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AUTOMORPHISMS OF A SURFACE OF GENERAL TYPE ACTING TRIVIALLY IN COHOMOLOGY

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Abstract. It is proved that, for a complex minimal smooth projective surface S of general type, any automorphism group of S, inducing trivial actions on the second rational cohomology of S, is isomorphic to a cyclic group of order less than five or the product of two groups of order two, provided that the Euler characteristic of the structure sheaf of S is larger than 188.

Introduction. It is well-known that, for a curve C of genus $g \geq 2$, the automorphism group $\mathrm{Aut}\, C$ acts faithfully on $H^1(C, \boldsymbol{Q})$.

The case of surfaces has been studied by many authors. For K3 and Enriques surfaces S, $\mathrm{Aut}\, S$ acts faithfully on $H^2(S, \boldsymbol{Z})$ (cf. [BR], [Ue]); and there exists an Enriques surface S for which $\mathrm{Aut}\, S$ does not act faithfully on $H^2(S, \boldsymbol{Q})$ (cf. [Pe]). For compact Kähler surfaces S with $h^0(T_S) = 0$ and the canonical linear system $|K_S|$ base point free, Peters [Pe] proved that, if a non-trivial $\sigma \in \mathrm{Aut}\, S$ acts trivially on $H^2(S, \boldsymbol{Q})$, then either $K_S^2 = 8\chi(\mathcal{O}_S)$ and the order $\mathrm{o}(\sigma)$ of σ is a power of 2 or $K_S^2 = 9\chi(\mathcal{O}_S)$ and $\mathrm{o}(\sigma)$ is a power of 3.

Taking the product of two hyperelliptic curves, one gets easily examples of surfaces of general type for which $\mathrm{Aut}\, S$ does not act faithfully on $H^2(S, \boldsymbol{Q})$. The aim of this paper is to prove the following

THEOREM A. *Let S be a complex minimal smooth projective surface of general type, and $\chi(\mathcal{O}_S)$ the Euler characteristic of the structure sheaf of S. Let $G \subset \mathrm{Aut}\, S$ be a subgroup of automorphisms acting trivially on $H^2(S, \boldsymbol{Q})$. If $\chi(\mathcal{O}_S) > 188$, then G is isomorphic to C_n $(n \leq 4)$ or $C_2 \times C_2$, where C_n is a cyclic group of order n.*

Theorem A is proved in Sections 2 through 4. Thanks to Beauville's theorem on the canonical map of S, the problem reduces to the analysis of the automorphisms of the canonical fiber surface $f : S \to B$, of genus $g \leq 5$. The main part of this paper is to treat the case $g = 3$ and G nonabelian of order 8 or 6. The idea of the proof is to prove the existence of a G-invariant irreducible curve (in a singular fiber of f) on which G acts faithfully and to analyze the action around it.

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We use standard notation as in [BPV] or [Ha]. In this paper we denote by C_n, D_{2n} and Q_8 the cyclic group of order n, the dihedral group of order $2n$, and the quaternion group of order 8.

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1. Preliminaries. For the reader's convenience, in this section we recall several results from the literature.

(1.1) Let S be a smooth complex projective surface of general type, with a fibration $f: S \to B$ of genus $g \geq 2$ over a smooth curve B. We assume that f is relatively minimal, that is, S has no (-1)-curves contained in a fiber of f. Denote by F the general fiber of f. Let K_S be the canonical divisor of S.

We say that f is a *hyperelliptic* (resp. *nonhyperelliptic*) *fibration* if F is a hyperelliptic (resp. nonhyperelliptic) curve. An irreducible curve C on S is *vertical* (with respect to f) if $f(C)$ is a point; otherwise, we say C is *horizonal*.

(1.2) Let $f: S \to B$ be a relatively minimal fibration of genus $g \geq 2$, and σ an involution of S inducing the trivial action on B. Let $u : \tilde{S} \to S$ be the blowup of all isolated fixed points of σ, and $\tilde{\sigma}$ the induced involution on $\tilde{S}$. Let $P_\sigma = \tilde{S}/\tilde{\sigma}$. Then f induces a fibration $h_\sigma: P_\sigma \to B$ of genus $g(F/\sigma)$ (not relatively minimal in general). We have a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\pi} & P_\sigma \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle h_\sigma} \\ S & \xrightarrow[f]{} & B. \end{array}$$

(1.2.1) If $\Gamma < f^*b$ ($b \in B$) is a σ-fixed curve, then from $(f \circ u)^*b = \pi^*(h_\sigma^* b)$, the coefficient of Γ in f^*b is divisible by 2. In particular, if f^*b is reduced, then σ acts nontrivially on any irreducible component of f^*b.

Let F' be a semistable fiber of f (i.e., F' is reduced with only nodes as singularities), and $p \in F'$ a node. We say that p is a *separating point* (resp. *nonseparating point*) of F', if $F' \setminus \{p\}$ is disconnected (resp. connected) as a topological space.

(1.3) (cf. [Ca, Lemma 2.4]) Let $f: S \to B$ and σ be as above, and F' a semistable singular fiber of f. If $p \in F'$ is an isolated fixed point of σ, then p is a node of F', and moreover if σ is a hyperelliptic involution of S, then p is a separating point of F'.

(1.4) Notation as in (1.2). If f is a relatively minimal hyperelliptic fibration, gluing the hyperelliptic involution of F gives an everywhere defined involution σ on S. Then $h_\sigma: P_\sigma \to B$ is a ruled surface. Let $(\tilde{R}, \tilde{\delta})$ be the double cover data corresponding to $\pi: \tilde{S} \to P_\sigma$. One has a minimal ruled surface P, and a (possibly singular) double cover data (R, δ) on P, satisfying the following conditions:

(i) There is a birational morphism $\phi: P_\sigma \to P$ such that $\tilde{R}$ is the reduced inverse image of R;

(ii) Let R_h be the sum of the nonvertical irreducible components of R. Then the singularities of R_h are at most of order $g+1$, and R^2 is the smallest among all such choices (cf. [X1, Lemma 6]). (P, R, δ) is called the *genus g data* corresponding to f.

(1.4.1) ([X1, Definition 5]) Let $f\colon S \to B$ be a hyperelliptic fibration corresponding to genus g data (P, R, δ). For any fiber F of f and $i = 3, \dots, g+2$, we define the i-singularity $s_i(F)$ of F as follows:

If i is odd, $s_i(F)$ equals the number of singularities of type $i \to i$ (that is, infinitely near points of multiplicity i) of R on the image of F.

If i is even, $s_i(F)$ equals the number of singularities of order i of R on the image of F, not belonging to a singularity of type $i-1 \to i-1$ or $i+1 \to i+1$.

The singularities $s_i(F)$ do not depend on the choice of the contraction map $\phi\colon\ P_\sigma \to P$ (cf. [X1, Lemma 8]). Clearly there are only a finite number of fibers F with $s_i(F) \neq 0$ for each i. A fiber F is *essential*, if $s_i(F) \neq 0$ for some i.

