A fixed points approach to stability of the Pexider equation

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Abstract

Using the fixed point theorem we establish the Hyers-Ulam-Rassias stability of the generalized Pexider functional equation

$$\frac{1}{\mid K \mid}\sum_{k\in K} f(x+k\cdot y) = g(x)+h(y), \quad x,y\in E$$

from a normed space E into a complete β-normed space F, where K is a finite abelian subgroup of the automorphism group of the group $(E,+)$.

2010 Mathematics Subject Classification. **39B82**. 39B52
Keywords. Stability of functional equation, fixed point, Pexider equation.

1 Introduction and preliminaries

Under what condition does there exist a group homomorphism near an approximate group homomorphism? This question concerning the stability of group homomorphisms was posed by S. M. Ulam [58]. In 1941, the Ulam's problem for the case of approximately additive mappings was solved by D. H. Hyers [21] on Banach spaces. In 1950 T. Aoki [2] provided a generalization of the Hyers' theorem for additive mappings and in 1978 Th. M. Rassias [47] generalized the Hyers' theorem for linear mappings by considering an unbounded Cauchy difference. The result of Rassias' theorem has been generalized by J.M. Rassias [44] and later by Găvruta [18] who permitted the Cauchy difference to be bounded by a general control function. Since then, the stability problems for several functional equations have been extensively investigated (cf. [16], [19], [23], [24], [25], [26], [27], [32], [41], [44], [45], [48], [49]).
Let E be a real vector space and F be a real Banach space. Let K be a finite abelian subgroup of $Aut(E)$ (the automorphism group of the group $(E,+)$, $|K|$ denotes the order of K. Writing the action of $k\in K$ on $x\in E$ as $k\cdot x$, we will say that $(f,g,h)\ : E\to F$ is a solution of the generalized Pexider functional equation, if

$$\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y) = g(x)+h(y), \quad x,y\in E \tag{1.1}$$

The generalized quadratic functional equation

$$\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y) = f(x)+f(y), \quad x,y\in E \tag{1.2}$$

Tbilisi Mathematical Journal 7(2) (2014), pp. 95–110.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 26 October 2014.
Accepted for publication: 02 December 2014.

and the generalized Jensen functional equation

$$\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y)=f(x),\quad x,y\in E \tag{1.3}$$

are particulars cases of equation (1.1).
The functional equations (1.1), (1.2) and (1.3) appeared in several works by H. Stetkær, see for example [55], [56] and [57]. We refer also to the recent studies by Ł. Radosław [50] and [51].
If we set $K=\{I,\sigma\}$, were $I\colon\ E\longrightarrow E$ denotes the identity function and σ denote an additive function of E, such that $\sigma(\sigma(x))=x$, for all $x\in E$ then equation (1.1) reduces to the Pexider functionals equations

$$f(x+y)+f(x+\sigma(y))=g(x)+h(y),\ x,y\in E, \tag{1.4}$$

$$f(x+y)=g(x)+h(y),\ x,y\in E,\quad (\sigma=I) \tag{1.5}$$

$$f(x+y)+f(x-y)=g(x)+h(y),\ x,y\in E,\ \ (\sigma=-I) \tag{1.6}$$

Y. H. Lee and K. W. Jung [33] obtained the Hyers-Ulam-Rassias of the Pexider functional equation (1.5). Jung [27] and Jung and Sahoo [30] investigated the Hyers-Ulam-Rassias stability of equation (1.6). Belaid et al. have proved the Hyers-Ulam stability of equation (1.1) and the Hyers-Ulam-Rassias stability of the functional equations (1.2), (1.3), (see [1], [11], [12] and [34]).
Recently, Radosław [50] obtained the Hyers-Ulam-Rassias stability of equation (1.1). In 2003 L. Cădariu and V. Radu [9] notice that a fixed point alternative method is very important for the solution of the Hyers-Ulam stability problem. Subsequently, this method was applied to investigate the Hyers-Ulam-Rassias stability for Jensen functional equation, as well as for the additive Cauchy functional equation [12] by considering a general control function $\varphi(x,y)$, with suitable properties, using such an elegant idea, several authors applied the method to investigate the stability of some functional equations, see for example [3], [4], [5], [6], [31], [35], [43].
In this paper, we will apply the fixed point method as in [9] to prove the Hyers-Ulam-Rassias stability of the functional equations (1.1), for a large classe of functions from a vector space E into complete β-normed space F.

Now, we recall one of fundamental results of fixed point theory.
Let X be a set. A function $d:X\times X\to[0,+\infty]$ is called a *generalized metric* on X if d satisfies the following:
(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(2) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Theorem 1.1. [15] Suppose we are given a complete generalized metric space (X,d) and a strictly contractive mapping $J:X\to X$, white the Lipshitz constant $L<1$. If there exists a nonnegative integer k such that $d(J^kx,J^{k+1}x)<+\infty$ for some $x\in X$, then the following are true:
(1) the sequence J^nx converges to a fixed point x^* of J;
(2) x^* is the unique fixed point of J in the set $Y=\{y\in X: d(J^kx,y)<+\infty\}$;
(3) $d(y,x^*)\le\frac{1}{1-L}d(y,Jy)$ for all $y\in Y$.

Throughout this paper, we fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Suppose E is a vector space over $\mathbb{K}$. A function $\|.\|_\beta \colon E \longrightarrow [0,\infty)$ is called a β-norm if and only if it satisfies

(1) $\|x\|_\beta = 0$, if and only if $x = 0$;
(2) $\|\lambda x\|_\beta = |\lambda|^\beta \|x\|_\beta$ for all $\lambda \in \mathbb{K}$ and all $x \in E$;
(3) $\|x+y\|_\beta \leq \|x\|_\beta + \|y\|_\beta$ for all $x, y \in E$.

2 Main results

In the following theorem, by using an idea of Cădariu and Radu [9, 12], we prove the Hyers-Ulam-Rassias stability of the generalized Pexider functional equation (1.1).

