Diaz-Metcalf and Pólya-Szegő type inequalities associated with Saigo fractional integral operator

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Abstract

This paper deals with the derivation of certain new Pólya–Szegő type inequalities by making use of the Saigo fractional integral operator. The results obtained cover the same kind of conclusions in the case of Riemann–Liouville and Erdélyi-Kober fractional integral operators.

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1 Introduction and results required

The authors of this research note realized recently a study [12] in which the cornerstone was the Saigo-type fractional integral [11] applied to some suitably bounded and/or integrable functions. Following the study on Saigo type fractional integral operator, firstly we shall derive certain new integral inequalities related to Diaz–Metcalf and Pólya–Szegő inequalities in Saigo type fractional integral setting together with related special cases which turn out to be the widely known classical inequalities by Rennie and Schweitzer.

The main building–block in both stories is the integral mean of a suitable input function h on a finite interval $[a, b]$ defined by

$$\mathscr{M}(h) = \frac{1}{b-a}\int_a^b h(x)\mathrm{d}x\,.$$

Its further specialized shapes adopted to the situations occur in the sequel.

The symbol $\chi_S(t)$ stands for the characteristic function of the set S, $\delta_{\lambda\mu}$ is the Kronecker symbol, while under $L^p_\varphi[A], p \in \mathbb{R}$ we mean the function space $\{h \mid \int_A |h(t)|^p\varphi(t)dt < \infty\}$.

1.1 Diaz - Metcalf weighted integral inequality.

The article [3] where the inequality initially appeared and e.g. the celebrated monograph [7] contain this classical result, also see [8] for the probabilistic point of view and equality analysis.

Let f, g be Borel - functions, such that

$$m \le f(x) \le M, \qquad 0 < n \le g(y) \le N, \qquad \text{a.e. } (x, y) \in [a, b] \times [c, d]. \tag{1.1}$$

Let $w(x, y)$ be a nonnegative normalized weight function for which $\mathrm{supp}(w) = [a, b] \times [c, d]$, i.e. $\int_a^b\int_c^d w(x, y)\mathrm{d}x\mathrm{d}y = 1$ and

$$\int_c^d w(x, y)\mathrm{d}y = w_1(x), \qquad \int_a^b w(x, y)\mathrm{d}x = w_2(y).$$

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Introducing the weighted integral mean with the weight p of the function h with a support being either a 1-, or on 2-dimensional rectangle, as the functional

$$\mathscr{M}_p(h) := \begin{cases} \displaystyle\int_a^b h(x)w_j(x)\,\mathrm{d}x, & p \equiv w_j,\ j=1,2\,, \\ \displaystyle\int_a^b \int_c^d h(x,y)w(x,y)\,\mathrm{d}x\mathrm{d}y, & p \equiv w\,, \end{cases} \tag{1.2}$$

the *weighted integral Diaz–Metcalf inequality* reads

$$\mathscr{M}_{w_1}(f^2) + \frac{mM}{nN}\mathscr{M}_{w_2}(g^2) \le \left(\frac{m}{N} + \frac{M}{n}\right)\mathscr{M}_w(fg). \tag{1.3}$$

Here the equality appears **iff** either **(i)** $m/N = M/n$ or **(ii)** $m/N < M/n$ and

$$\int_{\mathbb{I}_{x,y}} w(x,y)\mathrm{d}x\mathrm{d}y = 1,$$

where $\mathbb{I}_{x,y} := \{(x,y):\ f(x)/g(y) \in \{m/N, M/n\}\}$.

However, the specification $w(x,y) = (b-a)^{-1}\chi_{[a,b]^2}(x,y)\delta_{xy}$ infers the classical Diaz - Metcalf equal - weight integral inequality:

$$\mathscr{M}(f^2) + \frac{mM}{nN}\mathscr{M}(g^2) \le \left(\frac{m}{N} + \frac{M}{n}\right)\mathscr{M}(fg). \tag{1.4}$$

as $w_1(x) = (b-a)^{-1}\chi_{[a,b]}(x) = w_2(x)$. The equality in (1.4) holds **iff** either **(i)** $m/N = M/n$ or **(ii)** $m/N < M/n$ and

$$\int_{\mathbb{I}_x} w_1(x)dx = 1,$$

where $\mathbb{I}_x \equiv \mathbb{I}_{x,x},\ x \in [a,b]$, see also [3] and [7, p. 64].