(1.4.2) (Xiao [X1, Theorem 1]) Let $f\colon S \to B$ be the hyperelliptic fibration corresponding to genus g data (P, R, δ). If f has no essential fibers, then

$$K_S^2 = \frac{4g-4}{g}\chi(\mathcal{O}_S) - \frac{4(g^2-1)(g(B)-1)}{g}\,.$$

(1.5) (Reid [Re]) Let $f\colon S \to B$ be a nonhyperelliptic fibration of genus $g = 3$. Then the natural morphism of sheaves

$$r\colon S^2(f_*\omega_{S/B}) \to f_*\omega_{S/B}^2$$

is generically surjective. Let $\mathcal{M} = \operatorname{Coker} r$. Then $\mathcal{M} = \bigoplus_{b\in B}\mathcal{M}_b$, where $\mathcal{M}_b$ is the stalk of $\mathcal{M}$ at $b \in B$, which is an $\mathcal{O}_{B,b}$-module of finite length. Let $H(S/B, b) = \text{length}\,\mathcal{M}_b$. For any $b \in B$, if f^*b is a smooth nonhyperelliptic curve or an irreducible nonhyperelliptic curve with one node whose normalization is a curve of genus 2, then $H(S/B, b) = 0$. Using the Riemann-Roch theorem on S and the Leray spectral sequence, we have

$$K_S^2 = 3\chi(\mathcal{O}_S) + 10(g(B)-1) + \sum_{b\in B} H(S/B, b)\,.$$

For any normal surface X, we denote by $p_g(X)$ the geometric genus of a nonsingular model of X.

(1.6) (Beauville [Be]) Let S be a projective minimal nonsingular surface of general type with $\chi(\mathcal{O}_S) \geq 21$, and $\phi_S\colon S - - \to \boldsymbol{P}^{p_g(S)-1}$ the canonical map. There are two cases:

(1.6.1) ϕ_S is composed with a pencil. Then the moving part of $|K_S|$ is base point free. Let $f : S \to B$ be the fibration associated with ϕ_S, and g the genus of the general fiber of f. Then $2 \leq g \leq 5$ and $K_S^2 \geq (2g-2)(\chi(\mathcal{O}_S) - 2)$.

(1.6.2) $\dim \operatorname{Im}\phi_S = 2$. If $\chi(\mathcal{O}_S) \geq 31$, then either (i) $p_g(\operatorname{Im}\phi_S) = 0$ and $\deg\phi_S \leq 9$ or (ii) $p_g(\operatorname{Im}\phi_S) = p_g(S)$ and $\deg\phi_S \leq 3$.

(1.7) (Xiao [X2]) Let $f : S \to B$ be as in (1.6.1). Then $g(B) \leq 1$.

(1.8) (A special case of the logarithmic Miyaoka-Yau inequality. cf. [Sa]) Let S be a projective nonsingular complex surface of general type and $C \subset S$ a nonsingular curve. Then $K_S^2 \leq 9\chi(\mathcal{O}_S) + (g(C) - 1) - K_S C/4$.

(1.9) (Accola [Ac]) Let C be a curve of genus g, and $G \subset \mathrm{Aut}\, C$ a finite group. If G admits a partition, i.e., $G = \bigcup_{i=1}^{s} G_i$, where G_i are subgroups of G satisfying $G_i \cap G_j = \langle 1_G \rangle$ for all $i \neq j$, then

$$(s-1)g + |G|g(C/G) = \sum_{i=1}^{s} |G_i| g(C/G_i)\,.$$

For example, assume that $G = D_{2n}$ is a dihedral group of order $2n$. Let $\alpha \in G$ generate the cyclic subgroup of order n, and let $\beta \in G$ be an element of order 2 not in $\langle \alpha \rangle$. Then $\beta_i = \alpha^i \beta$ ($i = 1, 2, \ldots, n$) are elements in G not in $\langle \alpha \rangle$. So G admits a partition and we have

$$g + 2g(C/G) = g(C/\langle \alpha \rangle) + g(C/\langle \beta_1 \rangle) + g(C/\langle \beta_2 \rangle)\,.$$

(1.10) Let S be a smooth surface, $\sigma \in \mathrm{Aut}\, S$, and $p \in S$ a fixed point of σ. Then σ induces a linear action on the tangent space $T_p S$ of S at p. If this action is trivial, then σ is trivial.

A curve $C \subset S$ is *σ-invariant* (resp. *σ-fixed*), if $\sigma(C) = C$ (resp. $\sigma(p) = p$ for any $p \in C$).

(1.11) If a reduced σ-fixed curve C is singular, then σ is trivial. This follows from (1.10), since the induced action of σ on the tangent space at the singular point of C is trivial.

(1.12) Let C be a curve of genus g, and $G \subset \mathrm{Aut}\, C$ a finite group. If G has a fixed point, then G is cyclic.

(1.13) Let C be a curve of genus $g \geq 2$, and $G \subset \mathrm{Aut}\, C$ an abelian group. Assume that $g(C/G) = 1$. Let $\pi : C \to C/G$ be the quotient map. Let q_i ($i = 1, \ldots, k$) be the points over which π is ramified and r_i the ramification number of π over q_i. Then $k \geq 2$, and if $k = 2$ then $r_1 = r_2$. Indeed, G is an abelian quotient of $\pi_1(C/G \setminus \{q_1, \ldots, q_k\})$, which is generated by $\alpha, \beta, \gamma_1, \ldots, \gamma_k$ with one relation $\alpha\beta\alpha^{-1}\beta^{-1}\gamma_1 \cdots \gamma_k = 1$, where α and β are generators of $\pi_1(C/G)$ and γ_i is a small loop around q_i. Let $\bar{\gamma}_i$ be the image of γ_i in G. Then $\bar{\gamma}_i$ is of order r_i and $\bar{\gamma}_1 \cdots \bar{\gamma}_k = 1$.

(1.14) Let S be a smooth projective surface, and $G \subset \mathrm{Aut}\, S$ a finite subgroup such that G acts trivially on $H^2(S, \boldsymbol{Q})$. By the argument of [Pe, Lemma 2], we have that, if $p \in S$ a σ-fixed point for some $\mathrm{id} \neq \sigma \in G$, then either $p \in \mathrm{Bs}|K_S|$ (the base locus of $|K_S|$) or p is an isolated σ-fixed point. This implies:

(1.14.1) If $C \subset S$ is a σ-fixed curve for some $\mathrm{id} \neq \sigma \in G$, then $C \subset \mathrm{Bs}|K_S|$.

(1.14.2) If $C \subset S$ is a G-invariant curve, and $C \not\subset \mathrm{Bs}|K_S|$, then G acts faithfully on C, i.e., $G \hookrightarrow \mathrm{Aut}\, C$.

