Some approximation properties of generalized integral type operators

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Abstract

In this paper we introduce and study the Stancu type generalization of the integral type operators defined in (1.1). First, we obtain the moments of the operators and then prove the Voronovskaja type asymptotic theorem and basic convergence theorem. Next, the rate of convergence and weighted approximation for the above operators are discussed. Then, weighted L_p-approximation and pointwise estimates are studied. Further, we study the A-statistical convergence of these operators. Lastly, we give better estimations of the above operators using King type approach.

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1 Introduction

The approximation of functions by linear positive operators is an important research topic in the classical approximation theory and it also provides powerful tools to application areas such as computer-aided geometric design, numerical analysis and solutions of differential equations.
In the last five decades, several new operators of integral type have been introduced and their approximation properties were discussed by several researchers.
For $f \in C[0,\infty)$, Deniz et al. [4] introduced a positive linear operators as follows:

$$\widetilde{B}_n(f,x) = (n-1)\sum_{k=0}^{\infty} p_{n+2,k}(x)\int_0^{\infty} p_{n,k+1}(t)f(t)dt, \;\; x \in [0,\infty),\, n \in N \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$$

and $f : [0,\infty) \to R$ is an integrable function for which the integrals and the series above are convergent. They obtain different approximation properties for these operators.
In the recent years, Stancu type generalization of the certain operators introduced by several researchers and obtained different type of approximation properties of many operators, we refer some of the important papers in this direction as [1], [15], [18], [19], [22], [31], [32], [34] etc.

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Inspired by the above work, We introduce the Stancu type generalization of the operators (1.1) as follows:

$$\widetilde{B}_n^{(\alpha,\beta)}(f,x) = (n-1)\sum_{k=0}^{\infty} p_{n+2,k}(x)\int_0^{\infty} p_{n,k+1}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt. \tag{1.2}$$

Taking $\alpha = \beta = 0$ in (1.2), we get the operators $\widetilde{B}_n$ defined in (1.1).
The goal of the present paper is to study the basic convergence theorem, Voronovskaja-type asymptotic formula, rate of convergence, weighted approximation, weighted L_p-approximation, pointwise estimation and A-statistical convergence of the operators (1.2). Further, to obtain better approximation, we also propose the modification of the operators (1.2) using King type approach.

2 Auxiliary results

In this section we give some results about the operators $\widetilde{B}_n^{(\alpha,\beta)}$ useful in the main results.
Let $e_i(t) = t^i,\ i = 0, 1, 2.$

Lemma 2.1. [4] For $x \in [0,\infty)$ and $n > 3$, the operators $\widetilde{B}_n(f,x)$ defined in (1.1) have the following equalities

1. $\widetilde{B}_n(e_0,x) = 1;$
2. $\widetilde{B}_n(e_1,x) = \dfrac{(n+2)x+2}{n-2};$
3. $\widetilde{B}_n(e_2,x) = \dfrac{(n+2)(n+3)x^2 + 6(n+2)x + 6}{(n-3)(n-2)}.$

Lemma 2.2. For the operators $\widetilde{B}_n^{(\alpha,\beta)}(f,x)$ as defined in (1.2), the following equalities holds for $n > 3$

1. $\widetilde{B}_n^{(\alpha,\beta)}(e_0,x) = 1;$
2. $\widetilde{B}_n^{(\alpha,\beta)}(e_1,x) = \dfrac{n(n+2)x + n(\alpha+2) - 2}{(n-2)(n+\beta)};$
3. $\widetilde{B}_n^{(\alpha,\beta)}(e_2,x) = \dfrac{n^2(n+2)(n+3)x^2 + (8n^3+10n^2)x + 4n\alpha(n-3) + \alpha^2(n-3)(n-2)}{(n-3)(n-2)(n+\beta)^2}.$

Proof. For $x \in [0,\infty)$, in view of Lemma 2.1, we have

$$\widetilde{B}_n^{(\alpha,\beta)}(e_0,x) = 1.$$

The first order moment is given by

$$\begin{aligned}
\widetilde{B}_n^{(\alpha,\beta)}(e_1,x) &= (n-1)\sum_{k=0}^{\infty} p_{n+2,k}(x)\int_0^{\infty} p_{n,k+1}(t)\left(\frac{nt+\alpha}{n+\beta}\right) dt \\
&= \frac{n}{n+\beta}\widetilde{B}_n(e_1,x) + \frac{\alpha}{n+\beta} = \frac{n(n+2)x + n(\alpha+2) - 2}{(n-2)(n+\beta)}.
\end{aligned}$$

The second order moment is given by

$$\begin{aligned}
\widetilde{B}_n^{(\alpha,\beta)}(e_2,x) &= (n-1)\sum_{k=0}^{\infty} p_{n+2,k}(x)\int_0^{\infty} p_{n,k+1}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^2 dt\\
&= \left(\frac{n}{n+\beta}\right)^2 \widetilde{B}_n(e_2,x)+\frac{2n\alpha}{(n+\beta)^2}\widetilde{B}_n(e_1,x)+\left(\frac{\alpha}{n+\beta}\right)^2\\
&= \frac{n^2(n+2)(n+3)x^2+(8n^3+10n^2)x+4n\alpha(n-3)+\alpha^2(n-3)(n-2)}{(n-3)(n-2)(n+\beta)^2}.
\end{aligned}$$

Q.E.D.