1.2 Pólya - Szegő weighted integral inequality [7, 8].

Let f, g be positive Borel - functions which satisfy (1.1), that is

$$0 < m \le f(x) \le M, \qquad 0 < n \le g(y) \le N, \qquad \text{a.s. } (x,y) \in [a,b]\times[c,d],$$

and let $w(x,y)$ be a nonnegative weight function with $\mathrm{supp}(w) = [a,b]\times[c,d]$ for which

$$\int_c^d w(x,y)\mathrm{d}y = w_1(x), \qquad \int_a^b w(x,y)\mathrm{d}x = w_2(y).$$

Then it holds

$$\frac{\mathscr{M}_{w_1}(f^2)\mathscr{M}_{w_2}(g^2)}{\mathscr{M}_w(fg)} \le \frac14\left(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}}\right)^2. \tag{1.5}$$

The equality in (1.5) holds **iff** the following conditions are satisfied: $[a,b] \equiv [c,d]$ and

$$f(x) = m\chi_S(x) + M\chi_{[a,b]\setminus S}(x), \quad g(y) = N\chi_{[a,b]\setminus S}(y) + n\chi_S(y),$$

for a Borel set $S \subseteq [a,b]$ of Lebesgue measure

$$|S| = \frac{Mn(b-a)}{mN + Mn}, \tag{1.6}$$

moreover

$$w(x,y) = 0, \qquad (x,y) \in S^2 \cup ([a,b] \setminus S)^2. \tag{1.7}$$

When $[a,b]=[c,d]$ and $w(x,y) = (b-a)^{-1}\chi_{[a,b]^2}(x,y)\delta_{xy}$, we get

$$\frac{\mathscr{M}(f^2)\mathscr{M}(g^2)}{\big[\mathscr{M}(fg)\big]^2} \le \frac{1}{4}\left(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}}\right)^2 . \tag{1.8}$$

When f, g satisfy (1.6) and (1.7), then (1.8) becomes equality and *vice versa.* The inequality (1.8) originates back to Pólya and Szegő [9, p.81 & pp.251-252, Problem **93**] where, however, no equality analysis was performed.

We point out that Anber and Dahmani [1] obtained certain Pólya-Szegő–type inequalities by making use of Riemann-Liouville fractional integral operator.

Remark 1.1. According to the observation by Diaz and Metcalf [3, p. 417] it is clear that because (1.4) and since

$$\left(\mathscr{M}^{1/2}(f^2) - \sqrt{\frac{mN}{nM}}\,\mathscr{M}^{1/2}(f^2)\right)^2 \ge 0 \tag{1.9}$$

we immediately arrive at the Pólya–Szegő inequality (1.8). Modest adoptions of the same arguments lead from the weighted variant (1.3) and the previous relation (1.9) to the weighted Pólya–Szegő inequality (1.5) as well.

1.3 Fractional integration.

Following Saigo [11] we recall the definition of a fractional integration operator in the form considered by Saxena *et al.* [12].

Definition 1.2. Let $\Re\{\alpha\}, \eta > 0, \beta \in \mathbb{R}$. The Saigo fractional integral of the function f on $\mathbb{R}_+$ is defined as

$$I_{0,t}^{\alpha,\beta,\eta}[f] = \begin{cases} \dfrac{t^{-\beta}}{\Gamma(\alpha)} \displaystyle\int_0^1 (1-x)^{\alpha-1}\, {}_2F_1\Big[\begin{array}{c} \alpha+\beta,\, -\eta \\ \alpha \end{array} ; 1-x \Big]\, f(tx)\, \mathrm{d}x \\[2ex] \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} I_{0,t}^{\alpha+n,\beta-n,\eta-n}[f] \quad 0 < \Re\{\alpha\} + n \le 1,\, n \in \mathbb{N}, \end{cases} \tag{1.10}$$

where ${}_2F_1$ stands for the Gaussian hypergeometric function.

The Riemann–Liouville and Erdélyi–Kober fractional integral operators follow as special cases of (1.10), *viz.*

$$\begin{aligned} I_{0,t}^{\alpha}[f] &:= I_{0,t}^{\alpha,-\alpha,\eta}[f] = \frac{t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} f(tx)\, \mathrm{d}x \qquad \Re\{\alpha\} > 0, \\ I_{0,t}^{\alpha,\eta}[f] &:= I_{0,t}^{\alpha,0,\eta}[f] = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} x^\eta f(tx)\, \mathrm{d}x \qquad \Re\{\alpha\}, \eta > 0\,. \end{aligned}$$

For further details of Saigo fractional integral operator related with inequalities one can refer to the papers by Saxena *et al.* [12] and Jankov and Pogány [6].

