Some properties of certain subclasses of multivalent integral operators

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Abstract

For analytic function of the form $f_i(z) = z^p + \sum_{n=2} a_n^i z^n$, in the open unit disk, a class $\Gamma_\alpha^p(C_1, C_2; \gamma)$ is introduced and some properties for $\Gamma_\alpha^p(C_1, C_2; \gamma)$ of $f_i(z)$ in relation to coefficient bounds, convex conbination and convolution were obtained.

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1 Introduction

Let A denotes the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $U = \{z \in C : |z| < 1\}$ normalized with $f(0) = f'(0) - 1 = 0$ in the open disk,$U = \{z \in C : |z| < 1\}$. In [8], Seenivasagan gave a condition for the univalence of the integral operator

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{t=1}^{k} (\tfrac{f_i(z)}{t})^{\frac{1}{\alpha}} \, dt \right\}^{\frac{1}{\beta}}$$

where $f_i(z)$ is defined by

$$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n \tag{1.1}$$

while Makinde and Opoola in [5] obtained a condition for the starlikeness of the function

$$F_\alpha(z) = \int_0^z \prod_{t=1}^{k} (\frac{f_i(z)}{t})^{\frac{1}{\alpha}} dt, \quad \alpha \in C \tag{1.2}$$

where $f_i(z)$ is denoted by (1.1).

Also, Xiao-Feili et al introduced the class $L_1^*(\beta_1, \zeta_2, \lambda)$ which is a subclass of A such that

$$L_1^*(\beta_1, \beta_2, \lambda) = \left\{ f \in A : |\frac{f'(z) - 1}{\beta_1 f'(z) + \beta_2}| \leq \lambda \right\}, 0 \leq \beta_1 \leq 1, 0 < \beta_2 \leq 1; 0 < \lambda \leq 1$$

for some $\beta_1 \beta_2$ and some real λ. He further denoted T as the subclass of A consisting of the function of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0$$

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and $L^*(\beta_1, \beta_2, \lambda)$ denotes the subclass of $L_1^*(\beta_1, \beta_2, \lambda)$ which is given by

$$L^*(\beta_1, \beta_2, \lambda) = L_1^*(\beta_1, \beta_2, \lambda) \bigcap T$$

for some real number$0 \leq \beta_1 \leq 1, 0 < \beta_2 \leq 1; 0 < \lambda \leq 1$. The class $L^*(\beta_1, \beta_2, \lambda)$ was studied by Kim and Lee in [4], see also [1, 2, 9]. Moreover, Makinde and Oladipo in [7] introduced and studied the class $L_\alpha(\zeta_1, \zeta_2, \lambda)$.

Now, we defined $f_i(z)$ by

$$f_i(z) = z^p + \sum_{n=2}^{\infty} a_n^i z^n \tag{1.3}$$

Let $F_\alpha(z)$ be defined by (1.2), then

$$\frac{zF''_\alpha(z)}{F'_\alpha(z)} = \sum_{i=1}^{k} \frac{1}{\alpha}\left(\frac{zf'_i(z)}{f_i(z)} - 1\right)$$

Let $G(z)$ be denoted by

$$G(z) = \sum_{i=1}^{k} \frac{1}{\alpha}\left(\frac{zf'_i(z)}{f_i(z)} - 1\right)$$

We define

$$\Gamma_\alpha^p(C_1, C_2, \gamma) = \left\{ f_i \in A \left| \frac{G(z) + \frac{1}{\alpha} - 1}{\zeta_1(G(z) + \frac{1}{\alpha}) + \zeta_2} \right| \leq \gamma \right\} \tag{1.4}$$

for some complex $\zeta_1,\ \zeta_2,\ \alpha$ and for some real $\gamma,\ 0 \leq |\zeta_1| \leq 1,\ 0 < |\zeta_2| \leq 1,\ |\alpha| \leq 1$ and $0 < \gamma \leq 1$.

2 Main results

Theorem 2.1. Let $f_i(z)$ be as in (1.3) and $F_\alpha(z)$ be as in (1.2). Then, $f_i(z)$ is in the class $\Gamma_\alpha^p(C_1, C_2; \gamma)$ if and only if:

$$\sum_{i=1}^{n} \sum_{k=p+1}^{\infty} \{k[(1 + \gamma C_1) + \alpha(\gamma C_2 - 1)]\} |a_k^i| \leq \gamma |C_1 p + \alpha C_2| - |p - \alpha|, \tag{2.1}$$

$0 \leq C_1 \leq 1, 0 < C_1 \leq 1, 0 < \alpha \leq p.$

Proof. Using equations (1.1), (1.2) we have:

$$\begin{aligned} \left| \frac{G(z) + \frac{1}{\alpha} - 1}{C_1(G(z) + \frac{1}{\alpha}) + C_2} \right| &= \left| \frac{\sum_{i=1}^{n}(p-\alpha) + \sum_{k=p+1}^{\infty}(k-\alpha)a_k^i z^{k-p}}{\sum_{i=1}^{n}(C_1 p + \alpha C_2 + \sum_{k=p+1}^{\infty}(kC_1 + \alpha C_2)a_k^i z^{k-p})} \right| \\ &\leq \frac{|p-\alpha| + \sum_{i=1}^{n}\sum_{k=p+1}^{\infty}(k-\alpha)|a_k^i|}{|C_1 p + \alpha C_2| - \sum_{i=1}^{n}\sum_{k=p+1}^{\infty}(kC_1 + \alpha C_2)|a_k^i|}. \end{aligned}$$

Let $f_i(z)$ satisfies the inequality (2.1) then $f_i(z) \in \Gamma_\alpha^p(C_1, C_2)$.
Conversely, let $f_i(z) \in \Gamma_\alpha^p(C_1, C_2)$ then

$$\sum_{i=1}^{n} \sum_{k=p+1}^{\infty} \{k[(1 + \gamma C_1) + \alpha(\gamma C_2 - 1)]\} |a_k^i| \leq |C_1 p + \alpha C_2| - |p - \alpha|.$$

Q.E.D.