(1.15) Let S and G be as in (1.14). Assume that S has a fibration $f : S \to B$ and G induces the trivial action on B. If $p_g(S) > 0$ then $g(F/G) > 0$, where F is a general fiber of f. Indeed, We have $p_g(S/G) = \dim H^0(S, \omega_S)^G$ (cf. [Fr, p. 99]). By Hodge theory,

$H^0(S, \omega_S)^G = H^0(S, \omega_S)$. So $p_g(S/G) = p_g(S)$ and thus the general fiber of $S/G \to B$ is not rational if $p_g(S) > 0$.

2. First reductions. To prove Theorem A, let me start by fixing notation.

(2.1) Let S be a complex minimal nonsingular projective surface of general type with $\chi(\mathcal{O}_S) \geq 21$. Assume that the canonical map ϕ_S of S is composed with a pencil.

Let $G \subset \operatorname{Aut} S$ be a subgroup of automorphisms of S, inducing trivial actions on $H^2(S, \boldsymbol{Q})$.

Let M and Z be the moving part and the fixed part of $|K_S|$, respectively. By (1.6.1), $|M|$ has no base points. Let

$$\phi_S = \varphi \circ f\colon S \to B \to \operatorname{Im} \phi_S \subset \boldsymbol{P}^{p_g(S)-1}$$

be the Stein factorization of ϕ_S. We call $f\colon S \to B$ the *canonical fibration* associated with ϕ_S. Let F be a general fiber of f, and g the genus of F.

Let d and L be the degree and the hyperplane section of $\operatorname{Im} \phi_S$ in $\boldsymbol{P}^{p_g(S)-1}$ respectively. We have $\mathcal{O}_S(M) = f^*\varphi^*L$ and $M \sim_{\text{num}} \deg \varphi d F$. Note that $h^1(B, \varphi^*L) = 0$, since $g(B) \leq 1$ by (1.7), and $d \geq \operatorname{codim} \operatorname{Im} \phi_S + 1$ (cf. [Mu]). From

$$p_g(S) = h^0(S, \varphi^*L) = \deg(\varphi^*L) + 1 - g(B) + h^1(B, \varphi^*L) = \deg \varphi d + 1 - g(B)\,,$$

we get

$$\deg \varphi = 1 \quad \text{and} \tag{2.1.1}$$

$$d = \begin{cases} \chi(\mathcal{O}_S) & \text{if } g(B) = 1\,, \\ \chi(\mathcal{O}_S) - 2 + q(S) & \text{if } g(B) = 0\,. \end{cases} \tag{2.1.2}$$

(2.2) Since $H^0(S, \omega_S)$ is a direct factor of $H^2(S, \boldsymbol{C})$ by Hodge theory, G acts trivially on $H^0(S, \omega_S)$. This implies that G acts trivially on $\operatorname{Im} \phi_S$ and there is a homomorphism h of G into $\operatorname{Aut} B$. Since $\deg \varphi = 1$ (2.1.1), we have that $\operatorname{Ker} h = G$, i.e., G induces the trivial action on B, and $G \hookrightarrow \operatorname{Aut} F$ for a general fiber F of f.

NOTATION 2.3. Let $f\colon S \to B$ and G be as above.

(i) We write $Z = H + V$ and $H = n_1\Gamma_1 + n_2\Gamma_2 + \cdots$ with $n_1 \geq n_2 \geq \cdots$, where H (resp. V) is the horizontal part (resp. the vertical part) of Z, and Γ_i ($i = 1, 2, \ldots$) are the irreducible components of H, with n_i the multiplicity of Γ_i in H.

(ii) For a general fiber F of f, let R_F be the set of ramified points of the quotient map $F \to F/G$. For any two curves C and D on S, we denote by $C \cap D$ the set-theoretic intersection $\operatorname{supp} C \cap \operatorname{supp} D$.

LEMMA 2.4. *Let $f\colon S \to B$, H, Γ_i and G be as in* (2.1). *Let F be a general fiber of f. Then*

(2.4.1) $R_F \subset H \cap F$.

(2.4.2) *If $R_F = H \cap F$, then Γ_i is smooth for every i.*

PROOF. (i) Suppose that there is a point $p \in F$ such that $p \in R_F$ and $p \notin H \cap F$. Then there exists an element $\mathrm{id} \neq \sigma \in G$ such that p is σ-fixed. Since F is a general fiber, p is not an isolated fixed point of σ. So there exists a σ-fixed curve C passing through p. By (1.14.1), $C \subset \mathrm{Bs}|K_S|$. C is not vertical since F is a general fiber. So $C < H$, which contradicts the assumption that $p \notin H \cap F$.

(ii) For a general point $p \in \Gamma_i$, $p \in H \cap f^*(f(p)) = R_{f^*(f(p))}$. This implies there exists $\mathrm{id} \neq \sigma_p \in G$ such that p is σ_p-fixed. Since G is finite, there is a $\mathrm{id} \neq \sigma \in G$ such that Γ_i is σ-fixed. So Γ_i is smooth by (1.11). □

LEMMA 2.5. *Let $f: S \to B$, H, g and G be as in* (2.1). *Let F be a general fiber of f. If $2 \leq g \leq 4$, then either $|G| \leq 2g-2$, or G is nonabelian, G acts transitively on $H \cap F'$ for any fiber F' of f, and the only possibilities for the triple $(g, |G|, \#(H \cap F))$ are as follows*:

$$(3,\ 8,\ 4)\,, \quad (3,\ 6,\ 2)\,, \quad (4,\ 12,\ 6)\,, \quad (4,\ 8,\ 2)\,.$$

Moreover, if $(g, |G|, \#(H \cap F)) = (3,\ 8,\ 4)$ or $(4,\ 12,\ 6)$, then H is reduced and each irreducible component of H is smooth.

PROOF. For any point $p \in S$, let $\mathrm{stab}(p) = \{\tau \in G | \tau(p) = p\}$. If $r_p: = |\mathrm{stab}(p)| = 1$ for some $p \in H \cap F$, then $|G| \leq \#(H \cap F) \leq 2g-2$. So we can assume that $|\mathrm{stab}(p)| \geq 2$ for each $p \in H \cap F$. Let m be the number of orbits of $H \cap F$ under the action of G. Then by (2.4.1), the quotient map $\pi: F \to F/G$ has exactly m branch points. Using the Hurwitz formula for π, we get $|G| \leq 2g-2$ if either $g(F/G) \geq 2$, or $g(F/G) = 1$ and either G is abelian or $m \geq 2$. Hence we can assume that $g(F/G) = 1$ ($g(F/G) \neq 0$ by (1.15)), G is nonabelian and $m = 1$. Then G acts transitively on $H \cap F$ and hence on $H \cap F'$ for any fiber F' of f. In this case, for any point $p \in H \cap F$, we have $|G|/r_p = \#(H \cap F)$ and $\#(H \cap F) \mid 2g-2$. Using the Hurwitz formula for π again, we have $|G| = \#(H \cap F) + 2g - 2$. Note that G is nonabelian in this case, and we get that $(g, |G|, \#(H \cap F))$ equals one of the triples listed in the lemma. The last statement follows by (2.4.2). □

REMARK 2.6. If $(g, |G|, \#(H \cap F)) = (3,\ 6,\ 2)$, then either $H = 2\Gamma_1 + 2\Gamma_2$ or $H = 2\Gamma$, where Γ_i are sections of f and Γ is an irreducible smooth curve with $\Gamma F = 2$.

