Some different type integral inequalities concerning twice differentiable generalized relative semi-$(r; m, h)$-preinvex mappings

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Abstract

In this article, we first present some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-$(r; m, h)$-preinvex mappings. And then, a new identity concerning twice differentiable mappings defined on m-invex set is derived. By using the notion of generalized relative semi-$(r; m, h)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard, Ostrowski and Simpson type inequalities via fractional integrals are established. It is pointed out that some new special cases can be deduced from main results of the article.

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1 Introduction

The subsequent double inequality is known as Hermite-Hadamard inequality.

Theorem 1.1. Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex mapping on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the subsequent double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions (see [[2]-[25],[29],[30],[33],[36],[40],[41],[46],[47],[49],[50]]) and the references mentioned in these papers. Also the following result is known in the literature as the Ostrowski inequality [28], which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

Theorem 1.2. Let $f : I \longrightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a, b]. \tag{1.2}$$

The following inequality is well known in the literature as Simpson's inequality:

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Theorem 1.3. Let $f:[a,b]\longrightarrow\mathbb{R}$ be four time differentiable on the interval (a,b) and having the fourth derivative bounded on (a,b), that is $\|f^{(4)}\|_\infty=\sup_{x\in(a,b)}|f^{(4)}|<\infty$. Then, we have

$$\left|\int_a^b f(t)dt-\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\leq\frac{1}{2880}\|f^{(4)}\|_\infty(b-a)^5. \tag{1.3}$$

Inequality (1.3) gives an error bound for the classical Simpson quadrature formula, which is one of the most used quadrature formulae in practical applications.

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Ostrowski type inequalities (see [[9],[10],[43]]). For other recent results concerning Simpson type inequalities (see [[27],[39]]).

Let us evoke some definitions as follows.

Definition 1.4. [48] A set $M_\varphi\subseteq\mathbb{R}^n$ is named as a relative convex (φ-convex) set, if and only if, there exists a function $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that,

$$t\varphi(x)+(1-t)\varphi(y)\in M_\varphi,\ \forall\ x,y\in\mathbb{R}^n:\varphi(x),\varphi(y)\in M_\varphi, t\in[0,1]. \tag{1.4}$$

Definition 1.5. [48] A function f is named as a relative convex (φ-convex) function on a relative convex (φ-convex) set M_φ, if and only if, there exists a function $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that,

$$f(t\varphi(x)+(1-t)\varphi(y))\leq tf(\varphi(x))+(1-t)f(\varphi(y)), \tag{1.5}$$

$\forall\ x,y\in\mathbb{R}^n:\varphi(x),\varphi(y)\in M_\varphi, t\in[0,1]$.

Definition 1.6. [6] A non-negative function $f:I\subseteq\mathbb{R}\longrightarrow[0,+\infty)$ is said to be P-function, if

$$f(tx+(1-t)y)\leq f(x)+f(y),\quad \forall x,y\in I,\ t\in[0,1].$$

Definition 1.7. [1] A set $K\subseteq\mathbb{R}^n$ is said to be invex respecting the mapping $\eta:K\times K\longrightarrow\mathbb{R}^n$, if $x+t\eta(y,x)\in K$ for every $x,y\in K$ and $t\in[0,1]$.

Definition 1.8. [31] Let $h:[0,1]\longrightarrow\mathbb{R}$ be a non-negative function and $h\neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x+t\eta(y,x)\big)\leq h(1-t)f(x)+h(t)f(y) \tag{1.6}$$

for each $x,y\in K$ and $t\in[0,1]$ where $f(\cdot)>0$.

Clearly, when putting $h(t)=t$ in Definition 1.8, f becomes a preinvex function [38]. If the mapping $\eta(y,x)=y-x$ in Definition(1.8), then the non-negative function f reduces to h-convex mappings [45].

Definition 1.9. [44] Let $f:K\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a non-negative function, a function $f:K\longrightarrow\mathbb{R}$ is said to be a tgs-convex function on K if the inequality

$$f\big((1-t)x+ty\big)\leq t(1-t)[f(x)+f(y)] \tag{1.7}$$

holds for all $x,y\in K$ and $t\in(0,1)$.