Lemma 2.3. For $f \in C_B[0,\infty)$ (space of all real valued bounded and uniformly continuous functions on $[0,\infty)$ endowed with the norm $\| f \| = \sup\{|f(x)| : x \in [0,\infty)\}$), we have

$$\| \widetilde{B}_n^{(\alpha,\beta)}(f) \| \leq \| f \| .$$

Proof. Applying the definition (1.2) and Lemma 2.2, we get

$$\begin{aligned}
\|\widetilde{B}_n^{(\alpha,\beta)}(f)\| &\leq (n-1)\sum_{k=0}^{\infty} p_{n+2,k}(x)\int_0^{\infty} p_{n,k+1}(t)\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right| dt\\
&\leq \|f\|\widetilde{B}_n^{(\alpha,\beta)}(e_0,x)=\|f\|.
\end{aligned}$$

Q.E.D.

Remark 2.4. From Lemma 2.2 it follows

$$\begin{aligned}
\widetilde{B}_n^{(\alpha,\beta)}(t-x,x) &= \frac{(4n+2\beta-n\beta)x+n(\alpha+2)-2}{(n-2)(n+\beta)}\\
&= \xi_n^{(\alpha,\beta)}(x)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{B}_n^{(\alpha,\beta)}\left((t-x)^2,x\right) &= \left(\frac{2n^3+n^2(\beta^2-8\beta)-5n\beta^2}{(n-3)(n-2)(n+\beta)^2}\right)x^2\\
&\quad+\left(\frac{8n^3+10n^2+2(n-3)(n+\beta)(2-n(\alpha+2))}{(n-3)(n-2)(n+\beta)^2}\right)x\\
&\quad+\frac{4n\alpha(n-3)+\alpha^2(n-3)(n-2)}{(n-3)(n-2)(n+\beta)^2}\\
&= \zeta_n^{(\alpha,\beta)}(x).
\end{aligned}$$

3 Main results

In this section we give some approximation results in several settings. For the reader's convenience we split up this section in more subsections.

Theorem 1. Let $f \in C[0,\infty)$. Then $\lim_{n\to\infty} \widetilde{B}_n^{(\alpha,\beta)}(f,x) = f(x)$, uniformly in each compact subset of $[0,\infty)$.

Proof. In view of Lemma 2.2, we get

$$\lim_{n\to\infty} \widetilde{B}_n^{(\alpha,\beta)}(e_i,x) = x^i,\ i=0,1,2,$$

uniformly in each compact subset of $[0,\infty)$. Applying Bohman-Korovkin Theorem, it follows that $\lim_{n\to\infty} \widetilde{B}_n^{(\alpha,\beta)}(f,x) = f(x)$, uniformly in each compact subset of $[0,\infty)$. Q.E.D.

3.1 Voronovskaya-type theorem

In this section we prove Voronvoskaya-type asymptotic theorem for the operators $\widetilde{B}_n^{(\alpha,\beta)}$.

Theorem 2. Let $f \in C_B[0,\infty)$. If f', f'' exists at a fixed point $x \in [0,\infty)$, then we have

$$\lim_{n\to\infty} n\left(\widetilde{B}_n^{(\alpha,\beta)}(f,x) - f(x)\right) = ((4-\beta)x+\alpha+2)\, f'(x) + (x^2+(2-\alpha)x)f''(x).$$

Proof. Let $x \in [0,\infty)$ be fixed. By Taylor's expansion of f, we can write

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + r(t,x)(t-x)^2, \tag{3.1}$$

where the function $r(t,x)$ is the Peano form of remainder, $r(t,x) \in C_B[0,\infty)$ and $\lim_{t\to x} r(t,x) = 0$. Applying $\widetilde{B}_n^{(\alpha,\beta)}$ on both sides of (3.1), we have

$$\begin{aligned} n\left(\widetilde{B}_n^{(\alpha,\beta)}(f,x) - f(x)\right) &= nf'(x)\widetilde{B}_n^{(\alpha,\beta)}\left(t-x,x\right) + \frac{f''(x)}{2!}n\widetilde{B}_n^{(\alpha,\beta)}\left((t-x)^2,x\right) \\ &\quad + n\widetilde{B}_n^{(\alpha,\beta)}\left(r(t,x)(t-x)^2,x\right). \end{aligned}$$

In view of Remark 2.4, we have

$$\lim_{n\to\infty} n\widetilde{B}_n^{(\alpha,\beta)}\left(t-x,x\right) = ((4-\beta)x+\alpha+2) \tag{3.2}$$

and

$$\lim_{n\to\infty} n\widetilde{B}_n^{(\alpha,\beta)}\left((t-x)^2,x\right) = 2x^2 + (4-2\alpha)x. \tag{3.3}$$

Now, we shall show that

$$\lim_{n\to\infty} n\widetilde{B}_n^{(\alpha,\beta)}\left(r(t,x)(t-x)^2,x\right) = 0.$$

By using Cauchy-Schwarz inequality, we have

$$\widetilde{B}_n^{(\alpha,\beta)}\left(r(t,x)(t-x)^2,x\right) \leq \left(\widetilde{B}_n^{(\alpha,\beta)}(r^2(t,x),x)\right)^{1/2}\left(\widetilde{B}_n^{(\alpha,\beta)}((t-x)^4,x)\right)^{1/2}. \tag{3.4}$$

We observe that $r^2(x,x)=0$ and $r^2(.,x)\in C_B[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty}\widetilde{B}_n^{(\alpha,\beta)}(r^2(t,x),x)=r^2(x,x)=0. \tag{3.5}$$

Now, from (3.4) and (3.5) we obtain

$$\lim_{n\to\infty} n\widetilde{B}_n^{(\alpha,\beta)}\left(r(t,x)(t-x)^2,x\right)=0. \tag{3.6}$$

From (3.2), (3.3) and (3.6), we get the required result. Q.E.D.