Theorem 2.1. Let E be a vector space over $\mathbb{K}$ and let F be a complete β-normed space over $\mathbb{K}$. Let K be a finite abelian subgroup of the automorphism group of $(E,+)$. Let $f \colon E \longrightarrow F$ be a mapping for which there exists a function $\varphi : E \times F \to [0,\infty)$ and a constant $L < 1$, such that

$$\|\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y) - g(x) - h(y)\|_\beta \leq \varphi(x,y) \tag{2.1}$$

and

$$\sum_{k\in K} \varphi(x+k\cdot x, y+k\cdot y) \leq (|2K|)^\beta L\varphi(x,y) \tag{2.2}$$

for all $x, y \in E$. Then, there exists a unique solution $q \colon E \longrightarrow F$ of the generalierd quadratic functional equation (1.2) and a unique solution $j \colon E \longrightarrow F$ of the generalized Jensen functional equation (1.3) such that

$$\frac{1}{|K|}\sum_{k\in K} j(k\cdot x) = 0, \tag{2.3}$$

$$\|f(x) - q(x) - j(x) - g(0) - h(0)\|_\beta \leq \frac{2}{2^\beta}\frac{1}{1-L}\chi(x,x) + \frac{1}{2^\beta}\frac{1}{1-L}\psi(x,x), \tag{2.4}$$

$$\|g(x) - q(x) - j(x) - g(0)\|_\beta \leq \varphi(x,0) + \frac{2}{2^\beta}\frac{1}{1-L}\chi(x,x) + \frac{1}{2^\beta}\frac{1}{1-L}\psi(x,x) \tag{2.5}$$

and

$$\|h(x) - q(x) - h(0)\|_\beta \leq \frac{1}{2^\beta}\frac{1}{1-L}\psi(x,x) + \varphi(0,x) \tag{2.6}$$

for all $x \in E$, where

$$\chi(x,y) = \frac{|K|}{|K|^\beta}\varphi(0,y) + \varphi(x,y) + \varphi(x,0) + \varphi(0,y)$$
$$+\frac{1}{|K|^\beta}\sum_{k\in K}[\varphi(k\cdot x, y) + \varphi(k\cdot x, 0)]$$

and

$$\psi(x,y) = \frac{|K|}{|K|^\beta}\varphi(0,y) + \frac{1}{|K|^\beta}\sum_{k\in K}[\varphi(k\cdot x, y) + \varphi(k\cdot x, 0)].$$

Proof. Letting $y = 0$ in (2.1), to obtain

$$\|f(x) - g(x) - h(0)\|_\beta \le \varphi(x, 0) \tag{2.7}$$

for all $x \in E$. By using (2.7), (2.1) and the triangle inequality, we get

$$\|\frac{1}{|K|}\sum_{k\in K} f(x + k \cdot y) - f(x) - (h(y) - h(0))\|_\beta \le \|\frac{1}{|K|}\sum_{k\in K} f(x + k \cdot y) - g(x) - h(y)\|_\beta \tag{2.8}$$

$$+\|g(x) - f(x) + h(0)\|_\beta \le \varphi(x, y) + \varphi(x, 0)$$

for all $x, y \in E$. Replacing x by 0 in (2.1), we get

$$\|\frac{1}{|K|}\sum_{k\in K} f(k \cdot y) - g(0) - h(y)\|_\beta \le \varphi(0, y) \tag{2.9}$$

for all $y \in E$. So inequalities (2.8), (2.9) and the triangle inequality implies that

$$\|\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y)-f(x)-\frac{1}{|K|}\sum_{k\in K} f(k\cdot y)+g(0)+h(0)\|_\beta \le \frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y)-f(x)-(h(y)-h(0))\|_\beta \tag{2.10}$$

$$+\|\frac{1}{|K|}\sum_{k\in K} f(k \cdot y) - h(y) - g(0)\|_\beta \le \varphi(x, y) + \varphi(x, 0) + \varphi(0, y)$$

for all $x, y \in E$. Now, let

$$\varphi(x) = \frac{1}{|K|}\sum_{k\in K} f(k \cdot x) \tag{2.11}$$

for all $x \in E$. Then, φ satisfies

$$\frac{1}{|K|}\sum_{k\in K} \varphi(k \cdot x) = \varphi(x) \tag{2.12}$$

for all $x \in E$. Furthermore, in view of (2.10), (2.12) and the triangle inequality, we have

$$\|\frac{1}{|K|}\sum_{k'\in K} \varphi(x + k' \cdot y) - \varphi(x) - \varphi(y) + g(0) + h(0)\|_\beta \tag{2.13}$$

$$= \|\frac{1}{|K|}\sum_{k'\in K}\frac{1}{|K|}\sum_{k\in K} f(k \cdot x + kk' \cdot y) - \frac{1}{|K|}\sum_{k\in K} f(k \cdot x) - \frac{1}{|K|^2}\sum_{k,k'\in K} f(kk' \cdot y) + g(0) + h(0)\|_\beta$$

$$\le \frac{1}{|K|^\beta}\sum_{k\in K}\|\frac{1}{|K|}\sum_{k'\in K} f(k \cdot x + k' \cdot y) - f(k \cdot x) - \frac{1}{|K|}\sum_{k'\in K} f(k' \cdot y) + g(0) + h(0)\|_\beta$$

$$\le \frac{1}{|K|^\beta}\sum_{k\in K}[\varphi(k \cdot x, y) + \varphi(k \cdot x, 0)] + \frac{|K|}{|K|^\beta}\varphi(0, y) = \psi(x, y).$$

Since K is an abelian subgroup, so by using (2.2), we get

$$\sum_{k\in K} \psi(x + k \cdot x, y + k \cdot y) \le (2|K|)^\beta L\psi(x, y) \tag{2.14}$$

for all $x, y \in E$. Let us consider the set $X := \{g : E \longrightarrow F\}$ and introduce the *generalized metric* on X as follows:

$$d(g, h) = \inf\{C \in [0, \infty] : \|g(x) - h(x)\|_\beta \leq C\psi(x, x), \ \forall x \in E\}. \tag{2.15}$$