Definition 1.10. [[4],[28]] A function $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to MT-convex functions, if it is non-negative and $\forall\ x, y \in I$ and $t \in (0,1)$ satisfies the subsequent inequality:

$$f(tx+(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.8}$$

Definition 1.11. [34] A function: $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall x, y \in I$, and $t \in (0,1)$, among $m \in [0,1]$, satisfies the following inequality

$$f\big(tx+m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.9}$$

Definition 1.12. [37] Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta: K \times K \times (0,1] \longrightarrow \mathbb{R}$. A function $f: K \longrightarrow \mathbb{R}$ and $h_1, h_2: [0,1] \longrightarrow [0,+\infty)$, if

$$f\big(mx+t\eta(y,x,m)\big) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{1.10}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, together $m \in (0,1]$ is said to be generalized (m, h_1, h_2)-preinvex functions with respect to η.

We need the subsequent Riemann-Liouville fractional calculus background.

Definition 1.13. [29] Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

Note that $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of Riemann-Liouville fractional integrals, many authors extended to research Riemann-Liouville fractional inequalities via different classes of convex mappings: for generalizations, variations and new inequalities for them, (see [[29]-[39]]).

Let us recall the Gauss-Jacobi type quadrature formula as follows.

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k}f(\gamma_k) + R_m^{\star}|f|, \tag{1.11}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^{\star}|f|$ (see [42]).

In [26], Liu obtained integral inequalities for P-function related to the left-hand side of (1.11), and in [35], Özdemir et al. also presented several integral inequalities concerning the left-hand side of (1.11) via some kinds of convexity.

Motivated by the above literatures, the main objective of this article is to establish integral inequalities using two lemmas as auxiliary results for the left side of Gauss-Jacobi type quadrature formula and some new estimates on Hermite-Hadamard, Ostrowski and Simpson type inequalities via fractional integrals associated with generalized relative semi-$(r;m,h)$-preinvex mappings. It is pointed out that some new special cases will be deduced from main results of the article.

2 Main results

The following definitions will be used in this section.

Definition 2.1. [7] A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,mx) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y,mx)$ could reduce to $\eta(y,x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We next introduce generalized relative semi-$(r;m,h)$-preinvex mappings.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, $h : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. A mapping $f : K \longrightarrow (0,+\infty)$ is said to be generalized relative semi-$(r;m,h)$-preinvex, if

$$f\big(m\varphi(x) + t\eta(\varphi(y),\varphi(x),m)\big) \le M_r(h(t); f(x), f(y), m) \tag{2.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$ and for some fixed $m \in (0,1]$, where

$$M_r(h(t); f(x), f(y), m) := \begin{cases} \Big[mh(1-t)f^r(x) + h(t)f^r(y)\Big]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \\ \big[f(x)\big]^{mh(1-t)}\big[f(y)\big]^{h(t)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(x)$ and $f(y)$.

Remark 2.4. In Definition 2.3, if we choose $m = r = 1$, these definition reduces to the definition considered by Noor in [32] and Preda et. al. in [8].

Remark 2.5. In Definition 2.3, if we choose $r = 1$ and $\varphi(x) = x$, then we get Definition 1.12.

Remark 2.6. For $r = 1$, let us discuss some special cases in Definition 2.3 as follows.

(I) If taking $h(t) = t$, then we get generalized relative semi-m-preinvex mappings.
(II) If taking $h(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Breckner-preinvex mappings.
(III) If taking $h(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex mappings.
(IV) If taking $h(t) = 1$, then we get generalized relative semi-(m,P)-preinvex mappings.
(V) If taking $h(t) = t(1-t)$, then we get generalized relative semi-(m,tgs)-preinvex mappings.
(VI) If taking $h(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

We claim the following integral identity.

Lemma 2.7. Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0,1]$ and any fixed $p, q > 0$, we have

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\ &= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.\end{aligned}$$

Proof. It is easy to observe that

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\ &= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + t\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p \\ &\quad \times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - t\eta(\varphi(b), \varphi(a), m))^q \\ &\quad \times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt \\ &= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.\end{aligned}$$

This completes the proof of the lemma. Q.E.D.

With the help of Lemma 2.7, we have the following results.