It is worth mentioning the following result is given by Saxena *et al.* [12, p. 672, Theorem 2.1]. Let $\eta - \beta > 0, \mu > 0, \kappa + \alpha > 0$. Then there holds the formula

$$I^{\alpha,\beta,\eta}_{0,t}\big[x^{\mu-1}(t-x)^\kappa\big] = \frac{t^{\kappa+\mu-\beta-1}\mathrm{B}(\mu,\kappa+\alpha)}{\Gamma(\alpha)} \cdot {}_3F_2\Big[\begin{array}{c} \alpha+\beta,\, \kappa+\alpha,\, -\eta \\ \alpha,\; \kappa+\alpha+\mu \end{array};1\Big]\,,$$

where $\mathrm{B}(\cdot,\cdot)$ denotes the familiar Beta–function. When $\mu = 1, \kappa = 0$, employing [4, p. 806, Eq. 7.512^4] for $\Re(\alpha), \eta > 0, \beta < 1$, we get

$$I^{\alpha,\beta,\eta}_{0,t}[1] = \frac{t^{-\beta}}{\Gamma(\alpha+1)}\, {}_2F_1\Big[\begin{array}{c} \alpha+\beta,\, -\eta \\ \alpha+1 \end{array};1\Big] = \frac{\Gamma(\eta+1-\beta)\, t^{-\beta}}{\Gamma(1-\beta)\Gamma(\alpha+1+\eta)}$$

also see [12, p. 673, Remark 2.2].

1.4 On the Saigo–type weighted integral mean.

The Čebyšev functional for the Saigo type kernel contaning the Gaussian hypergeometric term ${}_2F_1$ has been introduced by Saxena *et al.* [12]. Comparing the weight function $w(x,y)$ (introduced in the section 1.1) to be a multiplicative separable function $w(x,y) = w_{\mathrm{S}}(x)w_{\mathrm{S}}(y), (x,y) \in [0,1]^2$, where

$$w_{\mathrm{S}}(x) := \frac{t^{-\beta}}{\Gamma(\alpha)\, I^{\alpha,\beta,\eta}_{0,t}[1]}\,(1-x)^{\alpha-1}\, {}_2F_1\Big[\begin{array}{c} \alpha+\beta,\, -\eta \\ \alpha+1 \end{array};1\Big]\,, \tag{1.11}$$

the marginal weight becomes t–free:

$$w_{\mathrm{S}}(x) = \frac{\alpha\,(1-x)^{\alpha-1}}{{}_2F_1\Big[\begin{array}{c} \alpha+\beta,\, -\eta \\ \alpha+1 \end{array};1\Big]}\, {}_2F_1\Big[\begin{array}{c} \alpha+\beta,\, -\eta \\ \alpha \end{array};1-x\Big].$$

The *Saigo-type weighted integral mean* (compare with (1.2)) of a suitably choosen input function f has been introduced by Saxena *et al.* [12, p. 673, Definition 2.3] in the form

$$\mathscr{M}^{\eta}_{\mathrm{S}}(f) = \mathscr{M}^{\alpha,\beta,\eta}_{0,t}(f) = \frac{I^{\alpha,\beta,\eta}_{0,t}[f]}{I^{\alpha,\beta,\eta}_{0,t}[1]} = \int_0^1 w_{\mathrm{S}}(x)\, f(tx)\,\mathrm{d}x, \qquad t > 0,$$

for all $\alpha, \eta > 0, \beta < 1$.

Further obvious reductions imply for $\beta = -\alpha$ the Riemann–Liouville fractional integral operator variant of the Diaz–Metcalf inequality, similarly $\beta = 0$ gives the Erdélyi–Kober case. In these cases, the associated fractional integral means become

$$\mathscr{M}_{\mathrm{RL}}(f) = \mathscr{M}^{\alpha,-\alpha,\eta}_{0,t}(f), \qquad \mathscr{M}_{\mathrm{EK}}(f) = \mathscr{M}^{\alpha,0,\eta}_{0,t}(f),$$

in which the defining weights are:

$$w_{\mathrm{RL}}(x) = \alpha(1-x)^{\alpha-1}, \qquad w_{\mathrm{EK}}(x) = \frac{\Gamma(1+\alpha+\eta)}{\Gamma(\alpha)\Gamma(1+\eta)}\, x^\eta(1-x)^{\alpha-1}.$$

In the RL and EK cases $\alpha, \eta > 0$.