PROPOSITION 2.7. *Let S be a complex minimal nonsingular projective surface of general type with $\chi(\mathcal{O}_S) \geq 21$, and let $G \subset \mathrm{Aut}\, S$ be a subgroup of automorphisms of S inducing trivial actions on $H^2(S, \boldsymbol{Q})$. Assume that the canonical map ϕ_S is composed with a pencil. Let $f: S \to B$ be the canonical fibration associated with ϕ_S, and g the genus of a general fiber of f. Furthermore, assume $\chi(\mathcal{O}_S) > 188$ if $g = 4$, and $\chi(\mathcal{O}_S) > 60$ if $g = 5$. Then $|G| \leq 4$.*

PROOF. By (1.6.1), $2 \leq g \leq 5$. If $g = 2$, we have $|G| \leq 2$ by Lemma 2.5. The proof of the case $3 \leq g \leq 5$ is longer and is postponed till the next two sections. □

PROOF OF THEOREM A. By Proposition 2.7, we can assume that ϕ_S is generically finite. Since $H^0(S, \omega_S)$ is a direct factor of $H^2(S, \boldsymbol{Q})$ by Hodge theory, G acts trivially on $H^0(S, \omega_S)$. This implies that G induces trivial actions on $\mathrm{Im}\, \phi_S$. So ϕ_S factors through the quotient map

$$\phi_S = \alpha \circ q : S \xrightarrow{q} S/G \xrightarrow{\alpha} \operatorname{Im} \phi_S .$$

Thus $|G| = \deg \phi_S / \deg \alpha$. Now if S is as in case (ii) of (1.6.2), then $|G| \leq 3$. If S is as in case (i) of (1.6.2), then $\deg \alpha \geq 2$, since $p_g(S/G) = p_g(S) \neq 0 = p_g(\operatorname{Im} \phi_S)$. So $|G| \leq \deg \phi_S / 2 \leq 9/2$. □

3. Proof of Proposition 2.7, the case $g = 3$.

LEMMA 3.1. *Let S be a complex nonsingular projective surface, and $G \subset \operatorname{Aut} S$ a subgroup of automorphisms of S inducing trivial actions on $H^2(S, \boldsymbol{Q})$. Let $C \subset S$ be an irreducible curve. If $C^2 < 0$, then C is G-invariant.*

PROOF. Indeed, if C is not σ-invariant for some $\mathrm{id} \neq \sigma \in G$, then $(\sigma^* C)C \geq 0$. On the other hand, $\sigma^* C$ is numerically equivalent to C, since G acts trivially on $\mathrm{NS}(S) \otimes \boldsymbol{Q} \hookrightarrow H^2(S, \boldsymbol{Q})$. So $(\sigma^* C)C = C^2 < 0$, a contradiction. □

LEMMA 3.2. *Let $f : S \to B$, H, g and G be as in* (2.1). *Assume that $g = 3$ and G is a nonabelian group of order* 8.

(i) *Let σ be the generator of the center of G, which is clearly a cyclic subgroup of order* 2. *Then H is σ-fixed* (*and hence smooth*), *and $G/\langle \sigma \rangle \hookrightarrow \operatorname{Aut} H$.*

(ii) *Let F' be a singular fiber of f and $C < F'$ an irreducible component with $CH \neq 0$. Then $G \hookrightarrow \operatorname{Aut} C$.*

PROOF. (i) Let F be a general fiber of f. Let $\bar{F} = F/\langle \sigma \rangle$ and $\bar{G} = G/\langle \sigma \rangle$. Since $g(F/G) = 1$ and $|\bar{G}| = 4$, using the Hurwitz formula for $\bar{F} \to \bar{F}/\bar{G} \simeq F/G$, we get $g(\bar{F}) = 1$. So σ has four fixed points on F. Since $\#(H \cap F) = 4$ in Lemma 2.5, by (2.4.1), H is σ-fixed and hence smooth by (1.11). Since G acts transitively on $H \cap F$ and $\#(H \cap F) = 4$, we have $G/\langle \sigma \rangle \hookrightarrow \operatorname{Aut} H$.

(ii) If F' is reducible, C is G-invariant by (3.1); if the reduced scheme F'_{red} of F' is irreducible, then $C = F'_{\mathrm{red}}$ is clearly G-invariant. So there is a homomorphism $h : G \to \operatorname{Aut} C$.

Let σ be as in (i). If $\sigma \in \operatorname{Ker} h$, then $C + H$ is σ-fixed. So σ is trivial by (1.11). This is impossible. Hence the lemma follows by showing that $\sigma \in \operatorname{Ker} h$ if $\operatorname{Ker} h$ is not trivial.

Suppose that $\operatorname{Ker} h$ is not trivial. If $G \simeq Q_8$, we have that $\sigma \in \operatorname{Ker} h$ since there is only one element of order 2 in Q_8. Now assume that $G \simeq D_8$. If $|\operatorname{Ker} h| = 2$, we get $\operatorname{Ker} h = \langle \sigma \rangle$ since a normal subgroup of order 2 must be contained in the center of G; If $|\operatorname{Ker} h| = 4$, let $\alpha \in G$ be an element of order 4. Then $\sigma = \alpha^2$ and $h(\alpha^2) = h(\alpha)^2 = \mathrm{id}$. So $\sigma \in \operatorname{Ker} h$. □

LEMMA 3.3. (i) *Let G be a nonabelian group of order* 8. *Assume that $G \hookrightarrow \operatorname{Aut} C$ for some smooth curve C of genus ≤ 1. Then $G \simeq D_8$. Moreover, if $g(C) = 1$, the elements of order* 4 *of G act freely on C.*

(ii) *If $G \simeq D_6 \hookrightarrow \operatorname{Aut} C$ for some smooth elliptic curve C, then the elements of order* 3 *of G act freely on C.*

PROOF. (i) If $C \simeq \boldsymbol{P}^1$, the lemma follows by the well known fact that a finite subgroup of Aut $\boldsymbol{P}^1$ is isomorphic to one of the following groups: C_n, D_{2n}, T_{12}, O_{24} and I_{60}, where T_{12}, O_{24} and I_{60} are the polyhedral groups of indicated orders.

If C is an elliptic curve, then $G = T \rtimes A$ (a semi-direct product), where T is a group of translations and $A \subset \operatorname{Aut} C$ is a subgroup preserving the group structure. If $T \simeq C_2$, then G must be abelian, which contradicts the assumption. Now assume that $|T| = 4$. Let $\alpha \in G$ be an element of order 4. Then it is easy to see that $\alpha^2 \in T$ since $|A| = 2$ in this case. So α^2 and hence α has no fixed points. This implies $\alpha \in T$. Hence $T \simeq C_4$, and the result follows.