3.2 Local approximation

For $C_B[0,\infty)$, let us consider the following K-functional:

$$K_2(f,\delta) \quad = \quad \inf_{g\in W^2}\{\| f-g \| +\delta \| g'' \|\},$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty): g', g''\in C_B[0,\infty)\}$. By, p. 177, Theorem 2.4 in [2], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\le C\omega_2(f,\sqrt{\delta}), \tag{3.7}$$

where

$$\omega_2(f,\sqrt{\delta}) \quad = \quad \sup_{0<|h|\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)} \mid f(x+2h)-2f(x+h)+f(x) \mid$$

is the second order modulus of smoothness of f. By

$$\omega_1(f,\delta) \quad = \quad \sup_{0<|h|\le\delta}\ \sup_{x\in[0,\infty)} \mid f(x+h)-f(x) \mid,$$

we denote the first order modulus of continuity of $f\in C_B[0,\infty)$.

Theorem 3. Let $f\in C_B[0,\infty)$. Then for every $x\in[0,\infty)$ and $n>3$, we have

$$\mid \widetilde{B}_n^{(\alpha,\beta)}(f,x)-f(x) \mid \quad \le \quad C\omega_2\left(f,\delta_n^{(\alpha,\beta)}(x)\right)+\omega_1\left(f,\xi_n^{(\alpha,\beta)}(x)\right),$$

where

$$\delta_n^{(\alpha,\beta)}(x) \quad = \quad \sqrt{\zeta_n^{(\alpha,\beta)}(x)+\left(\xi_n^{(\alpha,\beta)}(x)\right)^2}.$$

Proof. For $x\in[0,\infty)$, we consider the auxiliary operators $\widetilde{B}_n^{*(\alpha,\beta)}$ as follows:

$$\widetilde{B}_n^{*(\alpha,\beta)}(f,x) \quad = \quad \widetilde{B}_n^{(\alpha,\beta)}(f,x)-f\left(\frac{n(n+2)x+n(\alpha+2)-2}{(n-2)(n+\beta)}\right)+f(x). \tag{3.8}$$

From Lemma 2.2, we observe that the operators $\widetilde{B}_n^{*(\alpha,\beta)}$ are linear and preserve the linear functions. Hence

$$\widetilde{B}_n^{*(\alpha,\beta)}(t-x,x) \quad = \quad 0. \tag{3.9}$$

Let $h \in W^2$ and $x, t \in [0,\infty)$. By Taylor's expansion we have

$$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-v)h''(v)dv.$$

Applying $\widetilde{B}_n^{*(\alpha,\beta)}$ on both sides of the above equation and using (3.9), we get

$$\widetilde{B}_n^{*(\alpha,\beta)}(h,x) - h(x) = \widetilde{B}_n^{*(\alpha,\beta)}\left(\int_x^t (t-v)h''(v)dv, x\right).$$

Thus, by (3.8) we get

$$\begin{aligned}
|\widetilde{B}_n^{*(\alpha,\beta)}(h,x) - h(x)| &\leq \widetilde{B}_n^{(\alpha,\beta)}\left(\left|\int_x^t (t-v)h''(v)dv\right|, x\right) \\
&\quad + \left|\int_x^{\frac{n(n+2)x+n(\alpha+2)-2}{(n-2)(n+\beta)}} \left(\frac{n(n+2)x+n(\alpha+2)-2}{(n-2)(n+\beta)} - v\right)h''(v)dv\right| \\
&\leq \left(\zeta_n^{(\alpha,\beta)}(x) + \left(\xi_n^{(\alpha,\beta)}(x)\right)^2\right) \| h'' \| \\
&\leq \left(\delta_n^{(\alpha,\beta)}(x)\right)^2 \| h'' \| .
\end{aligned}$$

Since $|\widetilde{B}_n^{*(\alpha,\beta)}(f,x)| \leq \| f \|$, it follows,

$$\begin{aligned}
|\widetilde{B}_n^{(\alpha,\beta)}(f,x) - f(x)| &\leq |\widetilde{B}_n^{*(\alpha,\beta)}(f-h,x)| + |(f-h)(x)| + |\widetilde{B}_n^{*(\alpha,\beta)}(h,x) - h(x)| \\
&\quad + \left|f\left(\frac{n(n+2)x+n(\alpha+2)-2}{(n-2)(n+\beta)}\right) - f(x)\right| \\
&\leq \| f-h \| + \left(\delta_n^{(\alpha,\beta)}(x)\right)^2 \| h'' \| + \left|f\left(\frac{n(n+2)x+n(\alpha+2)-2}{(n-2)(n+\beta)}\right) - f(x)\right|.
\end{aligned}$$

Taking infimum over all $h \in W^2$, we get

$$| \widetilde{B}_n^{(\alpha,\beta)}(f,x) - f(x) | \leq K_2\left(f, \left(\delta_n^{(\alpha,\beta)}(x)\right)^2\right) + \omega_1\left(f, \xi_n^{(\alpha,\beta)}(x)\right).$$

In view of (3.7), we get

$$| \widetilde{B}_n^{(\alpha,\beta)}(f,x) - f(x) | \leq C\omega_2\left(f, \delta_n^{(\alpha,\beta)}(x)\right) + \omega_1\left(f, \xi_n^{(\alpha,\beta)}(x)\right),$$

which proves the theorem. Q.E.D.

3.3 Rate of convergence

In this section, we compute the rate of convergence of $\widetilde{B}_n^{(\alpha,\beta)}$ in terms of the modulus of continuity. Let $\omega_b(f,\delta)$ denote the usual modulus of continuity of f on the closed interval $[0,b], b > 0$ and it is given by the relation

$$\omega_b(f,\delta) = \sup_{|t-x|\leq\delta} \sup_{x,t\in[0,b]} |f(t) - f(x)|.$$

We observe that for a function $f \in C_B[0,\infty)$, the modulus of continuity $\omega_b(f,\delta)$ tends to zero. Now, we give a rate of convergence theorem for the operators $\widetilde{B}_n^{(\alpha,\beta)}$.