2 Diaz–Metcalf inequality for Saigo type operator

Now we present the Diaz–Metcalf type inequality for Saigo fractional integral.

Theorem 2.1. Let $f, g \in L^1_{w_{\rm S}}[0,t]$ and

$$0 < m \le f(x) \le M < \infty, \qquad 0 < n \le g(x) \le N < \infty; \qquad x \in [0,t].$$

Then for all $\Re\{\alpha\} > 0, \beta < 1, \eta > \beta - 1$, and for all $t > 0$ we have

$$\mathscr{M}_{\rm S}(f^2) + \frac{mM}{nN}\mathscr{M}_{\rm S}(g^2) \le \left(\frac{m}{N} + \frac{M}{n}\right)\mathscr{M}_{\rm S}(fg) \qquad \Re(t) > 0. \tag{2.1}$$

The equality in (2.1) holds **iff** $m/N = M/n$.

Proof. By the assumptions

$$\frac{m}{N} \le \frac{f(tx)}{g(tx)} \le \frac{M}{n} \qquad t > 0,\, x \in [0,1]\,,$$

which implies

$$\left(\frac{M}{n}g(tx) - f(tx)\right)\left(f(tx) - \frac{m}{N}g(tx)\right) \ge 0.$$

In turn, the expanded inequality

$$\left(\frac{M}{n} + \frac{m}{N}\right) f(tx)g(tx) \ge f^2(tx) + \frac{mM}{nN}\, g^2(tx)$$

multiplied by the Saigo's hypergeometric kernel

$$K_\eta(x) = \frac{t^{-\beta}}{\Gamma(\alpha)}(1-x)^{\alpha-1}\, {}_2F_1\left[\begin{matrix}\alpha+\beta,\, -\eta \\ \alpha\end{matrix}\; ; 1-x\right],$$

and integrated with respect to $x \in [0,1]$ becomes

$$I^{\alpha,\beta,\eta}_{0,t}[f^2] + \frac{mM}{nN}\, I^{\alpha,\beta,\eta}_{0,t}[g^2] \le \left(\frac{M}{n} + \frac{m}{N}\right) I^{\alpha,\beta,\eta}_{0,t}[fg]\,. \tag{2.2}$$

Dividing it by the normalizing factor $I^{\alpha,\beta,\eta}_{0,t}[1]$ we get the asserted inequality (2.1).

It is not difficult to see that (2.2) is equivalent to

$$I^{\alpha,\beta,\eta}_{0,t}\left[\left(f - \sqrt{\frac{mM}{nN}}\, g\right)^2 + 2\left\{\sqrt{\frac{mM}{nN}} - \frac{1}{2}\left(\frac{m}{N} + \frac{n}{M}\right)\right\} fg\right] \le 0\,.$$

Now, having in mind the Arithmetic mean–Geometric mean inequality, the equality analysis follows; see the end of the subsection 1.2. and also [3] and [7, p. 64]. Q.E.D.

Now, to obtain related Diaz–Metcalf like inequalities it is enough in (2.1) to specify $\beta = -\alpha$ and $\beta = 0$ respectively. However, we skip to expose these special cases.

Now, we give a modest extension of the Diaz–Metcalf inequality (2.1) considering *two different* weighted Saigo integral means which contain arbitrary independent scale parameters η and ξ. In the sequel we will denote

$$\mathscr{M}^\eta_{\rm S}(f) := \mathscr{M}^{\alpha,\beta,\eta}_{0,t}(f).$$

Theorem 2.2. Let f, g be integrable functions on $\mathbb{R}_+$ which satisfy (1.1). Then for all $\Re\{\alpha\} > 0, \beta < 1$; $\eta, \xi \in \mathbb{R}$ satisfying $\min\{\eta, \xi\} > \beta - 1$ we have

$$\mathscr{M}_{\rm S}^{\eta}(f^2) + \frac{mM}{nN}\mathscr{M}_{\rm S}^{\xi}(g^2) \le \left(\frac{m}{N} + \frac{M}{n}\right)\mathscr{M}_{\rm S}^{\eta}(f)\mathscr{M}_{\rm S}^{\xi}(g) \qquad \Re(t) > 0. \tag{2.3}$$