(ii) follows by an argument similar to that in (i). □

LEMMA 3.4. *Let $f: S \to B$, g and G be as in* (2.1). *Assume that $g = 3$.*
 (i) *If $G \simeq Q_8$, f is nonhyperelliptic.*
 (ii) *If $G \simeq D_8$ or D_6, f is hyperelliptic.*

PROOF. (i) Otherwise, let τ be the hyperelliptic involution of a general fiber F of f. Since $g(F/G) = 1$ by (1.15), we get $\tau \notin G$. This implies $G \hookrightarrow \operatorname{Aut} \boldsymbol{P}^1$, since Aut F is a $\langle\tau\rangle$-extension of a subgroup of Aut $\boldsymbol{P}^1$. This is impossible by Lemma 3.3.

(ii) Let F be a general fiber of f. If $G \simeq D_{2n}$ for $n = 3$ or 4, then by (1.9) we have $g(F) + 2g(F/D_{2n}) = g(F/\langle\alpha\rangle) + g(F/\langle\beta_1\rangle) + g(F/\langle\beta_2\rangle)$, where α and β_i are as in (1.9). Since $g(F/D_{2n}) = 1$ and $g(F/\langle\alpha\rangle) = 1$, we get $g(F/\langle\beta_i\rangle) = 2$. So F is étale over a curve of genus 2. This implies F is hyperelliptic by [Ac]. □

LEMMA 3.5. *Let $f: S \to B$ be a nonhyperelliptic fibration of genus* 3, *and $G \subset \operatorname{Aut} S$ a subgroup inducing the trivial action on B. Let F' be a fiber of f. Assume that $G \simeq Q_8$, and that F' is either a smooth hyperelliptic curve or a multiple fiber $2C$ with C smooth of genus* 2. *Then the kernel of the homomorphism $h: G \to \operatorname{Aut} F'_{\mathrm{red}}$ is not trivial.*

PROOF. Suppose that ker h is trivial. Denote by σ the unique element of order 2 in G.

First we assume that $F' = 2C$, where C is a smooth curve of genus 2. Let $p' = f(F')$ and fix a point $p \in B$ such that f^*p is smooth. Let $\tilde{B} \to B$ be a double cover ramified exactly at p and p', and let $\pi': \tilde{S} \to \tilde{B} \times_B S$ be the normalization. Then $\pi := p_2 \circ \pi': \tilde{S} \to S$ is ramified along f^*p, and $\tilde{f} := p_1 \circ \pi': \tilde{S} \to \tilde{B}$ is a fibration of genus 3, where p_1 and p_2 are the projections of $\tilde{B} \times_B S$ onto its factors. Let $\tilde{p}'$ be the inverse image of p'. Then $\tilde{F}' := \tilde{f}^*\tilde{p}'$ is a smooth hyperelliptic curve. Since G induces the trivial action on B, $\tilde{B} \times_B S \subset \tilde{B} \times S$ is G-invariant. So G acts on $\tilde{S}$, inducing the trivial action on $\tilde{B}$. We have $G \hookrightarrow \operatorname{Aut} \tilde{F}'$ if Ker h is trivial. Hence the lemma is reduced to the case when F' is a smooth hyperelliptic curve.

Now assume that F' is a smooth hyperelliptic curve. Let τ be the hyperelliptic involution of F'. If $\tau \notin G$, then $G \hookrightarrow \operatorname{Aut} \boldsymbol{P}^1$ since Aut F' is a $\langle\tau\rangle$-extension of a subgroup of Aut $\boldsymbol{P}^1$. This is impossible by Lemma 3.3. So we can assume that σ is the hyperelliptic involution of F'. Then there are eight σ-fixed points on F'. By (1.3), there exists a σ-fixed curve D passing through these points. Since $G \simeq Q_8 \hookrightarrow \operatorname{Aut} F'$ by assumption, we get $F' \neq D$. Now for a general fiber F, there are at least $\#(D \cap F) = DF = DF' \geq 8$ σ-fixed points.

This implies that σ is the hyperelliptic involution of F, contradicting the assumption that f is nonhyperelliptic. $\square$

LEMMA 3.6. *Let $f: S \to B$, H, g and G be as in* (2.1). *Assume that $g = 3$ and G is a nonabelian group of order* 8. *Let F' be a singular fiber of f and $C < F'$ an irreducible component. Denote by $\tilde{C}$ the normalization of C. If $g(\tilde{C}) \geq 2$, then F' belongs to one of the following possible types.*

(i) $F' = 2C$, *and C is smooth*;

(ii) $F' = C$ *is an irreducible curve with one node, and the normalization of F' is a curve of genus* 2;

(iii) $F' = C + D$, *where C and D are irreducible smooth curves meeting transversally at two points, and $g(C) = 2$ and $g(D) = 0$.*

PROOF. We have either $p_a(C) = 3$ or $C = \tilde{C}$. In the former case, $F' = C$ is an irreducible curve with one singularity, say q, and its normalization is a curve of genus 2. If $q \in F'$ is a cusp, the inverse image $\tilde{q}$ of the cusp $q \in F'$ under the normalization map is G-fixed. This implies G is cyclic by (1.12), a contradiction. So F' is of type (ii). In the latter case, since $K_S C = 2 - C^2 \geq 2$ and $K_S F' = 4$, we get either $C^2 = 0$ (F' is of type (i)) or $\mathrm{mult}_C F' = 1$. Now assume that $\mathrm{mult}_C F' = 1$. Then F' is 1-connected. Let $D < F'$ be an irreducible curve such that $DC > 0$. If $\#(D \cap C) = 1$, G is cyclic, a contradiction. So $\#(D \cap C) \geq 2$ and hence $DC \geq 2$. Note that $p_a(D + C) \leq 3$, hence we have $DC = 2$ and F' is of type (iii). $\square$

PROOF OF PROPOSITION 2.7, THE CASE $g = 3$. Let $f: S \to B$ be the canonical fibration associated with ϕ_S. By (2.2), G induces the trivial action on B, and $G \hookrightarrow \mathrm{Aut}\, F$, where F is a general fiber of f. Assume $g = 3$. By Lemma 2.5, if $|G| > 4$, then G is isomorphic to Q_8, D_8 or D_6. Now the result follows by the next claims. $\square$

CLAIM 3.7. *$G \simeq Q_8$ does not occur.*

PROOF. Suppose $G \simeq Q_8$. Then by Lemma 3.4, f is nonhyperelliptic. Since $g(B) \leq 1$ by (1.7), f has singular fibers.

Let F' be a singular fiber of f, and let $C < F'$ be an irreducible component such that $CH \neq 0$. By Lemma 3.2 (ii), we have $G \hookrightarrow \mathrm{Aut}\, C$. By Lemmas 3.3 (i), 3.5 and 3.6, we have that F' is of type (ii) or (iii) of (3.6).