Theorem 2.8. Let $k > 1$ and $0 < r \leq 1$. Suppose $h : [0,1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ is a continuous mapping on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. If $f^{\frac{k}{k-1}}$ is generalized relative semi-$(r; m, h)$-preinvex mappings on an open m-invex set K for some fixed $m \in (0,1]$, then for any fixed $p, q > 0$, we have

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\ &\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{1}{k}}(kp+1, kq+1) \Big[m f^{\frac{rk}{k-1}}(a) + f^{\frac{rk}{k-1}}(b) \Big]^{\frac{k-1}{rk}} \\ &\quad \times \left(\int_0^1 h^{\frac{1}{r}}(t)dt \right)^{\frac{k-1}{k}}.\end{aligned}$$

Proof. Let $k>1$ and $0<r\le 1$. Since $f^{\frac{k}{k-1}}$ is generalized relative semi-$(r;m,h)$-preinvex mappings on K, combining with Lemma 2.7, Hölder inequality, Minkowski inequality and properties of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\le|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}$$

$$\times\left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\right]^{\frac{k-1}{k}}$$

$$\le\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)$$

$$\times\left[\int_0^1\left[mh(1-t)f^{\frac{rk}{k-1}}(a)+h(t)f^{\frac{rk}{k-1}}(b)\right]^{\frac{1}{r}}dt\right]^{\frac{k-1}{k}}$$

$$\le\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h^{\frac{1}{r}}(1-t)f^{\frac{k}{k-1}}(a)dt\right)^r+\left(\int_0^1 h^{\frac{1}{r}}(t)f^{\frac{k}{k-1}}(b)dt\right)^r\right\}^{\frac{k-1}{rk}}$$

$$=\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[mf^{\frac{rk}{k-1}}(a)+f^{\frac{rk}{k-1}}(b)\right]^{\frac{k-1}{rk}}$$

$$\times\left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{k-1}{k}}.$$

So, the proof of this theorem is complete. Q.E.D.

We point out some special cases of Theorem 2.8.

Corollary 2.9. In Theorem 2.8 for $r=1$ and $h(t)=t^s$ where $s\in[0,1]$, we have the following inequality for generalized relative semi-(m,s)-Breckner-preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\le\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)}{s+1}\right]^{\frac{k-1}{k}}.$$

Corollary 2.10. In Theorem 2.8 for $r = 1$ and $h(t) = t^{-s}$ where $s \in [0, 1)$, we get the following inequality for generalized relative semi-(m, s)-Godunova-Levin-Dragomir preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp + 1, kq + 1)\left[\frac{mf^{\frac{k}{k-1}}(a) + f^{\frac{k}{k-1}}(b)}{1 - s}\right]^{\frac{k-1}{k}}.$$

Corollary 2.11. In Theorem 2.8 for $r = 1$ and $h(t) = t(1 - t)$, we obtain the following inequality for generalized relative semi-(m, tgs)-preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp + 1, kq + 1)\left[\frac{mf^{\frac{k}{k-1}}(a) + f^{\frac{k}{k-1}}(b)}{6}\right]^{\frac{k-1}{k}}.$$

Corollary 2.12. In Theorem 2.8 for $r = 1$ and $h(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, we deduce the following inequality for generalized relative semi-m-MT-preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$\leq \left(\frac{\pi}{4}\right)^{\frac{k-1}{k}} \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp + 1, kq + 1)\left[mf^{\frac{k}{k-1}}(a) + f^{\frac{k}{k-1}}(b)\right]^{\frac{k-1}{k}}.$$

Theorem 2.13. Let $l \geq 1$ and $0 < r \leq 1$. Suppose $h : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ is a continuous mapping on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. If f^l is generalized relative semi-$(r; m, h)$-preinvex mappings on an open m-invex set K for some fixed $m \in (0, 1]$, then for any fixed $p, q > 0$, we have