Theorem 4. Let $f \in C_B[0,\infty)$ and $\omega_{b+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,b+1] \subset [0,\infty)$, where $b>0$. Then, we have

$$|\widetilde{B}_n^{(\alpha,\beta)}(f,x)-f(x)| \leq 6M_f(1+b^2)\zeta_n^{(\alpha,\beta)}(b)+2\omega_{b+1}\left(f,\sqrt{\zeta_n^{(\alpha,\beta)}(b)}\right),$$

where M_f is a constant depending only on f.

Proof. For $x \in [0,b]$ and $t>b+1$. Since $t-x>1$, we have

$$|f(t)-f(x)| \leq M_f(2+x^2+t^2) \leq M_f(t-x)^2(2+3x^2+2(t-x)^2) \leq 6M_f(1+b^2)(t-x)^2.$$

For $x \in [0,b]$ and $t \leq b+1$, we have

$$|f(t)-f(x)| \leq \omega_{b+1}(f,|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta)$$

with $\delta>0$.
From the above, we have

$$|f(t)-f(x)| \leq 6M_f(1+b^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta),$$

for $x \in [0,b]$ and $t \geq 0$.
Applying $\widetilde{B}_n^{(\alpha,\beta)}$ and then Cauchy-Schwarz inequality to the above inequality, we get
$|\widetilde{B}_n^{(\alpha,\beta)}(f,x)-f(x)|$

$$\begin{aligned} &\leq 6M_f(1+b^2)\widetilde{B}_n^{(\alpha,\beta)}((t-x)^2,x)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}(\widetilde{B}_n^{(\alpha,\beta)}((t-x)^2,x))^{\frac{1}{2}}\right) \\ &\leq 6M_f(1+b^2)\zeta_n^{(\alpha,\beta)}(b)+2\omega_{b+1}\left(f,\sqrt{\zeta_n^{(\alpha,\beta)}(b)}\right). \end{aligned}$$

By choosing $\delta=\sqrt{\zeta_n^{(\alpha,\beta)}(b)}$, we obtain the desired result. Q.E.D.

3.4 Weighted approximation

In this section we give some weighted approximation properties of the operators $\widetilde{B}_n^{(\alpha,\beta)}$. We do this for the following class of continuous functions defined on $[0,\infty)$.
Let $B_\nu[0,\infty)$ denote the weighted space of real-valued functions f defined on $[0,\infty)$ with the property $|f(x)| \leq M_f\nu(x)$ for all $x \in [0,\infty)$, where $\nu(x)$ is a weight function and M_f is a constant depending on the function f. We also consider the weighted subspace $C_\nu[0,\infty)$ of $B_\nu[0,\infty)$ given by $C_\nu[0,\infty)=\{f \in B_\nu[0,\infty) : f \text{ is continuous on } [0,\infty)\}$ and $C_\nu^*[0,\infty)$ denotes the subspace of all functions $f \in C_\nu[0,\infty)$ for which $\lim\limits_{|x|\to\infty}\dfrac{f(x)}{\nu(x)}$ exists finitely.

It is obvious that $C^*_\nu[0,\infty) \subset C_\nu[0,\infty) \subset B_\nu[0,\infty)$. The space $B_\nu[0,\infty)$ is a normed linear space with the following norm:

$$\| f \|_\nu = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\nu(x)}.$$

The following results on the sequence of positive linear operators in these spaces are given in [5], [6].

Lemma 3.1. ([5], [6]) The sequence of positive linear operators $(L_n)_{n\geq 1}$ which act from $C_\nu[0,\infty)$ to $B_\nu[0,\infty)$ if and only if there exists a positive constant k such that $L_n(\nu,x) \leq k\nu(x)$, i.e. $\| L_n(\nu) \|_\nu \leq k$.

Theorem 5. ([5], [6]) Let $(L_n)_{n\geq 1}$ be the sequence of positive linear operators which act from $C_\nu[0,\infty)$ to $B_\nu[0,\infty)$ satisfying the conditions

$$\lim_{n\to\infty} \| L_n(t^k) - x^k \|_\nu = 0, \ k = 0,1,2,$$

then for any function $f \in C^*_\nu[0,\infty)$

$$\lim_{n\to\infty} \| L_n(f) - f \|_\nu = 0.$$

Lemma 3.2. Let $\nu(x) = 1 + x^2$ be a weight function. If $f \in C_\nu[0,\infty)$, then

$$\| \widetilde{B}_n^{(\alpha,\beta)}(\nu) \|_\nu \leq 1 + M.$$

Proof. Using Lemma 2.2, we have

$$\begin{aligned}\widetilde{B}_n^{(\alpha,\beta)}(\nu,x) \quad &= \quad 1 + \frac{n^2(n+2)(n+3)x^2}{(n-3)(n-2)(n+\beta)^2} + \frac{(8n^3+10n^2)x}{(n-3)(n-2)(n+\beta)^2} \\ &\quad + \frac{4n\alpha(n-3) + \alpha^2(n-3)(n-2)}{(n-3)(n-2)(n+\beta)^2}.\end{aligned}$$

Then

$$\| \widetilde{B}_n^{(\alpha,\beta)}(\nu) \|_\nu \quad \leq \quad 1 + \frac{n^2(n+2)(n+3) + 8n^3 + 10n^2 + 4n\alpha(n-3) + \alpha^2(n-3)(n-2)}{(n-3)(n-2)(n+\beta)^2}$$

there exists a positive constant M such that

$$\| \widetilde{B}_n^{(\alpha,\beta)}(\nu) \|_\nu \leq 1 + M$$

so the proof is completed. Q.E.D.

By using Lemma 3.2 we can easily see that the operators $\widetilde{B}_n^{(\alpha,\beta)}$ defined by (1.2) act from $C_\nu[0,\infty)$ to $B_\nu[0,\infty)$.