Proof. By the assumption (1.1) we obtain

$$\left(\frac{M}{n}g(ty) - f(tx)\right)\left(f(tx) - \frac{m}{N}g(ty)\right) \ge 0, \qquad t > 0;\ x, y \in [0,1].$$

The expanded inequality

$$\left(\frac{M}{n} + \frac{m}{N}\right) f(tx)g(ty) \ge f^2(tx) + \frac{mM}{nN}\, g^2(ty)$$

when multiplied by the product of two different–parameter Saigo's hypergeometric kernels

$$\frac{t^{-2\beta}}{\big[\Gamma(\alpha)\big]^2}\,\big[(1-x)(1-y)\big]^{\alpha-1}\,{}_2F_1\Big[\begin{array}{c}\alpha+\beta,\,-\eta\\ \alpha\end{array};1-x\Big]\,{}_2F_1\Big[\begin{array}{c}\alpha+\beta,\,-\xi\\ \alpha\end{array};1-y\Big],$$

and integrated with respect to x and y on the square $[0,1]^2$, becomes

$$I_{0,t}^{\alpha,\beta,\eta}[f^2]\,I_{0,t}^{\alpha,\beta,\xi}[1] + \frac{mM}{nN}\,I_{0,t}^{\alpha,\beta,\eta}[1]\,I_{0,t}^{\alpha,\beta,\xi}[g^2] \le \left(\frac{M}{n} + \frac{m}{N}\right) I_{0,t}^{\alpha,\beta,\eta}[f]\,I_{0,t}^{\alpha,\beta,\xi}[g]\,.$$

It is enough to renormalize the last display by $I_{0,t}^{\alpha,\beta,\eta}[1] \cdot I_{0,t}^{\alpha,\beta,\xi}[1]$ to obtain (2.3). Q.E.D.

For the allied RL and EK special cases of the Saigo–type Diaz–Metcalf inequality (2.3) we deduce the following forms:

$$\mathscr{M}_{\rm RL}^{\eta}(f^2) + \frac{mM}{nN}\mathscr{M}_{\rm RL}^{\xi}(g^2) \le \left(\frac{m}{N} + \frac{M}{n}\right)\mathscr{M}_{\rm RL}^{\eta}(f)\,\mathscr{M}_{\rm RL}^{\xi}(g),$$
$$\mathscr{M}_{\rm EK}^{\eta}(f^2) + \frac{mM}{nN}\mathscr{M}_{\rm EK}^{\xi}(g^2) \le \left(\frac{m}{N} + \frac{M}{n}\right)\mathscr{M}_{\rm EK}^{\eta}(f)\,\mathscr{M}_{\rm EK}^{\xi}(g),$$

for all $\Re(\alpha), \eta > 0$ and $t > 0$.

Comparing the inequalities' right–hand–side expressions in Theorems 1 and 2 we see that the terms $\mathscr{M}_{\rm S}(fg) \equiv \mathscr{M}_{\rm S}^{\eta}(fg)$ and $\mathscr{M}_{\rm S}^{\eta}(f)\mathscr{M}_{\rm S}^{\xi}(g)$ differ even in the case $\eta = \xi$. To discuss this question we recall the Čebyšev integral inequality [7, p. 40]: *If $f, g : [a,b] \to \mathbb{R}$ are integrable functions, both increasing or both decreasing, and $p : [a,b] \to \mathbb{R}_+$ is an integrable function, then*

$$\int_a^b p(x)f(x){\rm d}x \int_a^b p(x)g(x){\rm d}x \le \int_a^b p(x){\rm d}x \int_a^b p(x)f(x)g(x){\rm d}x\,.$$

Assume that $\eta = \xi$ and both functions f, g increase (decrease) simultaneously on $[0,t]$, $t > 0$, then choosing $p(x) = w_{\rm S}(x)$ (compare (1.11)), by the above listed Čebyšev integral inequality, we arrive at the classical Diaz–Metcalf inequality:

$$\mathscr{M}_{\rm S}(f^2) + \frac{mM}{nN}\mathscr{M}_{\rm S}(g^2) \le \left(\frac{m}{N} + \frac{M}{n}\right)\mathscr{M}_{\rm S}(fg)\,,$$

where, we transform the upper bound in (2.3). The same conclusions follow for the related Riemann–Liouville and Erdélyi–Kober integral means.