If F' is of type (iii) of (3.6), we have that either $D \not< V$ or $HD > 0$, where V is as in (2.1). Indeed, if both $D < V$ and $HD = 0$ hold, then $C \not< V$ and thus $VD < 0$. But from $0 = K_S D = (M + H + V)D$, we get $VD = 0$, a contradiction. Now by Lemma 3.2 (ii) and (1.14.2), $Q_8 \hookrightarrow \mathrm{Aut}\, D$. This is impossible by Lemma 3.3 (i).

Now if F' is of type (ii) of Lemma 3.6, we show that F' is nonhyperelliptic.

Let σ be the generator of the center of G. We have $G/\langle\sigma\rangle \simeq C_2 \times C_2$. First we claim that the node $q \in F'$ is an isolated σ-fixed point. Otherwise, there is a σ-fixed curve D passing through q. By (1.14.1), $D < H$. Since $q \in F'$ is G-fixed, $q \in H$ is $G/\langle\sigma\rangle$-fixed. By Lemma 3.2 (i) and (1.12), $G/\langle\sigma\rangle$ is cyclic, a contradiction.

Second, we claim that σ preserves the local two branches at q. Indeed, let $G' \subset G$ be the subgroup preserving the local two branches at q. Clearly G' is cyclic of order 4. Let α be a generator of G'. If $\sigma \notin G'$, then σ and α generate G, and it is easy to see that $\sigma\alpha\sigma = \alpha^{-1}$. This implies that $G \simeq D_8$, a contradiction.

Now we have that q is an isolated σ-fixed point and that σ preserves the local two branches at q. So $h_\sigma^*(f(q))$ consists of two irreducible smooth curves meeting transversally at two points, where $h_\sigma : P_\sigma \to B$ is as in (1.2). Since h_σ is of genus 1, σ is a hyperelliptic involution of the normalization $\tilde{F}'$ of F'. This implies that F' is a nonhyperelliptic fiber. Indeed, if there exists an involution τ on F' such that $F'/\langle\tau\rangle \simeq \boldsymbol{P}^1$, then τ exchanges the local two branches at q, and τ is a hyperelliptic involution of $\tilde{F}'$. This implies that $\sigma = \tau$ on F', which is absurd since one preserves the local two branches at q while the other not.

By the above argument, we have that any singular fiber F' of f is a nonhyperelliptic irreducible curve with one node. By Lemma 3.5 and (1.14.2), f has no smooth hyperelliptic fibers. Thus f has no fibers F' with non-vanishing $H(S/B, f(F'))$ (see (1.5) for the notation). By (1.5), we have that

$$K_S^2 = 3\chi(\mathcal{O}_S) + 10(g(B) - 1)\,.$$

We get a contradiction by (1.6.1). □

CLAIM 3.8. *$G \simeq D_8$ or D_6 does not occur.*

PROOF. Suppose $G \simeq D_6$. The proof of the case $G \simeq D_8$ is similar and is left to the reader. By Lemma 3.4, f is hyperelliptic. We will show that

(3.8.1) any singular fiber of f belongs to one of the following types:

(i) $F' = C$ is an irreducible curve with one node, and the normalization of F' is a curve of genus 2;

(ii) $F' = C$ is an irreducible curve with three nodes, and the normalization of F' is isomorphic to $\boldsymbol{P}^1$;

(iii) $F' = C + D$, where C and D are irreducible smooth curves meeting transversally at two points, and $g(C) = 2$ and $g(D) = 0$.

We note that, if F' belongs to one of the types (i)–(iii), the singularities $s_i(F') = 0$ for $i \geq 3$ (see (1.4.1) for the definition). (We check it when F' is of the type (iii); the other cases are similar. Let q_1 and q_2 be nodes of $F' = C + D$. Let τ be the hyperelliptic involution of f. Let the notation be as in (1.2) and (1.4). Since the dual graph of $h_\tau(f(F'))$ is a tree, we have $\tau q_1 = q_2$. Let $\bar{C}$ and $\bar{D}$ be the images of C and D under π, respectively. Then $h_\tau(f(F')) = \bar{C} + \bar{D}$ consists of two smooth rational curves meeting transversally at one point, and $\tilde{R}$ meets $\bar{C}$ (resp. $\bar{D}$) transversally at six (resp. two) points. By the choice of $\phi : P_\tau \to P$, ϕ contracts $\bar{D}$. Thus there is only one singular point of order 2 of R on the image of F' and hence by (1.4.1) $s_i(F') = 0$ for $i \geq 3$.)

Admitting (3.8.1) for the moment, we have that f has no essential fibers, and hence by (1.4)

$$K_S^2 = \frac{8}{3}\chi(\mathcal{O}_S) - \frac{32(g(B) - 1)}{3}\,.$$

On the other hand, by (1.6.1), $K_S^2 \geq 4(\chi(\mathcal{O}_S) - 2)$, a contradiction.

It remains to prove (3.8.1). Let $\alpha \in D_6$ (resp. $\sigma \in D_6$) be an element of order 3 (resp. 2). By the proof of Lemma 2.5, H is α-fixed. Let F' be a singular fiber of f. Let $C < F'$ be an irreducible component such that $CH \neq 0$, and $\tilde{C}$ the normalization of C. Then C is α-invariant by Lemma 3.1, and the homomorphism h of G into $\operatorname{Aut} C$ is injective. (Otherwise, $\operatorname{Ker} h = \langle\alpha\rangle$ or G since the nontrivial normal subgroup of G is $\langle\alpha\rangle$. Hence α is trivial on $C + H$, which is impossible by (1.11).) We distinguish two cases according to whether $f_H\colon H_{\mathrm{red}} \to B$ is étale at $H \cap F'$ or not.

Case 1. f_H is étale at $H \cap F'$. In this case $H \cap F'$ consists of two points, say p_1 and p_2. Since $HF' = 4$, by Remark 2.6, F' is smooth at these points. Since H is α-fixed, p_1 and p_2 are α-fixed. By the choice of C, there are at least two α-fixed points (p_1 and p_2) on it, and C is smooth at p_i for $i = 1$ and 2.

If $g(\tilde{C}) = 2$, then by the proof of Lemma 3.6, we have that F' is (i) or (iii).

If $g(\tilde{C}) = 1$, then by Lemma 3.3 (ii), we get a contradiction.

Now we assume $g(\tilde{C}) = 0$. We show that in this case either F' is of type (ii) or there exists a σ-fixed point $p \in C$ with $2 \nmid \operatorname{mult}_p F'$. We consider three cases according to the singularities of C.

(i) *There is a point* $p \in C_{\mathrm{sing}}$ *with* $\operatorname{mult}_p C \geq 3$. Then $p \in C$ is an ordinary triple point and $C \setminus \{p\}$ is smooth and $F' = C$. So p is σ-fixed and $\operatorname{mult}_p F' = 3$.

