilde{q} \in G$ such that $\pi(\tilde{q}) = q \in Q$. By the definition, $\mu(q) \in \mathrm{Out}(\Gamma)$ is defined to be the equivalence class determined by $\mathrm{conj}(\tilde{q}) \in \mathrm{Aut}(\Gamma)$, where $\mathrm{conj}(\tilde{q})$ denotes the automorphism of Γ defined by $\mathrm{conj}(\tilde{q})(\gamma) = \tilde{q} \cdot \gamma \cdot \tilde{q}^{-1}$ for any $\gamma \in \Gamma$. Since $\mu(q)$ is of infinite order in $\mathrm{Out}(\Gamma)$, $\mathrm{conj}(\tilde{q})$ is also of infinite order in $\mathrm{Aut}(\Gamma)$.

Now assume that $\varphi(\tilde{q})$ is of finite order. Thus, there is a natural number n such that $\varphi(\tilde{q})^n = 1$. Then we have

$$\begin{aligned} \varphi(\mathrm{conj}(\tilde{q})^n(\gamma)) &= \varphi(\tilde{q}^n \gamma \tilde{q}^{-n}) \\ &= \varphi(\tilde{q})^n \varphi(\gamma) \varphi(\tilde{q})^{-n} \\ &= \varphi(\gamma) \end{aligned}$$

for any $\gamma \in \Gamma$. However, since $\mathrm{conj}(\tilde{q})^n \neq 1$, $\mathrm{conj}(\tilde{q})^n(\gamma) \neq \gamma$ for some $\gamma \in \Gamma$. This implies that $\varphi|\Gamma = \varphi \circ \iota$ is not injective, a contradiction. □

Now we give the proof of Theorem II.

PROOF OF THEOREM II. To the contrary, assume that we have a homomorphism $\varphi : G \to \mathrm{Diff}^{\infty}_{+} S^1$ such that $e = \varphi^* E^{\infty}$. As in the proof of Lemma 5, by Ghys' rigidity theorem (Theorem 1) we can assume that $\varphi \circ \iota = \varphi \,|\, \Gamma$ is the faithful representation $\Gamma \to PSL(2, \boldsymbol{R})$ with respect to a hyperbolic metric on the surface Σ. From now on, we identify the fundamental group $\pi_1(\Sigma) = \Gamma$ with the image of this faithful representation. By the assumption, Lemmas 4 and 5, we have $q \in Q$ such that $\varphi(\tilde{q})$ is of infinite order in $\mathrm{Diff}^{\infty}_{+} S^1$, where $\tilde{q} \in G$ is an element such that $\pi(\tilde{q}) = q$. Note that if $\mu(q)$ is "orientation-reversing", then we may consider $q^2 \in Q$ instead of $q \in Q$. Then, $\mathrm{conj}(\tilde{q}) \in \mathrm{Aut}(\Gamma)$ is a representative of $\mu(q) \in \mathrm{Out}(\Gamma)$. Note that Γ is identified with a subgroup (a Fuchsian group) of $PSL(2, \boldsymbol{R})$, the group of isometries of $\boldsymbol{H}^2$. Recall that for any lift $\tilde{f} : \boldsymbol{H}^2 \to \boldsymbol{H}^2$ of any diffeomorphism $f \in \mathrm{Diff}^{\infty}_{+} \Sigma$, $\tilde{f}$ naturally extends to a homeomorphism on the circle at infinity S^1_{∞} (see [CB] for example).

CLAIM 1. *There exists a diffeomorphism $f \in \mathrm{Diff}^{\infty}_{+} \Sigma$ such that, as a homeomorphism on S^1_{∞},* $\mathrm{conj}(\tilde{q})(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$ *for any $\gamma \in \Gamma$, where $\tilde{f} : S^1_{\infty} \to S^1_{\infty}$ is the extension of a lift of f.*