3 On Pólya-Szegő inequality for Saigo type operator

The present topic is strongly connected in view of the Remark 1 to Diaz–Metcalf inequality chapter's matter.

Theorem 3.1. Let f, g be bounded functions on $\mathbb{R}_+$ which satisfy (1.1). Then for all $\Re(\alpha) > 0, \beta < 1, \eta > \beta - 1$, and for all $t > 0$ we have

$$\frac{\mathscr{M}_{\mathrm{S}}(f^2)\,\mathscr{M}_{\mathrm{S}}(g^2)}{\left[\mathscr{M}_{\mathrm{S}}(fg)\right]^2} \le \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2 . \tag{3.1}$$

Proof. Following the standard proof of the Pólya-Szegő inequality, applying the Arithmetic mean–Geometric mean inequality to the left–hand–side expression of (2.1) yields

$$\left(\frac{m}{N} + \frac{M}{n}\right)\mathscr{M}_{\mathrm{S}}(fg) \ge \mathscr{M}_{\mathrm{S}}(f^2) + \frac{mM}{nN}\mathscr{M}_{\mathrm{S}}(g^2) \ge 2\sqrt{\frac{mM}{nN}\,\mathscr{M}_{\mathrm{S}}(f^2)\,\mathscr{M}_{\mathrm{S}}(g^2)}\,;$$

obvious transformations of two utmost terms lead to the asserted inequality (3.1). Q.E.D.

For the equality analysis in this and similar cases we refer to [8, p. 117].

The related Pólya-Szegő type inequalities are valid as well for the Riemann–Liouville and Erdélyi–Kober integral operators. Actually, we get these special cases of Theorem 3 choosing $\beta = -\alpha$, and $\beta = 0$ respectively.

The corresponding counterpart of the Pólya-Szegő inequalities (1.5) and (1.8) considering two different weighted Saigo integral means which contain arbitrary independent scale parameters η and ξ, reads as follows.

Theorem 3.2. Let f, g be positive integrable functions on $\mathbb{R}_+$. When f, g satisfy (1.1) then for all $\Re(\alpha) > 0, \beta < 1, \min\{\eta, \xi\} > \beta - 1$ we have

$$\frac{\mathscr{M}^{\eta}_{\mathrm{S}}(f^2)\,\mathscr{M}^{\xi}_{\mathrm{S}}(g^2)}{\left[\mathscr{M}^{\eta}_{\mathrm{S}}(f)\,\mathscr{M}^{\xi}_{\mathrm{S}}(g)\right]^2} \le \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2 . \tag{3.2}$$

Proof. By virtue of the Arithmetic mean–Geometric mean inequality we transform the extended two parameter variant Diaz–Metcalf inequality (2.3) into

$$\left(\frac{m}{N} + \frac{M}{n}\right)\mathscr{M}^{\eta}_{\mathrm{S}}(f)\,\mathscr{M}^{\xi}_{\mathrm{S}}(g) \ge 2\sqrt{\frac{mM}{nN}\,\mathscr{M}^{\eta}_{\mathrm{S}}(f^2)\,\mathscr{M}^{\xi}_{\mathrm{S}}(g^2)}\,.$$

Now, obvious transformations result in the statement (3.2). Q.E.D.

We can also state the related Riemann–Liouville and Erdélyi–Kober variants of the last theorem, by taking $\beta = -\alpha$, and $\beta = 0$ respectively in Theorem 4, leaving these routine moves to the interested reader.

Moreover using $f \equiv g$ on the whole range of the functions, we deduce the following interesting result.

Corollary 3.3. Let f be integrable functions on $\mathbb{R}_+$ which satisfies (1.1). Then for all $\Re(\alpha) > 0, \beta < 1, \min\{\eta, \xi\} > \beta - 1$ we have

$$\frac{\mathscr{M}_{\rm S}^{\eta}(f^2)\,\mathscr{M}_{\rm S}^{\xi}(f^2)}{\big[\mathscr{M}_{\rm S}^{\eta}(f)\,\mathscr{M}_{\rm S}^{\xi}(f)\big]^2} \le \frac14\left(\frac{M}{m}+\frac{m}{M}\right)^2.$$