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p + 1, q + 1)$$
$$\times\left[mf^{rl}(a)I^r(h(t); r, p, q) + f^{rl}(b)I^r(h(t); r, q, p)\right]^{\frac{1}{rl}},$$

where

$$I(h(t); r, p, q) := \int_0^1 t^p(1 - t)^q h^{\frac{1}{r}}(1 - t)dt.$$

Proof. Let $l \geq 1$ and $0 < r \leq 1$. Since f^l is generalized relative semi-$(r;m,h)$-preinvex mappings on K, combining with Lemma 2.7, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$= \eta(\varphi(b),\varphi(a),m)^{p+q+1}\int_0^1 \left[t^p(1-t)^q\right]^{\frac{l-1}{l}}\left[t^p(1-t)^q\right]^{\frac{1}{l}}$$
$$\times f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}$$

$$\times\left[\int_0^1 t^p(1-t)^q f^l(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\right]^{\frac{1}{l}}$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left[\int_0^1 t^p(1-t)^q\left[mh(1-t)f^{rl}(a)+h(t)f^{rl}(b)\right]^{\frac{1}{r}}dt\right]^{\frac{1}{l}}$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}t^p(1-t)^q h^{\frac{1}{r}}(1-t)f^l(a)dt\right)^r+\left(\int_0^1 t^p(1-t)^q h^{\frac{1}{r}}(t)f^l(b)dt\right)^r\right\}^{\frac{1}{rl}}$$

$$= \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left[mf^{rl}(a)I^r(h(t);r,p,q)+f^{rl}(b)I^r(h(t);r,q,p)\right]^{\frac{1}{rl}}.$$

So, the proof of this theorem is complete. Q.E.D.

Let us discuss some special cases of Theorem 2.13.

Corollary 2.14. In Theorem 2.13 for $r=1$ and $h(t)=t^s$ with $s\in[0,1]$, one can get the following inequality for generalized relative semi-(m,s)-Breckner-preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left[mf^l(a)\beta(p+1,q+s+1)+f^l(b)\beta(q+1,p+s+1)\right]^{\frac{1}{l}}.$$

Corollary 2.15. In Theorem 2.13 for $r = 1$ and $h(t) = t^{-s}$ with $s \in (0,1]$, we deduce the following inequality for generalized relative semi-(m,s)-Godunova-Levin-Dragomir preinvex mappings:

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\ &\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\ &\quad\times\Big[mf^l(a)\beta(p+1,q-s+1)+f^l(b)\beta(q+1,p-s+1)\Big]^{\frac{1}{l}}.\end{aligned}$$

Corollary 2.16. In Theorem 2.13 for $r = 1$ and $h(t) = t(1-t)$, one can obtain the following inequality for generalized relative semi-(m,tgs)-preinvex mappings:

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\ &\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\beta^{\frac{1}{l}}(p+2,q+2)\Big[mf^l(a)+f^l(b)\Big]^{\frac{1}{l}}.\end{aligned}$$

Corollary 2.17. In Theorem 2.13 for $r = 1$ and $h(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, we derive the following inequality for generalized relative semi-m-MT-preinvex mappings:

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\ &\leq \left(\frac{1}{2}\right)^{\frac{1}{l}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\ &\quad\times\left[mf^l(a)\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)+f^l(b)\beta\left(q+\frac{1}{2},p+\frac{3}{2}\right)\right]^{\frac{1}{l}}.\end{aligned}$$

For establishing our second main results regarding generalizations of Hermite-Hadamard, Ostrowski and Simpson type inequalities associated with generalized relative semi-$(r;m,h)$-preinvexity via fractional integrals, we need the following new crucial lemma.

Lemma 2.18. Let $\varphi : I \longrightarrow K$ be a continuous function. Suppose $K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and let $\eta(\varphi(b),\varphi(a),m) > 0$. Assume that $f : K \longrightarrow \mathbb{R}$ be a twice differentiable mapping on K° and $f'' \in L_1(K)$. Then for any $\lambda \in [0,1]$ and $\alpha > 0$, the following identity holds:

$$\begin{aligned}&I_{f,\eta,\varphi}(x;\lambda,\alpha,m,a,b)\\ &=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 t(\lambda-t^\alpha)f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt\\ &+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 t(\lambda-t^\alpha)f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda-1}{\eta(\varphi(b),\varphi(a),m)}\Big\{\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))\\
&\quad +\eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))\Big\}\\
&\quad +\frac{1+\alpha-\lambda}{\eta(\varphi(b),\varphi(a),m)}\Big\{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))\\
&\quad +\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))\Big\}\\
&\quad +\frac{\lambda}{\eta(\varphi(b),\varphi(a),m)}\Big\{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a))+\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b))\Big\}\\
&\quad -\frac{\Gamma(\alpha+2)}{\eta(\varphi(b),\varphi(a),m)}\\
&\quad \times\Big[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))^-}f(m\varphi(a))+J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))^-}f(m\varphi(b))\Big]. \qquad (2.2)
\end{aligned}
$$

Proof. A simple proof of the equality (2.2) can be done by performing two integration by parts in the integrals above and changing the variable. The details are left to the interested reader. This completes the proof of the lemma. Q.E.D.