PROOF. Since the mapping class group $\mathcal{M}$ is isomorphic to a subgroup of $\mathrm{Out}(\Gamma)$ of index 2, we have a diffeomorphism $f \in \mathrm{Diff}^{\infty}_{+} \Sigma$ whose isotopy class $[f] \in \mathcal{M}$ corresponds to $\mu(q) \in \mathrm{Out}(\Gamma)$ through the isomorphism. Moreover, we can assume that the diffeomorphism f fixes a base point $* \in \Sigma$ and the induced homomorphism $f_* : \pi_1(\Sigma, *) \to \pi_1(\Sigma, *)$ is equal to $\mathrm{conj}(\tilde{q}) \in \mathrm{Aut}(\Gamma)$ (with the identification $\Gamma = \pi_1(\Sigma, *)$). From now on, we use the Poincaré disk model $\boldsymbol{D}$ instead of the upper half plane model $\boldsymbol{H}^2$. Thus we have the universal covering $\pi : \boldsymbol{D} \to \Sigma$. Choose a base point of $\boldsymbol{D}$, also denoted by $* \in \boldsymbol{D}$, which is a lift of the base point $* \in \Sigma$. For any $\gamma \in \pi_1(\Sigma, *)$, we choose a loop g at $* \in \Sigma$ representing γ. Then we have the lift $\tilde{g}_0$ of g begining at $* \in \boldsymbol{D}$. We also have the lifts $\tilde{g}_1$ and $\tilde{g}_{-1}$ of g, which begin at the terminal point of $\tilde{g}_0$ and end at the initial point of $\tilde{g}_0$, respectively. Iterating this procedure repeatedly forward and backward, we have two limit points on the circle at infinity. Denote the backward limit point by x_g^α and the forward limit point by x_g^ω. Namely, we choose the component of $\pi^{-1}(g)$ passing the base point $* \in \boldsymbol{D}$ and then we have its limit point $x_g^\alpha, x_g^\omega \in S^1_\infty$. Now we choose a unique lift $\tilde{f}$ of f such that $\tilde{f}(*) = *$ and $\tilde{f}(\tilde{g}) = \widetilde{f(g)}$, where $\widetilde{f(g)}$ is the lift of the loop $f(g)$ in Σ begining at $* \in \boldsymbol{D}$. It is now clear that the extension of $\tilde{f}$ to S^1_∞ sends x_g^α to $x_{f(g)}^\alpha$ and x_g^ω to $x_{f(g)}^\omega$. Therefore, as an action on S^1_∞, we have $\mathrm{conj}(\tilde{q})(\gamma) = f_*(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$. □

CLAIM 2. *Suppose that homeomorphisms h_1 and h_2 of S^1_∞ satisfy $h_1 \circ \gamma \circ h_1^{-1} = h_2 \circ \gamma \circ h_2^{-1}$ for any $\gamma \in \Gamma$. Then $h_1 = h_2$.*

PROOF. Set $h = h_1^{-1} \circ h_2$. Then the hypothesis implies that $\gamma \circ h = h \circ \gamma$ for any $\gamma \in \Gamma$ on S^1_∞. Each $\gamma \in \Gamma$ is a hyperbolic translation and it fixes exactly two points on S^1_∞. Since $\gamma \circ h = h \circ \gamma$, h also fixes the fixed points of each $\gamma \in \Gamma$. It can be easily seen that the union of the fixed point sets of all $\gamma \in \Gamma$ is dense in S^1_∞. Therefore h is the identity. □

Now, with the identification of Γ with the Fuchsian group in $PSL(2, \boldsymbol{R})$, φ sends $\tilde{q}\cdot\gamma\cdot\tilde{q}^{-1}$ to $\varphi(\tilde{q}) \cdot \gamma \cdot \varphi(\tilde{q})^{-1}$. Thus, by Claim 2, $\varphi(\tilde{q}) \in \mathrm{Diff}^\infty_+ S^1$ is equal to $\tilde{f}|S^1_\infty$. However, since f cannot be isotopic to a diffeomorphism of finite order, there is no hyperbolic metric on Σ which makes f an isometry (cf. Lemma 1). By Lemma 3, this implies that $\tilde{f}|S^1_\infty$ cannot be a diffeomorphism of the circle. This is a contradiction. □

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