Corollary 2.2. If $f_i(z) \in \Gamma_\alpha^p(C_1, C_2; \gamma)$, we have:

$$\sum_{i=1}^{n} \sum_{k=p+1}^{\infty} |a_k^i| \leq \frac{|C_1 p + \alpha C_2| - |p-\alpha|}{k[(1+\gamma C_1) + \alpha(\gamma C_2 - 1)]}.$$

Theorem 2.3. Let $f_i(z) \in \Gamma_\alpha^p(C_1, C_2)$ and $g_i(z)$ defined by $g_i(z) = z^p + \sum_{k=p+1}^{\infty} b_k^i z^k$ be in the same $\Gamma_\alpha^p(C_1, C_2)$. Then the function $h_i(z)$ defined by

$$h_i(z) = (1 - \lambda f_i(z) + \lambda g_i(z) = z^p + \sum_{k=p+1}^{\infty} C_k^i z^k$$

is also in the class $\Gamma_\alpha^p(C_1, C_2)$ where

$$C_k^i = (1-\lambda) a_k^i + \lambda b_k^i, \quad 0 \leq \lambda \leq 1.$$

Proof. Suppose each of $f_i(z)$ and $g_i(z)$ in the $\Gamma_\alpha^p(C_1, C_2; \lambda)$. Then we have

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{k=p+1}^{\infty}\{k[(1+\gamma C_1)+\alpha(\gamma C_2-1)]\}|C_k^i| &= \sum_{i=1}^{n}\sum_{k=p+1}^{\infty}\{k[(1+\gamma C_1)+\alpha(\gamma C_2-1)]\}|(1-\lambda)a_k^i+\lambda b_k^i| \\
&= (1-\lambda)\sum_{i=1}^{n}\sum_{k=p+1}^{\infty}\{k[(1+\gamma C_1)+\alpha(\gamma C_2-1)]\}|a_k^i| \\
&+ \lambda\sum_{i=1}^{n}\sum_{k=p+1}^{\infty}\{k[(1+\gamma C_1)+\alpha(\gamma C_2-1)]\}|b_k^i| \\
&\leq (1-\lambda)(|C_1p+\alpha C_2|-|p-\alpha|) \\
&+ \lambda(|C_1p+\alpha C_2|-|p-\alpha|) \\
&= |C_1p+\alpha C_2|-|p-\alpha|
\end{aligned}$$

which shows that the convex combination of $f_i(z)$ and $g_i(z)$ is in the class $\Gamma_\alpha^p(C_1, C_2; \gamma)$. Q.E.D.

Theorem 2.4. Let $f_i(z)$ be as in (3) and $F_\alpha(z)$ be as in (1.2) then the function $C_i(z)$ defined by

$$C_i(z) = z^p + \sum_{k=p+1}^{\infty} a_k^i b_k^i z^k$$

is in the class $\Gamma_\alpha^p(C_1, C_2; \gamma)$ if and only if

$$\sum_{i=1}^{n}\sum_{k=p+1}^{\infty}\{k[(1+\gamma C_1)+\alpha(\gamma C_2-1)]\}|a_k^i b_k^i| \leq \gamma|C_1p+\alpha C_2|-|p-\alpha|,$$

$0 \leq C_1 \leq 1, 0 < C_1 \leq 1, 0 < \alpha \leq p.$

Proof. Following the procedure of the proof of the Theorem 1, we obtain the result. Q.E.D.

Theorem 2.5. Let the function $\psi_i(z)$ be in the class $\Gamma_\alpha^p(C_1, C_2; \gamma)$ and the function $\zeta_i(z)$ defined by

$$\zeta_i(z) = z^p + \sum_{k=p+1}^{\infty} A_k^i B_k^i z^k$$

be in the same $\Gamma_\alpha^p(C_1, C_2; \gamma)$. Then the function $H(z)$ defined by

$$H(z) = (1-\lambda)\zeta_i(z) + \lambda\psi_i(z) = z^p + \sum_{k=p+1}^{\infty} C_k^i$$

is also in the class $\Gamma_\alpha^p(C_1, C_2; \gamma)$, where

$$C_k^i = (1-\lambda)a_k^i b_k^i + \lambda A_k^i B_k^i, \quad 0 \le \lambda \le 1.$$

Proof. The proof of the theorem is similar to that of Theorem 2, thus, we omit the proof. Q.E.D.

Corollary 2.6. Let $f_i(z)$ be as in (1.3) and F_α be as in (1.4). Then $f_i(z)$ is in the class $\Gamma_\alpha^p(C_1, C_2; \gamma)$ if and only if

$$\sum_{i=1}^{n} \sum_{k=p+1}^{\infty} \{k[(1+\gamma C_1) + \alpha(\gamma C_2 - 1)]\}|a_k^i| \le \gamma|C_1 p + \alpha C_2| - |p-\alpha|,$$

$0 \le C_1 \le 1, 0 < C_1 \le 1, 0 < \alpha \le p.$

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