Let f_n be a Cauchy sequence in (X, d). According to the definition of the Cauchy sequence, for any given $\varepsilon > 0$, there exists a positive integer N such that

$$d(f_n, f_m) \leq \varepsilon \tag{2.16}$$

for all integer m, n such that $m \geq N$ and $n \geq N$. That is, by considering the definition of the *generalized metric* d

$$\|f_m(x) - f_n(x)\|_\beta \leq \varepsilon\psi(x, x) \tag{2.17}$$

for all integer m, n such that $m \geq N$ and $n \geq N$, which implies that $f_n(x)$ is a Cauchy sequence in F, for any fixed $x \in E$. Since F is complete, $f_n(x)$ converges in F for each x in E. Hence, we can define a function $f : E \longrightarrow F$ by

$$f(x) = \lim_{n \longrightarrow \infty} f_n(x). \tag{2.18}$$

As a similar proof to [34], we consider the linear operator $J : X \to X$ such that

$$(Jh)(x) = \frac{1}{2|K|} \sum_{k \in K} h(x + k \cdot x) \tag{2.19}$$

for all $x \in E$. By induction, we can easily show that

$$(J^n h)(x) = \frac{1}{(2|K|)^n} \sum_{k_1, \ldots, k_n \in K} h\left(x + \sum_{i_j < i_{j+1}, k_{ij} \in \{k_1, \ldots, k_n\}} (k_{i_1} \ldots k_{i_p}) \cdot x\right) \tag{2.20}$$

for all integer n.
First, we assert that J is strictly contractive on X. Given g and h in X, let $C \in [0, \infty)$ be an arbitrary constant with $d(g, h) \leq C$, that is,

$$\|g(x) - h(x)\|_\beta \leq C\psi(x, x) \tag{2.21}$$

for all $x \in E$. So, it follows from (2.19), (2.14) and (2.21) we get

$$\begin{aligned}
\|(Jg)(x) - (Jh)(x)\|_\beta &= \|\frac{1}{2|K|} \sum_{k \in K} g(x + k \cdot x) - \frac{1}{2|K|} \sum_{k \in K} h(x + k \cdot x)\|_\beta \\
&= \frac{1}{(2|K|)^\beta} \|\sum_{k \in K} g(x + k \cdot x) - h(x + k \cdot x)\|_\beta \\
&\leq \frac{1}{(2|K|)^\beta} \sum_{k \in K} \|g(x + k \cdot x) - h(x + k \cdot x))\|_\beta \\
&\leq \frac{1}{(2|K|)^\beta} C \sum_{k \in K} \psi(x + k \cdot x, x + k \cdot x) \\
&\leq CL\psi(x, x)
\end{aligned}$$

for all $x \in E$, that is, $d(Jg, Jh) \leq LC$. Hence, we conclude that

$$d(Jg, Jh) \leq Ld(g, h)$$

for any $g, h \in X$. Now, we claim that

$$d(J(\varphi - g(0) - h(0), \varphi - g(0) - h(0)) < \infty. \tag{2.22}$$

By letting $y = x$ in (2.13), we obtain

$$\|(J(\varphi - g(0) - h(0)))(x) - (\varphi - g(0) - h(0))(x)\|_\beta = \frac{1}{2^\beta}\|\frac{1}{|K|}\sum_{k\in K}\varphi(x+k\cdot x) - 2\varphi(x) + g(0) + h(0)\|_\beta \leq \frac{1}{2^\beta}\psi(x, x) \tag{2.23}$$

for all $x \in E$, that is

$$d(J(\varphi - g(0) - h(0)), \varphi - g(0) - h(0)) \leq \frac{1}{2^\beta} < \infty \tag{2.24}$$

From Theorem 1.1, there exists a fixed point of J which is a function $q : E \to F$ such that $\lim_{n\longrightarrow\infty} d(J^n(\varphi - g(0) - h(0)), q) = 0$. Since $d(J^n(\varphi - g(0) - h(0)), q) \to 0$ as $n \to \infty$, there exists a sequence $\{C_n\}$ such that $\lim_{n\longrightarrow\infty} C_n = 0$ and $d(J^n\varphi - g(0) - h(0), q) \leq C_n$ for every $n \in \mathbb{N}$. Hence, from the definition of d, we get

$$\|(J^n(\varphi - g(0) - h(0)(x) - q(x)\|_\beta \leq C_n\psi(x, x) \tag{2.25}$$

for all $x \in E$. Therefore,

$$\lim_{n\to\infty} \|(J^n(\varphi - g(0) - h(0))(x) - q(x)\|_\beta = 0, \tag{2.26}$$

for all $x \in E$.

Now, if we put $\kappa(x) = \varphi(x) - g(0) - h(0)$, by using induction on n we prove the validity of following inequality

$$\|\frac{1}{|K|}\sum_{k\in K} J^n\kappa(x + k\cdot y) - J^n\kappa(x) - J^n\kappa(y)\|_\beta \leq L^n\psi(x, y). \tag{2.27}$$

In view of the commutativity of K the inequalities (2.13), (2.14) we have

$$\|\frac{1}{|K|}\sum_{k\in K} Jf(x + k\cdot y) - J\kappa(x) - J\kappa(y)\|_\beta$$

$$= \|\frac{1}{|K|}\sum_{k\in K}\frac{1}{2|K|}\sum_{k_1\in K}\kappa(x+k\cdot y+k_1\cdot x+k_1k\cdot y) - \frac{1}{2|K|}\sum_{k_1\in K}\kappa(x+k_1\cdot x) - \frac{1}{2|K|}\sum_{k_1\in K}\kappa(y+k_1\cdot y)\|_\beta$$

$$\leq \frac{1}{(2|K|^\beta)}\sum_{k_1\in K}\|\frac{1}{|K|}\sum_{k\in K}\kappa(x + k_1\cdot x + k\cdot(y + k_1\cdot y)) - \kappa(x + k_1\cdot x) - \kappa(y + k_1\cdot y)\|_\beta$$

$$\leq \frac{1}{(2|K|^\beta)}\sum_{k_1\in K}\psi(x + k_1\cdot x, y + k_1\cdot y) \leq \frac{1}{(2|K|)^\beta}(2|K|)^\beta L\psi(x, y) = L\psi(x, y).$$