(ii) $C_{\mathrm{sing}} \neq \emptyset$ *and for any point* $p \in C_{\mathrm{sing}}$, *with* $\operatorname{mult}_p C = 2$. Since p_1 and p_2 are α-fixed and α has exactly two fixed points on $\tilde{C} \simeq \boldsymbol{P}^1$, we have $\alpha(p) \neq p$ if $p \in C_{\mathrm{sing}}$. Hence either F' is of type (ii), or $F' = C$ is an irreducible curve with three cusps (say q_1, q_2 and q_3) and the normalization of F' is isomorphic to $\boldsymbol{P}^1$. In the latter case, let $\tilde{q}_i$ ($i = 1,\ 2,\ 3$) be the inverse image of q_i under the normalization map $\tilde{C} \to C$. Since $\{q_1, q_2, q_3\}$ is σ-invariant and there are exactly two σ-fixed points on $\tilde{C} \simeq \boldsymbol{P}^1$, there must be a point $\tilde{p} \in \tilde{C} \setminus \{q_1, q_2, q_3\}$ which is σ-fixed. Let p be the image of $\tilde{p}$ under the normalization map. Then p is σ-fixed and $\operatorname{mult}_p F' = 1$.

(iii) *C is a smooth rational curve.* Since F' is 1-connected, there is an irreducible curve $D < F'$ such that $DC > 0$. Since $D \cap C$ is α-invariant by Lemma 3.1, if $\#(D \cap C) \not\equiv 0 \pmod 3$, α has at least there fixed points (p_1, p_2 and a point in $D \cap C$) on C. This implies α is trivial on C and hence on $C + H$, which is impossible by (1.11). So we can assume $\#(D \cap C) \equiv 0 \pmod 3$. Since $p_a(C + D) \leq 3$, we have $DC \leq 4$. So $\#(D \cap C) = 3$. Since $D \cap C$ is σ-invariant and there are exactly two σ-fixed points on $\tilde{C} \simeq \boldsymbol{P}^1$, there is a point $p \in C \setminus D \cap C$ which is σ-fixed. We claim that $\operatorname{mult}_p F' = 1$. Otherwise, there is an irreducible curve $D' < F'$ passing through p. By the above argument, we can assume that $\#(D' \cap C) \equiv 0 \pmod 3$. This implies $p_a(C + D + D') > 3$, a contradiction.

Now by the above argument, we have that either F' is of type (ii) or there is a σ-fixed point $p \in C$ with $2 \nmid \operatorname{mult}_p F'$. In the latter case, let $u\colon \tilde{S} \to S$ be as in (1.2). If p is an isolated σ-fixed point, then the inverse image $E = u^{-1}(p)$ of p is a σ-fixed (-1)-curve, and the coefficient of E in $(f \circ u)^*(f(F'))$ is not divisible by 2. This is impossible by (1.2.1). So there is a σ-fixed curve D passing through p. Clearly $D \neq C$ by (1.11). By (1.14.1),

$D < \mathrm{Bs}|K_S|$, and hence $D < H$. Since α and σ generate G, this implies there is a G-fixed point $p' \in H \cap F$, and thus G is cyclic by (1.12), a contradiction.

Case 2. f_H is not étale at $H \cap F'$. Then $H \cap F'$ consists of one point, say p. By the choice of C, C passes through p. Since $G \hookrightarrow \mathrm{Aut}\ C$, by (1.12), $p \in C$ is a singular point.

Since $HF' = 4$, we have $\mathrm{mult}_C\, F' = 1$ and $\mathrm{mult}_p\, C = 2$. If $p \in C$ is a cusp, it is easy to see G is cyclic by (1.12). So we can assume that $p \in C$ is a node. Blowup S at p, and let E be the exceptional curve and $\tilde{H}$ the strict transform of H. If p is an ordinary node of C, then α preserves the local branches of C at p since the order of α is 3. So α preserves the three local branches of $C + H$ at p. This implies E and hence $E + \tilde{H}$ is α-fixed. By (1.11) α is trivial on S, a contradiction. Now we can assume that $p \in C$ is a node which can be resolved by at least two successive blowups. Then $p_a(C) \geq 2$ and $g(\tilde{C}) \leq 1$, where $\tilde{C}$ is the normalization of C. If $g(\tilde{C}) = 1$, by Lemma 3.3(ii), α is a translation of $\tilde{C}$, which is impossible since α preserves the local branches of C at p. Now we assume $g(\tilde{C}) = 0$. If F' is reducible, let D be an irreducible curve $D < F'$ such that $DC > 0$. Since $D \cap C$ is α-invariant by Lemma 3.1 and there are exactly two α-fixed points on $\tilde{C} \simeq \boldsymbol{P}^1$, we have $\#(D \cap C) \equiv 0 \pmod 3$. This implies $p_a(C + D) > 3$, a contradiction. Now we can assume $F' = C$. If there is a point $q \in C_{\mathrm{sing}} \setminus \{p\}$, then q is α-fixed and $\mathrm{milt}_q\, C = 2$ since $p_a(C) = 3$, and hence there are at least four α-fixed points on $\tilde{C}$. This implies α is trivial on $\tilde{C}$ and hence on C, a contradiction. So we can assume that $C \setminus \{p\}$ is smooth. Then p is σ-fixed. If σ preserves the local branches of C at p, then G also does. This implies G is cyclic by (1.12), a contradiction. So we can assume σ exchanges the local branches of C at p. This implies there are two σ-fixed points on $C \setminus \{p\}$. Now by the same argument as in the last paragraph of Case 1, we get a contradiction. This completes the proof of (3.8.1). □

4. Proof of Proposition 2.7, the case $g = 4, 5$.

LEMMA 4.1. *Let $f : S \to B$, H, Γ_i, g and G be as in* (2.1). *Assume that $g = 4$ and $|G| = 6$. Then H is reduced and Γ_i is nonsingular for every i.*

PROOF. Let F be a general fiber of f. Using the Hurwitz formula for $\pi : F \to F/G$, we get that $g(F/G) = 1$ (note that $g(F/G) \geq 1$ by (1.15)) and π has six ramification points. By (2.4.1), we have $\#(H \cap F) \geq 6$. This implies H is reduced. Since $\#(H \cap F) \leq 2g - 2 = 6$, we have $R_F = H \cap F$. By (2.4.2), Γ_i is nonsingular for every i. □

LEMMA 4.2. *Let S be a minimal surface whose canonical map is composed with a pencil, and $f : S \to B$ the associated canonical fibration of genus g. Assume that $g = 4$, and that the horizontal part H of the fixed part of $|K_S|$ is reduced and each irreducible component of H is nonsingular. Then $\chi(\mathcal{O}_S) \leq 188$.*

PROOF. Let the notation be as in (2.1). Under the assumption, we have

$$K_S \equiv M + \sum_{i=1}^{t} \Gamma_i + V\,, \quad (t \leq 6)\,.$$

Let $g_i = g(\Gamma_i)$. From $K_S\Gamma_i \geq M\Gamma_i + \Gamma_i^2$ and the adjunction formula for Γ_i, we get

$$K_S\Gamma_i \geq \frac{M\Gamma_i}{2} + g_i - 1\,. \tag{1}$$

So

$$\begin{aligned} K_S^2 &\geq K_S M + \sum K_S\Gamma_i \geq 6d + \frac{\sum M\Gamma_i}{2} + \sum(g_i - 1) \\ &= 9d + \sum(g_i - 1)\,. \qquad \left(\sum M\Gamma_i = MH = dFH = 6d\right) \end{aligned} \tag{2}$$