Using Lemma 2.18, we now state the following theorems for the corresponding version for power of second derivative.

Theorem 2.19. Suppose $h:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$ and let $\eta(\varphi(b),\varphi(a),m)>0$. Assume that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow(0,+\infty)$ be a twice differentiable mapping on K° (the interior of K). If $0<r\leq 1$ and $(f''(x))^q$ is generalized relative semi-$(r;m,h)$-preinvex mappings on K, $q>1$, $p^{-1}+q^{-1}=1$, then for any $\lambda\in[0,1]$ and $\alpha>0$, the following inequality for fractional integrals holds:

$$
\begin{aligned}
|I_{f,\eta,\varphi}(x;\lambda,\alpha,m,a,b)| &\leq \frac{B^{\frac{1}{p}}(\alpha,\lambda,p)}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{1}{q}}\\
&\quad\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m(f''(a))^{rq}+(f''(x))^{rq}\Big]^{\frac{1}{rq}} \qquad (2.3)\\
&\quad+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m(f''(b))^{rq}+(f''(x))^{rq}\Big]^{\frac{1}{rq}}\Bigg\},
\end{aligned}
$$

where

$$B(\alpha,\lambda,p):=\int_0^1|t(\lambda-t^{\alpha})|^p dt. \qquad (2.4)$$

Proof. Suppose that $q>1$ and $0<r\leq 1$. Using relation (2.2), generalized relative semi-$(r;m,h)$-preinvexity of $(f''(x))^q$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$
\begin{aligned}
&|I_{f,\eta,\varphi}(x;\lambda,\alpha,m,a,b)| \\
&\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |t(\lambda-t^{\alpha})||f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |t(\lambda-t^{\alpha})||f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt \\
&\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 |t(\lambda-t^{\alpha})|^p dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 \left(f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))\right)^q dt\right)^{\frac{1}{q}} \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 |t(\lambda-t^{\alpha})|^p dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 \left(f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))\right)^q dt\right)^{\frac{1}{q}} \\
&\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} B^{\frac{1}{p}}(\alpha,\lambda,p) \\
&\times\left(\int_0^1 \left[mh(1-t)(f''(a))^{rq}+h(t)(f''(x))^{rq}\right]^{\frac{1}{r}} dt\right)^{\frac{1}{q}} \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} B^{\frac{1}{p}}(\alpha,\lambda,p) \\
&\times\left(\int_0^1 \left[mh(1-t)(f''(b))^{rq}+h(t)(f''(x))^{rq}\right]^{\frac{1}{r}} dt\right)^{\frac{1}{q}} \\
&\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} B^{\frac{1}{p}}(\alpha,\lambda,p) \\
&\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h^{\frac{1}{r}}(1-t)(f''(a))^q dt\right)^r+\left(\int_0^1 h^{\frac{1}{r}}(t)(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}} \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} B^{\frac{1}{p}}(\alpha,\lambda,p) \\
&\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h^{\frac{1}{r}}(1-t)(f''(b))^q dt\right)^r+\left(\int_0^1 h^{\frac{1}{r}}(t)(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}}
\end{aligned}
$$

$$= \frac{B^{\frac{1}{p}}(\alpha,\lambda,p)}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{1}{q}}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m(f''(a))^{rq}+(f''(x))^{rq}\Big]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m(f''(b))^{rq}+(f''(x))^{rq}\Big]^{\frac{1}{rq}}\Bigg\}.$$

So, the proof of this theorem is complete. Q.E.D.

Let us discuss some special cases of Theorem 2.19.