Theorem 6. For every $f \in C^*_\nu[0,\infty)$, we have

$$\lim_{n\to\infty} \| \widetilde{B}_n^{(\alpha,\beta)}(f) - f \|_\nu = 0.$$

Proof. From [5], we know that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \| \widetilde{B}_n^{(\alpha,\beta)}(t^k) - x^k \|_\nu = 0,\ k = 0,1,2. \tag{3.10}$$

Since $\widetilde{B}_n^{(\alpha,\beta)}(1,x) = 1$, the condition in (3.10) holds for $k = 0$.
For $k = 1$, we have

$$\begin{aligned} \| \widetilde{B}_n^{(\alpha,\beta)}(t) - x \|_\nu &= \sup_{x\in[0,\infty)} \frac{|\widetilde{B}_n^{(\alpha,\beta)}(t,x) - x|}{1+x^2} \\ &\le \left| \frac{4n + 2\beta - n\beta + n(\alpha+2) - 2}{(n-2)(n+\beta)} \right| \end{aligned}$$

which implies that $\lim_{n\to\infty} \| \widetilde{B}_n^{(\alpha,\beta)}(t) - x \|_\nu = 0$.
Similarly, we can write for $k = 2$
$\| \widetilde{B}_n^{(\alpha,\beta)}(t^2) - x^2 \|_\nu$

$$\begin{aligned} &= \sup_{x\in[0,\infty)} \frac{|\widetilde{B}_n^{(\alpha,\beta)}(t^2,x) - x^2|}{1+x^2} \\ &\le \left| \frac{n^2(n+2)(n+3)}{(n-3)(n-2)(n+\beta)^2} - 1 \right| + \left| \frac{8n^3 + 10n^2 + 4n\alpha(n-3) + \alpha^2(n-3)(n-2)}{(n-3)(n-2)(n+\beta)^2} \right|, \end{aligned}$$

which implies that $\lim_{n\to\infty} \| \widetilde{B}_n^{(\alpha,\beta)}(t^2) - x^2 \|_\nu = 0$.
This completes the proof of theorem. Q.E.D.

3.5 Weighted L_p-approximation

Let w be positive continuous function on real axis $[0,\infty)$ satisfying the condition

$$\int_0^\infty x^{2p} w(x)dx < \infty.$$

We denote by $L_{p,w}[0,\infty)(1 \le p < \infty)$ the linear space of p-absolutely integrable on $[0,\infty)$ with respect to the weight function w

$$L_{p,w}[0,\infty) = \left\{ f : [0,\infty) \to R; \|f\|_{p,w} = \left(\int_0^\infty |f(x)|^p w(x)dx \right)^{\frac{1}{p}} < \infty \right\}.$$

Theorem 7. [7] Let $(L_n)_{n\ge1}$ be a uniformly bounded sequence of positive linear operators from $L_{p,w}[0,\infty)$ into $L_{p,w}[0,\infty)$, satisfying the conditions

$$\lim_{n\to\infty} \| L_n(t^k) - x^k \|_{p,w} = 0,\ k = 0,1,2. \tag{3.11}$$

Then for every $f \in L_{p,w}[0,\infty)$

$$\lim_{n\to\infty} \| L_n(f) - f \|_{p,w} = 0.$$

Now we choose $w(x)=\frac{1}{(1+x^{2r})^p}, 1\le p<\infty$ and consider analogue weighted L_p-space [4]:

$$L_{p,2r}[0,\infty)=\left\{f:[0,\infty)\to R; \|f\|_{p,2r}=\left(\int_0^\infty\left|\frac{f(x)}{1+x^{2r}}\right|^p dx\right)^{\frac{1}{p}}<\infty\right\}.$$

Theorem 8. For every $f\in L_{p,2r}[0,\infty), r>1$, we have

$$\lim_{n\to\infty}\| \widetilde{B}_n^{(\alpha,\beta)}(f)-f\|_{p,2r}=0.$$

Proof. Using the Theorem 7, we see that it is sufficient to verify the three conditions (3.11). Since $\widetilde{B}_n^{(\alpha,\beta)}(1,x)=1$, the first condition is obvious for $k=0$.
By Lemma 2.2, for $k=1$, we have

$$\left(\int_0^\infty\left|\frac{\widetilde{B}_n^{(\alpha,\beta)}(t,x)-x}{1+x^{2r}}\right|^p dx\right)^{\frac{1}{p}} \le \frac{4n+2\beta-n\beta}{(n-2)(n+\beta)}\left(\int_0^\infty\left|\frac{x}{1+x^{2r}}\right|^p dx\right)^{\frac{1}{p}} +\frac{n(\alpha+2)-2}{(n-2)(n+\beta)}\left(\int_0^\infty\left|\frac{1}{1+x^{2r}}\right|^p dx\right)^{\frac{1}{p}}$$

which implies that $\lim\limits_{n\to\infty}\| \widetilde{B}_n^{(\alpha,\beta)}(t)-x\|_{p,2r}=0$.
For $k=2$, we can write

$$\left(\int_0^\infty\left|\frac{\widetilde{B}_n^{(\alpha,\beta)}(t^2,x)-x^2}{1+x^{2r}}\right|^p dx\right)^{\frac{1}{p}} \le \left(\frac{n^2(n+2)(n+3)}{(n-3)(n-2)(n+\beta)^2}-1\right)\left(\int_0^\infty\left|\frac{x^2}{1+x^{2r}}\right|^p dx\right)^{\frac{1}{p}} +\frac{8n^3+10n^2}{(n-3)(n-2)(n+\beta)^2}\left(\int_0^\infty\left|\frac{x}{1+x^{2r}}\right|^p dx\right)^{\frac{1}{p}} +\frac{4n\alpha+\alpha^2(n-2)}{(n-2)(n+\beta)^2}\left(\int_0^\infty\left|\frac{1}{1+x^{2r}}\right|^p dx\right)^{\frac{1}{p}}$$

which implies that $\lim\limits_{n\to\infty}\| \widetilde{B}_n^{(\alpha,\beta)}(t^2)-x^2\|_{p,2r}=0$.
This completes the proof of theorem. Q.E.D.