This proves (2.27) for $n = 1$. Now, we assume that (2.27) is true for n. By using the commutativity of K, the inequalities (2.13), (2.14), we get

$$\|\frac{1}{|K|}\sum_{k\in K} J^{n+1}\kappa(x+k\cdot y) - J^{n+1}\kappa(x) - J^{n+1}\kappa(y) + g(0) + h(0)\|_\beta$$

$$= \|\frac{1}{|K|}\sum_{k\in K}\frac{1}{2|K|}\sum_{k'\in K} J^n\kappa(x+k\cdot y+k'\cdot x+k'k\cdot y)$$

$$-\frac{1}{2|K|}\sum_{k'\in K} J^n\kappa(x+k'\cdot x) - \frac{1}{2|K|}\sum_{k'\in K} J^n\kappa(y+k'\cdot y)\|_\beta$$

$$\leq \frac{1}{(2|K|)^\beta}\sum_{k'\in K}\|\frac{1}{|K|}\sum_{k\in K} J^n\kappa(x+k'\cdot x+k\cdot(y+k'\cdot y)) - J^n\kappa(x+k'\cdot x) - J^n\kappa(y+k'\cdot y)\|_\beta$$

$$\leq \frac{1}{(2|K|)^\beta}\sum_{k'\in K} L^n\psi(x+k'\cdot x, y+k'\cdot y) \leq L^{n+1}\psi(x,y),$$

which proves (2.27) for $n + 1$. Now, by letting $n \to \infty$, in (2.27), we obtain that q is a solution of equation (1.2). According to the fixed point theorem (Theorem 1.1, (3)) and inequality (2.24), we get

$$d(\varphi - g(0) - h(0), q) \leq \frac{1}{1-L} d(J(\varphi - g(0) - h(0)), \varphi - g(0) - h(0)) \leq \frac{1}{2^\beta(1-L)} \tag{2.28}$$

and so we have

$$\|\varphi(x) - q(x) - g(0) - h(0))\| \leq \frac{1}{2^\beta(1-L)}\psi(x,x) \tag{2.29}$$

for all $x \in E$. On the other hand if we put

$$\omega(x) = f(x) - \varphi(x) = f(x) - \frac{1}{|K|}\sum_{k\in K} f(k\cdot x) \tag{2.30}$$

for all $x \in E$, it follows from inequalities (2.10), (2.13) and the triangle inequality that

$$\|\frac{1}{|K|}\sum_{k'\in K}\omega(x+k'\cdot y) - \omega(x)\|_\beta \tag{2.31}$$

$$= \|\frac{1}{|K|}\sum_{k'\in K} f(x+k'\cdot y) - \frac{1}{|K|}\sum_{k\in K}\varphi(x+k\cdot y) - f(x) + \varphi(x)\|_\beta$$

$$\leq \| - \frac{1}{|K|}\sum_{k\in K}\varphi(x+k\cdot y) + \varphi(x) + \varphi(y) - g(0) - h(0)\|_\beta$$

$$+\|\frac{1}{|K|}\sum_{k'\in K} f(x+k'\cdot y) - f(x) - \frac{1}{|K|}\sum_{k'\in K} f(k'\cdot y) + g(0) + h(0)\|_\beta$$

$$\leq \frac{1}{|K|^{\beta}} \sum_{k\in K}[\varphi(k\cdot x, y)+\varphi(k\cdot x, 0)] + \frac{|K|}{|K|^{\beta}}\varphi(0,y)+\varphi(x,y)+\varphi(x,0)+\varphi(0,y) = \chi(x,y)$$

for all $x, y \in E$. By using the same definition for X as in the above proof, the *generalized metric* on X

$$d(g,h) = \inf\{C \in [0,\infty] : \ \|g(x)-h(x)\|_{\beta} \leq C\chi(x,x), \ \forall x \in E\}. \tag{2.32}$$

and some ideas of [34], we will prove that there exists a unique solution j of equation (1.3) such that

$$\|\omega(x) - j(x)\|_{\beta} \leq \frac{1}{1-L}\chi(x,x) \tag{2.33}$$

for all $x \in E$.
First, from (2.2) we can easily verify that $\chi(x,y)$ satisfies

$$\sum_{k\in K}\chi(x+k\cdot x, y+k\cdot y) \leq (2|K|)^{\beta}L\chi(x,y) \tag{2.34}$$

Let us consider the function $T : X \to X$ defined by

$$(Th)(x) = \frac{1}{|2K|}\sum_{k\in K} h(x+k\cdot x) \tag{2.35}$$

for all $x \in E$. Given $g, h \in X$ and $C \in [0,\infty]$ such that $d(g,h) \leq C$, so we get

$$\|(Tg)(x)-(Th)(x)\|_{\beta} = \|\frac{1}{|2K|}\sum_{k\in K} g(x+k\cdot x) - \frac{1}{|2K|}\sum_{k\in K} h(x+k\cdot x)\|_{\beta}$$

$$= \frac{1}{|2K|^{\beta}}\|\sum_{k\in K}[g(x+k\cdot x)-h(x+k\cdot x)]\|_{\beta}$$

$$\leq \frac{1}{|2K|^{\beta}}\sum_{k\in K}\|g(x+k\cdot x)-h(x+k\cdot x)\|_{\beta} \leq CL\chi(x,x)$$

for all $x \in E$. Hence, we see that $d(Tg, Th) \leq Ld(g,h)$ for all $g, h \in X$. So T is a strictly contractive operator.
Putting $y = x$ in (2.31), we have