Corollary 2.20. In Theorem 2.19, if we choose $x = \frac{a+b}{2}$, $m = 1$ and $\eta(\varphi(y),\varphi(x),1) = \varphi(y) - \varphi(x)$, we get the following generalized Simpson type inequality for fractional integrals:

$$\left|I_{f,\varphi}\left(\frac{a+b}{2};\lambda,\alpha,1,a,b\right)\right| \le \frac{B^{\frac{1}{p}}(\alpha,\lambda,p)}{(\varphi(b)-\varphi(a))}\left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{1}{q}}$$

$$\times\Bigg\{\left(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)\right)^{\alpha+2}\left[(f''(a))^{rq}+\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}\right]^{\frac{1}{rq}}$$

$$+\left(\varphi(b)-\varphi\left(\frac{a+b}{2}\right)\right)^{\alpha+2}\left[(f''(b))^{rq}+\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}\right]^{\frac{1}{rq}}\Bigg\}. \tag{2.5}$$

Corollary 2.21. In Theorem 2.19, if we choose $\lambda = m = 1$ and $\eta(\varphi(y),\varphi(x),1) = \varphi(y) - \varphi(x)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:

$$\big|I_{f,\varphi}(x;1,\alpha,1,a,b)\big| \le \left(\frac{\alpha}{2(\alpha+2)}\right)^{\frac{1}{p}}\left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{1}{q}}\frac{1}{(\varphi(b)-\varphi(a))}$$

$$\times\Bigg\{(\varphi(x)-\varphi(a))^{\alpha+2}\Big[(f''(a))^{rq}+(f''(x))^{rq}\Big]^{\frac{1}{rq}}$$

$$+(\varphi(b)-\varphi(x))^{\alpha+2}\Big[(f''(b))^{rq}+(f''(x))^{rq}\Big]^{\frac{1}{rq}}\Bigg\}. \tag{2.6}$$

Corollary 2.22. In Theorem 2.19, if we choose $\lambda = 0$, $m = 1$ and $\eta(\varphi(y),\varphi(x),1) = \varphi(y) - \varphi(x)$, we get the following generalized Ostrowski type inequality for fractional integrals:

$$\big|I_{f,\varphi}(x;0,\alpha,1,a,b)\big| \le \frac{1}{\big[p(\alpha+1)+1\big]^{\frac{1}{p}}}\left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{1}{q}}\frac{1}{(\varphi(b)-\varphi(a))}$$

$$\times\Bigg\{(\varphi(x)-\varphi(a))^{\alpha+2}\Big[(f''(a))^{rq}+(f''(x))^{rq}\Big]^{\frac{1}{rq}}$$

$$+ (\varphi(b) - \varphi(x))^{\alpha+2} \left[(f''(b))^{rq} + (f''(x))^{rq} \right]^{\frac{1}{rq}} \Bigg\}. \tag{2.7}$$

Theorem 2.23. Suppose $h : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and let $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0,+\infty)$ be a twice differentiable mapping on K°. If $0 < r \leq 1$ and $(f''(x))^q$ is generalized relative semi-$(r; m, h)$-preinvex mappings on K, $q \geq 1$, then for any $\lambda \in [0,1]$ and $\alpha > 0$, the following inequality for fractional integrals holds:

$$\begin{aligned} |I_{f,\eta,\varphi}(x; \lambda, \alpha, m, a, b)| &\leq \frac{E^{1-\frac{1}{q}}}{\eta(\varphi(b), \varphi(a), m)} \\ &\times \Bigg\{ |\eta(\varphi(x), \varphi(a), m)|^{\alpha+2} \Big[m (f''(a))^{rq} (F_1 + F_2)^r + (f''(x))^{rq} (G_1 + G_2)^r \Big]^{\frac{1}{rq}} \\ &+ |\eta(\varphi(x), \varphi(b), m)|^{\alpha+2} \Big[m (f''(b))^{rq} (F_1 + F_2)^r + (f''(x))^{rq} (G_1 + G_2)^r \Big]^{\frac{1}{rq}} \Bigg\}, \end{aligned} \tag{2.8}$$

where

$$E = E(\alpha, \lambda) := \frac{\alpha \lambda^{1+\frac{2}{\alpha}} + 1}{\alpha + 2} - \frac{\lambda}{2};$$