3.6 Pointwise estimates

In this section, we establish some pointwise estimates of the rate of convergence of the operators $\widetilde{B}_n^{(\alpha,\beta)}$. First, we give the relationship between the local smoothness of f and local approximation. We know that a function $f\in C[0,\infty)$ is in $\mathrm{Lip}_M(\eta)$ on E, $\eta\in(0,1]$, E$\subset[0,\infty)$ if it satisfies the condition

$$|f(t)-f(x)|\le M|t-x|^\eta,\ t\in[0,\infty)\ \ and\ \ x\in E,$$

where M is a constant depending only on η and f.

Theorem 9. Let $f \in C[0,\infty) \cap Lip_M(\eta)$, $E \subset [0,\infty)$ and $\eta \in (0,1]$. Then, we have

$$|\widetilde{B}_n^{(\alpha,\beta)}(f,x) - f(x)| \quad \leq \quad M\bigg(\Big(\zeta_n^{(\alpha,\beta)}(x)\Big)^{\eta/2} + 2d^{\eta}(x,E)\bigg), \quad x \in [0,\infty),$$

where M is a constant depending on η and f and $d(x,E)$ is the distance between x and E defined as

$$d(x,E) = \inf\{|t-x| : t \in E\}.$$

Proof. Let $\overline{E}$ be the closure of E in $[0,\infty)$. Then, there exists at least one point $x_0 \in \overline{E}$ such that

$$d(x,E) = |x - x_0|.$$

By our hypothesis and the monotonicity of $\widetilde{B}_n^{(\alpha,\beta)}$, we get

$$\begin{aligned} |\widetilde{B}_n^{(\alpha,\beta)}(f,x) - f(x)| &\leq \widetilde{B}_n^{(\alpha,\beta)}(|f(t) - f(x_0)|, x) + \widetilde{B}_n^{(\alpha,\beta)}(|f(x) - f(x_0)|, x) \\ &\leq M\left(\widetilde{B}_n^{(\alpha,\beta)}(|t - x_0|^{\eta}, x) + |x - x_0|^{\eta}\right) \\ &\leq M\left(\widetilde{B}_n^{(\alpha,\beta)}(|t - x|^{\eta}, x) + 2|x - x_0|^{\eta}\right). \end{aligned}$$

Now, applying Hölder's inequality with $p = \dfrac{2}{\eta}$ and $q = \dfrac{2}{2-\eta}$, we obtain

$$|\widetilde{B}_n^{(\alpha,\beta)}(f,x) - f(x)| \leq M\left(\left(\widetilde{B}_n^{(\alpha,\beta)}(|t-x|^2, x)\right)^{\eta/2} + 2d^{\eta}(x,E)\right),$$

from which the desired result immediate. Q.E.D.

For $a,b > 0$, Özarslan and Aktuğlu [29] consider the Lipschitz-type space with two parameters:

$$Lip_M^{(a,b)}(\eta) = \left(f \in C[0,\infty) : |f(t) - f(x)| \leq M\frac{|t-x|^{\eta}}{(t + ax^2 + bx)^{\eta/2}};\ x,t \in [0,\infty)\right),$$

where M is any positive constant and $0 < \eta \leq 1$.

Theorem 10. For $f \in Lip_M^{(a,b)}(\eta)$. Then, for all $x > 0$, we have

$$|\widetilde{B}_n^{(\alpha,\beta)}(f,x) - f(x)| \leq M\left(\frac{\zeta_n^{(\alpha,\beta)}(x)}{ax^2 + bx}\right)^{\eta/2}.$$

Proof. First we prove the theorem for $\eta = 1$. Then, for $f \in Lip_M^{(a,b)}(1)$, and $x \in [0,\infty)$, we have

$$\begin{aligned} |\widetilde{B}_n^{(\alpha,\beta)}(f,x) - f(x)| &\leq \widetilde{B}_n^{(\alpha,\beta)}(|f(t) - f(x)|, x) \\ &\leq M\widetilde{B}_n^{(\alpha,\beta)}\left(\frac{|t-x|}{(t + ax^2 + bx)^{1/2}}, x\right) \\ &\leq \frac{M}{(ax^2 + bx)^{1/2}}\widetilde{B}_n^{(\alpha,\beta)}(|t-x|, x). \end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}|\widetilde{B}_n^{(\alpha,\beta)}(f,x)-f(x)| &\leq \frac{M}{(ax^2+bx)^{1/2}}\left(\widetilde{B}_n^{(\alpha,\beta)}((t-x)^2,x)\right)^{1/2}\\ &\leq M\left(\frac{\zeta_n^{(\alpha,\beta)}(x)}{ax^2+bx}\right)^{1/2}.\end{aligned}$$

Thus the result holds for $\eta=1$.
Now, we prove that the result is true for $0<\eta<1$. Then, for $f\in Lip_M^{(a,b)}(\eta)$, and $x\in[0,\infty)$, we get

$$|\widetilde{B}_n^{(\alpha,\beta)}(f,x)-f(x)| \leq \frac{M}{(ax^2+bx)^{\eta/2}}\widetilde{B}_n^{(\alpha,\beta)}(|t-x|^\eta,x).$$

Taking $p=\frac{1}{\eta}$ and $q=\frac{2}{2-\eta}$, applying the Hölder's inequality, we have

$$|\widetilde{B}_n^{(\alpha,\beta)}(f,x)-f(x)|\leq \frac{M}{(ax^2+bx)^{\eta/2}}\left(\widetilde{B}_n^{(\alpha,\beta)}(|t-x|,x)\right)^{\eta}.$$

Finally by Cauchy-Schwarz inequality, we get

$$|\widetilde{B}_n^{(\alpha,\beta)}(f,x)-f(x)| \leq M\left(\frac{\zeta_n^{(\alpha,\beta)}(x)}{ax^2+bx}\right)^{\eta/2}.$$

Thus, the proof is completed. Q.E.D.