$$\|\frac{1}{|2K|}\sum_{k\in K}\omega(x+k\cdot x) - \frac{1}{2}\omega(x)\|_{\beta} \leq \frac{1}{2^{\beta}}\chi(x,x) \tag{2.36}$$

for all $x \in E$, so by the triangle inequality, we get

$$d(T\omega, \omega) \leq \frac{2}{2^{\beta}}. \tag{2.37}$$

From the fixed point theorem (Theorem 1.1), it follows that there exits a fixed point j of T in X such that

$$j(x) = \lim_{n\to\infty}\frac{1}{|2K|^{n}}\sum_{k_1,\ldots,k_n\in K}\omega\left(x+\sum_{i_j<i_{j+1},k_{ij}\in\{k_1,\ldots,k_n\}}[(k_{i_1})\cdots(k_{i_p})]\cdot x\right) \tag{2.38}$$

for all $x \in E$ and

$$d(\omega, j) \leq \frac{1}{1-L} d(T\omega, \omega). \tag{2.39}$$

So, it follows from (2.37) and (2.39) that

$$\|\omega(x) - j(x)\|_\beta \leq \frac{2}{2^\beta} \frac{1}{1-L} \chi(x, x) \tag{2.40}$$

for all $x \in E$.
By the same reasoning as in the above proof, one can show by induction that

$$\| \frac{1}{|K|} \sum_{k \in K} T^n \omega(x + k \cdot y) - T^n \omega(x) \|_\beta \leq L^n \chi(x, y) \tag{2.41}$$

for all $x, y \in E$ and for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.41), we get that j is a solution of the generalized Jensen functional equation (1.3).
From (2.11), (2.29) (2.30), (2.40) and the triangle inequality, we obtain

$$\|f(x) - q(x) - j(x) - g(0) - h(0)\|_\beta \leq \frac{2}{2^\beta} \frac{1}{1-L} \chi(x, x) + \frac{1}{2^\beta} \frac{1}{1-L} \psi(x, x), \tag{2.42}$$

$$\|g(x) - q(x) - j(x) - g(0)\|_\beta \leq \varphi(x, 0) + \frac{2}{2^\beta} \frac{1}{1-L} \chi(x, x) + \frac{1}{2^\beta} \frac{1}{1-L} \psi(x, x) \tag{2.43}$$

and

$$\|h(x) - q(x) - h(0)\|_\beta \leq \frac{1}{2^\beta} \frac{1}{1-L} \psi(x, x) + \varphi(0, x) \tag{2.44}$$

for all $x \in E$.
Finally, in the following we will verify that the solution j satisfies the condition

$$\frac{1}{|K|} \sum_{k \in K} j(k \cdot x) = 0 \tag{2.45}$$

for all $x \in E$ and we will prove the uniqueness of the solutions q and j which satisfy the inequalities (2.42) (2.43) and (2.44).
Due to definition of ω, we get $\frac{1}{|K|} \sum_{k \in K} \omega(k \cdot x) = 0$ for all $x \in E$, so we get $\frac{1}{|K|} \sum_{k \in K} T\omega(k \cdot x) = 0$, $\frac{1}{|K|} \sum_{k \in K} T^2\omega(k \cdot x) = 0, \ldots,$ $\frac{1}{|K|} \sum_{k \in K} T^n\omega(k \cdot x) = 0$. So, by letting $n \longrightarrow \infty$, we obtain the ralation (2.45).
Now, according to (2.44) and (2.2) we get by induction that

$$\|J^n(h - h(0))(x) - q(x)\|_\beta \leq L^n [\frac{1}{2^\beta} \frac{1}{1-L} \psi(x, x) + \varphi(0, x)] \tag{2.46}$$

for all $x \in E$ and for all $n \in \mathbb{N}$. So, by letting $n \longrightarrow \infty$, we get

$$\lim_{n \longrightarrow \infty} J^n(h - h(0))(x) = q(x) \tag{2.47}$$

for all $x \in E$, which proves uniqueness of q.
In a similar way, by induction we obtain

$$\|\Lambda^n(f-q-h(0)-g(0))(x)-j(x)\|_\beta \leq L^n[\frac{1}{1-L}\chi(x,x)+\frac{1}{2^\beta}\frac{1}{1-L}\psi(x,x)] \tag{2.48}$$

for all $x \in E$ and for all $n \in \mathbb{N}$, where

$$\Lambda l(x) = \frac{1}{|K|}\sum_{k\in K} l(x+k\cdot x).$$

Consequently, we have

$$\lim_{n\longrightarrow\infty} \Lambda^n(f-q-h(0)-g(0))(x) = j(x) \tag{2.49}$$

for all $x \in E$. This proves the uniqueness of the function j and this completes the proof of theorem.

In the following, we will investigate some special cases of Theorem 2.1, with the new weaker conditions. Q.E.D.

Corollary 2.2. Let E be a vector space over $\mathbb{K}$. Let K be a finite abelian subgroup of the automorphism group of $(E,+)$, Let $\alpha = \frac{\log(|K|)}{\log(2)}$. Fix a nonnegative real number β such that $\frac{\alpha}{\alpha+1} < \beta < 1$ and choose a number p with $0 < p < \beta + (\beta-1)\alpha$ and let F be a complete β-normed space over $\mathbb{K}$. If a function f: $E \longrightarrow F$ satisfies

$$\|\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y)-g(x)-h(y)\|_\beta \leq \theta(\|x\|^p+\|y\|^p) \tag{2.50}$$

and $\|x+k\cdot x\| \leq 2\|x\|$, for all $k \in K$, for all $x, y \in E$ and for some $\theta > 0$, then there exists a unique solution q: $E \longrightarrow F$ of the generalierd quadratic functional equation (1.2) and a unique solution j: $E \longrightarrow F$ of the generalized Jensen functional equation (1.3) such that

$$\frac{1}{|K|}\sum_{k\in K} j(k\cdot x) = 0, \tag{2.51}$$

$$\|f(x)-q(x)-j(x)-g(0)-h(0)\|_\beta \leq \frac{\theta}{2^\beta}\frac{(2|K|)^\beta}{(2|K|)^\beta-2^p|K|}\|x\|^p[\frac{|K|}{|K|^\beta}(4+4.3^p)+2+2.3^p] \tag{2.52}$$