$$F_1 = F_1(h(t); \alpha, \lambda, r) := \int_0^{\lambda^{\frac{1}{\alpha}}} t(\lambda - t^{\alpha}) h^{\frac{1}{r}}(1-t) dt;$$

$$F_2 = F_2(h(t); \alpha, \lambda, r) := \int_{\lambda^{\frac{1}{\alpha}}}^1 t(t^{\alpha} - \lambda) h^{\frac{1}{r}}(1-t) dt;$$

$$G_1 = G_1(h(t); \alpha, \lambda, r) := \int_0^{\lambda^{\frac{1}{\alpha}}} t(\lambda - t^{\alpha}) h^{\frac{1}{r}}(t) dt;$$

$$G_2 = G_2(h(t); \alpha, \lambda, r) := \int_{\lambda^{\frac{1}{\alpha}}}^1 t(t^{\alpha} - \lambda) h^{\frac{1}{r}}(t) dt.$$

Proof. Suppose that $q \geq 1$ and $0 < r \leq 1$. Using relation (2.2), generalized relative semi-$(r; m, h)$-preinvexity of $(f''(x))^q$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned} &|I_{f,\eta,\varphi}(x; \lambda, \alpha, m, a, b)| \\ \leq &\frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}}{|\eta(\varphi(b), \varphi(a), m)|} \int_0^1 |t(\lambda - t^{\alpha})| |f''(m\varphi(a) + t\eta(\varphi(x), \varphi(a), m))| dt \\ + &\frac{|\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{|\eta(\varphi(b), \varphi(a), m)|} \int_0^1 |t(\lambda - t^{\alpha})| |f''(m\varphi(b) + t\eta(\varphi(x), \varphi(b), m))| dt \\ &\leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 |t(\lambda - t^{\alpha})| dt \right)^{1-\frac{1}{q}} \end{aligned}$$

$$\times\left(\int_0^1 |t(\lambda-t^\alpha)|\left(f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))\right)^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 |t(\lambda-t^\alpha)|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 |t(\lambda-t^\alpha)|\left(f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}E^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 |t(\lambda-t^\alpha)|\Big[mh(1-t)(f''(a))^{rq}+h(t)(f''(x))^{rq}\Big]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}E^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 |t(\lambda-t^\alpha)|\Big[mh(1-t)(f''(b))^{rq}+h(t)(f''(x))^{rq}\Big]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}E^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h^{\frac{1}{r}}(1-t)|t(\lambda-t^\alpha)|(f''(a))^q dt\right)^r+\left(\int_0^1 h^{\frac{1}{r}}(t)|t(\lambda-t^\alpha)|(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}E^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h^{\frac{1}{r}}(1-t)|t(\lambda-t^\alpha)|(f''(b))^q dt\right)^r+\left(\int_0^1 h^{\frac{1}{r}}(t)|t(\lambda-t^\alpha)|(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}}$$

$$=\frac{E^{1-\frac{1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m(f''(a))^{rq}\left(F_1+F_2\right)^r+(f''(x))^{rq}\left(G_1+G_2\right)^r\Big]^{\frac{1}{rq}}\right.$$

$$\left.+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m(f''(b))^{rq}\left(F_1+F_2\right)^r+(f''(x))^{rq}\left(G_1+G_2\right)^r\Big]^{\frac{1}{rq}}\right\}.$$

So, the proof of this theorem is complete. Q.E.D.

Let us discuss some special cases of Theorem 2.23.

Corollary 2.24. In Theorem 2.23, if we choose $x = \frac{a+b}{2}$, $m = 1$ and $\eta(\varphi(y), \varphi(x), 1) = \varphi(y) - \varphi(x)$, we get the following generalized Simpson type inequality for fractional integrals:

$$\left|I_{f,\varphi}\left(\frac{a+b}{2}; \lambda, \alpha, 1, a, b\right)\right| \leq \frac{E^{1-\frac{1}{q}}}{(\varphi(b) - \varphi(a))}$$

$$\times \left\{ \left(\varphi\left(\frac{a+b}{2}\right) - \varphi(a)\right)^{\alpha+2} \left[(f''(a))^{rq} (F_1 + F_2)^r + \left(f''\left(\frac{a+b}{2}\right)\right)^{rq} (G_1 + G_2)^r \right]^{\frac{1}{rq}} \right.$$

$$\left. + \left(\varphi(b) - \varphi\left(\frac{a+b}{2}\right)\right)^{\alpha+2} \left[(f''(b))^{rq} (F_1 + F_2)^r + \left(f''\left(\frac{a+b}{2}\right)\right)^{rq} (G_1 + G_2)^r \right]^{\frac{1}{rq}} \right\}. \tag{2.9}$$