Finally, by $n = N$ (or by the equivalent specification $g \equiv 1$), we get

Corollary 3.4. Let f be integrable functions on $\mathbb{R}_+$ which satisfies (1.1). Then for all $\Re(\alpha) > 0, \beta < 1, \eta > \beta - 1$ we have

$$\mathscr{M}_{\rm S}^{\eta}(f^2) \le \frac14\left(\frac{M}{m}+\frac{m}{M}\right)^2 \big[\mathscr{M}_{\rm S}^{\eta}(f)\big]^2.$$

4 On Rennie and Schweitzer-type inequalities

The derived results have a set of consequent inequalities known under different names, like Kantorovič, Greub-Rheinboldt, Renie, Cassels, Schweitzer etc. inequalities, see e.g. [3, 5, 7, 8, 10, 13]. We presnt some of them in our Saigo integral operator setting.

Firstly, we present Rennie–type single–input–function inequalities whose origins can be found in [10], also see [7, 8]. The resulting inequalities are of the Rennie–type, see

Theorem 4.1. Let $F \in L^1_{w_{\rm S}}[0,t]$ and $0 < m \le F(x) \le M < \infty$, $x \in [0,t]$. Then for all $\Re\{\alpha\} > 0, \beta < 1, \eta > \beta - 1$, and for all $t > 0$ we have

$$\mathscr{M}_{\rm S}(F) + mM\,\mathscr{M}_{\rm S}(1/F) \le m + M \qquad \Re(t) > 0. \tag{4.1}$$

Moreover, for all $\Re\{\alpha\} > 0, \beta < 1;\ \eta, \xi \in \mathbb{R}$ such that $\min\{\eta, \xi\} > \beta - 1$, and for all $t > 0$ we have

$$\mathscr{M}_{\rm S}^{\eta}(F) + mM\,\mathscr{M}_{\rm S}^{\xi}(1/F) \le m + M\mathscr{M}_{\rm S}^{\eta}(F)\mathscr{M}_{\rm S}^{\xi}(1/F) \qquad \Re(t) > 0. \tag{4.2}$$

Proof. As F obeys $0 < m \le F(x) \le M < \infty$, we can write

$$\left(\frac{M}{F(tx)} - 1\right)(F(tx) - m) \ge 0.$$

The resulting expression

$$F(tx) + mM\,\frac{1}{F(tx)} \le m + M$$

multiplied by the Saigo's hypergeometric kernel $K_\eta(x)$ and integrated with respect to $x \in [0,1]$ gives via the normalization by $I_{0,t}^{\alpha,\beta,\eta}[1]$ the inequality (4.1).

Moreover, considering

$$\left(\frac{M}{F(ty)} - 1\right)(F(tx) - m) \ge 0,$$

we get

$$F(tx) + mM\,\frac{1}{F(ty)} \le m + M\,\frac{F(tx)}{F(ty)}.$$

The desired inequality (4.2) we obtain by multiplying the last display with $K_\eta(x)\,K_\xi(y)$, then integrating it on $(x,y) \in [0,1]^2$ and normalizing by $I_{0,t}^{\alpha,\beta,\eta}[1]\,I_{0,t}^{\alpha,\beta,\xi}[1]$. Q.E.D.

Finally, applying the Arithmetic mean–Geometric mean inequality to the left-hand side expression in (4.1), we easily complete the proof of the so–called *Schweitzer–type inequality*; the origins of which have been reported in [13], consult also [7, 8].

Theorem 4.2. Let $F \in L^1_{w_{\rm S}}[0,t]$ and $0 < m \le F(x) \le M < \infty$, $x \in [0,t]$. Then for all $\Re\{\alpha\} > 0, \beta < 1, \eta > \beta - 1$, and for all $t > 0$ we have

$$\mathscr{M}_{\rm S}(F)\,\mathscr{M}_{\rm S}(1/F) \le \frac{1}{4}\left(\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right)^2 \qquad \Re(t) > 0.$$

Remark 4.3. Let us remark that the same kind of procedure applied to the left–hand–side terms in (4.2) yields

$$\left[\sqrt{M}\,\big(\mathscr{M}_{\rm S}(F)\,\mathscr{M}_{\rm S}(1/F)\big)^{1/2} - \sqrt{m}\right]^2 \ge 0,$$

which is obviously not of any further interest.

Also it is worth to mention that a comprehensive equality analysis for Diaz–Metcalf inequality can be found in [2, 3], while further equivalence results have been established between Rennie's and the celebrated Diaz–Metcalf inequalities in [2].

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