$$\|g(x)-q(x)-j(x)-g(0)\|_\beta \leq \frac{\theta}{2^\beta}\frac{(2|K|)^\beta}{(2|K|)^\beta-2^p|K|}\|x\|^p[\frac{|K|}{|K|^\beta}(4+43^p)+2+2.3^p]+\theta\|x\|^p \tag{2.53}$$

and

$$\|h(x)-q(x)-h(0)\|_\beta \leq \frac{\theta}{2^\beta}\frac{(2|K|)^\beta}{(2|K|)^\beta-2^p|K|}\|x\|^p[\frac{|K|}{|K|^\beta}(2+2.3^p)]+\theta\|x\|^p \tag{2.54}$$

for all $x \in E$.

Corollary 2.3. Let E be a vector space over $\mathbb{K}$. Fix a nonnegative real number β less than 1 and choose a number p with $0 < p < 1$ and let F be a complete β-normed space over $\mathbb{K}$. If a function (f, g, h): $E \longrightarrow F$ satisfies

$$\|f(x+y) - g(x) - h(y)\|_\beta \leq \theta(\|x\|^p + \|y\|^p) \tag{2.55}$$

for all $x, y \in E$ and for some $\theta > 0$, then there exists an unique additive function a: $E \longrightarrow F$ such that

$$\|f(x) - a(x) - g(0) - h(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{2^\beta}{2^\beta - 2^p} \|x\|^p [6 + 6.3^p], \tag{2.56}$$

$$\|g(x) - a(x) - g(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{2^\beta}{2^\beta - 2^p} \|x\|^p [6 + 6.3^p] + \theta\|x\|^p \tag{2.57}$$

and

$$\|h(x) - a(x) - h(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{2^\beta}{2^\beta - 2^p} \|x\|^p [2 + 2.3^p] + \theta\|x\|^p \tag{2.58}$$

for all $x \in E$.

Corollary 2.4. Let E be a vector space over $\mathbb{K}$. Let $K = \{I, \sigma\}$ where σ is an volution of E ($\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in E$). Fix a nonnegative real number β such that $\frac{1}{2} < \beta < 1$ and choose a number p with $0 < p < 2\beta - 1$ and let F be a complete β-normed space over $\mathbb{K}$. If a function (f, g, h): $E \longrightarrow F$ satisfies

$$\|f(x+y) + f(x + \sigma(y)) - g(x) - h(y)\|_\beta \leq \theta(\|x\|^p + \|y\|^p) \tag{2.59}$$

and $\|x + \sigma(x)\| \leq 2\|x\|$, for all $x, y \in E$ and for some $\theta > 0$, then there exists a unique solution q: $E \longrightarrow F$ of the generalierd quadratic functional equation

$$f(x+y) + f(x + \sigma(y)) = 2f(x) + 2f(y), \quad x, y \in E \tag{2.60}$$

and a unique solution j: $E \longrightarrow F$ of the generalized Jensen functional equation

$$f(x+y) + f(x + \sigma(y)) = 2f(x), \quad x, y \in E \tag{2.61}$$

such that

$$j(\sigma(x)) = -j(x), \tag{2.62}$$

$$\|f(x) - q(x) - j(x) - g(0) - h(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{4^\beta}{4^\beta - 2.2^p} \|x\|^p [\frac{2}{2^\beta}(4 + 4.3^p) + 2 + 2.3^p] \tag{2.63}$$

$$\|g(x) - q(x) - j(x) - g(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{4^\beta}{4^\beta - 2^p.2} \|x\|^p [\frac{2}{2^\beta}(4 + 43^p) + 2 + 2.3^p] + \theta\|x\|^p \tag{2.64}$$

and

$$\|h(x) - q(x) - h(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{4^\beta}{4^\beta - 2^p.2} \|x\|^p [\frac{2}{2^\beta}(2 + 2.3^p)] + \theta\|x\|^p \tag{2.65}$$

for all $x \in E$.

Corollary 2.5. Let E be a vector space over $\mathbb{K}$ and let F be a complete β-normed space over $\mathbb{K}$. Let f: $E \longrightarrow F$ be a mapping for which there exists a function $\varphi : E \times F \to [0, \infty)$ and a constant $L < 1$, such that

$$\|f(x+y) + f(x+\sigma(y)) - g(x) - h(y)\|_\beta \le \varphi(x, y) \tag{2.66}$$

and

$$\varphi(2x, 2y) + \varphi(x+\sigma(x), y+\sigma(y)) \le 4^\beta L\varphi(x, y) \tag{2.67}$$

for all $x, y \in E$. Then, there exists a unique solution q: $E \longrightarrow F$ of the generalierd quadratic functional equation (2.62) and a unique solution j: $E \longrightarrow F$ of the generalized Jensen functional equation (2.63) such that

$$j(\sigma(x)) = -j(x), \tag{2.68}$$

$$\|f(x) - q(x) - j(x) - g(0) - h(0)\|_\beta \le \frac{2}{2^\beta}\frac{1}{1-L}\chi(x, x) + \frac{1}{2^\beta}\frac{1}{1-L}\psi(x, x), \tag{2.69}$$

$$\|g(x) - q(x) - j(x) - g(0)\|_\beta \le \varphi(x, 0) + \frac{2}{2^\beta}\frac{1}{1-L}\chi(x, x) + \frac{1}{2^\beta}\frac{1}{1-L}\psi(x, x) \tag{2.70}$$

and

$$\|h(x) - q(x) - h(0)\|_\beta \le \frac{1}{2^\beta}\frac{1}{1-L}\psi(x, x) + \varphi(0, x) \tag{2.71}$$

for all $x \in E$, where

$$\chi(x, y) = \frac{2}{2^\beta}\varphi(0, y) + \varphi(x, y) + \varphi(x, 0) + \varphi(0, y)$$
$$+\frac{1}{2^\beta}[\varphi(x, y) + \varphi(\sigma(x), y) + \varphi(x, 0) + \varphi(\sigma(x), 0)]$$

and

$$\psi(x, y) = \frac{2}{2^\beta}\varphi(0, y) + \frac{1}{2^\beta}[\varphi(x, y) + \varphi(\sigma(x), y) + \varphi(x, 0) + \varphi(\sigma(x), 0)].$$