Corollary 2.25. In Theorem 2.23, if we choose $\lambda = m = 1$ and $\eta(\varphi(y), \varphi(x), 1) = \varphi(y) - \varphi(x)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:

$$\left|I_{f,\varphi}(x; 1, \alpha, 1, a, b)\right| \leq \left(\frac{\alpha}{2(\alpha+2)}\right)^{1-\frac{1}{q}} \frac{1}{(\varphi(b) - \varphi(a))}$$

$$\times \left\{ (\varphi(x) - \varphi(a))^{\alpha+2} \left[(f''(a))^{rq} F^r + (f''(x))^{rq} G^r \right]^{\frac{1}{rq}} \right.$$

$$\left. + (\varphi(b) - \varphi(x))^{\alpha+2} \left[(f''(b))^{rq} F^r + (f''(x))^{rq} G^r \right]^{\frac{1}{rq}} \right\}, \tag{2.10}$$

where

$$F = F(h(t); \alpha, r) := \int_0^1 t(1 - t^\alpha) h^{\frac{1}{r}}(1-t) dt;$$

$$G = G(h(t); \alpha, r) := \int_0^1 t(1 - t^\alpha) h^{\frac{1}{r}}(t) dt.$$

Corollary 2.26. In Theorem 2.23, if we choose $\lambda = 0$, $m = 1$ and $\eta(\varphi(y), \varphi(x), 1) = \varphi(y) - \varphi(x)$, we get the following generalized Ostrowski type inequality for fractional integrals:

$$\left|I_{f,\varphi}(x; 0, \alpha, 1, a, b)\right| \leq \frac{1}{(\alpha+2)^{1-\frac{1}{q}}} \frac{1}{(\varphi(b) - \varphi(a))}$$

$$\times \left\{ (\varphi(x) - \varphi(a))^{\alpha+2} \left[(f''(a))^{rq} \overline{F}^r + (f''(x))^{rq} \overline{G}^r \right]^{\frac{1}{rq}} \right.$$

$$\left. + (\varphi(b) - \varphi(x))^{\alpha+2} \left[(f''(b))^{rq} \overline{F}^r + (f''(x))^{rq} \overline{G}^r \right]^{\frac{1}{rq}} \right\}, \tag{2.11}$$

where

$$\overline{F} = \overline{F}(h(t); \alpha, r) := \int_0^1 t^{\alpha+1} h^{\frac{1}{r}}(1-t) dt;$$

$$\overline{G} = \overline{G}(h(t); \alpha, r) := \int_0^1 t^{\alpha+1} h^{\frac{1}{r}}(t) dt.$$

Remark 2.27. In Theorems 2.19 and 2.23, if taking $r=1$ and $h(t)=t^s, h(t)=t^{-s}, h(t)=t(1-t)$ or $h(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then one can get some special fractional integral inequalities for generalized relative semi-(m,s)-Breckner-preinvex mappings, generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex mappings, generalized relative semi-(m,tgs)-preinvex mappings and generalized relative semi-m-MT-preinvex mappings, respectively. Also, using Theorems 2.19 and 2.23 some interesting applications to special means can be deduced. The details are left to the interested reader.

3 Conclusions

In this article, we first presented some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-$(r;m,h)$-preinvex mappings. And then, a new identity concerning twice differentiable mappings defined on m-invex set is derived. By using the notion of generalized relative semi-$(r;m,h)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard, Ostrowski and Simpson type inequalities via fractional integrals are established. It is pointed out that some new special cases are deduced from main results of the article. Motivated by this new interesting class of generalized relative semi-$(r;m,h)$-preinvex mappings we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard, Ostrowski and Simpson type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k-fractional integrals, local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p,q)-calculus, time scale calculus and conformable fractional integrals.

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