3.7 A-statistical approximation of Korovkin type

Let $A=(a_{nk}),(n,k\in N)$, be a non-negative infinite summability matrix. For a given sequence $x:=(x)_n$, the A-transform of x denoted by $Ax:((Ax)_n)$ is defined as

$$(Ax)_n=\sum_{k=1}^{\infty}a_{nk}x_k$$

provided the series converges for each n. A is said to be regular if $\lim_n(Ax)_n=L$ whenever $\lim_n x_n=L$. The sequence $x=(x)_n$ is said to be a A- statistically convergent to L i.e. $st_A-\lim_n(x)_n=L$ if for every $\varepsilon>0$, $\lim_n \sum_{k:|x_k-L|\geq\varepsilon} a_{nk}=0$. If we replace A by C_1 then A is a Cesáro matrix of order one and A- statistical convergence is reduced to the statistical convergence. Similarly, if $A=I$, the identity matrix, then A- statistical convergence coincides with the ordinary convergence. It is to be noted that the concept of A-statistical convergence may also be given in normed spaces. Many researchers have investigated the statistical convergence properties for several sequences and classes of linear positive operators (see [3], [8], [19], [24], [26]). In the following result we prove a weighted Korovkin theorem via A-statistical convergence.

Theorem 11. Let (a_{nk}) be a non-negative regular infinite summability matrix and $x \in [0, \infty)$. Then, for all $f \in C_{\nu}^{*}$, we have

$$st_A - \lim_{n} \| \widetilde{B}_n^{(\alpha,\beta)}(f) - f \|_{\nu} = 0.$$

Proof. From ([3] p. 195, Th. 6), it is enough to show that

$$st_A - \lim_{n} \| \widetilde{B}_n^{(\alpha,\beta)}(t^k) - x^k \|_{\nu} = 0, \ \ k = 0, 1, 2.$$

From Lemma 2.2, result holds for $k = 0$.
Again by using Lemma 2.2, we have

$$\begin{aligned} \| \widetilde{B}_n^{(\alpha,\beta)}(t) - x \|_{\nu} &\leq \frac{4n + 2\beta - n\beta}{(n-2)(n+\beta)} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{n(\alpha+2)-2}{(n-2)(n+\beta)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\ &\leq \frac{4n + 2\beta - n\beta + n(\alpha+2) - 2}{(n-2)(n+\beta)}. \end{aligned}$$

For any given $\varepsilon > 0$, let us define the following sets:

$$S := \left\{ n : \| \widetilde{B}_n^{(\alpha,\beta)}(t) - x \|_{\nu} \geq \varepsilon \right\},$$

$$S_1 := \left\{ n : \frac{4n + 2\beta - n\beta}{(n-2)(n+\beta)} \geq \frac{\varepsilon}{2} \right\}$$

and

$$S_2 := \left\{ n : \frac{n(\alpha+2)-2}{(n-2)(n+\beta)} \geq \frac{\varepsilon}{2} \right\}.$$

Then, we get $S \subseteq S_1 \cup S_2$ which implies that

$$\sum_{k\in S} a_{nk} \leq \sum_{k\in S_1} a_{nk} + \sum_{k\in S_2} a_{nk}$$

and hence

$$st_A - \lim_{n} \| \widetilde{B}_n^{(\alpha,\beta)}(t) - x \|_{\nu} = 0.$$

Similarly, we have

$$\begin{aligned} \| \widetilde{B}_n^{(\alpha,\beta)}(t^2) - x^2 \|_{\nu} &\leq \left(\frac{n^2(n+2)(n+3)}{(n-3)(n-2)(n+\beta)^2} - 1 \right) + \frac{8n^3 + 10n^2}{(n-3)(n-2)(n+\beta)^2} \\ &\quad + \frac{4n\alpha + \alpha^2(n-2)}{(n-2)(n+\beta)^2}. \end{aligned}$$

Now, we define the following sets:

$$U := \left\{ n : \| \tilde{B}_n^{(\alpha,\beta)}(t^2) - x^2 \|_{\nu} \geq \varepsilon \right\},$$

$$U_1 := \left\{ n : \left(\frac{n^2(n+2)(n+3)}{(n-3)(n-2)(n+\beta)^2} - 1 \right) \geq \frac{\varepsilon}{3} \right\},$$

$$U_2 := \left\{ n : \frac{8n^3 + 10n^2}{(n-3)(n-2)(n+\beta)^2} \geq \frac{\varepsilon}{3} \right\}$$

and

$$U_3 := \left\{ n : \frac{4n\alpha + \alpha^2(n-2)}{(n-2)(n+\beta)^2} \geq \frac{\varepsilon}{3} \right\}.$$

Then, we get $U \subseteq U_1 \cup U_2 \cup U_3$ which implies that

$$\sum_{k \in U} a_{nk} \leq \sum_{k \in U_1} a_{nk} + \sum_{k \in U_2} a_{nk} + \sum_{k \in U_3} a_{nk}$$

and hence

$$st_A - \lim_n \parallel \tilde{B}_n^{(\alpha,\beta)}(t^2) - x^2 \parallel_\nu = 0.$$

This completes the proof of the theorem. Q.E.D.