Corollary 2.6. Let E be a vector space over $\mathbb{K}$ and let F be a complete β-normed space over $\mathbb{K}$. Let f: $E \longrightarrow F$ be a mapping for which there exists a function $\varphi : E \times F \to [0, \infty)$ and a constant $L < 1$, such that

$$\|f(x+y) - g(x) - h(y)\|_\beta \le \varphi(x, y) \tag{2.72}$$

and

$$\varphi(2x, 2y) \le 2^\beta L\varphi(x, y) \tag{2.73}$$

for all $x, y \in E$. Then, there exists an unique additive function a: $E \longrightarrow F$ such that

$$\|f(x) - a(x) - g(0) - h(0)\|_\beta \le \frac{2}{2^\beta}\frac{1}{1-L}\chi(x, x) + \frac{1}{2^\beta}\frac{1}{1-L}\psi(x, x), \tag{2.74}$$

$$\|g(x) - a(x) - g(0)\|_\beta \le \varphi(x, 0) + \frac{2}{2^\beta}\frac{1}{1-L}\chi(x, x) + \frac{1}{2^\beta}\frac{1}{1-L}\psi(x, x) \tag{2.75}$$

and

$$\|h(x) - a(x) - h(0)\|_\beta \le \frac{1}{2^\beta}\frac{1}{1-L}\psi(x, x) + \varphi(0, x) \tag{2.76}$$

for all $x \in E$, where

$$\chi(x, y) = \varphi(0, y) + \varphi(x, y) + \varphi(x, 0) + \varphi(0, y) + [\varphi(x, y) + \varphi(x, 0)]$$

and

$$\psi(x, y) = \varphi(0, y) + [\varphi(x, y) + \varphi(x, 0)].$$

References

[1] M. Ait Sibaha, B. Bouikhalene and E. Elqorachi, Hyers-Ulam-Rassias stability of the K-quadratic functional equation. *J. Inequal. Pure and Appl. Math.* **8** (2007), Article 89.

[2] T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, **2** (1950), 64-66.

[3] M. Akkouchi, Hyers-Ulam-Rassias Stability of nonlinear Volterra integral equations via a fixed point approach, *Acta Universitatis Apulensis*, **26**, (2011), 257-266.

[4] M. Akkouchi, Stability of certain functional equations via a fixed point of ĆIRIĆ, *Filomat* **25:2**, (2011), 121-127.

[5] M.Akkouchi, A. Bounabat and M.H. Lalaoui Rhali, Fixed point approach to the stability of integral equation in the sense of Ulam-Hyers-Rassias, *Annales Mathematicae Silesianae* **25** (2011), 27-44

[6] M. Akkouchi and A. Ed-Darraz, On th stability of a generalized quadratic functional equation, *Acta Universitatis Apulensis*, **34**, (2013), 379-392.

[7] B. Bouikhalene, E. Elqorachi and Th. M. Rassias, On the Hyers-Ulam stability of approximately Pexider mappings. *Math. Inequal. Appl.*, **11** (2008), 805-818.

[8] J. Brzdęk, On a method of proving the Hyers-Ulam stability of functional equations on restricted domains, *Austr. J. of Math. Anal. Appl.*, **6** (1), Article 4 (2009), 1-10.

[9] L. Cădariu and V. Radu, Fixed points and the stability of Jensens functional equation, *Journal of Inequalities in Pure and Applied Mathematics*, vol. **4**, no. 1, (2003), article 4.

[10] L. Cădariu and V. Radu, On the stability of the Cauchy functional equation: a fixed point approach, *Grazer Math. Berichte*, vol. **346** (2004), 43-52.

[11] A. Charifi, B. Bouikhalene and E. Elqorachi, Hyers-Ulam-Rassias stability of a generalized Pexider functional equation, *Banach J. Math. Anal.*, **1** (2007), 176-185.

[12] A. Charifi, B. Bouikhalene, E. Elqorachi and A. Redouani, Hyers-Ulam-Rassias Stability of a generalized Jensen functional equation, *Australian J. Math. Anal. Appli.* **19** (2009), 1-16.

[13] Y. J. C. M. E. Gordji and S. Zolfaghari, Solutions and Stability of Generalized Mixed Type QC Functional Equations in Random Normed Spaces, Volume 2010, Article ID 403101, 16 pages.

[14] H.G. Dales and M.S. Moslehian, Stability of mappings on multi-normed spaces, *Glasgow Math. J.* **49** (2007), 321-332.

[15] J. B. Diaz and B. Margolis, A fixed point theorem of the alternative, for contractions on a generalized complete metric space, *Bulletin of the American Mathematical Society,* vol. **74** (1968), 305-309.

[16] G. L. Forti, Hyers-Ulam stability of functional equations in several variables, *Aequationes Math.*, **50** (1995), 143-190.

[17] G.-L. Forti and J. Sikorska, Variations on the Drygas equation and its stability, *Nonlinear Anal.: Theory, Meth. Appl.*, **74** (2011), 343-350.

[18] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.*, **184** (1994), 431-436.

[19] Z. Gajda, On stability of additive mappings, *Internat. J. Math. Math. Sci.*, **14** (1991), 431-434.

[20] M. E. Gordji, J. M. Rassias, M. B. Savadkouhi, Approximation of the Quadratic and Cubic Functional Equations in RN-spaces, *EUROPEAN J. PURE, APPL. MATH.,* **2** (2009), 494-507.

[21] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U. S. A.*, **27** (1941), 222-224.