On the other hand, using the logarithmic Miyaoka-Yau inequality for (S, Γ_i) (1.8), we have $K_S^2 \leq 9\chi(\mathcal{O}_S) + (g_i - 1) - K_S\Gamma_i/4$ for every i. Hence

$$\begin{aligned} K_S^2 &\leq 9\chi(\mathcal{O}_S) + \frac{\sum(g_i - 1)}{t} - \frac{\sum K_S\Gamma_i}{4t} \\ &\leq 9\chi(\mathcal{O}_S) + \frac{3\sum(g_i - 1)}{4t} - \frac{3d}{4t} \quad \text{(by (1))}\,. \end{aligned} \tag{3}$$

Combining (2) and (3), we get

$$3d \leq (4t - 3)\sum(1 - g_i) + 36t(\chi(\mathcal{O}_S) - d)\,.$$

Note that $t \leq 6$, and $d = \chi(\mathcal{O}_S)$ if $g(B) = 1$ and $d = \chi(\mathcal{O}_S) - 2 + q(S)$ if $g(B) = 0$ (2.1.2). Hence we get $\chi(\mathcal{O}_S) \leq 188$. □

PROOF OF PROPOSITION 2.7, THE CASE $g = 4$. Let $f : S \to B$ be the canonical fibration associated with ϕ_S, F the general fiber of f, and H the horizontal part of the fixed part of $|K_S|$. We have that G induces the trivial action on B, and $G \hookrightarrow \operatorname{Aut} F$ by (2.2). By Lemma 2.5, if $|G| > 4$ then either $|G| = 6$ or G is a nonabelian group of order 8 or 12 ($|G| \neq 5$ by the Hurwitz formula).

First we suppose that $|G| = 6$ or 12. Then by Lemmas 2.5 and 4.1, we have that H is reduced and each irreducible component of H is smooth. So by Lemma 4.2, $\chi(\mathcal{O}_S) \leq 188$, contradicting the assumption.

Second, we suppose that G is a nonabelian group of order 8. Then either $G \simeq D_8$ or $G \simeq Q_8$.

(i) The case $G \simeq D_8$ does not occur.

Otherwise, $D_8 \hookrightarrow \operatorname{Aut} F$ for a general fiber F of f. By (1.9), we have $4 + 2g(F/D_8) = g(F/\langle\alpha\rangle) + g(F/\langle\beta_1\rangle) + g(F/\langle\beta_2\rangle)$, where α, β_i are as in (1.9). But this is impossible since $g(F/D_8) = 1$, $g(F/\langle\alpha\rangle) = 1$, and $g(F/\langle\beta_i\rangle) \leq 2$ for every i by the Hurwitz formula.

(ii) The case $G \simeq Q_8$ does not occur.

Otherwise, let σ be a generator of $\operatorname{stab}(p)$ for some point $p \in H \cap F$. By the proof of Lemma 2.5, σ is of order 4. Consider the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\pi} & F/\langle\sigma\rangle \\ \downarrow & & \uparrow \lambda \\ C & \xrightarrow{:=} & F/\langle\sigma^2\rangle. \end{array}$$

Since the ramification index of π at $p \in F$ is 4, λ cannot be étale. This implies $g(C) = 2$.

Since Q_8 has only one element of order 2, $\langle \sigma^2 \rangle$ is a normal subgroup of Q_8 and $\overline{G} := Q_8/\langle \sigma^2 \rangle \simeq C_2 \times C_2$. Using the Hurwitz formula for $C \to C/G \simeq F/Q_8$, (note that $g(F/Q_8) = 1$,) by (1.13), we get $|\overline{G}| \leq 2$. This is a contradiction. □

PROOF OF PROPOSITION 2.7, THE CASE $g = 5$. Let $f : S \to B$ be the canonical fibration associated to ϕ_S, and F a general fiber of f. Let M, H, V, Γ_i, n_i and d be as in (2.1). Set $b = g(B)$. First we suppose that $n_1 < g$. Since $n_1 K_{S/B} + H + V$ is nef,

$$((n_1 + 1)K_S - M - n_1(2b - 2)F)H = (n_1 K_{S/B} + H + V)H \geq 0 .$$

So

$$K_S H \geq \frac{(2g-2)(d + n_1(2b-2))}{n_1 + 1} \geq \frac{(2g-2)(d + n_1(2b-2))}{g} .$$

On the other hand, using the Miyaoka-Yau inequality (cf. [Mi, Y]), we have

$$9\chi(\mathcal{O}_S) \geq K_S^2 = K_S(M + H + V) \geq (2g-2)d + K_S H .$$

Combining these two inequalities, we get $\chi(\mathcal{O}_S) \leq 34$, which contradicts the assumption.

Now we can assume that $n_1 \geq g$. Then Γ_1 is a section of f. This implies Γ_1 and hence the point $F \cap \Gamma_1 \in F$ is G-fixed. By (1.12), G is cyclic. Using the Hurwitz formula for $F \to F/G$, (note that $g(F/G) \geq 1$ (1.15) and by (1.13) when $g(F/G) = 1$) we get $G \simeq C_5$ and $\#(R \cap F) = 2$ if $|G| > 4$.

Now we prove that the case $G \simeq C_5$ does not occur. Otherwise, by (2.4.1), $\#(R \cap F) \geq 2$. Since $(H - n_1\Gamma_1)F = 8 - n_1 \leq 3$ and $|G| = 5$, we must have $\#(R \cap F) = 2$. So $H = n\Gamma_1 + (8-n)\Gamma_2$ with $5 \leq n \leq 7$ and $\Gamma_2 F = 1$. Since $\Gamma_1 + \Gamma_2$ is G-fixed, by (1.11), $\Gamma_1\Gamma_2 = 0$. From $K_S\Gamma_1 = (M + H + V)\Gamma_1 \geq d + n\Gamma_1^2$ and the adjunction formula for Γ_1, we get

$$K_S\Gamma_1 \geq \frac{d + n(2b-2)}{n+1} .$$

Similarly, we have

$$K_S\Gamma_2 \geq \frac{d + (8-n)(2b-2)}{9-n} .$$

Using the logarithmic Miyaoka-Yau inequality (1.8), we have

$$\begin{aligned} 9\chi(\mathcal{O}_S) + (b-1) - \frac{1}{4}K_S(\Gamma_1 + \Gamma_2) \geq K_S^2 &= K_S(M + n\Gamma_1 + (8-n)\Gamma_2 + V) \\ &\geq (2g-2)d + nK_S\Gamma_1 + (8-n)K_S\Gamma_2 . \end{aligned}$$

Combining these inequalities, we get $\chi(\mathcal{O}_S) \leq 60$, which contradicts the assumption. □

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