4 King's approach

To make the convergence faster, King [20] proposed an approach to modify the classical Bernstein polynomial, so that the sequence preserve test functions e_0 and e_2, where $e_i(t) = t^i, i = 0, 1, 2.$ After this approach many researcher contributed in this direction.
As the operator $\widetilde{B}_n^{(\alpha,\beta)}(f, x)$ defined in (1.2) preserve only the constant functions so further modification of these operators is proposed to be made so that the modified operators preserve the constant as well as linear functions.
For this purpose the modification of (1.2) is defined as

$$\widetilde{B}_n^{\star(\alpha,\beta)}(f, x) \quad = \quad (n-1) \sum_{k=0}^{\infty} p_{n+2,k}(r_n(x)) \int_0^\infty p_{n,k+1}(t) f\left(\frac{nt+\alpha}{n+\beta} \right) dt \tag{4.1}$$

where $r_n(x) = \frac{(n-2)(n+\beta)x - n(\alpha+2)+2}{n(n+2)}$ for $x \in I_n = [\frac{\alpha}{n+\beta}, \infty)$.

Lemma 4.1. For every $x \in I_n$, we have

1. $\widetilde{B}_n^{\star(\alpha,\beta)}(e_0, x) = 1;$
2. $\widetilde{B}_n^{\star(\alpha,\beta)}(e_1, x) = x;$
3. $\widetilde{B}_n^{\star(\alpha,\beta)}(e_2, x) = \frac{(n-2)(n+3)}{(n-3)(n+2)} x^2 + \frac{4n^2 - 2n^2\alpha - 6n\alpha + 12}{(n-3)(n+2)(n+\beta)} x + \frac{4n\alpha}{(n-2)(n+\beta)^2}$
$+ \frac{(6n^2 - n^2\alpha - 3n\alpha + 6)(2 - n(\alpha+2))}{(n-3)(n-2)(n+2)(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.$

Consequently, for each $x \in I_n$, we have the following equalities

$$\widetilde{B}_n^{\star(\alpha,\beta)}(t-x,x)=0$$

$$\begin{aligned}\widetilde{B}_n^{\star(\alpha,\beta)}((t-x)^2,x) &= \frac{2n}{(n-3)(n+2)}x^2+\frac{4n^2-2n^2\alpha-6n\alpha+12}{(n-3)(n+2)(n+\beta)}x+\frac{4n\alpha}{(n-2)(n+\beta)^2}\\ &\quad+\frac{(6n^2-n^2\alpha-3n\alpha+6)(2-n(\alpha+2))}{(n-3)(n-2)(n+2)(n+\beta)^2}+\frac{\alpha^2}{(n+\beta)^2}\\ &= \lambda_n^{(\alpha,\beta)}(x).\end{aligned}$$

Theorem 12. For $f \in C_B(I_n)$ and $x \in I_n$. Then for $n > 3$, there exists a positive constant C' such that

$$|\widetilde{B}_n^{\star(\alpha,\beta)}(f,x)-f(x)| \le C'\omega_2\left(f,\sqrt{\lambda_n^{(\alpha,\beta)}(x)}\right).$$

Proof. Let $h \in W^2$ and $x,t \in I_n$. Using the Taylor's expansion we have

$$h(t)=h(x)+(t-x)h'(x)+\int_x^t (t-v)h''(v)dv.$$

Applying $\widetilde{B}_n^{\star(\alpha,\beta)}$ on both sides and using Lemma 3.1, we get

$$\widetilde{B}_n^{\star(\alpha,\beta)}(h,x)-h(x) = \widetilde{B}_n^{\star(\alpha,\beta)}\left(\int_x^t (t-v)h''(v)dv,x\right).$$

Obviously, we have $\left|\int_x^t (t-v)h''(v)dv\right| \le (t-x)^2\|h''\|$.
Therefore

$$|\ \widetilde{B}_n^{\star(\alpha,\beta)}(h,x)-h(x)\ | \le \widetilde{B}_n^{\star(\alpha,\beta)}((t-x)^2,x)\ \|\ h''\ \|\ .$$

Since $|\ \widetilde{B}_n^{\star(\alpha,\beta)}(f,x)\ | \le \|f\|$, we get

$$\begin{aligned}|\ \widetilde{B}_n^{\star(\alpha,\beta)}(f,x)-f(x)\ | &\le |\ \widetilde{B}_n^{\star(\alpha,\beta)}(f-h,x)\ |+|\ (f-h)(x)\ |+|\ \widetilde{B}_n^{\star(\alpha,\beta)}(h,x)-h(x)\ |\\ &\le \|f-h\|+\lambda_n^{(\alpha,\beta)}(x)\|h''\|.\end{aligned}$$

Finally, taking the infimum over all $h \in W^2$ and using (3.7) we obtain

$$|\ \widetilde{B}_n^{\star(\alpha,\beta)}(f,x)-f(x)\ | \le C'\omega_2\left(f,\sqrt{\lambda_n^{(\alpha,\beta)}(x)}\right),$$

which proves the theorem. Q.E.D.

Theorem 13. Let $f \in C_B(I_n)$. If f'' exists at a fixed point $x \in I_n$, then we have

$$\lim_{n\to\infty} n\left(\widetilde{B}_n^{\star(\alpha,\beta)}(f,x)-f(x)\right)=x(x+2-\alpha)f''(x).$$

The proof follows along the lines of Theorem 2.

5 Conclusion

In this paper, we introduce the Stancu type generalization of the integral type operators defined in (1.1). The results of our lemmas and theorems are more general rather than the results of any other previous proved lemmas and theorems, which will be enrich the literate of classical approximation theory. The researchers and professionals working or intend to work in areas of analysis and its applications will find this research article to be quite useful. Consequently, the results so established may be found useful in several interesting situation appearing in the literature on Mathematical Analysis and Applied Mathematics.

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