[22] D. H. Hyers, G. Isac and Th. M. Rassias, On the asymptoticity aspect of Hyers-Ulam stability of mappings, *Proc. Amer. Math. Soc.*, **126** (1998), 425-430.

[23] D. H. Hyers, G. I. Isac and Th. M. Rassias, Stability of Functional Equations in Several Variables, *Birkhäuser, Basel,* 1998.

[24] D. H. Hyers and Th. M. Rassias, Approximate homomorphisms, *Aequationes Math.*, **44** (1992), 125-153.

[25] K.-W. Jun and Y.-H. Lee, A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation, *J. Math. Anal. Appl.*, **238** (1999), 305-315.

[26] S.-M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, *Springer, New York*, 2011.

[27] S.-M. Jung, Stability of the quadratic equation of Pexider type, *Abh. Math. Sem. Univ. Hamburg,* **70** (2000), 175-190.

[28] S.-M. Jung, M.S. Moslehian and P K. Sahoo, Stability of generalized Jensen equation on restricted domains, *J. Math Inequal.*, **4** (2010),191-206.

[29] S.-M. Jung and B. Kim, Local stability of the additive functional equation and its applications, *IJMMS*, (2003), 15-26.

[30] S.-M. Jung and P. K. Sahoo, Hyers-Ulam stability of the quadratic equation of Pexider type, *J. Korean Math. Soc.*, **38** (3) (2001), 645-656.

[31] S.-M. Jung and Zoon-Hee Lee, A Fixed Point Approach to the Stability of Quadratic Functional Equation with Involution, *Fixed Point Theory and Applications*, V (2008), Article ID 732086, 11 pages.

[32] Pl. Kannappan, Functional Equations and Inequalities with Applications, *Springer, New York*, 2009.

[33] Y. H. Lee and K. W. Jung, A generalization of the Hyers-Ulam-Rassias stability of the pexider equation, *J. Math. Anal. Appl.*, **246** (2000), 627-638.

[34] Y. Manar, E. Elqorachi and B. Bouikhalene, Fixed point and Hyers-Ulam-Rassias stability of the quadratic and Jensen functional equations *Nonlinear Funct. Anal. Appl.*, **15** (2) (2010), 273-284.

[35] M.S. Moslehian and A. Najati, Application of a fixed point theorem to a functional inequality, Fixed Point Theory **10** (2009), 141-149.

[36] M.S. Moslehian, The Jensen functional equation in non-Archimedean normed spaces, *J. Funct. Spaces Appl.*, **7** (2009), 13-24.

[37] M.S. Moslehian and Gh. Sadeghi, Stability of linear mappings in quasi-Banach modules, *Math. Inequal. Appl.*, **11** (2008), 549-557.

[38] A. Najati and M. B. Moghimi, Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces, *J. Math. Anal. Appl.*, **337** (2008), 399-415.

[39] A. Najati, On the stability of a quartic functional equation, *J. Math. Anal. Appl.*, **340** (2008), 569-574.

[40] A. Najati and C. Park, , Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation, *J. Math. Anal. Appl.*,

[41] C. Park, On the stability of the linear mapping in Banach modules, *J. Math. Anal. Appl.*, **275** (2002), 711-720.

[42] C. Park, HyersUlamRassias stability of homomorphisms in quasi-Banach algebras, *Bulletin Sci. Math.*, **132** 2008, 87-96.

[43] M. M. Pourpasha, J. M. Rassias, R. Saadati and S. M. Vaezpour, A fixed point approach to the stability of Pexider quadratic functional equation with involution, *J. Ineq. Appl.* V 2010 (2010) Article ID 839639, doi:10.1155/2010/839639.

[44] J.M. Rassias, On approximation of approximately linear mappings by linear mappings, *J. Funct. Anal.*, **46** (1982), 126-130.

[45] J. M. Rassias, Solution of a problem of Ulam, *J. Approx. Theory.*, **57** (1989), 268-273.

[46] J. M. Rassias, On the Ulam stability of mixed type mappings on restricted domains, *J. Math. Anal. Appl.*, **276** (2) (2002), 747-762.

[47] Th. M. Rassias, On the stability of linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.

[48] Th. M. Rassias, The problem of S. M. Ulam for approximately multiplicative mappings, *J. Math. Anal. Appl.*, **246** (2000), 352-378.

[49] Th. M. Rassias and J. Brzdęk, Functional Equations in Mathematical Analysis, *Springer, New York*, 2011.

[50] Radosław ł. The solution and the stability of the Pexiderized K-quardratic functional equation, 12th Debrecen-Katowice Winter Seminar on Functional Equationsand Inequalities, Hajdúszoboszló, Hungary, January 25-28, 2012.

[51] Radosław ł. Some generalization of Cauchys and the quadratic functional equations, *Aequationes Math.*, **83** (2012), 75-86.

[52] J. Schwaiger, The functional equation of homogeneity and its stability properties, *Österreich. Akad. Wiss. Math.-Natur, Kl, Sitzungsber. Abt.* **II** (1996), 205, 3-12.

[53] F. Skof, Approssimazione di funzioni δ-quadratic su dominio restretto, *Atti. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, **118** (1984), 58-70.

[54] F. Skof, Sull'approssimazione delle applicazioni localmente δ-additive, *Atti. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **117** (1983), 377-389.

[55] H. Stetkær, Functional equations on abelian groups with involution. *Aequationes Math.* **54** (1997), 144-172.

[56] H. Stetkær, Operator-valued spherical functions, *J. Funct. Anal.*, **224**, (2005), 338-351.

[57] H. Stetkær, Functional equations and matrix-valued spherical functions. *Aequationes Math.* **69** (2005), 271-292.

[58] S. M. Ulam, A Collection of Mathematical Problems, *Interscience Publ. New York,* 1961. Problems in Modern Mathematics, *Wiley, New York* 1964.

[59] D. Yang, Remarks on the stability of Drygas equation and the Pexider-quadratic equation, *Aequationes Math.*, **68** (2004), 108-116.